An explicit parallelization on products of spheres

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1 Introduction

It is a classical result in Algebraic Topology that spheres S^n are parallelizable if and only if their dimension is $n = 1, 3, 7$. As for the products of two or more spheres one has instead the following result ([Ker56]):

{teokervaire}

Theorem 1.1 (Kervaire) *The manifold $S^{n_1} \times \cdots \times S^{n_r}$, $r \geq 2$, is parallelizable if and only if at least one of the n_i is odd.*

Kervaire's proof does not provide an explicit parallelization on products of spheres. The only reference the author knows to provide explicit parallelizations is [Bru92], that considers the cases when one of the spheres is of dimension $1, 3, 5, 7$, and uses some specific arguments of these low dimensions. In [Bru92] the general case is left as an open problem.

The aim of this paper is to write an explicit orthonormal parallelization for all parallelizable products of spheres, using an explicit isomorphism with a trivial vector bundle obtained following a hint of [Hir88].

A description of some G-structures on $S^m \times S^n$ associated to this parallelization is given in [Par00].

2 An explicit parallelization $\mathcal{B}$ on $S^m \times S^1$

{genm1}

Denote by $x = (x_i)$ the coordinates on $\mathbb{R}^{m+1}$, and let $S^m \subset \mathbb{R}^{m+1}$ be given by

$$S^m \stackrel{\text{def}}{=} \{x = (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \text{ such that } |x|^2 = x_1^2 + \cdots + x_{m+1}^2 = 1\}.$$

The orthogonal projection of the standard coordinate frame $\{\partial_{x_i}\}_{i=1,\ldots,m+1}$ to the sphere plays an important role in the game, and deserves its own definition:

Definition 2.1 The $\imath^{th}$ *meridian vector field* M_i on S^m is

$$M_i \stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{x_i} \text{ on } S^m \qquad i = 1, \ldots, m+1.$$

□

Let M be the normal versor field of $S^m \subset \mathbb{R}^{m+1}$, that is,

$$M \stackrel{\text{def}}{=} \sum_{i=1}^{m+1} x_i \partial_{x_i}.$$

Since

$$\langle \partial_{x_i}, M \rangle = x_i \qquad i = 1, \ldots, m+1,$$

one obtains the following expression for M_i:

$$M_i = \partial_{x_i} - x_i M \qquad i = 1, \ldots, m+1, \tag{1}$$ {fundeq}

and thus

$$\langle M_i, M_j \rangle = \delta_{ij} - x_i x_j \qquad i, j = 1, \ldots, m+1. \tag{2}$$ {fundeqscal}

Let Γ be the cyclic infinite group of transformations of $\mathbb{R}^{m+1} - 0$ generated by the map $x \mapsto e^{2\pi} x$. Denote by H the corresponding diagonal real Hopf manifold, that is, the quotient manifold $(\mathbb{R}^{m+1} - 0)/\Gamma$: H turns out to be diffeomorphic to $S^m \times S^1$ by means of the map induced by the projection p:

$$\begin{array}{ccc} \mathbb{R}^{m+1} - 0 & \stackrel{p}{\longrightarrow} & S^m \times S^1 \\ x & \longmapsto & (x/|x|, \log|x| \mod 2\pi). \end{array}$$

The standard coordinate frame $\{\partial_{x_i}\}_{i=1,\ldots,m+1}$ on $\mathbb{R}^{m+1} - 0$ becomes Γ-equivariant when multiplied by the function $|x|$, whence it defines a parallelization on $S^m \times S^1$. This proves the following proposition...

Proposition 2.2 *$S^m \times S^1$ is parallelizable.*

...and enables us to give the following definition:

{Bdef}

Definition 2.3 Define $\mathcal{B} = \{b_i\}_{i=1,\ldots,m+1}$ as the frame on $S^m \times S^1$ induced by the Γ-equivariant frame $\{|x|\partial_{x_i}\}_{i=1,\ldots,m+1}$ on the universal covering $\mathbb{R}^{m+1}$ of $S^m \times S^1$ by means of p:

$$b_i \stackrel{\text{def}}{=} p_*(|x|\partial_{x_i}(x)) \qquad i = 1, \ldots, m+1.$$

□

The following theorem explicitly describes the frame $\mathcal{B}$:

{progeo}

Theorem 2.4 *Let M_i be the i^{th} meridian vector field on $S^m \subset \mathbb{R}^{m+1}$. Then*

$$b_i = M_i + x_i \partial_\theta \qquad i = 1, \ldots, m+1. \tag{3}$$ {progeofor}

Proof: Look at $S^m \times S^1$ as a Riemannian submanifold of $R^{m+1} \times S^1$, and in particular look at $T(S^m \times S^1) = TS^m \times TS^1$ as a Riemannian subbundle of $TR^{m+1}_{|S^m} \times TS^1$; this last is a trivial vector bundle and an orthonormal frame is $\{\partial_{x_1}, \ldots, \partial_{x_{m+1}}, \partial_\theta\}$. A computation then shows that

$$p_* = \frac{1}{|x|} \left((dx_1 - x_1\omega) \otimes \partial_{x_1} + \cdots + (dx_{m+1} - x_{m+1}\omega) \otimes \partial_{x_{m+1}} + |x|\omega \otimes \partial_\theta \right),$$

where ω is the 1-form given by

$$\omega \stackrel{\text{def}}{=} -d\left(\frac{1}{|x|}\right) = \frac{1}{|x|^2}\left(x_1 dx_1 + \cdots + x_{m+1} dx_{m+1}\right).$$

Whence, the frame $\mathcal{B}$ in the point $p(x) = (x/|x|, \log|x| \mod 2\pi)$ is given by

$$\frac{1}{|x|^2}\left(-x_1 x_i \partial_{x_1} + \cdots + (|x|^2 - x_i^2)\partial_{x_i} + \cdots - x_{m+1}x_i\partial_{x_{m+1}} + |x|x_i\partial_\theta\right) \qquad i = 1, \ldots, m+1,$$

that is, the frame $\mathcal{B}$ in the point $(x, \theta) \in S^m \times S^1$ is given by

$$\begin{aligned} b_i &= \left(-x_1 x_i \partial_{x_1} + \cdots + (1 - x_i^2)\partial_{x_i} + \cdots - x_{m+1}x_i\partial_{x_{m+1}} + x_i\partial_\theta\right) \\ &= \partial_{x_i} - x_i(x_1\partial_{x_1} + \cdots + x_{m+1}\partial_{x_{m+1}}) + x_i\partial_\theta \stackrel{(1)}{=} M_i + x_i\partial_\theta. \end{aligned} \qquad i = 1, \ldots, m+1. \tag{4}$$ {bruni1}

■

Remark 2.5 The notion of meridian vector field was given in [Bru92]: it was used to describe a parallelization on any product of a sphere by a parallelizable manifold. In this context, theorem 2.4 shows that the frame $\mathcal{B}$ given by definition 2.3 coincide with that of [Bru92]. □

{remor}

Remark 2.6 The frame $\mathcal{B}$ is orthonormal with respect to the product metric on $S^m \times S^1$ (use theorem 2.4 and formula (2)). □

The well-known bracket formula

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X \tag{5}$$ {kobano}

gives the brackets of $\mathcal{B}$:

$$[b_i, b_j] = x_i b_j - x_j b_i \qquad i, j = 1, \ldots, m+1. \tag{6}$$ {bracket1gen}

By means of theorem 2.4 and remark 2.6, one obtains the coframe $\mathcal{B}^* \stackrel{\text{def}}{=} \{b^i\}_{i=1,\dots,m+1}$ dual to $\mathcal{B}$ on $S^m \times S^1$:

$$b^i = dx_i + x_i d\theta \qquad i = 1, \dots, m+1. \tag{7}$$ {cobruni1}

Remark 2.7 Since

$$b_i = p_*(|x|\partial_{x_i}) \qquad i = 1, \dots, m+1,$$

the coframe $\mathcal{B}^*$ can be also described as the quotient of the Γ-invariant coframe on $\mathbb{R}^{m+1} - 0$ given by

$$\{|x|^{-1} dx_i\}_{i=1,\dots,m+1}.$$

□

A straightforward computation gives the structure equations for $\mathcal{B}$:

$$db^i = dx_i \wedge d\theta \stackrel{(7)}{=} b^i \wedge d\theta \qquad i = 1, \dots, m+1, \tag{8}$$

where the 1-form $d\theta$ is related to $\mathcal{B}^*$ by

$$d\theta = \sum_{i=1}^{m+1} x_i b^i.$$

The following lemma is trivial to prove, but will be useful:

{lemgen}

Lemma 2.8 *For each permutation π of $\{1, \dots, m+1\}$, the automorphism of $\mathbb{R}^{m+1} - 0$ given by $(x_1, \dots, x_{m+1}) \mapsto (x_{\pi(1)}, \dots, x_{\pi(m+1)})$ is Γ-equivariant. The induced diffeomorphism is*

$$\begin{array}{rccc} f_\pi : & S^m \times S^1 & \longrightarrow & S^m \times S^1 \\ & (x_1, \dots, x_{m+1}, \theta) & \longmapsto & (x_{\pi(1)}, \dots, x_{\pi(m+1)}, \theta), \end{array}$$

and $df_\pi(b_{\pi(i)}) = b_i$.

3 The Hopf fibration $S^3 \to S^2$ extends $\mathcal{B}$ to $S^m \times S^3$

{genm3}

Denote by $y = (y_j)$ the coordinates on $\mathbb{R}^4$, and let $S^3 \subset \mathbb{R}^4$ be given by

$$S^3 \stackrel{\text{def}}{=} \{y = (y_1, \dots, y_4) \in \mathbb{R}^4 \text{ such that } |y|^2 = y_1^2 + \dots + y_4^2 = 1\}.$$

Let $T = T_1, T_2, T_3$ be the vector fields on S^3 given by multiplication by $i, j, k \in \mathbb{H} = \mathbb{R}^4$ respectively, that is,

$$\begin{aligned} T = T_1 &= -y_2\partial_{y_1} + y_1\partial_{y_2} - y_4\partial_{y_3} + y_3\partial_{y_4}, \\ T_2 &= -y_3\partial_{y_1} + y_4\partial_{y_2} + y_1\partial_{y_3} - y_2\partial_{y_4}, \\ T_3 &= -y_4\partial_{y_1} - y_3\partial_{y_2} + y_2\partial_{y_3} + y_1\partial_{y_4}. \end{aligned} \tag{9}$$

It is well-known that S^3 can be foliated in S^1's, by means of the Hopf fibration $S^3 \rightarrow S^2$. Whence, one has a foliation of $S^m \times S^3$ in $S^m \times S^1$'s, and section 2 gives $m+1$ vector fields tangent to the leaves: they can be completed to a parallelization of $S^m \times S^3$ by means of a suitable parallelization of S^3, as it is now going to be shown in the following proposition:

{parm3}

Proposition 3.1 *$S^m \times S^3$ is parallelizable.*

Proof: One would like to use definition 2.3 to define $m+1$ vector fields on $S^m \times S^3$. The problem is that there is not a canonical identification of the fiber of the Hopf fibration $S^3 \rightarrow S^2$ with S^1, whence one has not a canonical angular coordinate on fibers. But formula (3) of theorem 2.4 only needs a unitary and tangent to fibers vector field on S^3 to be used: this is just what T is. Whence, define $\mathcal{B} \stackrel{\text{def}}{=} \{b_i\}_{i=1,\ldots,m+3}$ by

$$\begin{aligned} b_i &\stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \ldots, m+1, \\ b_{m+j} &\stackrel{\text{def}}{=} T_j \qquad j = 2, 3, \end{aligned} \tag{10}$$

{parsms3}

where M_i is the $\imath^{\text{th}}$ meridian vector field on S^m, to obtain the wished frame on $S^m \times S^3$. ■

{remor3}

Remark 3.2 The frame $\mathcal{B}$ is orthonormal with respect to the product metric on $S^m \times S^3$ (use formulas (10) and formula (2)). □

The same argument used in section 2 gives the brackets of $\mathcal{B}$:

$$\begin{aligned} [b_i, b_j] &= x_i b_j - x_j b_i \qquad i, j = 1, \ldots, m+1, \\ [b_i, b_{m+2}] &= -2x_i b_{m+3} \qquad i = 1, \ldots, m+1, \\ [b_i, b_{m+3}] &= 2x_i b_{m+2} \qquad i = 1, \ldots, m+1, \\ [b_{m+2}, b_{m+3}] &= -2T = -2 \sum_{i=1}^{m+1} x_i b_i. \end{aligned} \tag{11}$$

{bracket3gen}

Let $\tau = \tau_1, \tau_2, \tau_3$ be the 1-forms on $S^m \times S^3$ dual to $T = T_1, T_2, T_3$ respectively. The coframe $\mathcal{B}^* \stackrel{\text{def}}{=} \{b^i\}_{i=1,\ldots,m+3}$ is given by

$$\begin{aligned} b^i &= x_i \tau + dx_i \qquad i = 1, \ldots, m+1, \\ b^{m+j} &= \tau_j \qquad j = 2, 3. \end{aligned} \tag{12}$$

{cobruni3}

Differently from $S^m \times S^1$, the 1-form τ is not closed, so structure equations are a bit more complicated:

$$\begin{aligned} db^i &= b^i \wedge \tau + 2x_i b^{m+2} \wedge b^{m+3} \qquad i = 1, \ldots, m+1, \\ db^{m+2} &= 2b^{m+3} \wedge \tau, \\ db^{m+3} &= -2b^{m+2} \wedge \tau, \end{aligned} \tag{13}$$

{struc3}

where the 1-form τ is related to $\mathcal{B}^*$ by

$$\tau = \sum_{i=1}^{m+1} x_i b^i.$$

Remark 3.3 The same argument used above for $S^m \times S^3$ can be applied to the Hopf fibration $S^7 \to \mathbb{CP}^3$ to obtain a frame on $S^m \times S^7$. Nevertheless, formulas in this case are much more complicated. □

Proposition 3.1 and the previous remark can be easily generalized:

Theorem 3.4 ([Bru92]) *Let N^n be any parallelizable n-dimensional manifold. Then $S^m \times N$ is parallelizable.*

Proof: Let $T = T_1, T_2, \ldots, T_n$ be a frame on N. The required parallelization is thus given by

$$b_i \stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \ldots, m+1,$$
$$b_{m+j} \stackrel{\text{def}}{=} T_j \qquad j = 2, \ldots, n,$$

where M_i is the $\imath^{\text{th}}$ meridian vector field on S^m. ■

4 The general problem: when is a product of spheres parallelizable?

{bruintro}

The proof of the theorem of Kervaire cited in the introduction is here sketched:

Sketch of proof of theorem 1.1 (Kervaire):

1. Show by induction there exists an embedding of $S^{n_1} \times \cdots \times S^{n_r}$ in $\mathbb{R}^{n_1+\cdots+n_r+1}$. This is true for $r = 1$. Let

$$f = (f_1, \ldots, f_{n_1+\cdots+n_{r-1}+1})\colon S^{n_1} \times \cdots \times S^{n_{r-1}} \to \mathbb{R}^{n_1+\cdots+n_{r-1}+1}$$

   be the embedding given by the inductive hypothesis, where f is chosen in such a way that $f_1 \geq 0$. Let $u \in S^{n_1} \times \cdots \times S^{n_{r-1}}$, and let $(\xi_1, \ldots, \xi_{n_r+1}) \in S^{n_r}$: the embedding f is thus given by

$$\begin{array}{ccc} S^{n_1} \times \cdots \times S^{n_r} & \xrightarrow{f} & \mathbb{R}^{n_1+\cdots+n_r+1} \\ (u, (\xi_1, \ldots, \xi_{n_r+1})) & \longmapsto & (f_2(u), \ldots, f_{n_1+\cdots+n_{r-1}+1}(u), \xi_1\sqrt{f_1(u)}, \ldots, \xi_{n_r+1}\sqrt{f_1(u)}); \end{array}$$

2. suppose without any loss of generality that the odd dimension is not n_1, and observe that the degree of the Gauss map of the embedding f built in 1. is given by

$$\chi(D^{n_1+1} \times S^{n_2} \times \cdots \times S^{n_r}) = \chi(D^{n_1+1})\chi(S^{n_2}) \dots \chi(S^{n_r}) = 0,$$

where D^{n_1+1} denotes a topological disk of dimension $n_1 + 1$;

3. denote by $G_{k,n}$ and $V_{k,n}$ the Grassmannian and the Stiefel-Whitney manifold of oriented k-planes and oriented orthonormal frames in $\mathbb{R}^{k+n}$, respectively. The tangential map

$$S^{n_1} \times \cdots \times S^{n_r} \longrightarrow G_{n_1+\cdots+n_r,1}$$

is null-homotopic, since by 2. the Gauss map is;

4. last, denote by $P(S^{n_1} \times \cdots \times S^{n_r})$ the principal bundle of $S^{n_1} \times \cdots \times S^{n_r}$, and look at the following diagram to end the proof:

$$\begin{array}{ccc} P(S^{n_1} \times \cdots \times S^{n_r}) & \dashrightarrow & V_{n_1+\cdots+n_r,1} \\ \vdots & & \downarrow \\ S^{n_1} \times \cdots \times S^{n_r} & \longrightarrow & G_{n_1+\cdots+n_r,1} \end{array}$$

■

Note that, due to the homotopy theory considerations, the above proof is not very suitable to write down explicit parallelizations on products of spheres.

Another proof of Kervaire's theorem can be developed using a series of hints contained in the book [Hir88, exercises 3,4,5 and 6 of section 4.2]. Details of such a proof, as developed by the author, are given in the following.

In what follows, ε^k_B denotes the trivial vector bundle of rank k with base space B; moreover, whenever α is a vector bundle, $E(\alpha), p_\alpha, B(\alpha)$ denote the total space, the projection and the base space of α respectively.

{lembun}

Lemma 4.1 *Let α be a vector bundle. The Whitney sum $\alpha \oplus \varepsilon^k_{B(\alpha)}$ is described by*

$$E(\alpha \oplus \varepsilon^k_{B(\alpha)}) \simeq E(\alpha) \times \mathbb{R}^k,$$

$$p_{\alpha \oplus \varepsilon^k_{B(\alpha)}}(e, v) = p_\alpha(e),$$

$$B(\alpha \oplus \varepsilon^k_{B(\alpha)}) = B(\alpha).$$

Proof: The Whitney sum $\alpha \oplus \varepsilon^k_{B(\alpha)}$ is given by the pull-back of $\alpha \times \varepsilon^k_{B(\alpha)}$ by means of the diagonal map $B(\alpha) \to B(\alpha) \times B(\alpha)$ (see for instance [MS74, page 27]). Then

$$E(\alpha \oplus \varepsilon^k_{B(\alpha)}) = \{(e, b, v, b) \in E(\alpha) \times B(\alpha) \times \mathbb{R}^k \times B(\alpha) \text{ such that } p_\alpha(e) = b\}$$

and the thesis follows. ■

{lemmabundles}

Corollary 4.2 *Let α, β be vector bundles. Then, for any $k \geq 0$,*

$$\alpha \times (\beta \oplus \varepsilon^k_{B(\beta)}) \simeq (\alpha \oplus \varepsilon^k_{B(\alpha)}) \times \beta.$$

Proof: Observe that

$$E(\alpha \times (\beta \oplus \varepsilon^k_{B(\beta)})) \simeq E(\alpha) \times E(\beta \oplus \varepsilon^k_{B(\beta)}) \overset{4.1}{\simeq} E(\alpha) \times E(\beta) \times \mathbb{R}^k,$$
$$E(\alpha \oplus \varepsilon^k_{B(\alpha)}) \times \beta \simeq E(\alpha \oplus \varepsilon^k_{B(\alpha)}) \times E(\beta) \overset{4.1}{\simeq} E(\alpha) \times \mathbb{R}^k \times E(\beta),$$

and use the obvious isomorphism. ■

{probundles}

Theorem 4.3 *Suppose M^m and N^n satisfy the following properties:*

1. *$T(M) \oplus \varepsilon^1_M$ is trivial;*

2. *$T(N) \oplus \varepsilon^1_N$ is trivial;*

3. *there is a non-vanishing vector field on N.*

Then $M \times N$ is parallelizable.

Proof: Let ν be a complement in $T(N)$ of the non-vanishing vector field on N, that is,

$$T(N) \simeq \nu \oplus \varepsilon^1_N. \tag{14}$$ {c}

Then

$$\begin{aligned} T(M \times N) &\simeq T(M) \times T(N) \overset{(14)}{\simeq} T(M) \times (\nu \oplus \varepsilon^1_N) \\ &\overset{4.2}{\simeq} (T(M) \oplus \varepsilon^1_M) \times \nu \overset{1.}{\simeq} \varepsilon^{m+1}_M \times \nu \\ &\overset{4.2}{\simeq} \varepsilon^{m-1}_M \times (\nu \oplus \varepsilon^2_N) \overset{2.}{\simeq} \varepsilon^{m-1}_M \times \varepsilon^{n+1}_N \end{aligned} \tag{15}$$ {chain}

■

{ossshort}

Remark 4.4 Whenever N is itself parallelizable, formula (15) can be shortened:

$$\begin{aligned} T(M \times N) &\simeq T(M) \times T(N) \simeq T(M) \times \varepsilon^n_N \\ &\simeq (T(M) \oplus \varepsilon^1_M) \times \varepsilon^{n-1}_N \simeq \varepsilon^{m+1}_M \times \varepsilon^{n-1}_N. \end{aligned} \tag{16}$$ {chainshort}

□

The embedding $S^n \subset \mathbb{R}^{n+1}$ gives the triviality of $T(S^n) \oplus \varepsilon^1_{S^n}$; whenever n is odd, a non-vanishing vector field on $S^n \subset \mathbb{C}^{(n+1)/2}$ is given by the complex multiplication. Thus, the following:

{corbundles}

Corollary 4.5 *Let n be any positive odd integer. Then the manifold $S^m \times S^n$ is parallelizable.*

And finally:

Second proof of theorem 1.1: Apply $r-1$ times the corollary 4.2 to show that $T(S^{n_2} \times \cdots \times S^{n_r}) \oplus \varepsilon^1_{S^{n_2}\times\cdots\times S^{n_r}}$ is a trivial vector bundle, and use theorem 4.3. ■

5 An explicit parallelization $\mathcal{P}$ for products of 2 spheres

{secparexp}

An explicit parallelization $\mathcal{B}$ has already been found on $S^m \times S^n$, for $n = 1, 3, 7$, in the previous sections. Can one use theorem 4.3 to explicitly find a parallelization on any parallelizable $S^m \times S^n$? Answer is positive.

The trick in theorem 4.3 is simple: split TN by means of the never-vanishing vector field, then use the trivial summand to parallelize TM, and last detach a rank 2 trivial summand to parallelize the remaining part of TN. Remark 4.4 simply says that if N is itself parallelizable, one can avoid to detach the rank 2 trivial summand from M, using the parallelization of N instead.

Here and henceforth, n is supposed to be the odd dimension in $S^m \times S^n$.

Denote by $y = (y_j)$ the coordinates on $\mathbb{R}^{n+1}$, and let $S^n \subset \mathbb{R}^{n+1}$ be given by

$$S^n \stackrel{\text{def}}{=} \{y = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} \text{ such that } y_1^2 + \cdots + y_{n+1}^2 = 1\}.$$

Being n odd, a never-vanishing vector field, and hence a versor field, is defined on S^n: *here and henceforth, T denotes the versor field on S^n given by multiplication by i in $\mathbb{C}^{(n+1)/2}$, namely,*

$$T \stackrel{\text{def}}{=} -y_2\partial_{y_1} + y_1\partial_{y_2} + \cdots - y_{n+1}\partial_{y_n} + y_n\partial_{y_{n+1}}. \quad (17) \quad \{\text{eqtjexp}\}$$

When a shorter form of T is needed, t_j denotes the coordinates of T, that is,

$$T = \sum_{j=1}^{n+1} t_j \partial_{y_j} \quad (18)$$

where t_j is given by

$$t_j = \begin{cases} -y_{j+1} & \text{if } j \text{ is odd,} \\ y_{j-1} & \text{if } j \text{ is even.} \end{cases} \quad (19) \quad \{\text{eqtj}\}$$

Moreover, denote by N the normal versor field of $S^n \subset \mathbb{R}^{n+1}$ (recall that M denotes the normal versor field of $S^m \subset \mathbb{R}^{m+1}$):

$$N \stackrel{\text{def}}{=} \sum_{j=1}^{n+1} y_j \partial_{y_j}. \quad (20)$$

It is convenient to think of $T(S^m \times S^n) = TS^m \times TS^n$ as a Riemannian subbundle of $T\mathbb{R}^{m+1}_{|S^m} \times T\mathbb{R}^{n+1}_{|S^n}$; this last is trivial, and an orthonormal frame is $\{\partial_{x_1}, \ldots, \partial_{x_{m+1}}, \partial_{y_1}, \ldots, \partial_{y_{n+1}}\}$.

Denote by N_j the $\jmath^{\text{th}}$ meridian vector field on S^n (recall that M_i denotes the $\imath^{\text{th}}$ meridian vector field on S^m):

$$N_j \stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{y_j} \text{ on } S^n \qquad j = 1, \ldots, n+1.$$

The tangent space in a point $(x, y) \in S^m \times S^n$ is thus given by an euclidean vector subspace

$$T_x S^m \oplus T_y S^n \subset \mathbb{R}^{m+1} \oplus \mathbb{R}^{n+1},$$

which is generated by the $m + n + 2$ vectors $\{M_1(x), \ldots, M_{m+1}(x), N_1(y), \ldots, N_{n+1}(y)\}$.

One also has

$$T_x S^m \oplus \langle M(x) \rangle_{\mathbb{R}} = \mathbb{R}^{m+1} \qquad \text{and} \qquad T_y S^n \oplus \langle N(y) \rangle_{\mathbb{R}} = \mathbb{R}^{n+1}. \tag{21}$$ {uga}

As in formula (1), one obtains

$$\begin{aligned} \partial_{x_i} &= M_i + x_i M \qquad i = 1, \ldots, m+1, \\ \partial_{y_j} &= N_j + y_j N \qquad j = 1, \ldots, n+1. \end{aligned} \tag{22}$$ {fundeqold}

Moreover, denote by $T(y)^\perp$ the vector subspace of $T_y(S^n)$ which is orthogonal to $T(y)$:

$$\langle T(y) \rangle_{\mathbb{R}} \oplus T(y)^\perp = T_y S^n. \tag{23}$$ {ugb}

In what follows, some computation on the vector space $T_x(S^m) \oplus T_y(S^n)$ is done. For the sake of simplicity, the argument of vector fields is omitted, that is, T stands for $T(y)$, M stands for $M(x)$ etc...

Formula (15) in theorem 4.3 gives the following chain of pointwise isomorphisms:

$$\begin{aligned} T_x(S^m) \oplus T_y(S^n) &\stackrel{(23)}{=} T_x(S^m) \oplus \langle T \rangle_{\mathbb{R}} \oplus T^\perp \\ &\stackrel{\alpha}{\simeq} T_x(S^m) \oplus \langle M \rangle_{\mathbb{R}} \oplus T^\perp \\ &\stackrel{(21)}{=} \mathbb{R}^{m+1} \oplus T^\perp \\ &\stackrel{\beta}{\simeq} \mathbb{R}^{m-1} \oplus \langle N \rangle_{\mathbb{R}} \oplus \langle T \rangle_{\mathbb{R}} \oplus T^\perp \\ &\stackrel{(23)}{=} \mathbb{R}^{m-1} \oplus \langle N \rangle_{\mathbb{R}} \oplus T_y S^n \\ &\stackrel{(21)}{=} \mathbb{R}^{m-1} \oplus \mathbb{R}^{n+1}, \end{aligned} \tag{24}$$ {chainlinear}

where α is defined by

$$\alpha(T) \stackrel{\text{def}}{=} M,$$

and β is defined by

$$\beta(\partial_{x_m}) \stackrel{\text{def}}{=} N, \qquad \beta(\partial_{x_{m+1}}) \stackrel{\text{def}}{=} T.$$

Pulling back to $T_x(S^m) \oplus T_y(S^n)$ the $m-1$ generators $\{\partial_{x_1}, \ldots, \partial_{x_{m-1}}\}$ of $\mathbb{R}^{m-1}$ one obtains

$$\begin{aligned} \partial_{x_i} &\stackrel{(1)}{=} M_i + x_i M \\ &\stackrel{\alpha^{-1}}{\longmapsto} M_i + x_i T \end{aligned} \qquad i = 1, \ldots, m-1, \tag{25}$$ {framea}

whereas pulling back to $T_x(S^m) \oplus T_y(S^n)$ the $n+1$ generators $\{\partial_{y_1}, \ldots, \partial_{y_{n+1}}\}$ of $\mathbb{R}^{n+1}$ one obtains the more complicated formulas

$$\begin{aligned} \partial_{y_j} &\stackrel{(1)}{=} N_j + y_j N \\ &= \langle N_j, T\rangle T + (N_j - \langle N_j, T\rangle T) + y_j N \\ &\stackrel{\beta^{-1}}{\longmapsto} \langle N_j, T\rangle \partial_{x_{m+1}} + (N_j - \langle N_j, T\rangle T) + y_j \partial_{x_m} \qquad j = 1, \ldots, n+1. \\ &\stackrel{(1)}{=} \langle N_j, T\rangle (M_{m+1} + x_{m+1} M) + (N_j - \langle N_j, T\rangle T) + y_j (M_m + x_m M) \\ &\stackrel{\alpha^{-1}}{\longmapsto} \langle N_j, T\rangle (M_{m+1} + x_{m+1} T) + (N_j - \langle N_j, T\rangle T) + y_j (M_m + x_m T) \end{aligned} \tag{26}$$ {frameb}

The following theorem applies the above argument to $S^m \times S^n$, odd n, in order to obtain an explicit frame on it:

Theorem 5.1 *Let n be odd, and let $T = \sum_{j=1}^{n+1} t_j \partial_{y_j}$ be the tangent versor field on S^n given by formula (19). Also, let $\{M_i\}_{i=1,\ldots,m+1}$ and $\{N_j\}_{j=1,\ldots,n+1}$ be the meridian vector fields on S^m and S^n respectively. Last, let M and N be the normal versor fields of $S^m \subset \mathbb{R}^{m+1}$ and $S^n \subset \mathbb{R}^{n+1}$ respectively. The product $S^m \times S^n$ is parallelized by the frame $\mathcal{P} \stackrel{\text{def}}{=} \{p_1, \ldots, p_{m+n}\}$ given by*

$$\begin{aligned} p_i &\stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \ldots, m-1, \\ p_{m-1+j} &\stackrel{\text{def}}{=} y_j M_m + t_j M_{m+1} + (t_j x_{m+1} + y_j x_m - t_j) T + N_j \qquad j = 1, \ldots, n+1. \end{aligned} \tag{27}$$ {frame}

Moreover, $\mathcal{P}$ is orthonormal with respect to the standard metric on $S^m \times S^n$.

Proof: Observe that

$$\langle N_j, T\rangle \stackrel{(1)}{=} \langle \partial_{y_j} - y_j N, T\rangle = \langle \partial_{y_j}, T\rangle = t_j \qquad j = 1, \ldots, n+1$$

and use formulas (25) and (26) to obtain (27). The orthonormality can be proved by observing that both α and β in (24) are isometries. But one can also directly check the p_i's, taking into account formula (2). ■

6 The frames $\mathcal{P}$ and $\mathcal{B}$ on $S^m \times S^1$ and $S^m \times S^3$

If $n = 1, 3$ or 7, remark 4.4 can be used to obtain a parallelization simpler than $\mathcal{P}$ on $S^m \times S^n$. If $n = 1, 3$ this parallelization is just the one given in sections 2, 3 respectively, which was called $\mathcal{B}$. In this section relations between $\mathcal{B}$ and $\mathcal{P}$ are exploited.

Let $n = 1$. Formula (27) gives the frame $\mathcal{P} = \{p_1, \ldots, p_{m+1}\}$ on $S^m \times S^1$, whereas the frame $\mathcal{B}$ is given by formula (3). Clearly,

$$p_i = b_i \qquad i = 1, \ldots, m-1.$$

Since $\partial_\theta = -y_2\partial_{y_1} + y_1\partial_{y_2} = T$, one obtains

$$\langle N_1, \partial_\theta\rangle = \langle \partial_{y_1} - y_1 N, -y_2\partial_{y_1} + y_1\partial_{y_2}\rangle = -y_2,$$
$$\langle N_2, \partial_\theta\rangle = \langle \partial_{y_2} - y_2 N, -y_2\partial_{y_1} + y_1\partial_{y_2}\rangle = y_1,$$

and thus

$$N_1 = -y_2 T,$$
$$N_2 = y_1 T.$$

Whence

$$p_m = y_1(M_m + x_m T) - y_2(M_{m+1} + x_{m+1}T) + y_2 T - y_2 T = y_1 b_m - y_2 b_{m+1},$$
$$p_{m+1} = y_2(M_m + x_m T) + y_1(M_{m+1} + x_{m+1}T) - y_1 T + y_1 T = y_2 b_m + y_1 b_{m+1},$$

and one gets

$$\mathcal{P} = \mathcal{B}\left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & y_1 & y_2 \\ 0 & \cdots & 0 & -y_2 & y_1 \end{array}\right) \qquad (28) \quad \{\texttt{changebtop}\}$$

Brackets of $\mathcal{P}$ are thus easily obtained by means of formulas (28), (6) and (5):

$$\begin{aligned} [p_i, p_j] &= x_i p_j - x_j p_i \qquad i, j = 1, \ldots, m-1 \\ [p_i, p_m] &= (-x_m y_1 + x_{m+1} y_2) p_i + x_i p_m - x_i p_{m+1} \qquad i = 1, \ldots, m-1 \\ [p_i, p_{m+1}] &= (-x_m y_2 - x_{m+1} y_1) p_i + x_i p_m + x_i p_{m+1} \qquad i = 1, \ldots, m-1 \\ [p_m, p_{m+1}] &= (x_m(y_1 - y_2) - x_{m+1}(y_1 + y_2)) p_m + (x_m(y_1 + y_2) + x_{m+1}(y_1 - y_2)) p_{m+1} \end{aligned} \qquad (29) \quad \{\texttt{bracket1genp}\}$$

Formula (28) gives the frame $\mathcal{P}^*$ dual to $\mathcal{P}$:

$$p^i = b^i \qquad i = 1, \ldots, m-1,$$
$$p^m = y_1 b^m - y_2 b^{m+1},$$
$$p^{m+1} = y_2 b^m + y_1 b^{m+1}.$$

The structure equations for $\mathcal{P}$ are thus obtained by a straightforward computation:

$$\begin{aligned} dp^i &= dx_i \wedge \tau = p^i \wedge \tau \qquad i = 1, \ldots, m-1, \\ dp^m &= p^m \wedge \tau + p^{m+1} \wedge \tau, \\ dp^{m+1} &= p^{m+1} \wedge \tau - p^m \wedge \tau, \end{aligned} \tag{30}$$ {diff1genp}

where τ is given by

$$\tau = \sum_{i=1}^{m+1} x_i b^i = \sum_{i=1}^{m-1} x_i p^i + (x_m y_1 - x_{m+1} y_2) p^m + (x_m y_2 + x_{m+1} y_1) p^{m+1}.$$

Let $n = 3$. Formula (27) gives the frame $\mathcal{P} = \{p_1, \ldots, p_{m+3}\}$ on $S^m \times S^3$, whereas the frame $\mathcal{B}$ is given by formula (10). Clearly,

$$p_i = b_i \qquad i = 1, \ldots, m-1.$$

Denote by "$(*)_{j^{\text{th}}}$" the j^{th} coordinate of $*$. Since

$$\begin{aligned} \langle N_j - t_j T, T \rangle &= 0 \\ \langle N_j - t_j T, b_{m+2} \rangle &= (b_{m+2})_{j^{\text{th}}}, \qquad j = 1, \ldots, 4, \\ \langle N_j - t_j T, b_{m+3} \rangle &= (b_{m+3})_{j^{\text{th}}}, \end{aligned}$$

one gets

$$p_{m-1+j} = y_j b_m + t_j b_{m+1} + (b_{m+2})_{j^{\text{th}}} b_{m+2} + (b_{m+3})_{j^{\text{th}}} b_{m+3} \qquad j = 1, \ldots, 4.$$

Whence

$$\mathcal{P} = \mathcal{B} \left(\begin{array}{ccc|cccc} & & & 0 & 0 & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & \cdots & 0 & -y_2 & y_1 & -y_4 & y_3 \\ 0 & \cdots & 0 & -y_3 & y_4 & y_1 & -y_2 \\ 0 & \cdots & 0 & -y_4 & -y_3 & y_2 & y_1 \end{array} \right) \tag{31}$$ {changebtop3}

Brackets of $\mathcal{P}$ can be obtained by means of a not straightforward computation using formulas (31), (11) and (5). One can also refer to the next section, where general formulas for $\mathcal{P}$ are given.

Formula (31) gives the frame $\mathcal{P}^*$ dual to $\mathcal{P}$:

$$\mathcal{P}^* = \mathcal{B}^* \left(\begin{array}{ccc|cccc} & & & 0 & 0 & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & \cdots & 0 & -y_2 & y_1 & -y_4 & y_3 \\ 0 & \cdots & 0 & -y_3 & y_4 & y_1 & -y_2 \\ 0 & \cdots & 0 & -y_4 & -y_3 & y_2 & y_1 \end{array}\right)$$

7 General formulas for $\mathcal{P}$

Recall that $T = \sum_{j=1}^{n+1} t_j \partial_{y_j}$. Set

$$\begin{aligned} X_m &\stackrel{\text{def}}{=} M_m + x_m T, \\ X_{m+1} &\stackrel{\text{def}}{=} M_{m+1} + x_{m+1} T, \\ C_{j,k} &\stackrel{\text{def}}{=} y_j t_k - y_k t_j \qquad j,k = 1,\ldots,n+1, \\ D_{j,k} &\stackrel{\text{def}}{=} 2C_{j,k} \underbrace{\mp\delta_{k,j\pm 1}}_{j\ \frac{\text{odd}}{\text{even}}} \underbrace{\pm\delta_{j,k\pm 1}}_{k\ \frac{\text{odd}}{\text{even}}} \qquad j,k = 1,\ldots,n+1. \end{aligned}$$

Formulas (27) easily give

$$\begin{aligned} \sum_{j=1}^{n+1} y_j p_{m-1+j} &= M_m + x_m T = X_m, \\ \sum_{j=1}^{n+1} t_j p_{m-1+j} &= M_{m+1} + x_{m+1} T = X_{m+1}. \end{aligned}$$

A hard calculation then gives

$$
\begin{aligned}
[p_i, p_j] &= x_i p_j - x_j p_i \qquad i, j = 1, \dots, m-1, \\
[p_i, p_{m-1+j}] &= -(y_j x_m + t_j x_{m+1}) p_i \\
&\quad \underbrace{\mp x_i p_{m-1+j\pm 1}}_{j \, \frac{\text{odd}}{\text{even}}} + x_i y_j X_m + x_i t_j X_{m+1} \qquad i = 1, \dots, m-1, j = 1, \dots, n+1, \\
[p_{m-1+j}, p_{m-1+k}] &= D_{j,k} \sum_{i=1}^{m-1} x_i p_i + y_j p_{m-1+k} - y_k p_{m-1+j} \\
&\quad + (x_m D_{j,k} - x_{m+1} C_{j,k}) X_m + ((x_{m+1} - 1) D_{j,k} + x_m C_{j,k}) X_{m+1} \\
&\quad + \underbrace{(\mp y_j x_m \mp t_j x_{m+1} \pm t_j) p_{m-1+k\pm 1}}_{k \, \frac{\text{odd}}{\text{even}}} \\
&\quad + \underbrace{(\pm y_k x_m \pm t_k x_{m+1} \mp t_k) p_{m-1+j\pm 1}}_{j \, \frac{\text{odd}}{\text{even}}} \qquad j, k = 1, \dots, n+1.
\end{aligned}
\tag{32}
$$

{genbracket}

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