Structures related to parallelizations

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Abstract

A classical theorem of Kervaire states that products of spheres are parallelizable if and only if at least one of the factors has odd dimension. The versor field given by the complex multiplication on the odd-dimensional factor gives explicit parallelizations $\mathcal{B}$ (if the odd dimensions is $n = 1, 3$), $\mathcal{P}$ (for any odd n). In this paper we use $\mathcal{B}$, $\mathcal{P}$ to obtain orthogonal and symmetric orbits of $\mathcal{B}$, $\mathcal{P}$-invariant G-structures on $S^m \times S^n$, where $G = \mathrm{U}((m+n)/2)$, $\mathrm{Sp}((m+n)/4)$, G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$. This approach leads to an alternative description of classical structures as well as to new structures on products of spheres.

1 Introduction

In the classical paper [Ber55], M. Berger showed that the holonomy group of a not locally symmetric Riemannian manifold must act transitively on a sphere. Together with the isomorphisms $\mathrm{G}_2/\mathrm{SU}(3) \simeq S^6$, $\mathrm{Spin}(7)/\mathrm{G}_2 \simeq S^7$, this theorem gave rise to the problem, recently solved by D. Joyce, of finding examples of compact manifolds with holonomy G_2 and $\mathrm{Spin}(7)$ (see [Joy00], [Joy02]).

From a general point of view, given any Riemannian manifold M^d and a Lie group G that is the stabilizer of some tensor η on $\mathbb{R}^d$, that is, $G = \{g \in \mathrm{SO}(d) : g\eta = \eta\}$, a G-structure on M defines a global tensor η on M, and it can be shown that $\nabla\eta$ (the so-called *intrinsic torsion of the G-structure*) is a section of the vector bundle $\mathcal{W} \stackrel{\mathrm{def}}{=} T^* \otimes \mathfrak{g}^\perp$, where $\mathfrak{so}(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp$. The action of G splits $\mathcal{W}$ into irreducible components, say $\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_k$. G-structures on M can then be classified in at most 2^k classes, each class being

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given by the G-structures on M whose intrinsic torsion lifts to some subspace $\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l}$ of $\mathcal{W}$:

$$\begin{tikzcd}
\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l} \arrow[r, hook] & \mathcal{W} \arrow[d, shift right] \\
 & M \arrow[u, bend left, shift right, "\nabla\Phi"] \arrow[ul, dotted, bend left]
\end{tikzcd}$$

In this framework, the holonomy condition is the most restrictive, since M has holonomy group contained in G if and only if its intrinsic torsion is zero.

A. Gray and L. Hervella in [GH80] have considered the case $G = \mathrm{U}(n)$, that is, almost Hermitian structures. The space $\mathcal{W}$ splits in this case into four $\mathrm{U}(n)$-irreducible components, that give rise to exactly sixteen classes of almost Hermitian manifolds. Afterwards, M. Fernandez and Gray in [FG82] have treated the case $G = \mathrm{G}_2$, and Fernandez in [Fer86] the case $G = \mathrm{Spin}(7)$. In the former case, the G_2-irreducible components of $\mathcal{W}$ are four, giving rise to at most sixteen classes of G_2-manifolds, of which only nine was shown in [FG82] to be distinct; in the latter case, the $\mathrm{Spin}(7)$-irreducible components of $\mathcal{W}$ are two, giving rise to exactly four classes of $\mathrm{Spin}(7)$-manifolds. F. Cabrera (see [Cab95a] and [Cab96]) completed and refined the G_2 and $\mathrm{Spin}(7)$ classification: in particular, he showed that there are exactly fifteen distinct classes in the G_2 case (for connected manifold), and using the fact that the intrinsic torsion depends only on $d\eta$ and $d*\eta$, for the G_2 case, and only on $d\eta$, for the $\mathrm{Spin}(7)$ case (see [Sal89]), he gave an alternative characterization of each class. For instance, a G_2-structure belongs to the class $\mathcal{W}_4$ if and only if there exists a closed 1-form τ such that $d\eta = 3\tau \wedge \eta$ and $d*\eta = 4\tau \wedge \eta$; a $\mathrm{Spin}(7)$-structure belongs to the class $\mathcal{W}_2$ if and only if there exists a closed 1-form τ such that $d\eta = \tau \wedge \eta$ (these are the *locally conformal parallel structures*).

The following table (compare with [Sal00]) summarizes the situation (the weird G_2 and $\mathrm{Spin}(7)$ forms depend on the choice of the representation of G_2 and $\mathrm{Spin}(7)$ on $\mathbb{R}^7$ and $\mathbb{R}^8$ respectively):

d	Φ	G	k	# of classes
$2n$	Kähler form	$\mathrm{U}(n)$	4	16
7	locally: $\sum_{i\in\mathbb{Z}_7} e^{i,i+1,i+3}$	G_2	4	15
8	locally: $\lambda \wedge \sum_{i\in\mathbb{Z}_7} e^{i,i+1,i+3} - \sum_{i\in\mathbb{Z}_7} e^{i,i+2,i+3,i+4}$	$\mathrm{Spin}(7)$	2	4

At first, also $\mathrm{Spin}(9)$ appeared in Berger's list; but D. Alekseevskij proved (see [Ale68]) that any complete 16-dimensional Riemannian manifold with holonomy group contained in $\mathrm{Spin}(9)$ is either flat or isometric to the Cayley plane

$F_4/\mathrm{Spin}(9)$ or its noncompact dual. The study of Spin(9)-structures has been then neglected until December 1999, when T. Friedrich in [Fri01] pointed out that this is one of the three cases in which there is a notion of *weak holonomy* different from the classical notion of holonomy, the other two being $\mathrm{U}(n)$ and G_2. He started then to study such weak holonomy structures, developing a classification of Spin(9)-structures on sixteen-dimensional manifolds. This classification starts from the remark that the intrinsic torsion of a Spin(9)-structure can be replaced by a 1-form Γ taking values in $\Lambda^3(V^9)$, for a suitable defined vector bundle V^9 locally spanned by 9 auto-adjoint, anti-commuting real structures. The key point is that with this replacement one does not lose any information about the geometric type of the original Spin(9)-structure. The same point of view could be used to study G_2 and Spin(7)-structures, but it is especially useful for Spin(9)-structures, since the definition of the Spin(9)-invariant 8-form given in [BG72] is not easy to handle.

Since a product of spheres is parallelizable whenever one of its factors has odd dimension, it can in that case be equipped of any G-structure compatible with the dimension, and the properties of this structures depend on the choice of the parallelization.

In this paper the parallelizations $\mathcal{B}$ and $\mathcal{P}$ on $S^m \times S^n$, odd n, defined in [Par01a], are used to obtain G-structures where $G = \mathrm{U}((m+n)/2)$, if both the dimensions are odd, $G = \mathrm{Sp}((m+n)/4)$, if both the dimensions are odd and $m+n = 0 \mod 4$, $G = \mathrm{G}_2$, Spin(7), Spin(9) on the 7-dimensional, 8-dimensional, 16-dimensional products $S^m \times S^n$ respectively. Their orbits by the symmetric group $\mathfrak{S}_{m+n}$ and by the orthogonal group $\mathrm{O}(m+n)$ are then studied.

These results were announced in [Par01b].

Theorems 26, 28 show that isotopic almost complex structures are not necessarily isomorphic.

The symmetric orbits $\mathfrak{S}_{m+3}(I_\mathcal{B}, J_\mathcal{B}, K_\mathcal{B})$ on $S^m \times S^3$ and $\mathfrak{S}_{m+n}(I_\mathcal{P}, J_\mathcal{P}, K_\mathcal{P})$ on $S^m \times S^n$, $m+n = 0 \mod 4$, provide examples of non-integrable hyperhermitian structures (corollary 29), and the symmetric orbits $\mathfrak{S}_7\varphi_\mathcal{B}$, $\mathfrak{S}_8\phi_\mathcal{B}$ on $S^4 \times S^3$, $S^5 \times S^3$ respectively, $\mathfrak{S}_7\varphi_\mathcal{P}$, $\mathfrak{S}_8\phi_\mathcal{P}$ on $S^m \times S^n$, (n odd) $m+n = 7, 8$ respectively, provide examples of G_2 and Spin(7)-structures of general type (theorem 30).

The following theorems were conjectured using experimental data obtained by a computer calculation, and then proved by classical arguments: 3, 16, 26, 28, 34.

Since $\#(\mathfrak{S}_7 \cap \mathrm{G}_2) = 21$ and $\#(\mathfrak{S}_8 \cap \mathrm{Spin}(7)) = 168$, the symmetric orbits $\mathfrak{S}_7\varphi$ and $\mathfrak{S}_8\phi$ contains both $7!/21 = 8!/168 = 240$ different structures. This remark is useful to obtain an efficient implementation of all the computation involved in theorems 17, 30.

spheres (n odd)	G	frame	orbit	see
$S^m \times S^1_{m+1\equiv_2 0}$	$\mathrm{U}(\frac{m+1}{2})$	$\mathcal{B}$	$\mathrm{O}(m+1)I_{\mathcal{B}}$	theorem 32
$S^m \times S^1_{m+1\equiv_4 0}$	$\mathrm{Sp}(\frac{m+1}{4})$	$\mathcal{B}$	$\mathrm{O}(m+1)(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$	theorem 33
$S^6 \times S^1$	G_2	$\mathcal{B}$	$\mathrm{O}(7)\varphi_{\mathcal{B}}$	theorems 11, 22, 33
$S^7 \times S^1$	$\mathrm{Spin}(7)$	$\mathcal{B}$	$\mathrm{O}(8)\phi_{\mathcal{B}}$	theorems 11, 22, 33
$S^{15} \times S^1$	$\mathrm{Spin}(9)$	$\mathcal{B}$	$\mathrm{O}(16)\Phi_{\mathcal{B}}$	theorems 23, 24, 33
$S^m \times S^3_{m+3\equiv_2 0}$	$\mathrm{U}(\frac{m+3}{2})$	$\mathcal{B}$	$\mathfrak{S}_{m+3}I_{\mathcal{B}}$	theorem 28
$S^m \times S^3_{m+3\equiv_4 0}$	$\mathrm{Sp}(\frac{m+3}{4})$	$\mathcal{B}$	$\mathfrak{S}_{m+3}(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$	corollary 29
$S^4 \times S^3$	G_2	$\mathcal{B}$	$\mathfrak{S}_7\varphi_{\mathcal{B}}$	theorems 17, 30
$S^5 \times S^3$	$\mathrm{Spin}(7)$	$\mathcal{B}$	$\mathfrak{S}_8\phi_{\mathcal{B}}$	theorems 17, 30
$S^m \times S^n_{m+n\equiv_2 0}$	$\mathrm{U}(\frac{m+n}{2})$	$\mathcal{P}$	$\mathfrak{S}_{m+n}I_{\mathcal{P}}$	theorems 3, 26, 27
$S^m \times S^n_{m+n\equiv_4 0}$	$\mathrm{Sp}(\frac{m+n}{4})$	$\mathcal{P}$	$\mathfrak{S}_{m+n}(I_{\mathcal{P}}, J_{\mathcal{P}}, K_{\mathcal{P}})$	corollary 29
$S^m \times S^n_{m+n=7}$	G_2	$\mathcal{P}$	$\mathfrak{S}_7\varphi_{\mathcal{P}}$	theorems 17, 30
$S^m \times S^n_{m+n=8}$	$\mathrm{Spin}(7)$	$\mathcal{P}$	$\mathfrak{S}_8\phi_{\mathcal{P}}$	theorems 17, 30
$S^6 \times S^1$	G_2	$\mathcal{P}$	$\mathrm{O}(7)\varphi_{\mathcal{P}}$	theorems 16, 34

An index of the results in the paper.

2 Preliminaries

Recall the definitions of the frames $\mathcal{B}$ and $\mathcal{P}$ on $S^m \times S^n$, odd n (see [Par01a] for more details). Denote by $x = (x_i)$, $y = (y_j)$ the coordinates on $\mathbb{R}^{m+1}$, $\mathbb{R}^{n+1}$ and let S^m, S^n be the unit spheres in $\mathbb{R}^{m+1}$, $\mathbb{R}^{n+1}$ respectively. Look first to the cases $n = 1, 3$. The frame $\{|x|\partial_{x_i}\}_{i=1,\dots,m+1}$ on $\mathbb{R}^{m+1} \smallsetminus 0$ is projectable for the universal covering map $\mathbb{R}^{m+1} \smallsetminus 0 \to S^m \times S^1$ given by $p(x) = (x/|x|, \log|x| \bmod 2\pi)$, hence it defines a parallelization $\mathcal{B}$ on $S^m \times S^1$. The Hopf fibration of S^3 then gives a frame on $S^m \times S^3$, that is still denoted by $\mathcal{B}$.

In the general case the complex multiplication in $\mathbb{C}^{(n+1)/2} = \mathbb{R}^{n+1}$ induces a tangent versor field $T \stackrel{\text{def}}{=} \sum_{j=1}^{n+1} t_j \partial_{y_j}$ on S^n. Also, let $\{M_i\}_{i=1,\dots,m+1}$ and

$\{N_j\}_{j=1,\dots,n+1}$ be the meridian vector fields on S^m and S^n respectively, i. e.

$$M_i \overset{\text{def}}{=} \text{orthogonal projection of } \partial_{x_i} \text{ on } S^m \qquad i = 1, \dots, m+1,$$
$$N_j \overset{\text{def}}{=} \text{orthogonal projection of } \partial_{y_j} \text{ on } S^n \qquad j = 1, \dots, n+1.$$

Denote by $\mathcal{P}$ the parallelization on $S^m \times S^n$ given by the vector fields

$$\begin{aligned} p_i &\overset{\text{def}}{=} M_i + x_i T \qquad i = 1, \dots, m-1, \\ p_{m-1+j} &\overset{\text{def}}{=} y_j M_m + t_j M_{m+1} + (t_j x_{m+1} + y_j x_m - t_j) T + N_j \qquad j = 1, \dots, n+1. \end{aligned} \tag{1}$$

The frames $\mathcal{B}$ and $\mathcal{P}$ (the first defined only for $n = 1, 3$) are orthonormal with respect to the product metric on $S^m \times S^n$. Since we are going to consider only subgroups of the orthogonal group, then all the G-structures we look at in this paper are compatible with this metric.

The following table gives the relation between $\mathcal{P}$ and $\mathcal{B}$ when $n = 1, 3$:

$n = 1$	$n = 3$
$\mathcal{P} = \mathcal{B} \left(\begin{array}{ccc\|cc} & & & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & y_1 & y_2 \\ 0 & \cdots & 0 & -y_2 & y_1 \end{array} \right)$	$\mathcal{P} = \mathcal{B} \left(\begin{array}{ccc\|cccc} & & & 0 & 0 & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & \cdots & 0 & -y_2 & y_1 & -y_4 & y_3 \\ 0 & \cdots & 0 & -y_3 & y_4 & y_1 & -y_2 \\ 0 & \cdots & 0 & -y_4 & -y_3 & y_2 & y_1 \end{array} \right)$

(2)

The following are the structure equations, where τ is given by

$$\tau = \sum_{i=1}^{m+1} x_i b^i = \sum_{i=1}^{m-1} x_i p^i + (x_m y_1 - x_{m+1} y_2) p^m + (x_m y_2 + x_{m+1} y_1) p^{m+1}.$$

Note that the general formulas for $\mathcal{P}$, $n \geq 3$, are really complicated also to write down, and are therefore put at the end of the paper.

	$n=1$	$n=3$
$\mathcal{B}$	$db^i = \tau \wedge b^i, i=1,\dots,m+1$	$db^i = b^i \wedge \tau + 2x_i b^{m+2} \wedge b^{m+3}, i=1,\dots,m+1$ $db^{m+2} = 2b^{m+3} \wedge \tau$ $db^{m+3} = -2b^{m+2} \wedge \tau$
$\mathcal{P}$	$dp^i = p^i \wedge \tau, i=1,\dots,m-1$ $dp^m = p^m \wedge \tau + p^{m+1} \wedge \tau$ $dp^{m+1} = p^{m+1} \wedge \tau - p^m \wedge \tau$	see the general formula (A.1)

(3)

3 Almost Hermitian and hyperhermitian structures

Let m, n be odd. Let $\mathcal{C} \stackrel{\text{def}}{=} \{c_1, \dots, c_{m+n}\}$ be an *ordered* orthonormal basis of an Euclidean vector space V^{m+n}. The *Hermitian structure* $I_\mathcal{C}$ *on* V *associated to* $\mathcal{C}$ is given by $I_\mathcal{C}(c_{2i-1}) \stackrel{\text{def}}{=} c_{2i}$, $i = 1, \dots, (m+n)/2$. One thus obtain almost Hermitian structures $I_\mathcal{B}$ and $I_\mathcal{P}$ on $S^m \times S^n$, the former only defined when $n = 1, 3$.

The same way, if m, n are odd and $m + n = 0 \mod 4$, there are besides $I_\mathcal{C}$ the Hermitian structures $J_\mathcal{C}$, $K_\mathcal{C}$ given by

$$\begin{aligned} J_\mathcal{C}(c_{4i-3}) &\stackrel{\text{def}}{=} c_{4i-1} & K_\mathcal{C}(c_{4i-3}) &\stackrel{\text{def}}{=} c_{4i} \\ J_\mathcal{C}(c_{4i-2}) &\stackrel{\text{def}}{=} -c_{4i} & K_\mathcal{C}(c_{4i-2}) &\stackrel{\text{def}}{=} c_{4i-1} \end{aligned} \qquad i = 1, \dots, (m+n)/4.$$

The identity $I_\mathcal{C} J_\mathcal{C} = -J_\mathcal{C} I_\mathcal{C}$ shows that $(I_\mathcal{C}, J_\mathcal{C}, K_\mathcal{C})$ is a hyperhermitian structure on V, that is referred to as *the hyperhermitian structure associated to* $\mathcal{C}$. One thus obtain almost hyperhermitian structures $(I_\mathcal{B}, J_\mathcal{B}, K_\mathcal{B})$ and $(I_\mathcal{P}, J_\mathcal{P}, K_\mathcal{P})$ on $S^m \times S^n$, the former only defined when $n = 1, 3$.

We shall often forget to state "[...]where m, n are odd and $m + n = 0 \mod 4$[...]" and other similar conditions. Thus we here agree that whenever we write a statement about a product of two spheres then: (1) the right sphere has odd dimension, also if not specified; (2) all conditions necessary to make sensate the statement hold. For instance, the statement in proposition 4 "The almost hyperhermitian structures [...] on $S^m \times S^3$ [...]" become "Let $m+3 = 0 \mod 4$. The almost hyperhermitian structures [...] on $S^m \times S^3$ [...]".

The definition of $\mathcal{B}$ on $S^m \times S^1$ gives the following remark:

Remark 1 *The almost Hermitian structure $I_{\mathcal{B}}$ and the almost hyperhermitian structure $(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ on $S^m \times S^1$ coincide respectively with the Hopf Hermitian structure $I_{e^{2\pi}}$ and with the Hopf hyperhermitian structure $(I_{e^{2\pi}}, J_{e^{2\pi}}, K_{e^{2\pi}})$ induced by the multiplication by $e^{2\pi}$.*

Since the left-side matrix in formula (2) belongs to $\mathrm{U}((m+1)/2)$, but not to $\mathrm{Sp}((m+1)/4)$, the following remark does not hold in the almost hyperhermitian case:

Remark 2 *The almost Hermitian structure $I_{\mathcal{P}}$ on $S^m \times S^1$ coincide with the Hopf Hermitian structure $I_{e^{2\pi}}$ induced by the multiplication by $e^{2\pi}$.*

On each product $S^m \times S^n$ of two odd-dimensional spheres is defined a family of *Calabi–Eckmann complex structures*, parametrized by the moduli space of the torus $S^1 \times S^1$ (see [CE53]). The Calabi–Eckmann complex structure on $S^m \times S^n$ given by the non-real complex number τ is defined as follows: denote by S, T the versor field given by the complex multiplication on S^m, S^n respectively, and remark that the complex Hopf fibration induces a complex structure on their orthogonal complement (with respect to the product metric); then map S into $\mathfrak{Re}\,\tau S + \mathfrak{Im}\,\tau T$.

Only $\tau = \pm i$ gives thus Calabi–Eckmann Hermitian structures: here and henceforth, *denote by $I^{m,n}$ the Calabi–Eckmann Hermitian structure on $S^m \times S^n$ given by $\tau = -i$*. Therefore, $I^{m,n}(T) = S$. It is well-known that Calabi–Eckmann complex structures are a generalization of Hopf complex structures: in particular, using our notation, $I^{m,1} = I_{e^{2\pi}}$.

The following theorem was already proved in [Par01b]:

Theorem 3 *Let m, $n \geq 1$ be odd. Then the Calabi–Eckmann Hermitian structure $I^{m,n}$ on $S^m \times S^n$ coincide with the almost Hermitian structure $I_{\mathcal{P}}$ on $S^m \times S^n$ associated to $\mathcal{P}$.*

Using the previous theorem and the fact that the right-side matrix in formula (2) belongs to $\mathrm{U}((m+3)/2)$ and also to $\mathrm{Sp}((m+3)/4)$, one obtains:

Proposition 4 *The almost Hermitian structure $I_{\mathcal{B}}$ on $S^m \times S^3$ coincide with the Calabi–Eckmann Hermitian structure $I^{m,3}$. The almost hyperhermitian structures $(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ and $(I_{\mathcal{P}}, J_{\mathcal{P}}, K_{\mathcal{P}})$ on $S^m \times S^3$ coincide.*

The following theorem is a particular case of corollary 29 and is stated here for completeness:

Theorem 5 *On $S^m \times S^n$, for m, n odd and $m + n = 0 \mod 4$, the almost hyperhermitian structure $(I_{\mathcal{P}}, J_{\mathcal{P}}, K_{\mathcal{P}})$ is non-integrable.*

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Fig. 1. Multiplication table of the Cayley numbers.

4 Algebraic preliminaries: structures related to the octonions

Call $\{e_1, \dots, e_7\}$ the standard basis of $\mathbb{R}^7$, and $\{e^1, \dots, e^7\}$ the corresponding dual basis. Let $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$. Then the standard basis of $\mathbb{R}^8$ is $\{1, e_1, \dots, e_7\}$. Call $\{\lambda, e^1, \dots, e^7\}$ the corresponding dual basis, with an obvious misuse of notation. Let $\mathbb{O}$ be the non-associative normed algebra of Cayley numbers, that is, $\mathbb{R}^8$ equipped with the standard scalar product $\langle \cdot, \cdot \rangle$, and with multiplicative structure defined by (see [BG72]) $e_i^2 = -1$, $e_i e_j = -e_j e_i$ and $e_i e_{i+1} = e_{i+3}$ for all cyclic permutation of $\{i, i+1, i+3\}$, where indices run in $\mathbb{Z}_7$.

The multiplication table of $\mathbb{O}$ is given in figure 1.

Remark 6 *The standard quaternion subalgebra* $\mathbb{H}$ *of* $\mathbb{O}$ *is generated by* 1, e_1, e_2, e_4. *This choice is made (following for instance [BG72], [Gra77], [Mar81a], [FG82] or [Cab97]) in order to have a simpler definition of the forms associated to the* G_2 *and* $\mathrm{Spin}(7)$*-structures, to be considered on our products of spheres. An orthonormal basis* $\{1, e_1, \dots, e_7\}$ *of* $\mathbb{O}$ *satisfying the previous relations is called in many different ways: a* Cayley basis *(see [FG82], [Cab95a], [Cab95b], [Cab96] or [Cab97]), or also an* adapted basis *(see [Mar81a] or [Mar81b]) or again a* canonical basis *(see [BG72]). Last, some authors use the more classical (though more asymmetric) multiplication table given by choosing* $\{1, i, j, k, e, ie, je, ke\}$ *in place of* $\{1, e_1, \dots, e_7\}$ *(see [Mur89], [Mur92], [CMS96] or [FKMS97]).*

The 3-form φ on $\mathbb{R}^7 = \mathfrak{Im}(\mathbb{O})$ defined by $\varphi(x, y, z) \stackrel{\text{def}}{=} \langle x, yz \rangle$ can be computed using the table in figure 1:

$$\varphi = \sum_{i \in \mathbb{Z}_7} e^{i, i+1, i+3}. \tag{4}$$

Remark that G_2 is the stabilizer in $\mathrm{GL}(8)$ of φ. Thus, if $\mathcal{C}$ is any *ordered* orthonormal basis on an Euclidean vector space V of dimension 7, the above equation defines a G_2-*structure* $\varphi_{\mathcal{C}}$ *on* V *associated to* $\mathcal{C}$.

Denote by $*$ the Hodge star operator on $(\mathbb{R}^8, \langle\cdot,\cdot\rangle)$, where the positive orientation is given by $\{1, e_1, \ldots, e_7\}$. Since $\{1, e_1, \ldots, e_7\}$ is orthonormal, one obtains

$$*(\lambda \wedge \varphi) = -\sum_{i \in \mathbb{Z}_7} e^{i+2,i+4,i+5,i+6} = -\sum_{i \in \mathbb{Z}_7} e^{i,i+2,i+3,i+4}. \tag{5}$$

Define the 4-form ϕ on $\mathbb{R}^8$ by $\phi \stackrel{\text{def}}{=} \lambda \wedge \varphi + *(\lambda \wedge \varphi)$. Using (4) and (5) one gets

$$\phi = \lambda \wedge \sum_{i \in \mathbb{Z}_7} e^{i,i+1,i+3} - \sum_{i \in \mathbb{Z}_7} e^{i,i+2,i+3,i+4}. \tag{6}$$

The Lie group Spin(7) is the stabilizer of ϕ, thus if $\mathcal{C}$ is any *ordered* orthonormal basis on an Euclidean vector space V of dimension 8, the above equation defines a Spin(7)-*structure* $\phi_{\mathcal{C}}$ *on* V *associated to* $\mathcal{C}$.

5 G_2 and Spin(7)-structures

A G_2-structure on a seven-dimensional manifold M is a reduction of the structure group GL(7) to G_2. Since $\mathrm{G}_2 \subset \mathrm{SO}(7)$, a G_2-structure induces a metric.

Since G_2 is the stabilizer of φ, a G_2-structure gives a canonical identification of each tangent space with $\mathbb{R}^7$, in such a way that the local 3-form defined by (4) is actually global. Vice versa, if there exists on M a global 3-form that can be locally written as in (4), then M admits a G_2-structure. Hence, the G_2-structure is often identified with the 3-form.

Analogous statements hold for Spin(7)-structures: a Spin(7)-structure on an eight-dimensional manifold M is a reduction of the structure group GL(8) to $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Since Spin(7) is the stabilizer of ϕ, a Spin(7)-structure on M is often identified with a global 4-form that can be locally written as ϕ.

Definition 7 *Let M be a seven, eight-dimensional manifold with a* G_2*,* Spin(7)*-structure respectively. Let φ be its structure differential form and ∇ the Levi–Civita connection of the metric defined by φ. The structure is then said to be:* parallel *if $\nabla\varphi = 0$;* locally conformal parallel *if φ is locally conformal to local structures φ_α, which are parallel with respect to the local Levi–Civita connections they define.*

A G_2-structure is parallel if and only if $d\varphi = d{*}\varphi = 0$, and a Spin(7)-structure is parallel if and only if $d\phi = 0$ ([Sal89]). This facts can be used to characterize locally conformal parallel G_2 and Spin(7)-structures:

Theorem 8 *A* G_2*-structure φ on M^7 is locally conformal parallel if and only if there exists a closed $\tau \in \Omega^1(M)$ such that $d\varphi = 3\tau \wedge \varphi$, $d * \varphi = 4\tau \wedge *\varphi$.*

A Spin(7)*-structure* ϕ *on* M^8 *is locally conformal parallel if and only if there exists a closed* $\tau \in \Omega^1(M)$ *such that* $d\phi = \tau \wedge \phi$*.*

PROOF. Let φ be a locally conformal parallel G_2-structure. Then for each $x \in M$, there exist a neighborhood U of x and a map $\sigma \colon U \to \mathbb{R}$ such that the local G_2-structure $\varphi_U \stackrel{\text{def}}{=} e^{-3\sigma}\varphi_{|_U}$ is parallel with respect to its local Levi–Civita connection. One then obtains $d\varphi_U = d *_U \varphi_U = 0$, where $*_U$ is the local Hodge star-operator associated to φ_U, and using these relations together with $e^{4\sigma}*_U = e^{3\sigma}*$, one obtains $d\varphi_{|_U} = 3d\sigma \wedge \varphi_{|_U}$, $d * \varphi_{|_U} = 4d\sigma \wedge *\varphi_{|_U}$. The closed 1-form τ locally defined by $d\sigma$ is easily seen to be global. The reverse implication is obtained the same way, once observed that since τ is closed then there exist local maps $\sigma \colon U \to \mathbb{R}$ such that $\tau_{|_U} = d\sigma$. A similar argument proves the Spin(7) case. $\square$

Remark 9 *Let* φ *be a* G_2 *or a* Spin(7)*-structure. Using the local expression of* φ*, it can be shown that* $\alpha \wedge \varphi = 0$ *if and only if* $\alpha = 0$*, for any 2-form* α *on* M*. This means that the requirement of* τ *to be closed in the previous theorem can be dropped. Moreover, one can also modify the statement: in the* G_2 *case "[...] if and only if there exist* $\alpha, \beta \in \Omega^1(M)$ *such that* $d\varphi = \alpha \wedge \varphi$*,* $d*\varphi = \beta \wedge *\varphi$*", and then prove that* $-4\alpha = -3\beta = *(*d\varphi \wedge \varphi)$*; in the* Spin(7) *case "[...] if and only if there exist* $\alpha \in \Omega^1(M)$ *such that* $d\phi = \alpha \wedge \phi$*" and then prove that* $-7\alpha = *(*d\phi \wedge \phi)$*.*

Remark 10 *A parallel* G_2 *or* Spin(7)*-structure on a compact* M *gives a non-trivial element in* 3 *or* 4*-dimensional cohomology respectively (see [Bon66]).*

Let $\varphi_{\mathcal{B}}$, $\phi_{\mathcal{B}}$ be the G_2, Spin(7)-structure on $S^6 \times S^1$, $S^7 \times S^1$ respectively associated to $\mathcal{B}$, that is,

$$\varphi_{\mathcal{B}} \stackrel{\text{def}}{=} \sum_{i \in \mathbb{Z}_7} b^{i,i+1,i+3}, \qquad \phi_{\mathcal{B}} \stackrel{\text{def}}{=} b^8 \wedge \sum_{i \in \mathbb{Z}_7} b^{i,i+1,i+3} - \sum_{i \in \mathbb{Z}_7} b^{i,i+2,i+3,i+4}.$$

Theorem 11 $\varphi_{\mathcal{B}}$ *and* $\phi_{\mathcal{B}}$ *are locally conformal parallel. The local parallel structures are induced by* φ*,* ϕ *by* $p \colon \mathbb{R}^{m+1} \smallsetminus 0 \to S^m \times S^1$*, for* $m = 6,\ 7$ *respectively.*

PROOF. The 3-form $\varphi = \sum_{i \in \mathbb{Z}_7} dx_i \wedge dx_{i+1} \wedge dx_{i+3}$ is parallel, and on $\mathbb{R}^7 \smallsetminus 0$ it is globally conformal to the p-invariant 3-form

$$\varphi' = \frac{1}{|x|^3} \sum_{i \in \mathbb{Z}_7} dx_i \wedge dx_{i+1} \wedge dx_{i+3}.$$

Observe that $\mathbb{R}^7 \smallsetminus 0$ is locally diffeomorphic to $S^6 \times S^1$, and that φ' induces $\varphi_{\mathcal{B}}$, to end the proof in the G_2 case. The Spin(7) case is similar. $\square$

Remark 12 *By remark 10, $S^6 \times S^1$ and $S^7 \times S^1$ have no parallel* G_2 *and* $\mathrm{Spin}(7)$*-structures.*

Remark 13 *Since $\mathcal{B}$ is orthonormal, the metric induced on $S^6 \times S^1$, $S^7 \times S^1$ by means of $\varphi_{\mathcal{B}}$, $\phi_{\mathcal{B}}$ is the product metric.*

The same construction can be applied to seven and eight-dimensional products of spheres equipped with the frame $\mathcal{P}$. On $S^4 \times S^3$, $S^5 \times S^3$ also the frame $\mathcal{B}$ is available. One obtains G_2 and $\mathrm{Spin}(7)$-structures of *general type*. The rest of this section is devoted to recall what G_2 and $\mathrm{Spin}(7)$-structures of general type are. We do the discussion in the G_2 case, pointing out the main differences with the $\mathrm{Spin}(7)$ case.

Look at $\nabla\varphi$ as belonging to $\Omega^1(\Lambda^3 M) = \Gamma(T^*M \otimes \Lambda^3 M)$. The G_2-structure allows one to identify each tangent space with the standard 7-dimensional orthogonal representation of G_2. The induced action of G_2 splits each fiber of $T^*M \otimes \Lambda^3 M$ into irreducible components, giving rise to a splitting of $T^*M \otimes \Lambda^3 M$, and if $\nabla\varphi$ lifts to a particular component of this splitting, one says that φ belongs to the corresponding particular class. Actually, due to special properties of φ, it can be shown that $\nabla\varphi$ lifts always to a G_2-invariant subbundle $\mathcal{W}$ of $T^*M \otimes \Lambda^3 M$:

$$\begin{tikzcd}
\mathcal{W} \arrow[r, hook] \arrow[dr] & T^*M \otimes \Lambda^3 M \arrow[d] \\
& M \arrow[ul, bend left, "\nabla\varphi"]
\end{tikzcd}$$

As a consequence, the above splitting must be done on fibers of $\mathcal{W}$. The irreducible components of $\mathcal{W}$ turn out to be four, and they are classically denoted by $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$ for the components of rank 1, 14, 27, 7 respectively. The G_2-structure φ is then said *of type* $\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l}$ if $\nabla\varphi$ lifts to $\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l}$:

$$\begin{tikzcd}
\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l} \arrow[r, hook] & \mathcal{W} \arrow[d, shift left] \\
& M \arrow[u, shift left, bend left, "\nabla\varphi"] \arrow[ul, dotted, bend left]
\end{tikzcd}$$

In the $\mathrm{Spin}(7)$ case the irreducible components are only two, of rank 48 and 8. The $\mathrm{Spin}(7)$-structure ϕ is said *of type* $\mathcal{W}_1$, $\mathcal{W}_2$ if $\nabla\phi$ lifts to $\mathcal{W}_1$, $\mathcal{W}_2$ respectively:

$$\begin{tikzcd}
\mathcal{W}_1 \arrow[r, hook] & \mathcal{W} \arrow[d, shift left] & \mathcal{W}_2 \arrow[l, hook'] \\
& M \arrow[u, shift left, bend left, "\nabla\phi"] \arrow[ul, dotted, bend left] \arrow[ur, dotted, bend right] &
\end{tikzcd}$$

For more details, see [FG82] and [Fer86].

Definition 14 *If $\nabla\varphi$ does not lift to any between $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus$*

$\mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *then the* G_2*-structure* φ *is said to be* of general type. *If* $\nabla\phi$ *does not lift to neither* $\mathcal{W}_1$ *nor* $\mathcal{W}_2$ *then the* Spin(7)-*structure* ϕ *is said to be* of general type.

In [FG82], [Fer86] the irreducible components of $\mathcal{W}$ are explicitly given, but the defining relations are rather complicated. These relations can be simplified by looking at opportune G_2 and Spin(7)-equivariant maps. This is done in [Cab95a], [Cab96]. Here is a list of the resulting simplified relations restricted to the ones that will be useful in the following:

Theorem 15 ([Cab95a], [Cab96]) *A* G_2*-structure* φ *on a manifold* M *is of type:*

- $\mathcal{W}_4$ *if and only if there exist* α, $\beta \in \Omega^1(M)$ *such that* $d\varphi = \alpha \wedge \varphi$ *and* $d * \varphi = \beta \wedge *\varphi$ *(this class is needed in section 6);*
- $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ *if and only if* $(*d\varphi) \wedge \varphi = 0$*;*
- $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ *if and only if there exist* $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$ *such that* $d\varphi = \alpha \wedge \varphi + f * \varphi$*;*
- $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if there exists* $\beta \in \Omega^1(M)$ *such that* $d*\varphi = \beta \wedge *\varphi$*;*
- $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if* $d\varphi \wedge \varphi = 0$.

A Spin(7)*-structure* ϕ *on a manifold* M *is of type:*

- $\mathcal{W}_1$ *if and only if* $(*d\phi) \wedge \phi = 0$*;*
- $\mathcal{W}_2$ *if and only if there exists* $\alpha \in \Omega^1(M)$ *such that* $d\phi = \alpha \wedge \phi$.

Therefore, to check that a given G_2 or Spin(7)-structure is of general type, one must verify that none of the above relations is satisfied.

Theorem 16 *The* G_2*-structure* $\varphi_{\mathcal{P}}$ *associated to the frame* $\mathcal{P}$ *on* $S^6 \times S^1$ *is of general type.*

PROOF. The 3-form $\varphi_{\mathcal{P}}$ and the 4-form $*\varphi_{\mathcal{P}}$ are given by

$$\varphi_{\mathcal{P}} = \sum_{i \in \mathbb{Z}_7} p^{i,i+1,i+3}, \qquad *\varphi_{\mathcal{P}} = -\sum_{i \in \mathbb{Z}_7} p^{i,i+2,i+3,i+4}.$$

Using formulas (3) for $\mathcal{P}$ and $n = 1$ one obtains

$$\begin{aligned} d\varphi_{\mathcal{P}} &= 3\varphi_{\mathcal{P}} \wedge \tau - (p^{6,1,3} + p^{4,5,6} - p^{3,4,7} - p^{5,7,1}) \wedge \tau, \\ d * \varphi_{\mathcal{P}} &= -4 * \varphi_{\mathcal{P}} \wedge \tau - (p^{7,1,3} + p^{4,5,7} + p^{3,4,6} + p^{5,6,1}) \wedge p^2 \wedge \tau. \end{aligned}$$

A computation then shows that no relation of the previous theorem is satisfied, and $\varphi_{\mathcal{P}}$ is of general type. $\square$

The same result holds for seven and eight-dimensional product of spheres equipped with the frame $\mathcal{P}$, and for $S^4 \times S^3$, $S^5 \times S^3$ with the frame $\mathcal{B}$; but computation, being based on the general formulas (A.1), is much harder than before. Therefore, the following theorem was proved by a computer calculation:

Theorem 17 *The* G_2*-structures associated to the frames* $\mathcal{B}$ *and* $\mathcal{P}$ *on* $S^4 \times S^3$ *and to the frame* $\mathcal{P}$ *on* $S^2 \times S^5$ *are of general type. The* $\mathrm{Spin}(7)$*-structures associated to the frame* $\mathcal{P}$ *on* $S^7 \times S^1$*, to the frames* $\mathcal{B}$ *and* $\mathcal{P}$ *on* $S^5 \times S^3$ *and to the frame* $\mathcal{P}$ *on* $S^3 \times S^5$ *are of general type.*

6 Relations among the structures

A unified treatment of G_2 and Spin(7)-structures can be done by means of the vector cross product notion. A beautiful reference is [Gra69].

Definition 18 *Let* $(V, \langle \cdot, \cdot \rangle)$ *be an* n*-dimensional Euclidean vector space. An* r*-linear map* $P \colon V^r \to V$ $(1 \leq r \leq n)$ *is an* r-fold vector cross product *on* V *if* $P(v_1, \ldots, v_r)$ *is orthogonal to* $v_1, \ldots, v_r$ *and* $\langle P(v_1, \ldots, v_r), P(v_1, \ldots, v_r) \rangle = \det(\langle v_i, v_j \rangle)$.

In [BG67] vector cross products together with their automorphism groups are classified. Two of the four classes are strictly related to G_2 and Spin(7)-structures:

Proposition 19 ([Gra69, Proposition 2.2]) *A manifold* M *has a vector cross product of class (III), (IV) if and only if it has a* G_2, $\mathrm{Spin}(7)$*-structure respectively.*

In what follows we need the following corollary of theorems 8, 15:

Corollary 20 *A* G_2*-structure* φ *is of type* $\mathcal{W}_4$ *if and only if* φ *is locally conformal parallel. A* $\mathrm{Spin}(7)$*-structure* ϕ *is of type* $\mathcal{W}_2$ *if and only if* ϕ *is locally conformal parallel.*

The following theorem defines new vector cross products from old:

Theorem 21 ([Gra69, Theorem 2.6]) *Let* M *be an oriented hypersurface of* $\bar{M}$*, and let* N *be its unit normal vector field. Let* $\bar{P}$ *a differentiable* $(r+1)$*-fold vector cross product on* $\bar{M}$*. Then the map* P *given by* $P(X_1, \ldots, X_r) \stackrel{\text{def}}{=} \bar{P}(N, X_1, \ldots, X_r)$ *where* $X_1, \ldots, X_r \in \mathfrak{X}(M)$*, defines a vector cross product on* M*.*

Let us now describe how all these pieces collect together to give a Spin(7)-structure on $S^7 \times S^1$ and a G_2-structure on $S^6 \times S^1$. Firstly, use previous theorem with the Spin(7)-structure ϕ on $\mathbb{R}^8$ to obtain a G_2-structure φ_{S^7} on

S^7, that is described in [FG82]. Then the 4-form $\phi_{S^7\times S^1}$ on $S^7 \times S^1$ defined by $\phi_{S^7\times S^1} \stackrel{\text{def}}{=} d\theta \wedge \varphi_{S^7} + *\varphi_{S^7}$ is a locally conformal parallel Spin(7)-structure. (see [Cab95a] and use corollary 20). Using theorem 21 and proposition 19 one obtains a G_2-structure $\varphi_{S^6\times S^1}$ on $S^6 \times S^1 \subset S^7 \times S^1$, which turns out to be locally conformal parallel ([Cab97, Theorem 4.4] and corollary 20).

We now prove that $\varphi_{S^6\times S^1}$ and $\phi_{S^7\times S^1}$ are structures associated to $\mathcal{B}$:

Theorem 22 *The locally conformal parallel structures $\phi_{S^7\times S^1}$, $\varphi_{S^6\times S^1}$ coincide with $\phi_{\mathcal{B}}$, $\varphi_{\mathcal{B}}$ respectively.*

PROOF. Since $p\colon \mathbb{R}^8 \setminus 0 \to S^7 \times S^1$ is a local diffeomorphism and $p^*(\phi_{\mathcal{B}}) = |x|^{-4}\phi$, we are left to prove $p^*(\phi_{S^7\times S^1}) = |x|^{-4}\phi$. Consider the versor field $N \stackrel{\text{def}}{=} |x|^{-1}(x_1\partial_{x_1} + \cdots + x_8\partial_{x_8})$ on $\mathbb{R}^8 \setminus 0$ and its dual 1-form $n \in \Omega^1(\mathbb{R}^8 \setminus 0)$. Then a straightforward computation gives $p^*(\phi_{S^7\times S^1}) = |x|^{-4}(n \wedge i_N\phi + *(n \wedge i_N\phi))$, and since the action of Spin(7) on S^7 is transitive, one obtains $n \wedge i_N\phi + *(n \wedge i_N\phi) = \phi$. This completes the proof of the statement about Spin(7). To complete the proof, choose the embedding $S^6 \times S^1 \subset S^7 \times S^1$ given by $x_8 = 0$. The normal vector field is then $\partial_{x_8} = b_8$, and one obtains $\varphi_{S^6\times S^1} = i_{\partial_{x_8}}\phi_{S^7\times S^1} = i_{b_8}\phi_{\mathcal{B}} = \varphi_{\mathcal{B}}$. $\square$

7 Spin(9)-structures

A Spin(9)-structure on a sixteen-dimensional manifold M is a reduction of the structure group of M to $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$.

In [BG72] it is shown that Spin(9) is the stabilizer of a Spin(9)-invariant 8-form $\Phi \in \Lambda^8(\mathbb{R}^{16})$. This allows one to think a Spin(9)-structure on M^{16} as a global 8-form that can be locally written as Φ. In particular, on any parallelizable M^{16}, an explicit parallelization gives such a global 8-form. Therefore one can define Spin(9)-structures on $S^{15} \times S^1$, $S^{13} \times S^3$, $S^{11} \times S^5$, $S^9 \times S^7$, $S^7 \times S^9$, $S^5 \times S^{11}$, $S^3 \times S^{13}$, $S^1 \times S^{15}$ associated to the frames $\mathcal{P}$, $\mathcal{B}$ and denoted by $\Phi_{\mathcal{P}}$, $\Phi_{\mathcal{B}}$.

Theorem 23 *The* Spin(9)*-structure on $S^{15} \times S^1$ given by $\Phi_{\mathcal{B}}$ is locally conformal parallel. The local parallel* Spin(9)*-structures are induced by Φ by $p : \mathbb{R}^{16} \setminus 0 \to S^{15} \times S^1$.*

PROOF. It follows by the fact that $|x|^{-8}\Phi$ is p-invariant, globally conformal to Φ, and induces $\Phi_{\mathcal{B}}$. $\square$

Unfortunately, the 8-form Φ is not easy to handle, and this is probably one of the reasons why a Gray–Hervella-like classification of Spin(9)-structures was lacking until [Fri01].

In what follows, the construction given in [Fri01] is briefly described. Let $\mathcal{R}$ be a Spin(9)-structure on a 16-dimensional Riemannian manifold M^{16}, and denote by $\mathcal{F}(M)$ the principal orthonormal frame bundle. Then $\mathcal{R}$ is a subbundle of $\mathcal{F}(M)$:

$$\begin{tikzcd} \mathcal{R} \arrow[rr, hook] \arrow[dr] & & \mathcal{F}(M) \arrow[dl] \\ & M & \end{tikzcd}$$

The Levi–Civita connection $Z : T(\mathcal{F}(M)) \to \mathfrak{so}(16) = \mathfrak{spin}(9) \oplus \mathfrak{spin}(9)^{\perp}$ restricted to $T(\mathcal{R})$ decomposes into $Z^* \oplus \Gamma$, where Z^* is a connection in the principal Spin(9)-fibre bundle $\mathcal{R}$, and $\Gamma \in \Omega^1(\mathcal{R} \times_{\mathrm{Spin}(9)} \mathfrak{spin}(9)^{\perp}) = \Omega^1(\Lambda^3(V))$, where $V = V^9 \stackrel{\text{def}}{=} \mathcal{R} \times_{\mathrm{Spin}(9)} \mathbb{R}^9$. The irreducible components of $\Lambda^1(M) \otimes \Lambda^3(V)$ are described in [Fri01]. In particular, one component is the 16-dimensional representation $\Lambda^1(M)$, that defines the *nearly parallel* Spin(9)-*structures*. The action of Spin(9) on S^{15} is transitive, with isotropy subgroup Spin(7), and this allows to define the principal Spin(7)-fibre bundle $\mathcal{R}_{S^{15}\times S^1}$ $\mathrm{Spin}(9) \times S^1 \to S^{15} \times S^1$, that in [Fri01] is shown to be actually a nearly parallel $\mathrm{Spin}(7) \subset \mathrm{Spin}(9)$-structure on $S^{15} \times S^1$.

Theorem 24 *The nearly parallel* Spin(9)*-structure* $\mathcal{R}_{S^{15}\times S^1}$ *and the locally conformal parallel* Spin(9)*-structure* $\Phi_{\mathcal{B}}$ *on* $S^{15} \times S^1$ *are the same.*

PROOF. Consider the following diagram of $\mathrm{Spin}(7) \subset \mathrm{Spin}(9)$-structures:

$$\begin{tikzcd} & \mathrm{Spin}(9) \times \mathbb{R}^+ \arrow[r] \arrow[d, "\mathcal{R}'"'] & \mathrm{Spin}(9) \times S^1 \arrow[d, "\mathcal{R}"] \\ \mathbb{R}^{16} \setminus 0 \arrow[r, "\alpha"] & \dfrac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)} \times \mathbb{R}^+ \arrow[r, "\beta"] & \dfrac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)} \times S^1 \end{tikzcd}$$

where $\alpha(x) = (x/|x|, |x|)$ and $\beta([g], \rho) = ([g], \log \rho \mod 2\pi)$. Then $\beta \circ \alpha = p : \mathbb{R}^{16} \setminus 0 \to S^{15} \times S^1$ and the map $\alpha^{-1} \circ \mathcal{R}' : \mathrm{Spin}(9) \times \mathbb{R}^+ \to \mathbb{R}^{16} \setminus 0$ is a $\mathrm{Spin}(7) \subset \mathrm{Spin}(9)$-structure on $\mathbb{R}^{16} \setminus 0$. The pull-back $(\beta \circ \alpha)^* \Phi_{\mathcal{B}} \in \Omega^8(\mathbb{R}^{16} \setminus 0)$ gives by definition the admissible frame $\{|x|\partial_{x_1}, \ldots, |x|\partial_{x_{16}}\}$. A direct computation shows that this frame is admissible also for $\alpha^{-1} \circ \mathcal{R}'$. $\square$

8 Orthogonal and symmetric orbits

The representations of $\mathfrak{S}_{m+n}$ and $\mathrm{O}(m+n)$ on $\mathbb{R}^{m+n}$ give symmetric and orthogonal orbits of G-structures on $S^m \times S^n$. In this section we describe the following orbits:

- $\mathfrak{S}_{m+n} I_{\mathcal{P}}$, $\mathfrak{S}_{m+n}(I_{\mathcal{P}}, J_{\mathcal{P}}, K_{\mathcal{P}})$ on $S^m \times S^n$; $\mathfrak{S}_7 \varphi_{\mathcal{P}}$ on $S^6 \times S^1$, $S^4 \times S^3$, $S^2 \times S^5$; $\mathfrak{S}_8 \phi_{\mathcal{P}}$ on $S^7 \times S^1$, $S^5 \times S^3$, $S^3 \times S^5$, $S^1 \times S^7$;
- $\mathfrak{S}_{m+3} I_{\mathcal{B}}$, $\mathfrak{S}_{m+3}(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ on $S^m \times S^3$; $\mathfrak{S}_7 \varphi_{\mathcal{B}}$ on $S^4 \times S^3$; $\mathfrak{S}_8 \phi_{\mathcal{B}}$ on $S^5 \times S^3$;
- $\mathrm{O}(m+1) I_{\mathcal{B}}$, $\mathrm{O}(m+1)(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ on $S^m \times S^1$; $\mathrm{O}(7)\varphi_{\mathcal{B}}$ on $S^6 \times S^1$; $\mathrm{O}(8)\phi_{\mathcal{B}}$ on $S^7 \times S^1$; $\mathrm{O}(16)\Phi_{\mathcal{B}}$ on $S^{15} \times S^1$;
- $\mathrm{O}(7)\varphi_{\mathcal{P}}$ on $S^6 \times S^1$.

8.1 The symmetric orbit

The following is a particular case of theorem 32:

Lemma 25 *All almost Hermitian structures on $S^m \times S^1$ in $\mathfrak{S}_{m+1} I_{\mathcal{B}}$ are biholomorphic to the Hopf Hermitian structure $I_{e^{2\pi}}$.*

The following theorem splits the symmetric orbit of almost Hermitian structures associated to $\mathcal{P}$ in integrable and non-integrable ones:

Theorem 26 *Let m, n be odd. An almost Hermitian structure $I \in \mathfrak{S}_{m+n} I_{\mathcal{P}}$ on $S^m \times S^n$ is integrable if and only if*

$$I(p_{m-1+j}) = \pm p_{m+j} \qquad j \text{ odd},\ j = 1, \dots, n+1, \tag{7}$$

where the sign is the same for all j.

PROOF. Firstly, the if part. Taking $-I$ in case, one can suppose all signs in (7) to be positive. Suppose $I = \pi I_{\mathcal{P}}$ for an element π of $\mathfrak{S}_{m+n}$. Then there exist a set of $(n+1)/2$ pairs of elements of $\{1, \dots, m+n\}$ of the form $(\mathrm{odd}, \mathrm{odd}{+}1)$ which is invariant for π, and this gives $\tilde{\pi} \in \mathfrak{S}_{m+1}$, where $\tilde{\pi}(m) = m$, $\tilde{\pi}(m+1) = m+1$. Let S^1 be the fiber of the Hopf fibration of S^n, and consider the frame $\mathcal{B} = \{b_1, \dots, b_{m+1}\}$ on $S^m \times S^1$. Writing $\mathcal{B}$ in the basis $\mathcal{P}$ we obtain $\pi I_{\mathcal{P}}(b_m) = b_{m+1}$, hence $\pi I_{\mathcal{P}}$ coincide with $\tilde{\pi} I_{\mathcal{B}}$ on $S^m \times S^1$. Then lemma 25 implies that $\pi I_{\mathcal{P}}$ is integrable on $S^m \times S^1$.

Define the versor field $\tilde{\pi}(S)$ on S^m by $\tilde{\pi}(S) \stackrel{\text{def}}{=} -x_{\tilde{\pi}(2)} \partial_{x_{\tilde{\pi}(1)}} + x_{\tilde{\pi}(1)} \partial_{x_{\tilde{\pi}(2)}} + \dots - x_{\tilde{\pi}(m+1)} \partial_{x_{\tilde{\pi}(m)}} + x_{\tilde{\pi}(m)} \partial_{x_{\tilde{\pi}(m+1)}}$. Let S^1 be the orbit of $\tilde{\pi}(S)$ in S^m, and let $\tilde{\mathcal{B}} = \{\tilde{b}_1, \dots, \tilde{b}_{n+1}\}$ be the frame on $S^1 \times S^n$ given by $\tilde{b}_j \stackrel{\text{def}}{=} N_j - y_j \tilde{\pi}(S)$, $j =$

$1,\dots,n+1$. One obtains $N_j - y_j\tilde{\pi}(S) = p_{m-1+j} - y_j b_m - t_j b_{m+1} + t_j T - y_j\tilde{\pi}(S)$. Then, using the fact that $\pi I_{\mathcal{P}}(T) = \tilde{\pi}(S)$, we get $\pi I_{\mathcal{P}}(\tilde{b}_j) = \tilde{b}_{j+1}$ for odd $j = 1,\dots,n+1$. Namely, $\pi I_{\mathcal{P}}$ coincide with $I_{\tilde{\mathcal{B}}} = I_{e^{2\pi}}$ on $S^1 \times S^n$, and it is thus integrable on $S^1 \times S^n$. A straightforward computation show that the torsion of $\pi I_{\mathcal{P}}$ is zero when evaluated on $(b_i, \tilde{b}_j)$, and since $\mathcal{B} \cup \tilde{\mathcal{B}}$ spans $T(S^m \times S^n)$, the proof of the if part is completed.

Secondly, the only if part: it is given by a case by case computation, here sketched, which uses formulas (A.1). Suppose that condition (7) is not satisfied. Then, taking $-I$ in case, there exists an odd $j \in \{1,\dots,n+1\}$ such that one of the following conditions holds:

(1) there exist i, $k \in \{1,\dots,m-1\}$, $i \neq k$ such that

$$I(p_{m-1+j}) = p_i \qquad \text{and} \qquad I(p_{m+j}) = \pm p_k;$$

(2) there exist $i \in \{1,\dots,m-1\}$, $k \in \{1,\dots,n+1\}$, $k \neq j$, $j+1$ such that

$$I(p_{m-1+j}) = p_i \qquad \text{and} \qquad I(p_{m+j}) = \pm p_{m-1+k};$$

(3) there exist $i \in \{1,\dots,n+1\}$, $k \in \{1,\dots,m-1\}$, $i \neq j$, $j+1$ such that

$$I(p_{m-1+j}) = p_{m-1+i} \qquad \text{and} \qquad I(p_{m+j}) = \pm p_k;$$

(4) there exist i, $k \in \{1,\dots,n+1\}$, i, $k \neq j$, $j+1$, $i \neq k$ such that

$$I(p_{m-1+j}) = p_{m-1+i} \qquad \text{and} \qquad I(p_{m+j}) = \pm p_{m-1+k}.$$

The torsion tensor can then be computed in each case, using formulas (A.1), and in particular one obtains (t_j are the coordinates of T):

(1) $\langle N(p_{m-1+j}, p_{m+j}), p_k \rangle = 2(\pm x_i(1 - y_j^2 - y_{j+1}^2) + x_k(1 - 2(y_j^2 + y_{j+1}^2))) \neq 0$;
(2) $\langle N(p_{m-1+j}, p_{m+j}), p_i \rangle (y_j = y_{j+1} = y_k = 0, t_k = 1) = 2(x_i \mp x_{m+1}) \neq 0$;
(3) $\langle N(p_{m-1+j}, p_{m+j}), p_k \rangle (y_j = y_{j+1} = y_i = 0, t_i = 1) = 2(x_k \pm x_{m+1}) \neq 0$;
(4) $\langle N(p_{m-1+j}, p_{m+j}), p_{m-1+i} \rangle (y_j = y_{j+1} = t_i = x_m = 0, y_i = x_{m+1} = 1) = \mp 2t_k \neq 0$,

which concludes the proof. $\square$

Theorem 27 *All integrable almost Hermitian structures on $S^m \times S^n$ in $\mathfrak{S}_{m+n} I_{\mathcal{P}}$ are biholomorphic to the Calabi–Eckmann Hermitian structure $I^{m,n}$.*

PROOF. Let $\pi I_{\mathcal{P}} \in \mathfrak{S}_{m+n} I_{\mathcal{P}}$ be integrable. Let $\tilde{\pi}$ be the element of $\mathfrak{S}_{m+1}$ built in proof of theorem 26. Then the map $(x_1,\dots,x_{m+1},y) \mapsto (x_{\tilde{\pi}(1)},\dots,x_{\tilde{\pi}(m+1)},y)$ is a biholomorphism between $\pi I_{\mathcal{P}}$ and $I_{\mathcal{P}}$, and theorem 3 ends the proof. $\square$

Theorem 28 *An almost Hermitian structure $I \in \mathfrak{S}_{m+3} I_{\mathcal{B}}$ on $S^m \times S^3$ is integrable if and only if $I(b_{2n-1}) = \pm b_{2n}$. In this case, it is biholomorphic to $I^{m,3}$.*

PROOF. Suppose that $I(b_{2n-1}) \neq \pm b_{2n}$. Then there exist $i \neq j \in \{1, \ldots, 2n-2\}$ such that $I(b_{2n-1}) = b_i$, $I(b_{2n}) = \pm b_j$. Then $N(b_{2n-1}, b_{2n}) = 2(\pm x_i b_j \mp x_j b_i + 2\sum_{k=1}^{2n-2} x_k b_k - 2x_j b_j - 2x_i b_i) \neq 0$, showing that I is non-integrable. To prove the reverse implication, suppose that $\pi I_{\mathcal{B}} \in \mathfrak{S}_{m+3} I_{\mathcal{B}}$ satisfies $\pi I_{\mathcal{B}}(b_{2n-1}) = \pm b_{2n}$. Then $\pi I_{\mathcal{P}}$ satisfies the hypothesis of theorem 26 for $n = 3$, and using the right-side matrix in formula (2) we obtain that $\pi\mathcal{P}$ and $\pi\mathcal{B}$ differ by a unitary matrix, which implies $\pi I_{\mathcal{B}} = \pi I_{\mathcal{P}}$. □

Corollary 29 *All almost hyperhermitian structures on $S^m \times S^n$, $S^m \times S^3$ in $\mathfrak{S}_{m+n}(I_{\mathcal{P}}, J_{\mathcal{P}}, K_{\mathcal{P}})$, $\mathfrak{S}_{m+3}(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ respectively are non-integrable.*

PROOF. It follows by theorems 26, 28 once observed that a 2-dimensional distribution cannot be invariant for a hyperhermitian structure. □

It should be here remarked that, since an explicit expression for the Spin(9)-invariant 8-form is still lacking, it was not possible to apply the G_2 and Spin(7) techniques to the symmetric orbit of Spin(9)-structures.

Theorem 30 *The symmetric orbits $\mathfrak{S}_7 \varphi_{\mathcal{P}}$ of G_2-structures on $S^6 \times S^1$, $S^4 \times S^3$, $S^2 \times S^5$, $\mathfrak{S}_8 \phi_{\mathcal{P}}$ of Spin(7)-structures on $S^7 \times S^1$, $S^5 \times S^3$, $S^3 \times S^5$, $S^1 \times S^7$, $\mathfrak{S}_7 \varphi_{\mathcal{B}}$ of G_2-structures on $S^4 \times S^3$, $\mathfrak{S}_8 \phi_{\mathcal{B}}$ of Spin(7)-structures on $S^5 \times S^3$ are of general type.*

PROOF. Use theorem 15 together with the structure equations (3). □

8.2 The orthogonal orbit

This section is devoted to prove results about the orthogonal orbits $\mathrm{O}(m+1)I_{\mathcal{B}}$, $\mathrm{O}(m+1)(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$, $\mathrm{O}(7)\varphi_{\mathcal{B}}$, $\mathrm{O}(8)\phi_{\mathcal{B}}$, $\mathrm{O}(16)\Phi_{\mathcal{B}}$, $\mathrm{O}(7)\varphi_{\mathcal{P}}$ on $S^m \times S^1$.

The following lemma is trivial to prove but useful:

Lemma 31 *Let $A \in \mathrm{O}(m+1)$. Then $A : \mathbb{R}^{m+1} \smallsetminus 0 \to \mathbb{R}^{m+1} \smallsetminus 0$ is p-invariant, and the induced diffeomorphism $f_A \colon S^m \times S^1 \to S^m \times S^1$ is given by $(x, \theta) \mapsto (A(x), \theta)$. Moreover, the matrix of df_A with respect to the basis $\mathcal{B}$ on $S^m \times S^1$ is A.*

Theorem 32 *All almost Hermitian structures on $S^m \times S^1$ in the orthogonal orbit* $\mathrm{O}(m+1)I_{\mathcal{B}}$ *are biholomorphic to the Hopf Hermitian structure* $I_{e^{2\pi}}$.

PROOF. Let $I_{A(\mathcal{B})} \in \mathrm{O}(m+1)I_{\mathcal{B}}$. Then the matrix of $I_{A(\mathcal{B})} \circ df_A$ with respect to the basis $\mathcal{B}$ is $[I_{A(\mathcal{B})} \circ df_A]_{\mathcal{B}} = AIA^{-1}A = AI = [df_A \circ I_{\mathcal{B}}]_{\mathcal{B}}$, and the conclusion follows by remark 1. $\square$

Clearly, the proof does not rely on properties of $\mathrm{U}((m+1)/2)$, and it works fine for all the other G-structures:

Theorem 33 *All almost hyperhermitian structures on $S^m \times S^1$ in the orthogonal orbit* $\mathrm{O}(m+1)(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ *are equivalent to the Hopf hyperhermitian structure* $(I_{e^{2\pi}}, J_{e^{2\pi}}, K_{e^{2\pi}})$*. The* G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$*-structures on* $S^6 \times S^1$, $S^7 \times S^1$, $S^{15} \times S^1$ *in the orthogonal orbits* $\mathrm{O}(7)\varphi_{\mathcal{B}}$, $\mathrm{O}(8)\phi_{\mathcal{B}}$, $\mathrm{O}(16)\Phi_{\mathcal{B}}$ *are isomorphic to* $\varphi_{\mathcal{B}}$, $\phi_{\mathcal{B}}$, $\Phi_{\mathcal{B}}$ *respectively.*

Since the lemma does not hold for the frame $\mathcal{P}$ on $S^m \times S^1$, because of the twisting of p_m, p_{m+1}, the following theorem is not trivial:

Theorem 34 *The* G_2*-structures on $S^6 \times S^1$ in the orthogonal orbit* $\mathrm{O}(7)\varphi_{\mathcal{P}}$ *are of general type.*

PROOF. Let $A = (a_{i,j}) \in \mathrm{SO}(7)$, and denote by $\{q^1, \dots, q^7\}$ the coframe on $S^6 \times S^1$ induced by A, that is, $q^i \stackrel{\text{def}}{=} \sum_{j=1}^7 a_{i,j} p^j$. Let $\tau = -y_2 dy_1 + y_1 dy_2$ be the usual 1-form on $S^6 \times S^1$, and u_i its coordinates with respect to $\{q^1, \dots, q^7\}$. Then

$$\varphi_{A(\mathcal{P})} = \sum_{i \in \mathbb{Z}_7} q^{i,i+1,i+3}, \qquad *\varphi_{A(\mathcal{P})} = -\sum_{i \in \mathbb{Z}_7} q^{i,i+2,i+3,i+4},$$

and using the structure equations (3) one obtains

$$\begin{aligned}
d\varphi_{A(\mathcal{P})} &= 3\varphi_{A(\mathcal{P})} \wedge \tau \\
&\quad + \sum_{i \in \mathbb{Z}_7} ((a_{i,6}p^7 - a_{i,7}p^6) q^{i+1,i+3} - (a_{i+1,6}p^7 - a_{i+1,7}p^6) q^{i,i+3} \\
&\quad + (a_{i+3,6}p^7 - a_{i+3,7}p^6) q^{i,i+1}) \wedge \tau \\
d * \varphi_{A(\mathcal{P})} &= -4 * \varphi_{A(\mathcal{P})} \wedge \tau \\
&\quad + \sum_{i \in \mathbb{Z}_7} ((a_{i,6}p^7 - a_{i,7}p^6) q^{i+2,i+3,i+4} - (a_{i+2,6}p^7 - a_{i+2,7}p^6) q^{i,i+3,i+4} \\
&\quad + (a_{i+3,6}p^7 - a_{i+3,7}p^6) q^{i,i+2,i+4} - (a_{i+4,6}p^7 - a_{i+4,7}p^6) q^{i,i+2,i+3}) \wedge \tau.
\end{aligned}$$

The 3-form $*d\varphi_{A(\mathcal{P})}$ is not so easy to write. Let $\alpha_{i,j} \stackrel{\text{def}}{=} a_{i,6}a_{j,7} - a_{i,7}a_{j,6}$. Then by a long calculation one obtains

$$
\begin{aligned}
&* d\varphi_{A(\mathcal{P})} = \sum_{i\in\mathbb{Z}_7} [\\
&(-3u_{i+2} - u_i(-\alpha_{i,i+2} + \alpha_{i+4,i+3} + \alpha_{i+5,i+1}) + u_{i+3}(-\alpha_{i+6,i+1} + \alpha_{i+3,i+2} + \alpha_{i+4,i}) \\
&+ u_{i+1}(\alpha_{i+6,i+3} + \alpha_{i+1,i+2} + \alpha_{i+5,i}))q^{i+4,i+5,i+6} \\
&+ (3u_{i+4} - u_i(\alpha_{i,i+4} + \alpha_{i+2,i+3} + \alpha_{i+6,i+1}) + u_{i+1}(-\alpha_{i+1,i+4} - \alpha_{i+5,i+3} + \alpha_{i+6,i}) \\
&- u_{i+3}(-\alpha_{i+5,i+1} - \alpha_{i+2,i} + \alpha_{i+3,i+4}))q^{i+2,i+5,i+6} \\
&+ (-3u_{i+5} + u_{i+3}(\alpha_{i+4,i+1} + \alpha_{i+6,i} + \alpha_{i+3,i+5}) + u_i(\alpha_{i,i+5} + \alpha_{i+2,i+1} - \alpha_{i+6,i+3}) \\
&- u_{i+1}(\alpha_{i+2,i} + \alpha_{i+4,i+3} - \alpha_{i+1,i+5}))q^{i+2,i+4,i+6} \\
&+ (3u_{i+6} - u_{i+3}(-\alpha_{i+5,i} + \alpha_{i+2,i+1} + \alpha_{i+3,i+6}) - u_i(\alpha_{i+5,i+3} + \alpha_{i,i+6} - \alpha_{i+4,i+1}) \\
&+ u_{i+1}(-\alpha_{i+4,i} - \alpha_{i+1,i+6} + \alpha_{i+2,i+3}))q^{i+2,i+4,i+5} \\
&+ (u_{i+6}(-\alpha_{i,i+2} + \alpha_{i+4,i+3} + \alpha_{i+5,i+1}) - u_{i+3}(\alpha_{i+5,i+2} + \alpha_{i,i+1} + \alpha_{i+4,i+6}) \\
&+ u_{i+2}(\alpha_{i+5,i+3} + \alpha_{i,i+6} - \alpha_{i+4,i+1}) - u_{i+1}(-\alpha_{i,i+3} - \alpha_{i+4,i+2} + \alpha_{i+5,i+6}))q^{i,i+4,i+5}].
\end{aligned}
$$

Now use theorem 15 to check which classes $\varphi_{A(\mathcal{P})}$ belongs to. As for the class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, one obtains

$$0 = d\varphi_{A(\mathcal{P})} \wedge \varphi_{A(\mathcal{P})} = \sigma \wedge \tau$$

where σ is a 6-form on $S^6 \times S^1$ with constant coefficients with respect to $\mathcal{P}$, and this is easily seen to be impossible. The existence of a 1-form β on $S^6 \times S^1$ such that $d * \varphi_{A(\mathcal{P})} = \beta \wedge \varphi_{A(\mathcal{P})}$ implies that

$$\alpha_{i,i+1} + \alpha_{i+5,i+2} - \alpha_{i+6,i+4} = 0 \qquad i \in \mathbb{Z}_7.$$

But this system has no solution, hence $\varphi_{A(\mathcal{P})}$ does not belong to the class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. The above system comes out also requiring the existence of a 1-form α and a function f on $S^6 \times S^1$ such that $d\varphi_{A(\mathcal{P})} = \alpha \wedge \varphi_{A(\mathcal{P})} + f * \varphi_{A(\mathcal{P})}$, hence $\varphi_{A(\mathcal{P})}$ does not belong to $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$. Finally, $*d\varphi_{A(\mathcal{P})} \wedge \varphi_{A(\mathcal{P})} \neq 0$ by a direct computation.

If $\det A = -1$, some signs in formulas are reversed, but the same impossible conditions are obtained. $\square$

A Structure equations for $\mathcal{P}$ on $S^m \times S^n$

Using the following abbreviations:

$$
\begin{aligned}
X_m &\stackrel{\text{def}}{=} \sum_{j=1}^{n+1} y_j p_{m-1+j}, \\
X_{m+1} &\stackrel{\text{def}}{=} \sum_{j=1}^{n+1} t_j p_{m-1+j}, \\
C_{j,k} &\stackrel{\text{def}}{=} y_j t_k - y_k t_j \qquad j,k = 1,\dots,n+1, \\
D_{j,k} &\stackrel{\text{def}}{=} 2C_{j,k} \underbrace{\mp\delta_{k,j\pm 1}}_{j\ \frac{\text{odd}}{\text{even}}} \underbrace{\pm\delta_{j,k\pm 1}}_{k\ \frac{\text{odd}}{\text{even}}} \qquad j,k = 1,\dots,n+1.
\end{aligned}
$$

one can write down the general formulas:

$$
\begin{aligned}
[p_i, p_j] &= x_i p_j - x_j p_i \qquad i,j = 1,\dots,m-1, \\
[p_i, p_{m-1+j}] &= -(y_j x_m + t_j x_{m+1}) p_i \\
&\quad \underbrace{\mp x_i p_{m-1+j\pm 1}}_{j\ \frac{\text{odd}}{\text{even}}} + x_i y_j X_m + x_i t_j X_{m+1} \qquad i = 1,\dots,m-1, j = 1,\dots,n+1, \\
[p_{m-1+j}, p_{m-1+k}] &= D_{j,k} \sum_{i=1}^{m-1} x_i p_i + y_j p_{m-1+k} - y_k p_{m-1+j} \\
&\quad + (x_m D_{j,k} - x_{m+1} C_{j,k}) X_m + ((x_{m+1} - 1) D_{j,k} + x_m C_{j,k}) X_{m+1} \\
&\quad + \underbrace{(\mp y_j x_m \mp t_j x_{m+1} \pm t_j) p_{m-1+k\pm 1}}_{k\ \frac{\text{odd}}{\text{even}}} \\
&\quad + \underbrace{(\pm y_k x_m \pm t_k x_{m+1} \mp t_k) p_{m-1+j\pm 1}}_{j\ \frac{\text{odd}}{\text{even}}} \qquad j,k = 1,\dots,n+1.
\end{aligned}
\tag{A.1}
$$

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