Locally conformal Kähler reduction

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Abstract

We define reduction of locally conformal Kähler manifolds, considered as conformal Hermitian manifolds, and we show its equivalence with an unpublished construction given by Biquard and Gauduchon. We show the compatibility between this reduction and Kähler reduction of the universal cover. By a recent result of Kamishima and the second author, in the Vaisman case (that is, when a metric in the conformal class has parallel Lee form) if the manifold is compact its universal cover comes equipped with the structure of Kähler cone over a Sasaki compact manifold. We show the compatibility between our reduction and Sasaki reduction, hence describing a subgroup of automorphisms whose action causes reduction to bear a Vaisman structure. Then we apply this theory to construct a wide class of Vaisman manifolds.

Keywords: locally conformal Kähler manifold, Vaisman manifold, Sasaki manifold, Lee form, momentum map, Hamiltonian action, reduction, conformal geometry.
AMS 2000 subject classification: 53C55, 53D20, 53C25.

1 Introduction

Since 1974 when the classical reduction procedure of S. Lie was formulated in modern terms by J. Marsden and A. Weinstein for symplectic structures, this technique was extended to other various geometric structures defined by a closed form. Extending the equivariant symplectic reduction to Kähler manifolds was most natural: one only showed the almost complex structure was also projectable. Generalizations to hyperkähler and quaternion Kähler geometry followed. The extension to contact geometry is also natural and can be understood via the symplectization of a contact manifold. In each case, the momentum map is produced by a Lie group acting by specific automorphisms of the structure.

A *locally conformal Kähler manifold* is a conformal Hermitian manifold $(M, [g], J)$ such that for one (and hence for all) metric g in the conformal class the corresponding Kähler form Ω satisfies $d\Omega = \omega \wedge \Omega$, where ω is a closed 1-form. This is equivalent to the existence of an atlas such that the restriction of g to any chart is conformal to a Kähler metric.

The 1-form $\omega \in \Omega^1(M)$ was introduced by H.-C. Lee in [Lee43], and it is therefore called the *Lee form* of the Hermitian structure (g, J).

It was not obvious how to produce a quotient construction in conformal geometry. The first published result we are aware of belongs to S. Haller and T. Rybicki who proposed in [HR01] a reduction for locally conformal symplectic structures. Their technique is essentially local: they reduce the local symplectic structures, then glue the local reduced structures. But even earlier, since 1998, an unpublished paper by O. Biquard and P. Gauduchon proposed a quotient construction for locally conformal Kähler manifolds [BG98]. Their construction relies heavily on the language and techniques of conformal geometry as developed, for example, in [CP99]. The key point is the fact that a locally conformal Kähler structure can be seen as a closed 2-form with values in a vector bundle (of densities).

Our starting point was the paper [HR01]. Following the lines of the Kähler reduction, we verified that the complex structure of a locally conformal Kähler manifold can be projected to the quotient. In section 3 of this paper we construct the momentum map associated to an action by locally conformal Kähler automorphisms, lying on the notion of *twisted Hamiltonian action* given by I. Vaisman in [Vai85]. In section 4 we extend Haller-Rybicki construction to the complex setting. Then, in section 5 we present, rather in detail, due to

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its very restricted previous circulation, the Biquard-Gauduchon construction. The main result of this section proves the equivalence between the Biquard-Gauduchon reduction and ours.

The universal cover of a locally conformal Kähler manifold has a natural (global) homothetic Kähler structure. We exploit this fact in section 6 in order to relate locally conformal Kähler reduction to the Kähler reduction of its universal cover.

The study of locally conformal Kähler manifolds started in the field of Hermitian manifolds. Most of the known examples of locally conformal Kähler metrics are on compact manifolds and enjoy the additional property of having parallel Lee form with respect to the Levi-Civita connection. Locally conformal Kähler metrics with parallel Lee form were first introduced and studied by I. Vaisman in [Vai79, Vai82], so we call *Vaisman metric* a locally conformal Kähler metric with this property. Manifolds bearing a Vaisman metric show a rich geometry. Such are the Hopf surfaces $H_{\alpha,\beta}$ described in [GO98], all diffeomorphic with $S^1 \times S^3$ (see also [Par99]). More generally, I. Vaisman firstly showed that on the product $S^1 \times S^{2n+1}$ given as a quotient of $\mathbb{C}^n \setminus 0$ by the cyclic infinite group spanned by $z \mapsto \alpha z$, where $z \in \mathbb{C}^n \setminus 0$ and $|\alpha| \neq 1$, the projection of the metric $|z|^{-2} \sum dz^i \otimes d\bar{z}^i$ is locally conformal Kähler with parallel Lee form $-|z|^{-2} \sum (z_i d\bar{z}_i + \bar{z}_i dz_i)$. The complete list of compact complex locally conformal Kähler surfaces admitting parallel Lee form was given by F. Belgun in [Bel00] where it is also proved the existence of some compact complex surfaces which do not admit any locally conformal Kähler metric.

The definition of Vaisman metric is *not* invariant up to conformal changes. A conformally equivalent notion of Vaisman manifold is still missing, but a recent result by Kamishima and the second author in [KO01] provides one in the compact case, generalizing the one first proposed by Belgun in [Bel00] in the case of surfaces. We develop this notion in section 7 where we analyze reduction in this case.

Vaisman geometry is closely related with Sasaki geometry. In this case the picture turns out to be the following. The category of ordinary locally conformal Kähler manifolds can be seen as the image of the category of pairs (K, Γ) of homothetic Kähler manifolds with a subgroup Γ of homotheties acting freely and properly discontinuously, with morphisms given by homothetic Kähler morphisms equivariant by the actions. What we prove in section 6 is that under the functor associating to (K, Γ) the locally conformal Kähler manifold K/Γ Hamiltonian actions go to twisted Hamiltonian actions, and vice versa, see Theorem 6.5. So the images of subgroups producing Kähler reduction actually are subgroups producing locally conformal Kähler reduction (up to topological conditions), and vice versa. The same way the category of (compact) Vaisman manifolds can be seen as the image of the category of pairs (W, Γ), with W a Sasaki manifold and Γ a subgroup of (proper) homotheties of the Kähler cone $W \times \mathbb{R}$ acting freely and properly discontinuously, with morphisms given by Sasaki morphisms equivariant by the actions. The functor associating to (W, Γ) the Vaisman manifold $(W \times \mathbb{R})/\Gamma$ is surjective on objects but not on morphisms: we call *Vaisman morphisms* the ones in the image. What we prove in section 7 is that, up to topological conditions, subgroups of Sasaki automorphisms producing Sasaki reduction go to subgroups producing Vaisman reduction, and vice versa. This is particularly remarkable since, up to topological conditions, Sasaki reduction applies to *any* subgroup of automorphisms, that is, the momentum map is *always* defined. So we obtain that reduction by Vaisman automorphisms is always defined (up to topological conditions) and produces Vaisman manifolds.

This allows building a wide set of Vaisman manifolds, reduced by circle actions on Hopf manifolds, in section 8.

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2 Locally conformal Kähler manifolds

Let (M, J) be any almost-complex n-manifold, $n \geq 4$, let g be a Hermitian metric on (M, J). Let Ω be the Kähler form defined by $\Omega(X, Y) \stackrel{\text{def}}{=} g(JX, Y)$. The map $L\colon \Omega^1(M) \to \Omega^3(M)$ given by the wedging with Ω is injective, so that the g-orthogonal splitting $\Omega^3(M) = \mathrm{Im}L \oplus (\mathrm{Im}L)_0$ induces a well-defined $\omega \in \Omega^1(M)$ given by the relation $d\Omega = \omega \wedge \Omega + (d\Omega)_0$. The 1-form $\omega \in \Omega^1(M)$ is called the *Lee form* of the almost-Hermitian structure (g, J).

A relevant notion in this setting is that of *twisted differential.* Given a p-form ψ its twisted differential is the $(p+1)$-form

$$d^\omega \psi \stackrel{\text{def}}{=} d\psi - \omega \wedge \psi.$$

Remark that $d^\omega \circ d^\omega = 0$ if and only if $d\omega = 0$.

A Hermitian metric g on a complex manifold (M, J) is said to be *locally conformal Kähler* if g is (locally) conformal to local Kähler metrics. In this case the local forms $d\alpha_U$ coming from the local conformal factors e^{α_U} paste to a global form ω satisfying $d\Omega = \omega \wedge \Omega$. Vice versa this last equation together with $d\omega = 0$ characterizes the locally conformal Kähler metrics. In other words a Hermitian metric is locally conformal Kähler if and only if

$$d^\omega \circ d^\omega = 0 \qquad \text{and} \qquad d^\omega \Omega = 0. \tag{1}$$

Definition 2.1 A conformal Hermitian manifold $(M, [g], J)$ of complex dimension bigger than 1 is said to be a *locally conformal Kähler manifold* if one (and hence all of) the metrics in $[g]$ is locally conformal Kähler.

Remark 2.2 If, in particular, the Lee form of one (and hence all) of the metrics in $[g]$ is exact, then the manifold is said to be *globally conformal Kähler.* This is in fact equivalent to requiring that in the conformal class there exists a Kähler metric, that is, any metric in $[g]$ is globally conformal to a Kähler metric. From [Vai80] it is known that for compact manifolds possessing a Kähler structure forbids existence of locally non-globally conformal Kähler structures, so the two worlds are generally considered as disjoint. In this paper, however, the two notions behave the same way, so we consider the global case as a subclass of the local case.

From now on, let $(M, [g], J)$ be a locally conformal Kähler manifold.

Not unlike the Kähler case, locally conformal Kähler manifolds come equipped with a notable subset of $\mathfrak{X}(M)$: given a smooth function f the *associated Hamiltonian vector field* is the Ω-dual of df, and *Hamiltonian vector fields* are vector fields that admit such a presentation. But the notion that works for reduction, as shown in [HR01], is the one given in [Vai85] obtained by twisting the classical. Given f its *associated twisted Hamiltonian vector field* is the Ω-dual of $d^\omega f$. The subset of $\mathfrak{X}(M)$ of *twisted Hamiltonian vector fields* is that of vector fields admitting such a presentation.

Remark 2.3 If M is not globally conformal Kähler the function associating to f its twisted Hamiltonian vector field is injective. Indeed $d^\omega f = 0$ implies $\omega = d \log |f|$ on $f \neq 0$, so either $f \equiv 0$ or ω is exact.

Define a *twisted Poisson bracket* on $C^\infty(M)$ by

$$\{f_1, f_2\} \stackrel{\text{def}}{=} \Omega(\sharp d^\omega f_1, \sharp d^\omega f_2) \tag{2}$$

The relation

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = d^\omega \Omega(\sharp d^\omega f_1, \sharp d^\omega f_2, \sharp d^\omega f_3) = 0 \tag{3}$$

proves that this bracket turns $C^\infty(M)$ into a Lie algebra. Remark that the first equality in (3) holds generally on any almost-Hermitian manifold (M, g, J) under the only assumption $d\omega = 0$.

Remark 2.4 Remark that the notion of Hamiltonian vector field is invariant up to conformal change of the metric, even though the function (possibly, the functions) associated to a Hamiltonian vector field changes by the conformal factor. A straightforward computation shows in fact that, if $\Omega' = e^\alpha \Omega$ and $\omega' = \omega + d\alpha$ is the corresponding Lee form, the following relations hold

$$d^\omega f = e^{-\alpha} d^{\omega'}(e^\alpha f)$$

$$\sharp_\Omega d^\omega f = \sharp_{\Omega'} d^{\omega'}(e^\alpha f)$$

$$\{e^\alpha f_1, e^\alpha f_2\}^{\Omega'} = e^\alpha \{f_1, f_2\}^\Omega$$

so that multiplication by e^α yields an isomorphism between $(C^\infty(M), \{\ ,\ \}^\Omega)$ and $(C^\infty(M), \{\ ,\ \}^{\Omega'})$ commuting with the corresponding maps $\sharp_\Omega d^\omega$ and $\sharp_{\Omega'} d^{\omega'}$ in the space of twisted Hamiltonian vector fields. In particular if M is globally conformal Kähler, then the twisted Hamiltonian vector fields of M coincide with the ordinary Hamiltonian vector fields, since the Lee form of a Kähler metric is 0.

Definition 2.5 Given two locally conformal Kähler manifolds $(M, [g], J)$ and $(M', [g'], J)$ a smooth map h from M to M' is a *locally conformal Kähler morphism* if $h^*J' = J$ and $[h^*g'] = [g]$. We denote by $\mathrm{Aut}(M, [g], J)$, or briefly by $\mathrm{Aut}(M)$, the group of locally conformal Kähler automorphisms of $(M, [g], J)$.

The group $\mathrm{Aut}(M)$ is a Lie group, contained as a subgroup in the complex Lie group of biholomorphisms of (M, J). However, unlike the Riemannian case, the Lie algebra of $\mathrm{Aut}(M)$ is *not* closed for the complex structure. This will be used in the sequel.

3 The locally conformal Kähler momentum map

In this paper we consider (connected) Lie subgroups G of $\mathrm{Aut}(M)$.

Remark 3.1 It follows from [MPPS97] that whenever a locally conformal Kähler manifold M is compact, the group $\mathrm{Aut}(M)$ coincides with the isometries of the *Gauduchon metric* in the conformal class, that is, the one whose Lee form is coclosed. Hence, in particular, $\mathrm{Aut}(M)$ is compact. More generally if a subgroup G of $\mathrm{Aut}(M)$ is compact then by using the Haar integral one obtains a metric in the conformal class such that G is contained in the group of its isometries. So the case when G is *not* constituted by isometries of a specific metric can only happen if both M and G are non-compact.

Throughout the paper we identify fundamental vector fields with elements X of the Lie algebra $\mathfrak{g}$ of G, so that if $x \in M$ then $\mathfrak{g}(x)$ means $T_x(Gx)$.

Imitating the terminology established in [MS95], we call the action of G *weakly twisted Hamiltonian* if the associated infinitesimal action is of twisted Hamiltonian vector fields, that is, if there exists a (linear) map $\mu^\cdot \colon \mathfrak{g} \to C^\infty(M)$ such that $\iota_X \Omega = d^\omega \mu^X$ for fundamental vector fields $X \in \mathfrak{g}$, and *twisted Hamiltonian* if μ can be chosen to be a Lie algebra homomorphism with respect to the Poisson bracket (2). In this case we say that the Lie algebra homomorphism μ is a *momentum map* for the action of G, or, equivalently, with the same name and symbol we refer to the induced map $\mu \colon M \to \mathfrak{g}^*$ given by $\langle \mu(x), X \rangle \stackrel{\text{def}}{=} \mu^X(x)$, for $X \in \mathfrak{g}$ and carets denoting the evaluation.

Remark 3.2 Note that the property of an action of being twisted Hamiltonian is a property of the conformal structure, even though the Poisson structure on $C^\infty(M)$ is not conformally invariant, see Remark 2.4. If $g' = e^\alpha g$ then $\mu^{\omega'} = e^\alpha \mu^\omega$. In particular the preimage of 0 is well-defined.

Remark 3.3 Remark that μ is not equivariant for the standard coadjoint action on $\mathfrak{g}^*$. It is known from [HR01] that by modifying the coadjoint action by means of the conformal factors arising from $h^*g \sim g$ one can force μ to be equivariant.

On $\mu^{-1}(0)$ the twisted differential of the associated twisted Hamiltonian functions $\mu(\mathfrak{g})$ coincides with the ordinary differential, since $d^\omega_x \mu^X = d_x \mu^X - \mu^X(x)\omega_x$ for $X \in \mathfrak{g}$, $x \in \mu^{-1}(0)$. Thus, if the action is twisted Hamiltonian, then the functions in $\mu(\mathfrak{g})$ vanish on the whole orbit of $x \in \mu^{-1}(0)$, since for $x \in \mu^{-1}(0)$ and $Y \in \mathfrak{g}(x)$

$$d_x \mu^X(Y) = d^\omega_x \mu^X(Y) = \Omega(\sharp d^\omega \mu^X, \sharp d^\omega \mu^Y)(x) = \{\mu^X, \mu^Y\}(x) = \mu^{[X,Y]}(x) = 0,$$

that is to say, $\mu^{-1}(0)$ is closed for the action of G.

Moreover, if 0 is a regular value for μ, then $T(\mu^{-1}(0))^{\perp_\Omega} = \mathfrak{g}$, since for any $x \in \mu^{-1}(0)$, $X \in \mathfrak{g}, V \in \mathfrak{X}(\mu^{-1}(0))$ we have

$$\Omega(X, V)(x) = d^\omega_x \mu^X(V) = d_x \mu^X(V) = 0.$$

Thus we say that $\mu^{-1}(0)$ is a *coisotropic* submanifold of M.

In the next section we show how to obtain a locally conformal Kähler structure on $\mu^{-1}(0)/G$ under the additional hypothesis of it being a manifold. But we remark here that, due to the missing equivariance of μ, a non-zero reduction is not straightforward.

Remark 3.4 We give a brief description of the existence and unicity problem for momentum maps. Suppose the action is weakly twisted Hamiltonian, and choose a linear map $\mu^{\cdot}\colon \mathfrak{g} \to C^\infty(M)$. Denote by N the kernel of $d^\omega\colon C^\infty(M) \to \Omega^1(M)$. The obstruction for μ to be a Lie algebra homomorphism is given by the map $\tau\colon \mathfrak{g} \times \mathfrak{g} \to N$ sending (X, Y) into $\{\mu^X, \mu^Y\} - \mu^{[X,Y]}$, which can be shown to live in $H^2(\mathfrak{g}, N)$, and this cohomology class vanishes whenever the action is twisted Hamiltonian. If this is the case, then momentum maps are parameterized by $H^1(\mathfrak{g}, N)$. If $(M, [g], J)$ is non-globally conformal Kähler, then $N = 0$, see Remark 2.3. Then, in particular, a weakly Hamiltonian action on a compact non-Kähler locally conformal Kähler manifold always admits a unique momentum map.

In the following we will often need a technical lemma we prove here once and for all. If g and g' are tensors on the same manifold we write $g \sim g'$ if they are conformal to each other.

Lemma 3.5 *Let M be a manifold, let $\{U_i\}_{i\in\mathcal{I}}$ be a locally finite open covering. Let $\{\rho_i\}$ be a partition of unity relative to $\{U_i\}$. The following three facts hold.*

i) Let g and g' be two tensors globally defined on M and such that for any i

$$g|_{U_i} \sim g'|_{U_i};$$

then g and g' are globally conformal.

ii) Let $\{g_i\}$ be a collection of local tensors, where g_i is defined on U_i, such that whenever $U_i \cap U_j \neq \emptyset$

$$g_i|_{U_i\cap U_j} \sim g_j|_{U_i\cap U_j};$$

then the tensor $g \overset{\text{def}}{=} \sum_i \rho_i g_i$ is globally defined on M and $g|_{U_i}$ is locally conformal to g_i.

iii) Let $\{g_i\}$ and g be as in ii). If g' is a global tensor such that $g'|_{U_i}$ is locally conformal to g_i, then g and g' are globally conformal.

Proof: First prove *i)*. Let e^{α_i} be the conformal factor such that

$$g|_{U_i} = e^{\alpha_i} g'|_{U_i};$$

then recalling that $\sum_i \rho_i = 1$ one obtains

$$g = (\sum_i \rho_i e^{\alpha_i}) g'.$$

Now turn to *ii)*. For any $x \in M$ let U_x be a neighborhood of x which is completely contained in any U_i that contains x, let U_{i_x} be one of them and $e^{\alpha_{x,i}}$ be the conformal factor between g_{i_x} and g_i, defined on $U_{i_x} \cap U_i$ which contains U_x: then the following holds

$$g|_{U_x} = (\sum_i \rho_i e^{\alpha_{x,i}}) g_{i_x}.$$

Finally *i)* and *ii)* imply *iii)*. ■

Remark 3.6 Using a more sophisticated argument it is proved in [HR01] that in case *ii)* one obtains $g|_{U_i} \sim g_i$.

4 The reduction theorem

Theorem 4.1 *Let $(M, [g], J)$ be a locally conformal Kähler manifold. Let G be a Lie subgroup of* $\mathrm{Aut}(M)$ *whose action is twisted Hamiltonian and is free and proper on $\mu^{-1}(0)$, 0 being a regular value for the momentum map μ. Then there exists a locally conformal Kähler structure $([\bar{g}], \bar{J})$ on $\mu^{-1}(0)/G$, uniquely determined by the condition $\pi^*\bar{g} \sim i^*g$, where i denotes the inclusion of $\mu^{-1}(0)$ into M and π denotes the projection of $\mu^{-1}(0)$ onto its quotient.*

Proof: Since $\mu^{-1}(0)$ is coisotropic, and its isotropic leaves are the orbits of G, the $[g]$-orthogonal splitting $T_xM = E_x \oplus \mathfrak{g}(x) \oplus J\mathfrak{g}(x)$ holds, where E_x is the $[g]$-orthogonal complement of $\mathfrak{g}(x)$ in $T_x(\mu^{-1}(0))$. This shows that E is a complex subbundle of TM and, since J is constant along $\mathfrak{g}$, it induces an almost complex structure $\bar{J}$ on $\mu^{-1}(0)/G$. This is proven to be integrable the same way as in the Kähler case, by computing the Nijenhuis tensor of $\bar{J}$ and recalling that $\pi_*[V,W] = [\pi_*V, \pi_*W]$ for projectable vector fields V, W.

Take an open cover $\mathcal{U}$ of $\mu^{-1}(0)/G$ that trivializes the G-principal bundle $\pi\colon \mu^{-1}(0) \to \mu^{-1}(0)/G$ and for each $U \in \mathcal{U}$ choose a local section s_U of π.

Fix an open set U. On its preimage we have two horizontal distributions: the (global) already defined distribution E, $[g]$-horizontal, and the tangent distribution S_U to $s_U(U)$, translated along the fibres by means of the action of G to give a distribution on the whole preimage of U. Remark that S_U cannot be chosen to coincide with E in general, since S_U is obviously a (local) foliation, whereas E is not integrable in general.

Given a vector field $\bar{V}$ on U denote by V its $[g]$-horizontal lifting. Then for any $\bar{V}$ the vector fields V and $J(V)$ are projectable and $\bar{J}(\bar{V}) = \pi_*J(V)$. Moreover denote by $V + \nu_V$ the lifting of $\bar{V}$ tangent to S_U, so that $ds_U(\bar{V}) = V + \nu_V$.(1) Remark that ν_V is a vertical vector field on $\pi^{-1}(U)$, and that clearly $V + \nu_V$ is projectable itself: more explicitly, for a generic $x \in \pi^{-1}(U)$,

$$(V + \nu_V)_x = (h_x^{-1})_* d_{\pi(x)} s_U(\bar{V}_{\pi(x)})$$

where by h_x we denote the element of G that takes x in $s_U(\pi(x))$.

Now define a local 2-form $\bar{\Omega}_U \overset{\text{def}}{=} s_U^* i^*\Omega$ on U. Since vertical vector fields are Ω-orthogonal to any vector field on $\pi^{-1}(U)$, this definition implies that for any pair $(\bar{V}, \bar{W})$ of vector fields on U

$$\begin{aligned}
\bar{\Omega}_U(\bar{V}, \bar{W}) &= s_U^* i^*\Omega(\bar{V}, \bar{W}) \\
&= i^*\Omega(V + \nu_V, W + \nu_W) \\
&= i^*\Omega(V, W).
\end{aligned}$$

Since $i^*\Omega$ is compatible with J and positive, the local form $\bar{\Omega}_U$ easily turns out to be compatible with $\bar{J}$, since

$$\begin{aligned}
\bar{\Omega}_U(\bar{J}(\bar{V}), \bar{J}(\bar{W})) &= s_U^* i^*\Omega(\bar{J}(\bar{V}), \bar{J}(\bar{W})) \\
&= i^*\Omega(ds_U(\pi_*J(V)), ds_U(\pi_*J(W))) \\
&= i^*\Omega(J(V) + \nu_{J(V)}, J(W) + \nu_{J(W)}) \\
&= i^*\Omega(J(V), J(W)) \\
&= i^*\Omega(V, W) \\
&= \bar{\Omega}_U(\bar{V}, \bar{W})
\end{aligned}$$

and the same way one shows that $\bar{\Omega}_U$ is positive.

Denote by $\bar{g}_U$ the corresponding local Hermitian metric, which is then locally conformal Kähler.

We want now to show that $\pi^*\bar{\Omega}_U$ is conformally equivalent to $i^*\Omega$ on $\pi^{-1}(U)$.

So consider a pair of generic (that is, non necessarily projectable) vector fields $(\tilde{V}, \tilde{W})$ on $\pi^{-1}(U)$. For any $x \in \pi^{-1}(U)$ denote by V^x the projectable vector field such that V^x_x coincide with $\tilde{V}_x$, that is $V^x_y \overset{\text{def}}{=} (h_{x,y}^{-1})_* \tilde{V}_{h_x^{-1} s_U(\pi(y))}$, where by $h_{x,y}$ we denote the element of G that takes y in $h_x^{-1} s_U(\pi(y))$. Similarly define W^x, and call $(\bar{V}^x, \bar{W}^x)$ the projected vector fields on U. We then have

$$\begin{aligned}
\pi^*\bar{\Omega}_U(\tilde{V}_x, \tilde{W}_x) &= \bar{\Omega}_U(\pi_*\tilde{V}^x_x, \pi_*\tilde{W}^x_x) \\
&= \bar{\Omega}_U(\bar{V}^x_{\pi(x)}, \bar{W}^x_{\pi(x)}) \\
&= i^*\Omega(V^x_{s_U(\pi(x))}, W^x_{s_U(\pi(x))}).
\end{aligned}$$

By evaluating the projectable vector field V^x in the point $y = s_U(\pi(x))$ one obtains the following

$$\begin{aligned}
\pi^*\bar{\Omega}_U(\tilde{V}_x, \tilde{W}_x) &= i^*\Omega((h_x)_*\tilde{V}_x, (h_x)_*\tilde{W}_x) \\
&= h_x^* i^*\Omega(\tilde{V}_x, \tilde{W}_x).
\end{aligned}$$

Now remark that h_x is a conformal map, hence there exists a smooth function α_x such that $h_x^* i^*\Omega(\tilde{V}_x, \tilde{W}_x) = \alpha_x(x) i^*\Omega(\tilde{V}_x, \tilde{W}_x)$. But by construction the function $x \mapsto \alpha_x(x)$ is smooth, so the two 2-forms are conformally equivalent.

(1) To be precise we should write this expression in the form $ds_U(\bar{V}) = V \circ s_U + \nu_V \circ s_U$.

Then, if $U, U' \in \mathcal{U}$ overlap, we obtain on their intersection that $\bar{\Omega}_U$ is conformally equivalent to $\bar{\Omega}_{U'}$:

$$\bar{\Omega}_{U'} = s_{U'}^* i^* \Omega \sim s_{U'}^* \pi^* \bar{\Omega}_U = \bar{\Omega}_U.$$

We use a partition of unity $\{\rho_U\}$ to glue all together these local forms, obtaining a global 2-form

$$\bar{\Omega} = \sum_{U \in \mathcal{U}} \rho_U \bar{\Omega}_U$$

on $\mu^{-1}(0)/G$ which, by Lemma 3.5, is locally conformal to any $\bar{\Omega}_U$.

This implies that $\bar{\Omega}$ is still compatible with $\bar{J}$ and positive, and therefore induces a global Hermitian metric $\bar{g}$ on $\mu^{-1}(0)/G$ which is locally conformal Kähler because it is locally conformal to the locally conformal Kähler metrics $\bar{g}_U$ on U. This ends the existence part.

If g' is any locally conformal Kähler metric on $\mu^{-1}(0)/G$ such that $\pi^* g' \sim i^* g$, then for any $x \in \mu^{-1}(0)/G$ on $U_x \subset U$ we obtain $g'|_{U_x} = s_U^* \pi^* g'|_{U_x} \sim s_U^* i^* g|_{U_x} = \bar{g}_U|_{U_x} \sim \bar{g}|_{U_x}$. So the globally defined metrics g and g', being locally conformal, are in fact conformal, by Lemma 3.5. The claim then follows. ■

Remark 4.2 If $\mu^{-1}(0)/G$ has real dimension two then reduction equips it with a complex structure and a conformal family of Kähler metrics.

Remark 4.3 Let us note by passing that the zero level set offers a natural example of CR-submanifold of M (see [DO98]). Indeed, the tangent space in each point splits as a direct orthogonal sum of a J-invariant and a J-anti-invariant distribution: $T_x(\mu^{-1}(0)) = E_x \oplus \mathfrak{g}(x)$. A result of D. Blair and B. Y. Chen states that the anti-invariant distribution of a CR-submanifold in a locally conformal Kähler manifold is integrable. In our case, this is trivially true because the anti-invariant distribution is just a copy of the Lie algebra of G.

5 Conformal setting and the Biquard-Gauduchon construction

In defining the reduced locally conformal Kähler structure on $\mu^{-1}(0)/G$ we used a specific metric in the conformal class $[g]$, to obtain a conformal class $[\bar{g}]$. In this section we present a more intrinsic construction for the locally conformal Kähler reduction, due to O. Biquard and P. Gauduchon, which makes use of the language of conformal geometry. To this aim we mainly fill in details and reorganize material contained in [CP99] and in the unpublished paper [BG98].

Moreover we prove that the two constructions are in fact the same, by showing in Lemma 5.2 and its consequences the correspondence between representatives and intrinsic objects.

Let V be a real n-dimensional vector space, and t a real number. The 1-dimensional vector space L_V^t of *densities of weight* t on V is the vector space of maps $l\colon (\Lambda^n V) \setminus 0 \to \mathbb{R}$ satisfying $l(\lambda w) = |\lambda|^{-t/n} l(w)$ if $\lambda \in \mathbb{R} \setminus 0$ and $w \in (\Lambda^n V) \setminus 0$. We say that a density l is *positive* if it takes only positive real values. For positive integers t we have $L_V^t = L_V^1 \otimes \cdots \otimes L_V^1$ and for negative integers t we have $L_V^t = (L_V^1)^* \otimes \cdots \otimes (L_V^1)^*$. Thus, given an element l of L_V^1, we denote by l^t the corresponding element of L_V^t under these canonical identifications, for any t integer.

Remark that $\Lambda^{n+d}(V \oplus \mathbb{R}^d) \simeq \Lambda^n V$, and this gives a canonical isomorphism between $L_{V \oplus \mathbb{R}^d}^t$ and L_V^t:

$$l \in L_V^t \mapsto \operatorname{sgn}(l) l^{\frac{n}{n+d}} \in L_{V \oplus \mathbb{R}^d}^t. \tag{4}$$

To any Euclidean metric g on the vector space V we associate the positive element l_g^t of L_V^t which sends the length-one element of $(\Lambda^n V) \setminus 0$ to 1. Then under a homothety $e^\alpha g$ of the metric we have $l_{e^\alpha g}^t = e^{-t\alpha/2} l_g^t$, and the positive definite element $g \otimes l_g^2$ of $S^2 V \otimes L_V^2$ only depends on the homothety class c of g.

Conversely, given an element c of $S^2 V \otimes L_V^2$, we can associate to any positive element l of L_V^1 the element $c \otimes l^{-2}$ of $S^2 V \otimes L_V^2 \otimes L_V^{-2} = S^2 V$, and if c is positive definite so is $c \otimes l^{-2}$, which therefore defines a Euclidean metric on V. If, moreover, c satisfies the normalization condition $l_{c \otimes l^{-2}}^2 = l^2$ for one (and hence for all) positive element l of L_V^1, then the correspondence between such c's and the homothety classes of g is bijective.

For any vector bundle $E \to M$, define the associated *density line bundle* $L_E^t \to M$ as the bundle whose fiber over $x \in M$ is the 1-dimensional vector space $L_{E_x}^t$. If n is the rank of E, then L_E^t can be globally defined as the fibred product $\mathrm{P}(E) \times_G L_{\mathbb{R}^n}^t$, where $\mathrm{P}(E)$ denotes the principal bundle associated to E with structure group $G \subset \mathrm{GL}(n)$, and an element A of G acts on $L_{\mathbb{R}^n}^t$ by multiplication by $|\det A|^{t/n}$. Remark that, in particular, L_E^t has the same principal bundle as E, for any $t \in \mathbb{R}$.

The above construction identifies conformal classes of metrics on E with normalized positive defined sections of $S^2E \otimes L^2_E$. In particular, if $E = TM$, the conformal class of a Riemannian metric can be thought of as a normalized positive defined section c of $S^2M \otimes L^2_M$, where we denote L^t_{TM} by L^t_M.

A trivialization (usually positive) of L^1_M is called a *gauge* or also a *length scale.*

This way, on a conformal manifold (M, c), we have a Riemannian metric whenever we fix a gauge. As a terminology, instead of saying "...take a gauge l, and let $g \stackrel{\mathrm{def}}{=} c \otimes l^{-2}$..." we shall say "...let g be a metric in the conformal class c...".

Since a connection on M means a connection on $\mathrm{GL}(M)$ and $\mathrm{GL}(M)$ is also the principal bundle of L^t_M, a connection on M induces a connection on L^t_M, for any $t \in \mathbb{R}$. Vice versa, suppose a connection ∇ on L^1_M is given. Then we can use a conformal version of the six-terms formula to define a connection on M, still denoted by ∇, which is compatible with c:

$$2c(\nabla_X Y, Z) = \nabla_X c(Y, Z) + \nabla_Y c(X, Z) - \nabla_Z c(X, Y) + c([X, Y], Z) - c([X, Z], Y) - c([Y, Z], X), \quad (5)$$

where both members are sections of L^2_M.

This way one proves the fundamental theorem of conformal geometry:

Theorem 5.1 (Weyl) *Let (M, c) be a conformal manifold. There is an affine bijection between connections on L^1_M and torsion-free connections on M preserving c.*

Torsion-free compatible connections on a conformal manifold are called Weyl connections. In contrast with the Riemannian case, the previous theorem says in particular that on a conformal manifold there is not a uniquely defined torsion-free compatible connection.

In this setting a *conformal almost-Hermitian manifold* is a conformal manifold (M, c) together with an almost-complex structure J on M compatible with one (and hence with all) metric in the conformal class.

Let (M, c, J) be a conformal almost-Hermitian manifold. We then have a non-degenerate fundamental form Ω taking values in L^2_M, that is, $\Omega(X, Y) \stackrel{\mathrm{def}}{=} c(JX, Y) \in \Gamma(L^2_M)$, for $X, Y \in \mathfrak{X}(M)$. For any metric g defining c, with corresponding fundamental form Ω_g, we have $\Omega = \Omega_g \otimes l^2_g$. The notion of Lee form ω_g of the almost-Hermitian metric g on (M, J) is clearly dependent on the metric, but a straightforward computation shows that the connection ∇ on L^1_M given by $\nabla_X l_g \stackrel{\mathrm{def}}{=} (-1/2)\omega_g(X) l_g$ does not depend on the choice of g in the conformal class c.

The fundamental theorem of conformal geometry gives then a torsion-free compatible connection on M, which is called the *canonical Weyl connection* of the conformal almost-Hermitian manifold (M, c, J). We denote simply by ∇ this connection on M, and we use the same symbol for the induced connection on L^t_M, for any $t \in \mathbb{R}$. In particular, the constant $-1/2$ in the definition of ∇ was chosen in order that $\nabla l^2_g = -\omega_g \otimes l^2_g$.

Thus, given any L^t_M-valued tensor ψ on a conformal almost-Hermitian manifold, we can differentiate it with respect to the canonical Weyl connection, and any choice of a metric g in the conformal class c gives a corresponding real valued tensor ψ_g. The following Lemma links this intrinsic point of view with the gauge-dependant setting of almost-Hermitian manifolds. We state it only for L^2_M-valued differential forms, because this is the only case we need.

Lemma 5.2 (Equivalence lemma) *Let (M, c, J) be a conformal almost-Hermitian manifold, with canonical Weyl connection ∇. Let ψ be a p-form taking values in L^2_M. Then for any metric g in the conformal class c we have*

$$d^\nabla \psi = d^{\omega_g} \psi_g \otimes l^2_g.$$

Proof:

$$\begin{aligned} d^\nabla \psi = d^\nabla(\psi_g \otimes l^2_g) &= d\psi_g \otimes l^2_g + (-1)^{|\psi_g|} \psi_g \wedge \nabla l^2_g \\ &= d\psi_g \otimes l^2_g - (-1)^{|\psi_g|} \psi_g \wedge \omega_g \otimes l^2_g = d\psi_g \otimes l^2_g - \omega_g \wedge \psi_g \otimes l^2_g = d^{\omega_g} \psi_g \otimes l^2_g. \end{aligned}$$

■

Using the equivalence Lemma we obtain in particular

$$d^\nabla \Omega = d^{\omega_g} \Omega_g \otimes l^2_g. \quad (6)$$

Since the Weyl connection is compatible with c, we have also

$$0 = \nabla c = \nabla(g \otimes l^2_g) = \nabla g \otimes l^2_g + g \otimes \nabla l^2_g = \nabla g \otimes l^2_g - g \otimes \omega_g \otimes l^2_g = (\nabla g - \omega_g \otimes g) \otimes l^2_g. \quad (7)$$

Theorem 5.3 *Let (M, c, J) be a conformal almost-Hermitian manifold, and let ∇ be the canonical Weyl connection. Let g be any metric in the conformal class c. Then:*

i) ∇ preserves J if and only if J is integrable and $(d\Omega_g)_0 = 0$;

ii) the curvature $R^\nabla = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ of ∇ is given by $R^\nabla l_g^2 = d\omega_g \otimes l_g^2$.

Proof: For any complex connection ∇ the following formula holds, linking the torsion T of ∇ with the torsion N of J:

$$T(JX, JY) - J(T(JX, Y)) - J(T(X, JY)) - T(X, Y) = -N(X, Y).$$

Since Weyl connections are torsion free, if we find any complex Weyl connection then J is integrable. We want to show that, if the canonical Weyl connection is complex, then also $(d\Omega_g)_0 = 0$. Denote by A the alternation operator and by C the contraction such that $\Omega_g = C(J \otimes g)$, then

$$\begin{aligned} d\Omega_g &= A(\nabla\Omega_g) = A(\nabla C(J \otimes g)) = A(C(J \otimes \nabla g)) = A(C(J \otimes \omega_g \otimes g)) \\ &= A(\omega_g \otimes C(J \otimes g)) = A(\omega_g \otimes \Omega_g) = \omega_g \wedge \Omega_g, \end{aligned}$$

where we have used formula (7) to obtain $\nabla g = \omega_g \otimes g$.

Suppose now that $(d\Omega_g)_0 = 0$ and that J is integrable. Then using the conformal six-terms formula (5) we obtain the following conformal version of a classical formula in Hermitian geometry (see [KN69, p. 148]):

$$4c((\nabla_X J)Y, Z) = 6d^\nabla\Omega(X, JY, JZ) - 6d^\nabla\Omega(X, Y, Z),$$

and this shows that $c((\nabla_X J)Y, Z) = 0$ if $d^\nabla\Omega = 0$. But this last condition is equivalent, by formula (6), to $(d\Omega_g)_0 = 0$, and the claim then follows from the non-degeneracy of c. As for the curvature R^∇ of ∇, using equivalence Lemma we obtain

$$R^\nabla l_g^2 = -d^\nabla(\nabla l_g^2) = -d^\nabla(-\omega_g \otimes l_g^2) = d^{\omega_g}\omega_g \otimes l_g^2 = d\omega_g \otimes l_g^2.$$

■

Since a locally conformal Kähler manifold is a conformal Hermitian manifold (M, c, J) such that $(d\Omega_g)_0 = 0$ and $d\omega_g = 0$, for one (and then for all) choice of metric g in the conformal class c (compare with formula (1)), we can give the following intrinsic characterization of locally conformal Kähler manifolds:

Corollary 5.4 *Let (M, c, J) be a conformal Hermitian manifold. Denote by Ω the L_M^2-valued fundamental form, and let ∇ be the canonical Weyl connection. Then (M, c, J) is locally conformal Kähler if and only if ∇ is flat and Ω is d^∇-closed.*

Moreover, Theorem 5.3 gives also the following

Corollary 5.5 *On a locally conformal Kähler manifold the canonical Weyl connection preserves the complex structure.*

Unless otherwise stated, from now on we consider locally conformal Kähler manifolds (M, c, J).

A locally conformal Kähler manifold (M, c, J) comes then naturally equipped with a closed 2-form Ω, the only difference from the Kähler case being that Ω now takes values in L_M^2. We want go further with this analogy.

Define the pairing $\sharp\colon \Omega^1(L_M^2) \to \mathfrak{X}(M)$ by $\iota_{\sharp\alpha}\Omega = \alpha$, and use it to define a Poisson bracket on $\Gamma(L_M^2)$ by $\{f_1, f_2\} \stackrel{\text{def}}{=} \Omega(\sharp\nabla f_1, \sharp\nabla f_2)$. Using Lemma 5.2 and formula (3), one shows the relation

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = d^\nabla\Omega(\sharp\nabla f_1, \sharp\nabla f_2, \sharp\nabla f_3) = 0, \tag{8}$$

proving that this bracket turns $\Gamma(L_M^2)$ into a Lie algebra. Remark that, as formula (3), the first equality in (8) holds generally on conformal almost-Hermitian manifolds such that the canonical Weyl connection is flat.

We finally describe the intrinsic version of $\mathrm{Aut}(M)$. If l is a section of L_M^t and h is a diffeomorphism of M, then the section h^*l of L_M^t is given by

$$(h^*l)_x \stackrel{\text{def}}{=} l_{h(x)} \circ (h_*)_x, \tag{9}$$

that is, if $x \in M$ and $w \in \Lambda^n(T_xM) \setminus 0$, we have $(h^*l)_x(w) \overset{\text{def}}{=} l_{h(x)}((h_*)_x w)$. Recall that for any x the differential induces the map $(h_*)_x \colon \Lambda^n(T_xM) \to \Lambda^n(T_{h(x)}M)$ which is in fact a linear map between 1-dimensional vector spaces. Whenever a metric g is fixed, a trivialization w^g of $\Lambda^n(TM)$ associating to x the length-one element w^g_x is defined, hence one can associate to any diffeomorphism h a never-vanishing smooth function d^g_h defined by

$$h_* w^g_x = d^g_h(x) w^g_{h(x)},$$

so the following derivation rule holds for l^t_g:

$$\begin{aligned}
(h^* l^t_g)_x(w^g_x) &= (l^t_g)_{h(x)}(h_* w^g_x) \\
&= (l^t_g)_{h(x)}(d^g_h(x) w^g_{h(x)}) \\
&= |d^g_h(x)|^{-\frac{t}{n}} (l^t_g)_{h(x)}(w^g_{h(x)}) = |d^g_h(x)|^{-\frac{t}{n}},
\end{aligned}$$

that is, in short, $h^* l^t_g = |d^g_h|^{-\frac{t}{n}} l^t_g$.

For any diffeomorphism of M we then define h^*c in the obvious way, that is, $h^*c = h^*g \otimes h^* l^2_g$. Since

$$d^{e^f g}_h = e^{(n/2)f \circ h} e^{(-n/2)f} d^g_h,$$

this definition does not depend on the choice of the gauge g, and gives the intrinsic notion of $\mathrm{Aut}(M)$ as follows.

Proposition 5.6 *A diffeomorphism h of a conformal manifold (M, c) preserves c if and only if it is a conformal transformation of one (and hence of all) metric g in the conformal class c.*

Proof: Indeed, $h^*g = e^\alpha g$ implies $d^g_h = e^{n\alpha/2}$, so $h^* l^2_g = e^{-\alpha} l^2_g$, and then $h^*c = h^*(g \otimes l^2_g) = h^*g \otimes h^* l^2_g = g \otimes l^2_g = c$. Vice versa $h^*c = c$ implies $h^*g \otimes h^* l^2_g = g \otimes l^2_g$, hence $(|d^g_h|^{-\frac{2}{n}} h^*g) \otimes l^2_g = g \otimes l^2_g$, that is $h^*g = |d^g_h|^{\frac{2}{n}} g$. ■

Lemma 5.7 *The Weyl connection of a conformal almost-Hermitian manifold (M, c, J) is invariant for* $\mathrm{Aut}(M)$*, that is, $h_* \nabla_V W = \nabla_V W$ whenever $h_* V = V$ and $h_* W = W$.*

Proof: This is because $\mathrm{Aut}(M)$ preserves c and J, and ∇ is defined just using these ingredients. More formally, we want to show that, if $h \in \mathrm{Aut}(M)$ and V, W, Z are h-invariant vector fields, then $c(\nabla_V W, Z) = c(h_* \nabla_V W, Z)$. But we have

$$c(\nabla_V W, Z) = (h^*c)(\nabla_V W, Z) = h^*(c(h_* \nabla_V W, h_* Z)) = h^*(c(h_* \nabla_V W, Z)),$$

where we used the general property that if ψ is any tensor field of type $(r, 0)$ and $X_1, \ldots, X_r$ are vector fields, then $h^*(\psi(h_* X_1, \ldots, h_* X_r)) = (h^*\psi)(X_1, \ldots, X_r)$. We are therefore only left to show that $c(\nabla_V W, Z)$ is h-invariant for all $h \in \mathrm{Aut}(M)$, that is, we are left to show that the second side of the conformal six-terms formula (5) is h-invariant for all $h \in \mathrm{Aut}(M)$. But it turns out that each summand of (5) is h-invariant. We show this only on its first and fourth summand, the others being similar: the first summand

$$\begin{aligned}
h^* \nabla_V c(W, Z) &= h^* \nabla_V (g(W, Z) l^2_g) \\
&= h^*(V g(W, Z) l^2_g) + h^*(g(W, Z) \nabla_V l^2_g) \\
&= h^* V g(W, Z) h^* l^2_g - h^*(g(W, Z)) h^*(\omega_g(V)) h^* l^2_g \\
&= V((h^*g)(W, Z)) l^2_{h^*g} - (h^*g)(W, Z) \omega_{h^*g}(V) l^2_{h^*g} \\
&= \nabla_V((h^*g)(W, Z) l^2_{h^*g}) = \nabla_V c(W, Z),
\end{aligned}$$

where we have used that V and h commute on $C^\infty(M)$, since V is h-invariant, that $h^* l^2_g = l^2_{h^*g}$ and that $h^* \omega_g = \omega_{h^*g}$. The fourth summand is

$$h^*(c([V, W], Z)) = h^*(g([V, W], Z) l^2_g) = h^*(g([V, W], Z)) h^* l^2_g = (h^*g)([V, W], Z) l^2_{h^*g} = c([V, W], Z),$$

where we have used the already cited properties and that the Lie bracket of invariant vector fields is invariant. ■

Corollary 5.8 *Let (M, c, J) be a conformal almost-Hermitian manifold with Weyl connection ∇, and let $G \subset \mathrm{Aut}(M)$. If V, W, Z are G-invariant vector fields on M, then $c(\nabla_V W, Z)$ is G-invariant.*

Let G be a Lie subgroup of $\mathrm{Aut}(M)$, as in section 3. The momentum map can then be defined as a homomorphism of Lie algebras $\mu^{\cdot} : \mathfrak{g} \to \Gamma(L^2_M)$ such that $\iota_X \Omega = d^\nabla \mu^X$. We also denote by μ the corresponding element of $\Gamma(\mathfrak{g}^* \otimes L^2_M)$ given by $\langle \mu(x), X\rangle = \mu^X(x)$, carets denoting the evaluation.

Remark 5.9 In [BG98] the existence of such a homomorphism of Lie algebras is shown to imply the condition

$$-\frac{1}{2}\omega_g(X) + \frac{1}{n}\operatorname{div}_g X = 0$$

on any fundamental vector field X. This is equivalent to the condition

$$\mathcal{L}_X \Omega_g - \omega_g(X)\Omega_g = 0$$

one finds in [HR01], since $\mathcal{L}_X \Omega_g = ((2/n)\operatorname{div}_g X)\Omega_g$.

If we choose a metric g in the conformal class c, then $\mu^X = \mu^X_g l^2_g$, where $\mu^{\cdot}_g : \mathfrak{g} \to C^\infty(M)$.

Theorem 5.10 *The map $\mu^{\cdot} : \mathfrak{g} \to \Gamma(L^2_M)$ is a momentum map if and only if $\mu^{\cdot}_g : \mathfrak{g} \to C^\infty(M)$ is a momentum map as in section 3.*

Proof: Use Lemma 5.2 to compute $d^\nabla \mu^X$ with respect to the fixed gauge:

$$d^\nabla \mu^X = d^{\omega_g}\mu^X_g \otimes l^2_g,$$

so that $d^\nabla \mu^X = \iota_X(\Omega_g \otimes l^2_g) = \iota_X \Omega_g \otimes l^2_g$ if and only if $\iota_X \Omega_g = d^{\omega_g}\mu^X_g$. We then have to check that μ is a Lie algebra homomorphism if and only if μ_g is. But this is a direct consequence of Lemma 5.2, and of the fact that $\sharp\alpha = \sharp_g \alpha_g$:

$$\{f_1, f_2\} = \Omega(\sharp\nabla f_1, \sharp\nabla f_2) = \Omega_g(\sharp\nabla f_1, \sharp\nabla f_2) \otimes l^2_g = \Omega_g(\sharp_g d^{\omega_g} f_{1,g}, \sharp_g d^{\omega_g} f_{2,g}) \otimes l^2_g = \{f_{1,g}, f_{2,g}\}_g \otimes l^2_g.$$

$\blacksquare$

Remark 5.11 The previous theorem allows using all proofs of section 3 as proofs in this conformal setting, just fixing a gauge. In particular, the zero set $\mu^{-1}(0)$, where 0 denotes the zero section of $\mathfrak{g}^* \otimes L^2_M$, is the zero set of any μ_g, and it is therefore closed with respect to the action of G and coisotropic with respect to Ω. Moreover, the assumption of 0 being a regular value for μ_g translates into the assumption that the zero section be transverse to μ, and under this assumption the isotropic foliation is given exactly by fundamental vector fields $\mathfrak{g}$.

Theorem 5.12 (Biquard & Gauduchon, [BG98]) *Let (M, c, J) be a locally conformal Kähler manifold. Let G be a Lie subgroup of $\mathrm{Aut}(M)$ whose action admits a momentum map $\mu : \mathfrak{g} \to \Gamma(L^2_M)$. Suppose that G acts freely and properly on $\mu^{-1}(0)$, 0 denoting the zero section of $\mathfrak{g}^* \otimes L^2_M$, and suppose that μ is transverse to this zero section. Then there exists a locally conformal Kähler structure $(\bar{c}, \bar{J})$ on $\mu^{-1}(0)/G$.*

Proof: Due to Lemma 5.2 and to Theorem 5.10, this theorem can be viewed at as a translation of Theorem 4.1 in the conformal language. From this point of view, the theorem was already proved.

We want here to give an intrinsic proof, using the characterization of locally conformal Kähler manifolds given by corollary 5.4.

Take the c-orthogonal decomposition $T_x M = E_x \oplus \mathfrak{g}(x) \oplus J\mathfrak{g}(x)$, where E_x is the c-orthogonal complement of $\mathfrak{g}(x)$ in $T_x(\mu^{-1}(0))$. We obtain a vector bundle $E \to \mu^{-1}(0)$ of rank $n - 2\dim G$.

First we need to relate $L^t_{\mu^{-1}(0)/G}$ with L^t_E. Remark that $E/G \to \mu^{-1}(0)/G$ is isomorphic as a bundle to the tangent bundle of $\mu^{-1}(0)/G$, by means of $\pi_*|_E$. On its side L^t_E/G is isomorphic to $GL(E/G) \times_{GL(n-2\dim G)} L^t_{\mathbb{R}^{n-2\dim G}}$, since the actions of G and of $GL(n-2\dim G)$ on $GL(E)$ commute, that is, if $g \in G$, $\gamma \in GL(n-2\dim G)$ and $p \in GL(E)$, then $g_*(p\gamma) = (g_* p)\gamma$. This means that $L^t_{\mu^{-1}(0)/G}$ is isomorphic to L^t_E/G, the isomorphism being explicitly given by sending an element l of $L^2_{\mu^{-1}(0)/G,\bar{x}}$ to $[l \circ \pi_{*,x}]$, where $\pi(x) = \bar{x}$, and the action of G on L^t_E being given by (9).

Now remark that the canonical splitting $TM = E \oplus \mathfrak{g} \oplus J\mathfrak{g}$ gives an isomorphism of $L^2_M|_{\mu^{-1}(0)}$ with L^2_E, by formula (4), and this isomorphism is G-equivariant.

We therefore think of elements of $L^2_{\mu^{-1}(0)/G}$ as equivalence classes of elements of $L^2_M|_{\mu^{-1}(0)}$.

During the proof of this theorem, we denote by $\bar V, \bar W, \dots$ vector fields on $\mu^{-1}(0)/G$, and by $V, W, \dots$ their lifts to E. Note that $V, W, \dots$ are G-invariant vector fields.

Define $\bar c(\bar V, \bar W)$ to be the projection to $L^2_M|_{\mu^{-1}(0)}/G$ of the section $c(V, W)$, that is

$$(\bar c(\bar V, \bar W))_{\bar x} \stackrel{\text{def}}{=} [c(V,W)_x] \in (L^2_M|_{\mu^{-1}(0)})_x/G \simeq (L^2_{\mu^{-1}(0)/G})_{\bar x}$$

where x is an element in $\pi^{-1}(\bar x)$. The choice of x is irrelevant, since $h^*(c(V,W)) = h^*(c(h_*V, h_*W)) = (h^*c)(V,W) = c(V,W)$.

We have thus defined an almost-Hermitian conformal manifold $(\mu^{-1}(0)/G, \bar c, \bar J)$. In order to show that it is locally conformal Kähler we compute its canonical Weyl connection, and then use corollary 5.4.

Let ∇^E be the orthogonal projection of ∇ from $T(\mu^{-1}(0))$ to E. Since by Lemma 5.7 the Weyl connection ∇ is invariant for $\mathrm{Aut}(M)$, we have that $\nabla^E_V W$ is a projectable vector field. Define

$$\bar\nabla_{\bar V}\bar W \stackrel{\text{def}}{=} \pi_* \nabla^E_V W. \tag{10}$$

The torsion $T^{\bar\nabla}_{\bar V, \bar W}$ of $\bar\nabla$ is just $\pi_* T^{\nabla^E}_{V,W} = 0$. Moreover, $\bar\nabla$ is compatible with $\bar J$:

$$\begin{aligned}(\bar\nabla_{\bar V}\bar J)\bar W &= \bar\nabla_{\bar V}(\bar J \bar W) - \bar J \bar\nabla_{\bar V}\bar W\\ &= \pi_*\nabla^E_V(JW) - \bar J \pi_* \nabla^E_V W\\ &= \pi_*(\nabla^E_V(JW) - J\nabla^E_V W) = \pi_*(\nabla_V J)^E W = 0.\end{aligned}$$

Eventually, Theorem 5.3 proves that $\bar J$ is integrable.

Look at the Weyl connection ∇ on L^2_M as a map $\nabla_V \colon \Gamma(L^2_M|_{\mu^{-1}(0)}) \to \Gamma(L^2_M|_{\mu^{-1}(0)})$, and remark that the $\mathrm{Aut}(M)$-invariance of V implies that ∇_V is G-equivariant, thus defines a connection on $L^2_{\mu^{-1}(0)/G}$. We denote it again by $\bar\nabla$:

$$\bar\nabla_{\bar V}[l] \stackrel{\text{def}}{=} [\nabla_V l] \in L^2_M|_{\mu^{-1}(0)}/G.$$

Using the conformal six-terms formula (5) and corollary 5.8, we see that the connection $\bar\nabla$ on $\mu^{-1}(0)/G$ defined by (10) is the associated Weyl connection, which is therefore the canonical Weyl connection of $(\mu^{-1}(0)/G, \bar c, \bar J)$.

The curvature $R^{\bar\nabla}$ is given by

$$R^{\bar\nabla}_{\bar V, \bar W}[l] = -d^{\bar\nabla}\bar\nabla[l](\bar V, \bar W) = -[d^\nabla \nabla l(V,W)] = [R^\nabla_{V,W}] = 0.$$

Finally, denoting by $\bar\Omega$ the $L^2_{\mu^{-1}(0)/G}$-valued fundamental form of $(\mu^{-1}(0)/G, \bar c, \bar J)$, we have $\pi^*(d^{\bar\nabla}\bar\Omega) = d^\nabla\Omega = 0$, thus $d^{\bar\nabla}\bar\Omega = 0$, and corollary 5.4 says that $(\mu^{-1}(0)/G, \bar c, \bar J)$ is locally conformal Kähler. ∎

6 Compatibility with Kähler reduction

In this section we analyze the relation between locally conformal Kähler reduction of a manifold and Kähler reduction of a covering. We refer to [Fut88] for the Kähler reduction.

As a first step we show that the two notions of reduction on globally conformal Kähler manifolds are compatible.

Proposition 6.1 *Let $(M, [g], J)$ be a globally conformal Kähler manifold and denote by g a Kähler metric. Let $G \subset \mathrm{Aut}(M)$ a subgroup satisfying the hypothesis of the reduction theorem and which moreover is composed by isometries with respect to g. Denote by $(\mu^{-1}(0)/G, [\bar g], \bar J)$ the reduced locally conformal Kähler manifold. Then the action of G is Hamiltonian for g, the submanifold $\mu^{-1}(0)$ is the same as in the Kähler reduction and the conformal class of the reduced Kähler metric is $[\bar g]$. So, in particular, the reduced manifold is globally conformal Kähler.*

Proof: As the action of G is twisted Hamiltonian for $[g]$ Remarks 2.4 and 3.2 imply that it is Hamiltonian for g. Moreover, the subspace $\mu^{-1}(0)$ is the same for both notions. The construction of the almost-complex structure on the quotient is the same in the two cases, so $\bar J$ is defined. Denote by $\tilde\Omega$ the Kähler form that the Kähler reduction provides on $\mu^{-1}(0)/G$. Then $\pi^*\tilde\Omega = i^*\Omega$, so the claim follows by the uniqueness part of the reduction theorem. ∎

Example 6.2 If $(M, [g], J)$ is a globally conformal Kähler manifold the reduced structure is not necessarily globally conformal Kähler. Actually, any locally conformal Kähler manifold $(M, [g], J)$ can be seen as a reduction of a globally conformal manifold. Indeed, consider the universal covering $\tilde{M}$ of M equipped with its pulled-back locally conformal Kähler structure, which is globally conformal Kähler since $\tilde{M}$ is simply connected. This covering manifold can be considered to be acted on by the discrete group of holomorphic conformal maps $G \stackrel{\text{def}}{=} \pi_1(M)$, which, having trivial associated infinitesimal action, is clearly Hamiltonian, with trivial momentum map: hence $\mu^{-1}(0) = \tilde{M}$ and $\mu^{-1}(0)/G = M$.

We now concentrate our attention to the structure of the universal cover $\tilde{M}$ of a locally conformal Kähler manifold $(M, [g], J)$.

Remark 6.3 The pull-back by the covering map p of any metric of $[g]$ is globally conformal Kähler since $\tilde{M}$ is simply connected. It is easy to show that on any complex manifold Z such that $\dim_{\mathbb{C}}(Z) \geq 2$ if two Kähler metrics are conformal then their conformal factor is constant. In our case remark that the pull-back of any metric in $[g]$ is conformal to a Kähler metric $\tilde{g}$ by

$$\tilde{g} = e^{-\tau} p^* g$$

where τ satisfies $d\tau = \omega_{\tilde{g}} = p^*\omega_g$ and is then only defined up to adding a constant. What is remarkable is that the action of $\pi_1(M)$ on $\tilde{M}$ is by homotheties of the Kähler metrics (we fix points in M and in $\tilde{M}$ in order to have this action well-defined). Moreover any element of $\mathrm{Aut}(M)$ lifts to a homothety of the Kähler metrics of $\tilde{M}$, if $\dim_{\mathbb{C}}(M) \geq 2$. This is in fact an equivalent definition of locally conformal Kähler manifolds (see [Vai82] and [DO98]).

With this model in mind, we define a *homothetic Kähler manifold* as a triple $(K, \langle g \rangle, J)$, where (K, g, J) is a Kähler manifold and $\langle g \rangle$ denotes the set of metrics differing from g by multiplication for a positive factor. We define $\mathcal{H}(K)$ to be the group of biholomorphisms of K such that $f^*g = \lambda g$, $\lambda \in \mathbb{R}_+$, and we call such a map a *homothety* of K of *dilation factor* λ. The dilation factor does not depend on the choice of g in $\langle g \rangle$, so a homomorphism ρ is defined from $\mathcal{H}(K)$ to $\mathbb{R}_+$ associating to any homothety its dilation factor (see also [KO01]). Note that $\ker \rho$ is the subgroup of $\mathcal{H}(K)$ containing the maps that are isometries of one and then all of the metrics in $\langle g \rangle$. If K is given as a globally conformal Kähler manifold $(K, [g], J)$, then $\mathcal{H}(K)$ can be considered as the well-defined subgroup of $\mathrm{Aut}(K)$ of homotheties with respect to the Kähler metrics in $[g]$. We now give a condition for a locally conformal Kähler manifold covered by a globally conformal one to be globally conformal Kähler.

Proposition 6.4 *Given a globally conformal Kähler manifold $(\tilde{M}, [\tilde{g}], J)$ and a subgroup Γ of $\mathrm{Aut}(\tilde{M})$ acting freely and properly discontinuously, the quotient $M \stackrel{\text{def}}{=} \tilde{M}/\Gamma$ (with its naturally induced complex structure) comes equipped with a locally conformal Kähler structure $[g]$ uniquely determined by the condition $[p^*g] = [\tilde{g}]$, where p denotes the covering map $\tilde{M} \to M$.*

Assume now that $\Gamma \subset \mathcal{H}(\tilde{M})$. Then the induced structure is globally conformal Kähler if and only if $\rho(\Gamma) = 1$.

Proof: The action of Γ can be seen as satisfying the hypothesis of the reduction theorem, so the first claim follows. However we give a straightforward construction.

Let $\tilde{g}$ be one of the Kähler metrics of the structure of $\tilde{M}$. Given an atlas $\{U_i\}$ for the covering map p, induce a local Kähler metric g_i on any U_i by projecting $\tilde{g}$ restricted to one of the connected components of $p^{-1}(U_i)$. Then g_i and g_j differ by a conformal map on $U_i \cap U_j$, hence by a partition of unity of $\{U_i\}$ one can glue the set $\{g_i\}$ to a global metric g which is locally conformal the g_i's, see Lemma 3.5, hence is locally conformal Kähler. The conformal class of g is uniquely defined by this construction. Moreover, p^*g is conformal to $\tilde{g}$, as they are conformal on each component of the covering $\{p^*(U_i)\}$ and again Lemma 3.5 holds. If g' is a Hermitian metric on M such that p^*g' is conformal to $\tilde{g}$, then on each U_i the restricted metric $g'|_{U_i}$ is conformal to g_i hence to $g|_{U_i}$, so g and g' are conformal, again see Lemma 3.5.

Now assume that $\Gamma \subset \mathcal{H}(K)$, and that $\rho(\Gamma) \neq 1$. Then Γ is *not* contained in the isometries of any Kähler metric of $\tilde{M}$. If in the class of $[g]$ there existed a Kähler metric $\bar{g}$ then its pull-back $p^*\bar{g}$ would belong to $\langle \tilde{g} \rangle$. But $p^*\bar{g}$ being a pull-back implies that Γ acts with isometries with respect to it, which is absurd since $\rho(\Gamma) \neq 1$. Conversely, if $\rho(\Gamma) = 1$ then p is a Riemannian covering space and g itself is Kähler. Hence the induced locally conformal Kähler structure is globally conformal Kähler if and only if $\rho(\Gamma) = 1$. ∎

This allows, under a natural condition, to compute locally conformal Kähler reduction as having a Kähler reduction as covering space. First remark that any group $G \subseteq \mathrm{Aut}(M)$ lifts to subgroups $\tilde{G} \subseteq \mathcal{H}(\tilde{M})$ all having the property that $p \circ \tilde{G} = G$.

Theorem 6.5 *Let $(M, [g], J)$ be a locally conformal Kähler manifold, let $G \subset \mathrm{Aut}(M)$ be a subgroup satisfying the hypothesis of the reduction theorem, and admitting a lifting $\tilde{G}$ such that $\rho(\tilde{G}) = 1$. Then the Kähler reduction is defined, with momentum map denoted by $\mu_{\tilde{M}}$, $\tilde{G}$ commutes with the action of $\pi_1(M)$, and the following equality of locally conformal Kähler structures holds:*

$$\mu^{-1}(0)/G \simeq (\mu_{\tilde{M}}^{-1}(0)/\tilde{G})/\pi_1(M). \tag{11}$$

Conversely, let $\tilde{G}$ be a subgroup of isometries of a homothetic Kähler manifold $(\tilde{M}, \langle\tilde{g}\rangle, J)$ of complex dimension bigger than 1 satisfying the hypothesis of Kähler reduction and commuting with the action of a subgroup $\Gamma \subset \mathcal{H}(\tilde{M})$ acting freely and properly discontinuously and such that $\rho(\Gamma) \neq 1$. Moreover, assume that Γ acts freely and properly discontinuously on $\mu_{\tilde{M}}^{-1}(0)/G$. Then $\tilde{G}$ induces a subgroup G of $\mathrm{Aut}(M)$, M being the locally conformal Kähler manifold $\tilde{M}/\Gamma$, which satisfies the hypothesis of the reduction theorem, and the isomorphism (11) holds.

Proof: To show that Kähler reduction is defined, one has to show that the action of $\tilde{G}$ is Hamiltonian with respect to the globally conformal Kähler structure of $\tilde{M}$. First remark that the Lie algebra of $\tilde{G}$ coincides with that of G, that we denote by $\mathfrak{g}$ as usual, and that the fundamental vector field associated with $X \in \mathfrak{g}$ on $\tilde{M}$ is p^*X, where, as we claimed, we identify X with its associated fundamental field on M. Fix a metric $g \in [g]$ with Lee form ω and fundamental form Ω and let μ be the momentum map for g. Then we claim that

$$\begin{array}{rccc} \mu_{\tilde{M}}^{\cdot}: & \mathfrak{g} & \longrightarrow & C^\infty(\tilde{M}) \\ & X & \longmapsto & p^*\mu^X \end{array}$$

is a momentum map for the action of $\tilde{G}$ on $\tilde{M}$ with respect to the globally conformal Kähler metric p^*g. Indeed

$$\begin{aligned} d^{p^*\omega}\mu_{\tilde{M}}^X &= d^{p^*\omega}(p^*\mu^X) \\ &= p^*d^\omega\mu^X \\ &= p^*\iota_X\Omega \\ &= \iota_{p^*X}p^*\Omega. \end{aligned}$$

The same way one shows that $\mu_{\tilde{M}}$ is a homomorphism of Poisson algebras, since such is μ_M. But now recall that from Remark 3.2 the property of an action to be twisted Hamiltonian is a conformal one, so the action of G is also twisted Hamiltonian for the Kähler metrics conformal to p^*g, and is then ordinarily Hamiltonian for these Kähler metrics from Proposition 6.1. This in turn implies, since $\rho(G) = 1$, that Kähler reduction is defined and $\mu^{-1}(0)$ is diffeomorphic to $\mu_{\tilde{M}}^{-1}(0)/\pi_1(M)$.

As the action of $\tilde{G}$ is induced by p, it commutes with the action of $\pi_1(M)$, so the following diagram of differentiable manifolds commutes:

$$\begin{tikzcd} \mu_{\tilde{M}}^{-1} \arrow[rr] \arrow[d, "p"] & & \mu_{\tilde{M}}^{-1}(0)/\tilde{G} \arrow[d] \\ \mu^{-1}(0) \arrow[r] & (\mu_{\tilde{M}}^{-1}(0)/\tilde{G})/\pi_1(M) \simeq \mu^{-1}(0)/G. & \end{tikzcd}$$

Moreover the locally conformal Kähler structures induced on $\mu^{-1}(0)/G$, as covered by the Kähler reduction $\mu_{\tilde{M}}^{-1}(0)/G$ and as locally conformal Kähler reduction, are easily seen to coincide, and this ends the first part of the proof.

Conversely note that, as in Remark 6.3, if τ is such that $p^*\omega = d\tau$, then $e^{-\tau}p^*g$ is Kähler, hence conformal to $\tilde{g}$. So Γ acts as isometries of $e^\tau\tilde{g}$. We claim that $e^\tau\mu_{\tilde{M}}$ is Γ-invariant, where by $\mu_{\tilde{M}}$ we denote the Kähler momentum map. Postponing for the moment the proof, this defines the locally conformal Kähler momentum map as $\mu^X \stackrel{\text{def}}{=} (e^\tau\mu_{\tilde{M}}^X)/\Gamma$, where we identify the Lee algebra of G with that of $\tilde{G}$. This induced momentum map is easily shown to be a homomorphism with respect to the Poisson structure. Moreover $\mu_M^{-1}(0) \simeq \mu_{\tilde{M}}^{-1}(0)/\Gamma$.

Finally since the action of Γ on $\mu_{\tilde{M}}^{-1}(0)/G$ is free and properly discontinuous, $(\mu_{\tilde{M}}^{-1}(0)/G)/\Gamma$ is a manifold, and since the diagram

$$\begin{tikzcd} \mu_{\tilde{M}}^{-1}(0) \arrow[r] \arrow[d, "p"] & \mu_{\tilde{M}}^{-1}(0)/G \arrow[d] \\ \mu_{M}^{-1}(0) \arrow[r] & (\mu_{\tilde{M}}^{-1}(0)/G)/\Gamma = \mu_{M}^{-1}(0)/G \end{tikzcd}$$

commutes, the action of G on $\mu_M^{-1}(0)$ is proper and free, that is, G satisfies the hypothesis of the reduction theorem. So the first part of the theorem implies the second.

We are left to prove the claim. For simplicity we write μ instead of $\mu_{\tilde{M}}$. First we show that for any $\gamma \in \Gamma$ and $X \in \mathfrak{g}$ we have $\gamma^*\mu^X = \rho(\gamma)\mu^X$. Indeed, recalling that $\gamma^* X = X$

$$\begin{aligned} d\gamma^*\mu^X &= \gamma^* d\mu^X \\ &= \gamma^*\iota_X\tilde{\Omega} \\ &= \iota_X\gamma^*\tilde{\Omega} \\ &= \rho(\gamma)\iota_X\tilde{\Omega} \\ &= \rho(\gamma)d\mu^X, \end{aligned}$$

hence $\gamma^*\mu^X - \rho(\gamma)\mu^X$ is constant on M, and is equal to 0 since it is so on $\mu^{-1}(0)$. So $\gamma^*(e^\tau\mu^X) = e^{\gamma^*\tau}\rho(\gamma)\mu^X$. But now recall that, from one side, the formula $\gamma^* e^\tau\tilde{\Omega} = e^{\gamma^*\tau}\rho(\gamma)\tilde{\Omega}$ holds true, from the other that Γ acts as isometries of $e^\tau\tilde{g}$, hence $e^{\gamma^*\tau}\rho(\gamma) = 1$, so the claim is true. ∎

7 Reduction of compact Vaisman manifolds

7.1 A conformal definition of compact Vaisman manifolds

The original definition of Vaisman manifold is relative to a Hermitian manifold: the metric of a Hermitian manifold (M, g, J) is a *Vaisman metric* if it is locally conformal Kähler with ω non-exact and if $\nabla^g\omega_g = 0$, where ∇^g is the Levi-Civita connection.

Definition 7.1 A conformal Hermitian manifold $(M, [g], J)$ is a *Vaisman manifold* if it is a locally conformal Kähler manifold, non globally conformal Kähler and admitting a Vaisman metric in $[g]$.

The condition on the parallelism of the Lee form is *not* invariant up to conformal changes of metric, and there is not in the literature a conformally invariant criterion to decide whether a given locally conformal Kähler manifold is Vaisman.

Such a criterion was recently given in [KO01] in the case of compact locally conformal Kähler manifolds. Here we shall use it to derive a presentation for compact Vaisman manifolds that behaves effectively with respect to reduction.

The construction strictly links Vaisman geometry with Sasaki geometry. We start with the following definition-proposition, which is equivalent to the standard one. On this subject see [Bla02, BG99].

Definition 7.2 Let (W, g_W, η) be a Riemannian manifold of odd dimension bigger than 1 with a contact form η such that on the distribution $\eta = 0$ the $(1,1)$-tensor J that associates to a vector field V the vector field $\sharp_g\iota_V d\eta$ satisfies $J^2 = -1$. Call ζ the Reeb vector field of η. Define on the cone $W \times \mathbb{R}$ the metric $g = e^t dt \otimes dt + e^t\pi^* g_W$ and the complex structure that extends J associating to dt the vector field $\pi^*\zeta$, π being the projection of $W \times \mathbb{R}$ to W. This is equivalent to assigning to $W \times \mathbb{R}$ the same J and the compatible symplectic form $\Omega \overset{\text{def}}{=} d(e^t\pi^*\eta)$. Then we say that (W, g_W, η) is a *Sasaki manifold* if its cone $(W \times \mathbb{R}, e^t dt \otimes dt + e^t\pi^* g_W, J)$ is Kähler.

The standard example is that of the odd-dimensional sphere contained in $\mathbb{C}^n \setminus 0$, with $n \geq 2$. The usual Kähler metric $\sum dz_i \otimes d\bar{z}_i$ associated to the complex structure of $\mathbb{C}^n \setminus 0$ restricts to the sphere to a Riemannian metric and a CR-structure, respectively, that give to S^{2n-1} the Sasaki structure whose cone is $\mathbb{C}^n \setminus 0$ itself, via the identification $(x,t) \mapsto e^{t/2}x$.

It is well-known that the conformal metric $|z|^{-2}\sum dz_i \otimes d\bar{z}_i$ has parallel Lee form. This property extends to every Kähler cone, as is implicit in [KO01].

Lemma 7.3 *The Kähler cone $(W\times\mathbb{R}, g, J)$ of a Sasaki manifold admits the metric $\tilde{g} = 2e^{-t}g$ in its conformal class such that $\nabla^{\tilde{g}}\omega_{\tilde{g}} = 0$. In particular the Lee form of $\tilde{g}$ is $-dt$.*

Proof: Recall that the fundamental form Ω of g is Kähler, so the fundamental form $\tilde{\Omega} = 2e^{-t}\Omega$ of $\tilde{g}$ is such that $d\tilde{\Omega} = -2e^{-t}dt\wedge\Omega = -dt\wedge\Omega$. So $-dt$ is the Lee form of $\tilde{g}$. Remark that

$$\tilde{g} = 2dt\otimes dt + 2\pi^* g_W$$

This shows that ∂_t is twice the metric dual of $-dt$. Recall that for any 1-form σ the following holds

$$2\tilde{g}(\nabla^{\tilde{g}}_X\sigma^\sharp, Z) = (\mathcal{L}_{\sigma^\sharp}\tilde{g})(X,Z) + d\sigma(X,Z)$$

where $\mathcal{L}$ denotes the Lie derivative. So, in our case, since dt is closed, we only have to show that ∂_t is Killing. But this is true since $\mathcal{L}_{\partial_t}\tilde{g} = 2\mathcal{L}_{\partial_t}dt\otimes dt$, and $\mathcal{L}_{\partial_t}dt = d\iota_{\partial_t}dt = 0$. ■

The following also follows from computations developed in [KO01].

Proposition 7.4 *Let (W, g_W, η) be a Sasaki manifold, let Γ be a subgroup of $\mathcal{H}(W\times\mathbb{R})$ acting freely and properly discontinuously on $W\times\mathbb{R}$, in such a way that $\rho(\Gamma)\neq 1$ and for any $\gamma\in\Gamma$*

$$\gamma\circ\phi_t = \phi_t\circ\gamma,$$

that is, Γ commutes with the real flow generated by ∂_t.

Then the induced locally conformal Kähler structure on $M \overset{\text{def}}{=} (W\times\mathbb{R})/\Gamma$ is a Vaisman structure, not globally conformal Kähler.

Proof: Since $\Gamma\subset\mathcal{H}(W\times\mathbb{R})$ and $\rho(\Gamma)\neq 1$ the quotient has a locally non globally conformal Kähler structure, recall Proposition 6.4.

To show that the structure is Vaisman we show that Γ acts by isometries of the metric $\tilde{g} = 2e^{-t}g_{W\times\mathbb{R}}$, where by $g_{W\times\mathbb{R}}$ we denote the cone metric on $W\times\mathbb{R}$. This is equivalent to show that Γ acts by symplectomorphisms of the conformal Kähler form $2e^{-t}d(e^t\pi^*\eta)$.

We claim that for any $\gamma\in\Gamma$ the following properties hold:

$$\gamma^*\pi^*\eta = \pi^*\eta$$
$$\gamma^* e^t = \rho(\gamma)e^t.$$

For this, first note that γ commuting with the real natural flow implies $\gamma_*\partial_t = \partial_t$, it being holomorphic implies $\gamma_* J\partial_t = J\partial_t$ and it being conformal implies $\gamma_*\langle\partial_t, J\partial_t\rangle^\perp = \langle\partial_t, J\partial_t\rangle^\perp$. Now remark that for $X\in\langle\partial_t, J\partial_t\rangle^\perp$ $\pi_* X\in \text{Null}\ \eta$, so

$$\gamma^*\pi^*\eta(X) = \eta(\pi_*\gamma_* X) = 0.$$

Moreover $1 = \eta(\zeta) = \eta(\pi_*(J\partial_t)) = \eta(\pi_*(\gamma_* J\partial_t) = \gamma^*\pi^*\eta(J\partial_t)$ and this implies the first claim. Now recall that $\pi^*\eta = e^{-t}\iota_{\partial_t}\Omega$, so

$$\begin{aligned}\gamma^*(e^t)\rho(\gamma)\iota_{\partial_t}\Omega &= \gamma^*(e^{-t}\iota_{\partial_t}\Omega)\\ &= \gamma^*\pi^*\eta\\ &= \pi * \eta\\ &= e^{-t}\iota_{\partial_t}\Omega\end{aligned}$$

which shows the second claim.

Then it follows

$$\begin{aligned}\gamma^*(2e^{-t}d(e^t\pi^*\eta)) &= 2\rho(\gamma)^{-1}e^{-t}d\gamma^*(e^t\pi^*\eta)\\ &= 2\rho(\gamma)^{-1}e^{-t}\rho(\gamma)d(e^t\gamma^*\pi^*\eta)\\ &= 2e^{-t}d(e^t\pi^*\eta).\end{aligned}$$

So $\tilde{g}$ factors through the action of Γ, hence inducing g_M on M which, by Lemma 7.3, is Vaisman, and belongs to the locally conformal Kähler structure of M since $p^*g_M = \tilde{g}\sim g_{W\times\mathbb{R}}$. ■

The characterization given in [KO01] shows in fact that *any* compact Vaisman manifold is produced this way. We briefly recall this construction, since some details which are less relevant in that work become necessary in this one, so we need to express them explicitly.

Remark that a vector field V generating a 1-parameter subgroup of $\text{Aut}(M)$ does *not* imply that the flow of JV is contained in $\text{Aut}(M)$. If this happens, the set of the flows of the subalgebra generated by V and JV is a Lie subgroup of $\text{Aut}(M)$ of real dimension 2 that has a structure of complex Lie group of dimension 1. This motivates the following definition:

Definition 7.5 ([KO01]) A *holomorphic conformal flow* on a locally conformal Kähler manifold $(M, [g], J)$ is a 1-dimensional complex Lie subgroup of the biholomorphisms of (M, J) which is contained in $\mathrm{Aut}(M)$.

Remark 7.6 The field ∂_t on a Kähler cone of a Sasaki manifold generates a holomorphic conformal flow. Its flow $\phi_s(w,t) = (w, t+s)$ is in fact contained in $\mathcal{H}(W \times \mathbb{R})$, and satisfies $\rho(\phi_s) = e^s$, since $\pi\phi_s = \pi$ and

$$\begin{aligned}\phi_s^*(d(e^t\pi^*\eta)) &= d(e^{t+s}\phi_s^*\pi^*\eta) \\ &= e^s d(e^t\pi^*\eta).\end{aligned}$$

The flow of $J\partial_t$, which is a vector field that restricts to the Reeb vector field of W, which is a Killing vector field of W, generates isometries of $W \times \mathbb{R}$. We call the real flow generated by ∂_t the *natural real flow* and the holomorphic conformal flow generated by ∂_t the *natural holomorphic flow* of the Kähler cone.

Finally remark that for a biholomorphism h of a Hermitian manifold to commute with the flow of a vector field V it is necessary and sufficient that it commutes with the whole holomorphic flow, since $h^*V = V$ is equivalent to $h^*JV = JV$. So if a holomorphic conformal flow $\mathcal{C}$ is defined on a locally conformal Kähler manifold saying that it is preserved by an automorphism h is equivalent to saying that h preserves a real generator of $\mathcal{C}$.

Theorem 7.7 (Kamishima & Ornea, [KO01]) *Let $(M, [g], J)$ be a compact, connected, non-Kähler, locally conformal Kähler manifold of complex dimension $n \geq 2$. Then $(M, [g], J)$ is Vaisman if and only if* $\mathrm{Aut}(M)$ *admits a holomorphic conformal flow.*

Proof: For the technical lemmas we refer directly to the cited paper.

First, if M is a Vaisman manifold, then the dual vector field $\sharp\omega$ of its Vaisman metric generates a holomorphic conformal flow, that is, both the flow of $\omega^\sharp$ and the flow of $J\omega^\sharp$ belong to $\mathrm{Aut}(M)$, see [DO98].

On the opposite direction let $\mathcal{C}$ be the holomorphic conformal flow on M. Fix a lift $\tilde{\mathcal{C}}$ of $\mathcal{C}$ to $\tilde{M}$. One proves (Lemma 2.1) that $\rho(\tilde{\mathcal{C}}) = \mathbb{R}_+$. Choose a vector field ξ on $\tilde{M}$ such that the flow $\{\psi_t\}$ of $-J\xi$ is contained in $\ker \rho|_{\tilde{\mathcal{C}}}$. Remark that the flow $\{\phi_t\}$ of ξ is also contained in $\tilde{\mathcal{C}}$, since $\tilde{\mathcal{C}}$ is a holomorphic conformal flow, and that $t \mapsto \rho(\phi_t))$ is surjective. Choose $\tilde{\Omega}$ in the homothety class of Kähler forms on $\tilde{M}$ in such a way that this homomorphism is $t \mapsto e^t$, that is for any t

$$\phi_t^*\tilde{\Omega} = e^t\tilde{\Omega} \qquad \psi_t^*\tilde{\Omega} = \tilde{\Omega}.$$

In particular the subgroup $\{\phi_t\}$ of $\tilde{\mathcal{C}}$ is isomorphic to $\mathbb{R}$.

At this step the hypothesis of compactness of M is crucial: using this fact one proves that the action of $\{\phi_t\}$ is free and proper (Lemma 2.2). In particular ξ is never vanishing.

Define the smooth map

$$\begin{array}{rccc} s: & \tilde{M} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \tilde{\Omega}(J\xi_x, \xi_x) \end{array}$$

and remark that 1 is a regular value of s, that $s^{-1}(1)$ is non empty, hence is a regular submanifold of $\tilde{M}$ that we denote by W (Proposition 2.2). Note that W is the submanifolds of those points where ξ has unitary norm. In particular one proves that if x is in W then $d_x s(\xi_x) = 1$, so ξ is transversal to W.

Denote by i the inclusion of W in M. It turns then out that $(W, i^*\iota_\xi\tilde{\Omega}, i^*g)$ is a connected Sasaki manifold, and that

$$\begin{array}{rccc} H: & W \times \mathbb{R} & \longrightarrow & \tilde{M} \\ & (w,t) & \longmapsto & \phi_t(w) \end{array}$$

is an isometry with respect to the Kähler cone structure on $W \times \mathbb{R}$.

One is left to show that $\pi_1(M)$ satisfies the conditions of Proposition 7.4. Indeed $\rho(\pi_1(M)) \neq 1$ since M is non Kähler, and $\pi_1(M)$ commutes with the real flow generated by ∂_t since this last factors to M. ■

The proof of this theorem proves in particular the following fact.

Corollary 7.8 *Any compact Vaisman manifold $(M, [g], J)$ can be presented as a pair (W, Γ) where W is a compact Sasaki manifold and $\Gamma \subset \mathcal{H}(W \times \mathbb{R})$ such that M is isomorphic as a locally conformal Kähler manifold to $(W \times \mathbb{R})/\Gamma$. Moreover W can be chosen to be simply connected, hence $W \times \mathbb{R}$ is the universal covering of M and Γ is isomorphic to $\pi_1(M)$.*

Remark 7.9 This can be reformulated in the following way. Consider the collection of pairs (W, Γ) as in the previous corollary. Given a Sasaki morphism one naturally induces a morphism on the Kähler cones by composing with identity on the factor $\mathbb{R}$. So define the category $\mathcal{S}$ of pairs (W, Γ) by considering as morphisms between (W, Γ) and (W', Γ') those Sasaki morphisms (i.e. isometries preserving the contact form) which induce between $W \times \mathbb{R}$ and $W' \times \mathbb{R}$ morphisms which are equivariant with respect to the actions of Γ and Γ'. Define a functor between this category and the category of Vaisman manifolds (with morphisms given by holomorphic, conformal maps) by associating to (W, Γ) the manifold $(W \times \mathbb{R})/\Gamma$ and to a morphism the induced morphism between the quotients. This functor is surjective on the objects of the subcategory of compact Vaisman manifolds, but not on the morphisms.

We say that (W, Γ) is a *presentation* of M if M is in the image of (W, Γ) by this functor. This construction suggests the following definition.

Definition 7.10 Let $(M, [g], J)$ be a compact Vaisman manifold. A locally conformal Kähler morphism is a *Vaisman morphism* if it belongs to the image of the functor from $\mathcal{S}$ to Vaisman manifolds. In particular *Vaisman automorphisms* are a subgroup of $\mathrm{Aut}(M, [g], J)$.

Remark that Vaisman morphisms are the set of locally conformal Kähler morphisms h that commute with the *Vaisman real flow*, that is, the real flow generated by the Lee field, and that admit a lifting $\tilde{h} \in \mathcal{H}(\tilde{M})$ such that $\rho(\tilde{h}) = 1$.

Remark 7.11 It must be noted that whenever $(M, [g], J)$ is Vaisman and compact the Gauduchon metric is the Vaisman metric (well-defined up to homothety). This implies that $\mathrm{Aut}(M)$ *coincides* with the holomorphic isometries of the Vaisman metric in this case. So Vaisman automorphisms coincide with the set of those h being isometries of the Vaisman metric that commute with the Vaisman real flow and admitting a lifting $\tilde{h} \in \mathcal{H}(\tilde{M})$ such that $\rho(\tilde{h}) = 1$.

Remark 7.12 In passing we remark that the categories we are talking about all admit a forgetting functor in corresponding "symplectic" categories, respectively locally conformal symplectic manifolds, homothetic symplectic manifolds and contact manifolds (with the functor given by symplectic cone construction). By associating to the pairs (W, Γ) the locally conformal symplectic manifolds $(W \times \mathbb{R})/\Gamma$ one obtains a subcategory that might represent the symplectic version of Vaisman manifolds. See [Vai85] as a leading reference on locally conformal symplectic manifolds, in particular see the notion of locally conformal symplectic manifolds of first kind.

7.2 Reduction for compact Vaisman manifolds

It is noted in [BG98] that G acting by isometries with respect to a Vaisman metric g does not imply that the reduced metric is Vaisman, since ω_g being parallel with respect to the Levi-Civita connection of g does not imply its restriction to $\mu^{-1}(0)$ being parallel.

We prove that our reduction is compatible with Sasaki reduction, see [GO01], and thus show in Theorem 7.15 that reduction of compact Vaisman manifolds by the action of Vaisman automorphisms (whose action results to be always twisted Hamiltonian) produces Vaisman manifolds. Given a Sasaki manifold (W, η, J) we call *Sasaki isomorphisms* and denote by $\mathrm{Isom}(W)$ the diffeomorphisms of W preserving both the metric and the contact form.

Theorem 7.13 *Let $((W, \eta, J), \Gamma)$ be a pair in the category $\mathcal{S}$ and denote by M the associated Vaisman manifold. Let $G \subset \mathrm{Isom}(W)$ be a subgroup satisfying the hypothesis of Sasaki reduction. Then G can be considered as a subgroup of $\mathcal{H}(W \times \mathbb{R})$. Assume that the action of G commutes with that of Γ, and that Γ acts freely and properly discontinuously on the Kähler cone $(\mu_W^{-1}(0)/G) \times \mathbb{R}$.*

Then G induces a subgroup of $\mathrm{Aut}(M)$ satisfying the hypothesis of the reduction theorem, and the reduced manifold is isomorphic with $((\mu_W^{-1}(0)/G) \times \mathbb{R})/\Gamma$. In particular the reduced manifold is Vaisman.

Proof: We first prove that the induced action satisfies the hypothesis of the Kähler reduction. The momentum map μ_W is

$$\begin{array}{rccc} \mu_W : & \mathfrak{g} & \longrightarrow & C^\infty(W) \\ & X & \longmapsto & \iota_X \eta. \end{array}$$

Remark that the fundamental field associated to $X \in \mathfrak{g}$ on $W \times \mathbb{R}$ is projectable, so we can define $\mu^X_{W\times\mathbb{R}} \stackrel{\text{def}}{=} e^t\pi^*\iota_{\pi_* X}\eta$. To show that this is a momentum map for the action of G on $W \times \mathbb{R}$ we directly compute

$$\begin{aligned}
d\mu^X_{W\times\mathbb{R}} &= d(e^t\pi^*\iota_{\pi_* X}\eta) \\
&= d(e^t\iota_X\pi^*\eta) \\
&= d(\iota_X e^t\pi^*\eta) \\
&= \iota_X d(e^t\pi^*\eta) - \mathcal{L}_X(e^t\pi^*\eta) \\
&= \iota_X\Omega
\end{aligned}$$

where $\mathcal{L}_X(e^t\pi^*\eta) = 0$ comes from the properties of the action. So the action of G is weakly Hamiltonian. Moreover

$$\begin{aligned}
\{\mu^X_{W\times\mathbb{R}}, \mu^Y_{W\times\mathbb{R}}\} &= \Omega(X, Y)) \\
&= d(e^t\pi^*\eta)(X, Y) \\
&= Y(e^t\pi^*\eta(X)) - X(e^t\pi^*\eta(Y)) + \iota_{[X,Y]}e^t\pi^*\eta \\
&= 0 + 0 + e^t\pi^*\iota\pi_*[X, Y]\eta \\
&= \mu^{[X,Y]}_{W\times\mathbb{R}}
\end{aligned}$$

where $Y(e^t\pi^*\eta(X)) = X(e^t\pi^*\eta(Y)) = 0$ is due to the properties of the action. The action of G is then Hamiltonian, and Kähler reduction is defined.

So $\mu^{-1}_{W\times\mathbb{R}}(0) \simeq (\mu^{-1}_W(0)\times\mathbb{R})$, and the action of G being proper and free on $\mu^{-1}_W(0)$ implies it having the same properties on $\mu^{-1}_{W\times\mathbb{R}}(0)$. Moreover the action of Γ being free and properly discontinuous on $(\mu^{-1}_W(0)/G) \times \mathbb{R} = \mu^{-1}_{W\times\mathbb{R}}(0)/G$ implies one can apply Theorem 6.5, so the isomorphism is proven. Then by applying Proposition 7.4 one sees that the reduced manifold is Vaisman. The theorem then follows. ∎

Remark 7.14 The previous theorem also applies to non-compact Vaisman manifolds of the form $(W\times\mathbb{R})/\Gamma$.

But now recall that Sasaki reduction, as contact reduction in fact, does *not* need a notion of Hamiltonian action, that is, the action of any subroup of Sasaki isometries lead to a momentum map, hence to reduction, up to topological conditions, that is $\mu^{-1}(0)$ being non-empty, 0 being a regular value for μ, and G acting freely and properly discontinuously on $\mu^{-1}(0)$. This implies the main result of this section.

Theorem 7.15 *Let $(M, [g], J)$ be a compact Vaisman manifold. Let $G \subset \mathrm{Aut}(M)$ be a subgroup of Vaisman automorphisms. Then the action of G on M is twisted Hamiltonian. If $\mu^{-1}(0)$ is non-empty, 0 is a regular value for μ and G acts freely and properly on $\mu^{-1}(0)$, then the reduced manifold is Vaisman.*

Moreover for any (W, Γ) presentation of M and for any $\tilde{G}$ subgroup of isometries of W in the preimage of G, the group $\tilde{G}$ induces a Sasaki reduction $\mu^{-1}_W(0)/\tilde{G}$ and the Vaisman reduced manifold is isomorphic to $((\mu^{-1}_W(0)/\tilde{G}) \times \mathbb{R})/\Gamma$.

Proof: The automorphisms of M lift to a subgroup $\tilde{G}$ of $\mathcal{H}(W \times \mathbb{R})$ of maps commuting with the natural flow and such that $\rho(\tilde{G}) = 1$. By definition of Vaisman automorphisms $\tilde{G}$ is contained in $\mathrm{Isom}(W)$. So the compatibility between Sasaki and Kähler reduction of [GO01] applies, hence $\mu^{-1}_{W\times\mathbb{R}}(0)/\tilde{G}$ is isomorphic with the Kähler cone $(\mu^{-1}_W(0)/\tilde{G}) \times \mathbb{R}$.

Moreover Γ acts freely and properly discontinuously on it, since the quotient $\mu^{-1}_M(0)/G$ is a manifold, and commutes with its natural real flow. Then Theorem 7.13 applies, and this proves the theorem. ∎

8 A class of examples: weighted actions on Hopf manifolds

We apply the theorem in last section to the simple case when $\Gamma = \mathbb{Z}$ is contained as a discrete subgroup of the natural holomorphic flow of $W \times \mathbb{R}$. In particular the Vaisman manifold topologically is simply $W \times S^1$. This nevertheless covers much of the already known examples of Vaisman manifolds, as was shown in [KO01]. We briefly review the definition of those manifolds.

First consider S^{2n-1} equipped with the CR structure J coming from $\mathbb{C}^n$. The action on $\mathbb{C}^n \setminus 0$ of the cyclic group Γ_α generated by $z \mapsto \alpha z$ (for any $\alpha \in \mathbb{C}$ such that $|\alpha| > 1$) produces the so-called *standard Hopf manifolds.* For any $\{c_1, \ldots, c_n\} \in (S^1)^n$ and any set $A \stackrel{\text{def}}{=} \{a_1, \ldots, a_n\}$ of real numbers such that $0 <$

$a_1 \le \cdots \le a_n$ the action of the cyclic group $\Gamma_{\{c_1,\ldots,c_n\},A} \subset \mathcal{C}$ generated by $(z_1 \ldots z_n) \mapsto (e^{a_1}c_1z_1,\ldots,e^{a_n}c_nz_n)$ produces the complex manifolds usually called *non-standard Hopf manifold.*

To endow the complex Hopf manifolds with a Vaisman structure let η_0 be the Sasaki structure coming from the differential 2-form $\Omega = -i\sum dz_i \wedge d\bar{z}_i$ of $\mathbb{C}^n$. The action of any Γ_α is by homotheties for the cone structure, hence induces Vaisman structures $((S^{2n-1},\eta_0,J),\Gamma_\alpha)$ on standard Hopf manifolds. More generally, for any $A = \{a_1,\ldots,a_n\}$ of real numbers such that $0 < a_1 \le \cdots \le a_n$ let η_A be defined the following way:

$$\eta_A \stackrel{\text{def}}{=} \frac{1}{\sum a_i|z_i|^2}\eta_0.$$

Fixed A one obtains that for any $\{c_1,\ldots,c_n\} \in (S^1)^n$ the action of $\Gamma_{\{c_1,\ldots,c_n\},A}$ is by homotheties of the corresponding cone structure on $\mathbb{C}^n \setminus 0$, hence inducing Vaisman structures

$$((S^{2n-1},\eta_A,J),\Gamma_{\{c_1,\ldots,c_n\},A})$$

on the non-standard Hopf manifolds (see [KO01]).

We now analyze reduction of Hopf manifolds by means of circle actions: in fact by acting on (S^{2n-1},η_A,J) by a circle of Sasaki isometries, if $n > 2$, we generate, for every single $\Gamma_{\{c_1,\ldots,c_n\},A}$, a Vaisman reduced manifold of dimension $2n-2$, whose underlying manifold is the product of the Sasaki reduced manifold with S^1.

Remark that the contact structures of the Sasaki manifolds (S^{2n-1},η_A,J) all coincide. Denote by $\mathrm{Cont}(S^{2n-1})$ the set of contact automorphisms of S^{2n-1}, which coincide with restrictions of biholomorphisms of $\mathbb{C}^n$.

For any $\Lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1\ldots\lambda_n \neq 0$ let $G_\Lambda \subset \mathrm{Cont}(S^{2n-1})$ be the subgroup of those maps $h_{\Lambda,t}$, $t \in \mathbb{R}$, such that

$$h_{\Lambda,t}(z_1,\ldots,z_n) = (e^{i\lambda_1 t}z_1,\ldots,e^{i\lambda_n t}z_n).$$

Remark that any G_Λ is composed by holomorphic isometries of the standard Kähler structure. Moreover a direct computation shows that its action on S^{2n-1} is by isometries for any of the η_A. We call the action of G_Λ *weighted* by the *weights* $(\lambda_1,\ldots,\lambda_n)$. It is easy to see that G_Λ is isomorphic to the circle $\mathbb{R}/2\pi\mathbb{Z}$.

The corresponding momentum map for the Sasaki manifold (S^{2n-1},η_A,J) is defined by:

$$\mu^1_\Lambda(z) = H_\Lambda(z) \stackrel{\text{def}}{=} \frac{1}{2(\sum a_i|z_i|^2)}(\lambda_1|z_1|^2 + \cdots + \lambda_n|z_n|^2).$$

So a Sasaki reduction is defined whenever the weights are such that $\mu^{-1}(0)$ is not empty and the action on $\mu^{-1}(0)$ is free and proper. The condition that $\mu^{-1}(0)$ is not empty is equivalent to requiring that the signs of the λ_i are not all the same.

Let $k = k(\Lambda) \in \{1,\ldots,n-1\}$ be the number of negative weights of Λ, and assume the negative weights are the first k. Without loss of generality we might assume that $k \le n/2$. Then there is a diffeomorphism

$$\begin{array}{cccc} \Phi_\Lambda : & S^{2k-1}\times S^{2n-2k-1} & \longrightarrow & \mu^{-1}(0) \\ & ((\xi_1,\ldots,\xi_k),(\zeta_1,\ldots,\zeta_{n-k})) & \longmapsto & (\frac{\xi_1}{\sqrt{-\lambda_1}},\ldots,\frac{\xi_k}{\sqrt{-\lambda_k}},\frac{\zeta_1}{\sqrt{\lambda_{k+1}}},\ldots,\frac{\zeta_{n-k}}{\sqrt{\lambda_n}}) \end{array}$$

equivariant with respect to the action

$$w_{\Lambda,t}((\xi_1,\ldots,\xi_k),(\zeta_1,\ldots,\zeta_{n-k})) = ((e^{i\lambda_1 t}\xi_1,\ldots,e^{i\lambda_k t}\xi_k),(e^{i\lambda_{k+1}t}\zeta_1,\ldots,e^{i\lambda_n t}\zeta_{n-k}))$$

from one side and the action of G_Λ on $\mu^{-1}(0)$ from the other: $h_{\Lambda,t}\circ\Phi_\Lambda = \Phi_\Lambda\circ w_{\Lambda,t}$.

Call $S(\Lambda)$ the quotient of this action. Since G_Λ is compact for any Λ, this will generally be an orbifold. The following holds.

Proposition 8.1 *The circle G_Λ acts freely on $S^{2k-1}\times S^{2n-2k-1}$ if and only if*

$$\gcd(\lambda_m,\lambda_j) = 1$$

for all $m \le k$ and all $j \ge k+1$, that is, if and only if the positive weights are coprime with the negative ones.

Then Theorem 7.13 implies the following.

Corollary 8.2 *For any Λ as in Proposition 8.1, for any $(a_1,\ldots,a_n) \in \mathbb{R}^n$ such that $0 < a_1 \le \cdots \le a_n$ and for any $(c_1,\ldots,c_n) \in (S^1)^n$ there exists a Vaisman structure on $S(\Lambda)\times S^1$, each being the reduction by the action of G_Λ of the complex Hopf manifold associated to $((a_1,\ldots,a_n),(c_1,\ldots,c_n))$ endowed with its Vaisman structure.*

Quite surprisingly we were able to find very little information in literature on the topological type of the reductions $S(\Lambda)$, even though the weighted circle actions on $\mathbb{C}^n \setminus 0$ are natural examples of Hamiltonian circle actions. In the following examples we list the topological type of some cases.

Example 8.3 Assume that $n \geq 2$, $k = 1$, that is, $\lambda_1 < 0$, $\lambda_i > 0, i = 2, \ldots, n$. Then the space $\mu^{-1}(0)$ is diffeomorphic to $S^1 \times S^{2n-3}$.

One easily shows that $S(-1, 1, \ldots, 1)$ is S^{2n-3}. One can also show that the Sasaki structure reduced from the standard is again the standard one, so any standard Hopf manifold comes as a reduction of the corresponding Hopf manifold of higher dimension. This is also shown in [BG98].

In turn, as shown in [GO01], for any positive integer p, $S(-p, 1, \ldots, 1)$ is diffeomorphic to $S^{2n-3}/\mathbb{Z}_p$, so we obtain a family of Vaisman structures on $(S^{2n-3}/\mathbb{Z}_p) \times S^1$. In particular for $n = 3$ we obtain Sasaki structures on lens spaces of the form $L(p, 1)$, hence Vaisman structures on complex surfaces diffeomorphic with $L(p, 1) \times S^1$.

Example 8.4 If $n = 4$, $k = 2$, then $\mu^{-1}(0)$ is diffeomorphic to $S^3 \times S^3$. In particular $S(-1, -1, 1, 1)$ is known to be $S^3 \times S^2$, see [GO01]. So we obtain a family of Vaisman structures on $S^3 \times S^2 \times S^1$. It is interesting to note that reducing from the standard structure one obtains a manifold that also bears a semi-Kähler structure, when seen as twistor space of the standard Hopf surface.

Example 8.5 More generally if $n = 2k = 4s$, $\mu^{-1}(0)$ is diffeomorphic to $S^{4s-1} \times S^{4s-1}$. In analogy with the case $n = 4$ treated in [GO01] apply the diffeomorphism

$$\begin{array}{ccc} S^{2k-1} \times S^{2k-1} & \longrightarrow & S^{2k-1} \times S^{2k-1} \\ ((\xi_1, \ldots, \xi_k), (\zeta_1, \ldots, \zeta_k)) & \longmapsto & ((\xi_1\zeta_k + \overline{\xi_2\zeta_{k-1}}, \xi_1\zeta_{k-1} - \overline{\xi_2\zeta_k}, \ldots, \xi_{k-1}\zeta_2 + \overline{\xi_k\zeta_1}, \xi_{k-1}\zeta_k - \overline{\xi_k\zeta_2}), (\zeta_1, \ldots, \zeta_k)) \end{array}$$

and remark that it is equivariant with respect to the action of $G_{(-1,\ldots,-1,1,\ldots,1)}$ on the first space and the action on the second space given by the product of the trivial action on the first factor and the Hopf action on the second factor. This proves that $S(-1, \ldots, -1, 1, \ldots, 1)$ is diffeomorphic to $S^{4s-1} \times \mathbb{CP}^{2s-1}$. Thus we obtain families of Vaisman structures on $S^{4s-1} \times \mathbb{CP}^{2s-1} \times S^1$.

Example 8.6 If the weights on one of the spheres coincide, then the action factors through the corresponding projective space. For example for any positive integer p the manifold $S(-p, -p, -p, \lambda_4, \lambda_5)$ is a fiber bundle over $\mathbb{CP}^2$.

The general tools one might use to investigate the manifolds $S(\Lambda)$ are the homotopy exact sequence and the (co)homology Thom-Gysin sequence of the fibration $S^1 \hookrightarrow S^{2k-1} \times S^{2n-2k-1} \to S(\Lambda)$.

From the homotopy exact sequence it is easy to obtain that for $k > 1$ the manifolds $S(\Lambda)$ are simply connected. In the case of Example 8.4 the manifolds $S(\Lambda)$ are then simply connected 5-manifolds, which we recall were completely classified up to diffeomorphism in [Bar65]. As for the case of $n = 5$, $k > 1$, an important class of simply connected 7-manifold is described in [CE00].

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