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Hermitian and special structures on products of spheres

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Hermitian and special structures on products of spheres

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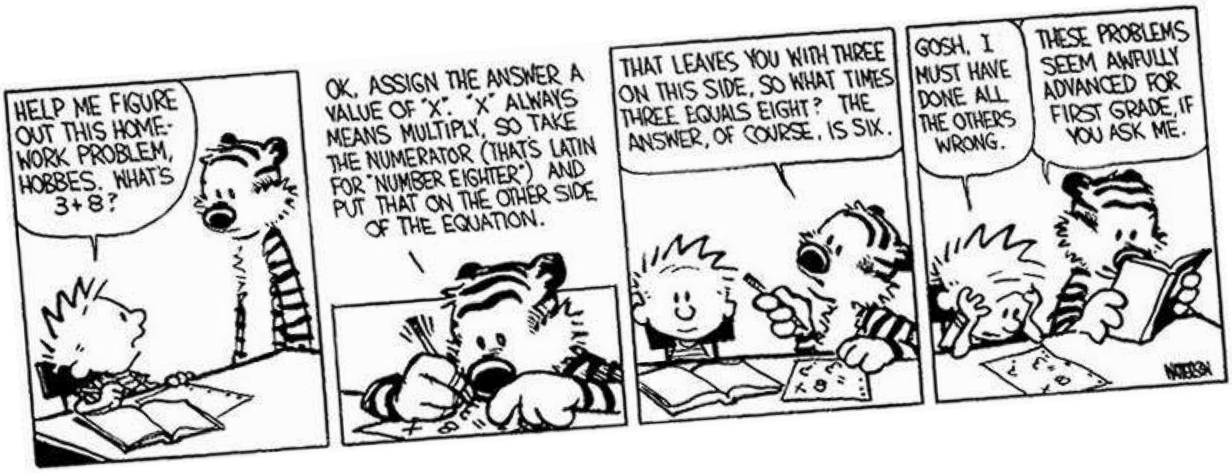
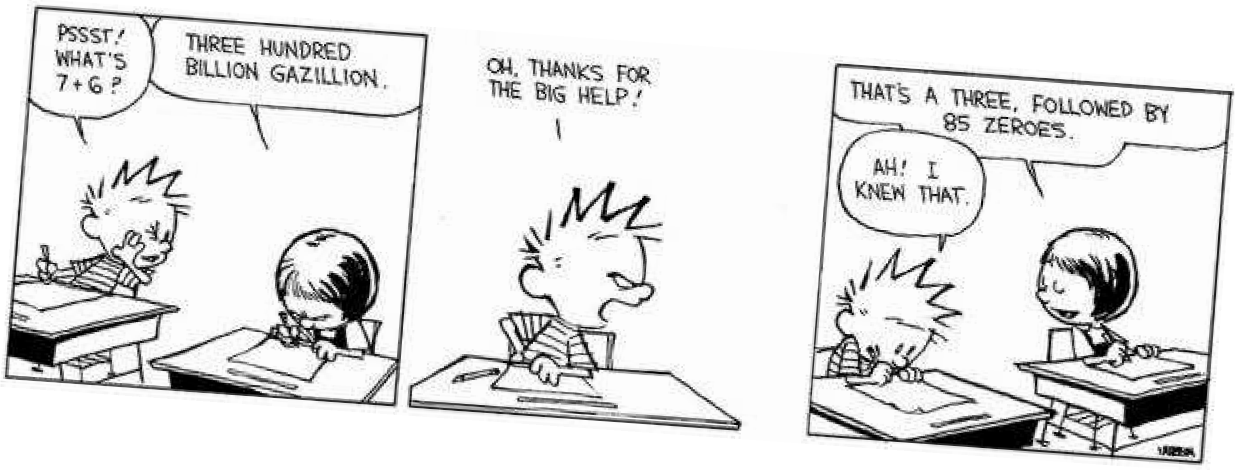
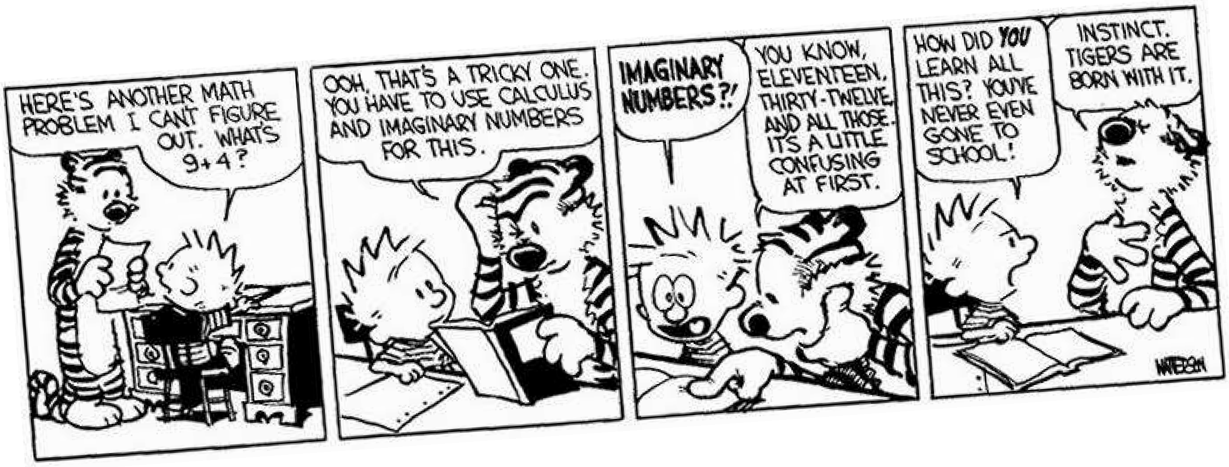






Introduction

It is a classical result in Algebraic Topology that spheres S^n are parallelizable if and only if their dimension is $n = 1$, 3 or 7. For products of two or more spheres the class of parallelizable manifolds is much wider. In fact, a striking result in this respect is the following theorem of M. Kervaire (see [Ker56]):

Theorem The manifold $S^{n_1} \times \cdots \times S^{n_r}$, $r \geq 2$, is parallelizable if and only if at least one of the n_i is odd.

Unlikely the case of a single sphere, where an explicit parallelization is given by the corresponding structure of division algebra of $\mathbb{R}^{n+1}$, an explicit parallelization on $S^{n_1} \times \cdots \times S^{n_r}$ is not straightforward. The only results known to the author in this direction were obtained by M. Bruni in [Bru92], where explicit parallelizations are provided only in the cases when one of the spheres is of dimension 1, 3, 5 or 7, using specific properties of these low dimensions.

In this thesis the problem of writing an explicit parallelization on $S^{n_1} \times \cdots \times S^{n_r}$ is solved in the general case, that is, an explicit $(m+n)$-uple of orthonormal vector fields is obtained on $S^m \times S^n$, in terms of the standard coordinates in $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$; for products of more than two spheres an inductive argument is used to extend the construction.

This parallelization is then exploited to define some significant G-structures on the products of spheres of suitable dimension, and to describe their differential properties.

The groups G considered are: $G = \mathrm{U}((m+n)/2)$, if both the dimensions are odd (that is, almost-Hermitian structures on $S^m \times S^n$); $G = \mathrm{Sp}((m+n)/4)$, if both the dimensions are odd and $m + n = 0 \mod 4$ (that is, almost-hyperhermitian structures on $S^m \times S^n$); $G = \mathrm{G}_2$, Spin(7) and Spin(9) on the 7-dimensional, 8-dimensional and 16-dimensional products $S^m \times S^n$ respectively. In some cases, the structures associated with the parallelization turn out to be classical structures, of which is then provided an explicit description (for instance, when G is the unitary group this approach recover the Calabi-Eckmann Hermitian structures). In the other cases, this construction provides new structures on products of spheres.

The action of the symmetric group on the parallelization gives rise to new structures. For each of the above groups G, the elements of this symmetric orbit of G-structures are described, and some remarkable differential properties are obtained; in some cases the whole orthogonal orbit is considered.

Among all parallelizable products of two spheres, $S^3 \times S^1$ is the lowest dimensional non toral case: its Lie group structure gives an explicit parallelization that was used in [Gau81] by P. Gauduchon to describe all diagonal Hopf complex structures $S^3 \times S^1$ can be equipped of. The same parallelization turns out to be useful to describe a family of locally conformal Kähler metrics on $S^3 \times S^1$ equipped with the structure of diagonal Hopf surface.

Locally conformal Kähler metrics on Hopf surfaces

The study of metrics on complex surfaces arose in the sixties out of the following question: which compact complex surfaces admit a Kählerian metric? It is a classical theorem (see for instance [BPV84, pages 266–269]) of Complex Geometry that all the complex surfaces with even first Betti number do admit a Kähler metric. This theorem, whose classical approach was through Kodaira's classification of minimal complex surfaces, has been recently proved by direct methods independently by N. Buchdahl and A. Lamari in [Buc99] and [Lam99] respectively.

Is there a weakened version of the Kähler hypothesis that one can hope to prove for surfaces with odd first Betti number? The notion of locally conformal Kähler manifold was introduced in this context by I. Vaisman in [Vai76]. Nevertheless, until 1998 there were very few examples of locally conformal Kähler manifolds, namely *some* Hopf surfaces, some Inoue surfaces and manifolds of type $(G/\Lambda) \times S^1$, where G is a nilpotent or solvable group. Recent relevant results were obtained by P. Gauduchon and L. Ornea in the paper [GO98], where they showed that *every* primary Hopf surface is locally conformal Kähler by writing a (family of) locally conformal Kähler metric (with parallel Lee form) for diagonal Hopf surfaces, and then deforming it for those of class 0 (as remarked by the authors, the argument used in [GO98] follows some suggestions of C. LeBrun). Even more recent results were obtained by F. A. Belgun in [Bel99] and [Bel00] where he classified the locally conformal Kähler surfaces with parallel Lee form, showed that also secondary Hopf surfaces are locally conformal Kähler and proved that some Inoue surfaces do not admit any locally conformal Kähler metric, settling thus at the same time a question raised by F. Tricerri in [Tri82], and the question whether any non-Kähler complex surface admits a locally conformal Kähler metric, raised by I. Vaisman in [Vai87]. A reference to local conformal Kähler geometry is the book [DO98] written by S. Dragomir and L. Ornea.

In chapter 1 it is shown that the metrics written in [GO98] for diagonal Hopf surfaces $H_{\alpha,\beta}$ are the only ones with parallel Lee form in a family of locally conformal Kähler metrics parametrized by the smooth positive functions on S^1 (theorem 1.2.4, remark 1.2.5 and theorem 1.2.7). This result is obtained by a slight modification of a technique developed for the simpler case $|\alpha| = |\beta|$, that leads to the family of locally conformal Kähler metrics given by formula (1.13), and, for any $k > 0$, to the following invariant metric on $\mathbb{C}^2 - 0$ with parallel Lee form (see theorem 1.2.2):

$$(\|z_1\|^2 + \|z_2\|^2)^{-2} \Big((kz_1\bar{z}_1 + z_2\bar{z}_2)\, dz_1 \otimes d\bar{z}_1 + (k-1)\, z_2\bar{z}_1 dz_1 \otimes d\bar{z}_2 + (k-1)\, z_1\bar{z}_2 dz_2 \otimes d\bar{z}_1 + (z_1\bar{z}_1 + kz_2\bar{z}_2)\, dz_2 \otimes d\bar{z}_2 \Big).$$

In section 1.3 four distributions canonically associated to the family of locally conformal Kähler metrics are described in detail. They are all shown to be integrable, and necessary and sufficient conditions for compactness of leaves are written (theorems 1.3.2, 1.3.3 and 1.3.6). In section 1.4 it is shown that when the foliation $\mathcal{E}_{\alpha,\beta}$ has all compact leaves -and this happens, according to theorem 1.3.6, if and only if $\alpha^m = \beta^n$ for some integers n and m-, the leaf space can be identified with $\mathbb{CP}^1$ in such a way that the canonical projection is a holomorphic map (theorem 1.4.1). This means that, whenever $H_{\alpha,\beta}$ is elliptic, the ellipticity is explicitly given by the foliation $\mathcal{E}_{\alpha,\beta}$. In section 1.5 it is shown that $\mathcal{E}_{\alpha,\beta}$ is quasi-regular (regular) if and only if the Hopf surface is elliptic (diagonal), and the corresponding structure of orbifold with two conical points on the leaf space is described (theorem 1.5.1).

Explicit parallelizations on products of spheres

As pointed out by M. Bruni in [Bru92], it is useful to write explicit parallelizations on parallelizable products of spheres: chapter 2 is devoted to solve this problem, that is, to write explicit parallelizations on $S^m \times S^n$, for any odd n. The never-vanishing vector field on the odd-dimensional sphere is used to write an explicit isomorphism between $T(S^m \times S^n)$ and $S^m \times S^n \times \mathbb{R}^{m+n}$, following a hint of [Hir88]. An explicit orthonormal parallelization $\mathcal{P}$ on $S^m \times S^n$ is then obtained pulling back the standard basis of $\mathbb{R}^{m+n}$ (theorem 2.4.1). Whenever S^n is itself parallelizable, a similar construction provides a frame $\mathcal{B}$ simpler than $\mathcal{P}$: this is exploited for $n = 1, 3$ (definition 2.1.3 and theorem 2.1.4 for $S^m \times S^1$, theorem 2.2.1 for $S^m \times S^3$), and coincide with the parallelizations given in [Bru92]. The last part of the chapter is devoted to write the structure equations for the parallelizations, and to exploit the relations between $\mathcal{B}$ and $\mathcal{P}$, whenever both are defined.

Structures on products of spheres

In the classical paper [Ber55], M. Berger showed that the holonomy group of a not locally symmetric Riemannian manifold must act transitively on a sphere. Together with the isomorphisms

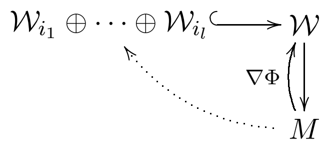
$$\mathrm{G}_2/\mathrm{SU}(3) \simeq S^6, \qquad \mathrm{Spin}(7)/\mathrm{G}_2 \simeq S^7,$$

this theorem gave rise to the problem, recently solved by D. Joyce, of finding examples of compact manifolds with holonomy G_2 and Spin(7) (see [Joy00]).

From a general point of view, given any Riemannian manifold M^d and a Lie group G that is the stabilizer of some tensor η on $\mathbb{R}^d$, that is,

$$G = \{g \in \mathrm{SO}(d) \text{ such that } g \cdot \eta = \eta\},$$

a G-structure on M defines a global tensor η on M, and it can be shown that $\nabla\eta$ (the so-called *intrinsic torsion of the G-structure*) is a section of the vector bundle $\mathcal{W} \stackrel{\text{def}}{=} T^* \otimes \mathfrak{g}^\perp$, where $\mathfrak{so}(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp$. The action of G splits $\mathcal{W}$ into irreducible components, say $\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_k$. G-structures on M can then be classified in at most 2^k classes, each class being given by the G-structures on M whose intrinsic torsion lifts to some subspace $\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l}$ of $\mathcal{W}$:



In this framework, the holonomy condition is the most restrictive, since M has holonomy group contained in G if and only if its intrinsic torsion is zero.

The first to deal in a systematic way with this kind of classification have been A. Gray and L. Hervella in [GH80], where they considered the case $G = \mathrm{U}(n)$, that is, almost-Hermitian structures. The space $\mathcal{W}$ splits in this case into four $\mathrm{U}(n)$-irreducible components, that give rise to exactly sixteen classes of almost Hermitian manifolds. Afterwards, M. Fernandez and Gray in [FG82] have treated the case $G = \mathrm{G}_2$, and Fernandez in [Fer86] the case $G = \mathrm{Spin}(7)$. In the former case, the G_2-irreducible components of $\mathcal{W}$ are four, giving rise to at most sixteen classes of G_2-manifolds, of which only nine was shown in [FG82] to be distinct; in the latter case, the Spin(7)-irreducible components of $\mathcal{W}$ are two, giving rise to exactly four classes of Spin(7)-manifolds. F. Cabrera (see [Cab96] and [Cab95a]) completed and refined the G_2 and Spin(7) classification: in particular, he showed that there are exactly fifteen distinct classes in the G_2 case (for connected manifold), and using the fact that the intrinsic torsion depends only on $d\eta$ and $d * \eta$, for the G_2 case, and only on $d\eta$, for the Spin(7) case, he characterized each class in a way simpler than in [FG82] and in [Fer86]. For instance, a G_2-structure belongs to the class $\mathcal{W}_4$ if and only if there exists a closed

1-form τ such that $d\eta = 3\tau \wedge \eta$ and $d * \eta = 4\tau \wedge \eta$; a Spin(7)-structure belongs to the class $\mathcal{W}_2$ if and only if there exists a closed 1-form τ such that $d\eta = \tau \wedge \eta$ (these are the *locally conformal parallel structures*).

The following table (compare with [Sal00]) summarizes the situation (the weird G_2 and Spin(7) forms depend on the choice of the representation of G_2 and Spin(7) on $\mathbb{R}^7$ and $\mathbb{R}^8$ respectively):

d	Φ	G	k	# of classes
$2n$	Kähler form	$U(n)$	4	16
7	locally: $\sum_{i\in\mathbb{Z}/(7)} e^{i,i+1,i+3}$	G_2	4	15
8	locally: $\lambda \wedge \sum_{i\in\mathbb{Z}/(7)} e^{i,i+1,i+3} - \sum_{i\in\mathbb{Z}/(7)} e^{i,i+2,i+3,i+4}$	Spin(7)	2	4

At first, also Spin(9) appeared in Berger's list; but D. Alekseevskij stated and R. Brown, A. Gray proved (see [Ale68] and [BG72]) that any complete 16-dimensional Riemannian manifold with holonomy group contained in Spin(9) is either flat or isometric to the Cayley plane F_4/Spin(9) or its noncompact dual. The study of Spin(9)-structures has been then neglected until december 1999, when T. Friedrich in [Fri99] pointed out that this is one of the three cases in which there is a notion of *weak holonomy* different from the classical notion of holonomy, the other two being $U(n)$ and G_2. He started then to study such weak holonomy structures, developing a Gray-Hervella-like classification of Spin(9)-structures on sixteen-dimensional manifolds. This classification starts from the remark that the intrinsic torsion of a Spin(9)-structure can be replaced by a 1-form Γ taking values in $\Lambda^3(V^9)$, for a suitable defined vector bundle V^9 locally spanned by 9 auto-adjoint, anti-commuting real structures. The key point is that with this replacement one does not lose any information about the geometric type of the original Spin(9)-structure. The same point of view could be used to study G_2 and Spin(7)-structures, but it is especially useful for Spin(9)-structures, since the definition of the Spin(9)-invariant 8-form given in [BG72] is not easy to handle.

Chapter 3 is devoted to study some properties of G-structures on products of two spheres. More precisely, these properties are integrability for almost-Hermitian and almost-hyperhermitian structures, and Gray-Hervella-like classification for G_2, Spin(7) and Spin(9)-structures. It turns out that on $S^{2n-1} \times S^1$ the frames $\mathcal{B}$ and $\mathcal{P}$ defined in chapter 2 give rise to the same integrable Hermitian structure of diagonal Hopf complex manifold (remark 3.3.1), and on $S^{2n-3} \times S^3$ the frames $\mathcal{B}$, $\mathcal{P}$ give rise to the same integrable Hermitian structure of Calabi-Eckmann manifold (theorem 3.4.2 and remark 3.4.3). This facts suggest that the frame $\mathcal{P}$ could be used to give an alternative definition of Calabi-Eckmann Hermitian structures, and this is the matter of theorem 3.5.1. The theorems for the Hermitian case are then applied to the hyperhermitian case, showing that the almost-hyperhermitian structure canonically associated to $\mathcal{B}$ on $S^{4n-1} \times S^1$ is the integrable hyperhermitian structure of diagonal Hopf hypercomplex manifold, whereas all other $Sp(n)$-structures canonically associated to the parallelizations are non integrable (remark 3.6.1 and theorem 3.6.2). The frame $\mathcal{B}$ defines locally conformal parallel G_2, Spin(7), Spin(9)-structures on $S^6 \times S^1$, $S^7 \times S^1$, $S^{15} \times S^1$ respectively (theorems 3.8.5, 3.9.5, 3.11.1). In the theorems 3.10.11 and 3.11.2 it is shown that these are the same structures defined in [Cab97], [Cab95a] and [Fri99]: the frame $\mathcal{B}$ provides then an alternative definition for classical special structures on $S^6 \times S^1$, $S^7 \times S^1$ and $S^{15} \times S^1$. Theorems 3.8.11 and 3.9.11 provide examples of G_2 and Spin(7)-structures of general type (that is, structures whose intrinsic torsion does not lift to any proper invariant subbundle of $\mathcal{W}$) on $S^6 \times S^1$, $S^4 \times S^3$, $S^2 \times S^5$ (G_2-structures) and on $S^7 \times S^1$, $S^5 \times S^3$, $S^3 \times S^5$, $S^1 \times S^7$ (Spin(7)-structures). These examples are new, to the knowledge of the author.

In chapter 4 the standard orthogonal representation of $\mathrm{O}(m+n)$ on $\mathbb{R}^{m+n}$ is used to provide more G-structures on $S^m \times S^n$, odd n, using a fixed orthonormal parallelization. Since the frame $\mathcal{B}$ on $S^m \times S^1$ is conformally induced by the standard frame on the universal covering space $\mathbb{R}^{m+1} - 0$, the orthogonal action gives isomorphic structures (theorems 4.4.2, 4.4.3). This argument is specific for the frame $\mathcal{B}$ on $S^m \times S^1$, and it does not fit for other parallelizations. Nevertheless, in theorem 4.4.4 it is proved that the G_2-structures on $S^6 \times S^1$ in the orthogonal orbit $\mathrm{O}(7) \cdot \varphi_{\mathcal{P}}$ are all of general type.

In the rest of the chapter the attention is restricted to the symmetric group $\mathfrak{S}_{m+n} \subset \mathrm{O}(m+n)$: one obtains in this way the families $\mathcal{I}_{\mathcal{P}}$, $\mathcal{H}_{\mathcal{P}}$, $\mathcal{G}_{\mathcal{P}}$, $\mathcal{S}_{\mathcal{P}}$, $\mathcal{N}_{\mathcal{P}}$ of almost-Hermitian, almost-hyperhermitian, G_2, Spin(7), Spin(9)-structures respectively, on products $S^m \times S^n$ of suitable dimension. If $n = 1$, 3, the corresponding families $\mathcal{I}_{\mathcal{B}}$, $\mathcal{H}_{\mathcal{B}}$, $\mathcal{G}_{\mathcal{B}}$, $\mathcal{S}_{\mathcal{B}}$, $\mathcal{N}_{\mathcal{B}}$ are also defined. The statements about properties of G-structures in the above symmetric orbits are obtained by a computer calculation, and in the typical but simplest cases a classical proof is also given.

It should be remarked that, except for the cases $S^m \times S^1$ with the frame $\mathcal{B}$, at least in the $\mathrm{U}(n)$ case the symmetric orbit does not contain all isomorphic structures, since it contains both integrable and non-integrable Hermitian structures (theorems 4.1.2, 4.1.4, 4.1.7, 4.1.9). The symmetric orbits $\mathcal{H}_{\mathcal{B}}$ on $S^{4n-3} \times S^3$ and $\mathcal{H}_{\mathcal{P}}$ on $S^m \times S^n$, $m + n = 0 \mod 4$, provide examples of non-integrable hyperhermitian structures (theorem 4.2.2), and the symmetric orbits $\mathcal{G}_{\mathcal{B}}$, $\mathcal{S}_{\mathcal{B}}$ on $S^4 \times S^3$, $S^5 \times S^3$ respectively, $\mathcal{G}_{\mathcal{P}}$, $\mathcal{S}_{\mathcal{P}}$ on $S^m \times S^n$, (n odd) $m + n = 7$, 8 respectively, provide examples of G_2 and Spin(7)-structures of general type (theorems 4.3.2, 4.3.3).

About computations

The following theorems were conjectured using experimental data obtained by a computer calculation, and then proved by classical arguments: 3.4.1, 3.4.2, 3.5.1, 3.8.10, 4.1.2, 4.1.4, 4.1.7, 4.1.9, 4.3.2, 4.4.4.

Since $\#(\mathfrak{S}_7 \cap \mathrm{G}_2) = 21$ and $\#(\mathfrak{S}_8 \cap \mathrm{Spin}(7)) = 168$, the symmetric orbits $\mathcal{G}$ and $\mathcal{S}$ contains both $7!/21 = 8!/168 = 240$ different structures. This remark is useful to obtain an efficient implementation of all the computation involved in the following theorems, that are proved by such a computation: 3.8.11, 3.9.11, 4.3.3.

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(1) Unità di Roma 1, responsabile Stefano Marchiafava

Contents

Chapter 1

Locally conformal Kähler metrics and elliptic fibrations

1.1 Preliminaries

A Hermitian manifold (M^{2n}, J, g) is called *locally conformal Kähler*, briefly *l.c.K.*, if there exists an open covering $\{U_i\}_{i\in I}$ of M and a family $\{f_i\}_{i\in I}$ of smooth functions $f_i\colon U_i \to \mathbb{R}$ such that the metrics g_i on U_i given by

$$g_i \stackrel{\text{def}}{=} e^{-f_i} g_{|_{U_i}}$$

are Kählerian metrics. The following relation holds on U_i between the fundamental forms Ω_i and $\Omega_{|_{U_i}}$ respectively of g_i and $g_{|_{U_i}}$:

$$\Omega_i = e^{-f_i}\Omega_{|_{U_i}},$$

so the *Lee form* ω locally defined by

$$\omega_{|_{U_i}} \stackrel{\text{def}}{=} df_i \tag{1.1}$$

is in fact global, and satisfies $d\Omega = \omega \wedge \Omega$. The manifold (M, J, g) is then l.c.K. if and only if there exists a global closed 1-form ω such that

$$d\Omega = \omega \wedge \Omega$$

(see for instance the recent book [DO98]).

As Kodaira defined in [Kod66, 10], a *Hopf surface* is a complex compact surface H whose universal covering is $\mathbb{C}^2 - 0$. If $\pi_1(H) \simeq \mathbb{Z}$, then H is called a *primary* Hopf surface. Kodaira showed that every primary Hopf surface can be obtained as

$$\frac{\mathbb{C}^2 - 0}{\langle f \rangle}, \qquad f(z_1, z_2) \stackrel{\text{def}}{=} (\alpha z_1 + \lambda z_2^m, \beta z_2),$$

where m is a positive integer and α, β, λ are complex numbers such that

$$(\alpha - \beta^m)\lambda = 0 \qquad \text{and} \qquad |\alpha| \geq |\beta| > 1.$$

Write $H_{\alpha,\beta,\lambda,m}$ for the generic primary Hopf surface. If $\lambda \neq 0$, then

$$f(z_1, z_2) = (\beta^m z_1 + \lambda z_2^m, \beta z_2)$$

and the surface $H_{\beta,\lambda,m} \stackrel{\text{def}}{=} H_{\beta^m,\beta,\lambda,m}$ is called *of class* 0, while if $\lambda = 0$, then

$$f(z_1, z_2) = (\alpha z_1, \beta z_2)$$

and the surface $H_{\alpha,\beta} \stackrel{\text{def}}{=} H_{\alpha,\beta,0,m}$ is called *of class* 1 (this terminology refers to the notion of *Kähler rank* as given in [HL83, § 9]).

A globally conformal Kähler metric on $\mathbb{C}^2 - 0$ (that is, of the form $e^{-f}g$ where $f \colon \mathbb{C}^2 - 0 \to \mathbb{R}$ and g is Kähler), which is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1 + \lambda z_2^m, \beta z_2)$, defines a l.c.K. metric on $H_{\alpha,\beta,\lambda,m}$: this is the case for the metric

$$\frac{dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2}{z_1\bar{z}_1 + z_2\bar{z}_2} \tag{1.2}$$

which is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$ (and so defines a l.c.K. metric on $H_{\alpha,\beta}$) whenever $|\alpha| = |\beta|$. The Lee form of this metric is parallel for the Levi-Civita connection (see [Vai79]).

In [Vai82], I. Vaisman called *generalized Hopf (g.H.)* manifolds those l.c.K. manifolds (M, J, g) with a parallel Lee form. Recently, since F. A. Belgun proved that primary Hopf surfaces of class 0 do not admit any generalized Hopf structure (see [Bel00]), some authors (see for instance [DO98, GO98]) decided to use the term *Vaisman manifold* instead.

Definition 1.1.1 A *Vaisman manifold* is a l.c.K. manifold (M, J, g) with parallel Lee form with respect to the Levi-Civita connection of g.

Define the operator d^c by $d^c(f)(X) \stackrel{\text{def}}{=} -df(J(X))$ for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, and call *potential* on the open set $\mathcal{U}$ of the complex manifold (M, J) a map $f \colon \mathcal{U} \to \mathbb{R}$ such that the 2-form on $\mathcal{U}$ of type $(1, 1)$ given by $(dd^c f)/2$ is positive: namely, such that the bilinear map g on $\mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U})$ given by

$$g(X, Y) \stackrel{\text{def}}{=} -\frac{dd^c f}{2}(J(X), Y)$$

is a (Kählerian) metric on $\mathcal{U}$.

Take the potential $\Phi_{\alpha,\beta} \colon \mathbb{C}^2 - 0 \to \mathbb{R}$ given by

$$\Phi_{\alpha,\beta}(z_1, z_2) \stackrel{\text{def}}{=} e^{\frac{(\log|\alpha| + \log|\beta|)\theta}{2\pi}} \tag{1.3}$$

where θ is given by

$$\frac{|z_1|^2}{e^{\frac{\theta \log|\alpha|}{\pi}}} + \frac{|z_2|^2}{e^{\frac{\theta \log|\beta|}{\pi}}} = 1. \tag{1.4}$$

In [GO98] the following theorem is proved:

Theorem 1.1.2 ([GO98, Proposition 1 and Corollary 1]) *The metric associated to the* 2*-form of type* $(1,1)$ *on* $\mathbb{C}^2 - 0$

$$\frac{dd^c\Phi_{\alpha,\beta}}{2\Phi_{\alpha,\beta}}$$

is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$*. The induced metric on* $H_{\alpha,\beta}$ *is Vaisman for every* α *and* β*.*

1.2 Some metrics on $S^1 \times S^3$

1.2.1 Definitions, notations and preliminary tools

Look at the 3-sphere as

$$S^3 \stackrel{\text{def}}{=} \left\{(\xi_1, \xi_2) \in \mathbb{C}^2 : |\xi_1|^2 + |\xi_2|^2 = 1\right\}$$

and at S^1 as the quotient of $\mathbb{R}$ by the map $\theta \mapsto \theta + 2\pi$. The manifolds $S^1 \times S^3$ and $H_{\alpha,\beta}$ are diffeomorphic (see [Kat75, theorem 9]) by means of the map $F_{\alpha,\beta}$ induced by F by the diagram

$$\begin{array}{ccc} \mathbb{R} \times S^3 & \stackrel{F}{\longrightarrow} & \mathbb{C}^2 - 0 \\ h\big\downarrow & & \big\downarrow f \\ \mathbb{R} \times S^3 & \stackrel{F}{\longrightarrow} & \mathbb{C}^2 - 0 \end{array}$$

where

$$h(\theta, (\xi_1, \xi_2)) \stackrel{\text{def}}{=} (\theta + 2\pi, (\xi_1, \xi_2)),$$
$$f(\xi_1, \xi_2) \stackrel{\text{def}}{=} (\alpha\xi_1, \beta\xi_2),$$
$$F(\theta, (\xi_1, \xi_2)) \stackrel{\text{def}}{=} (e^{\frac{\theta \log \alpha}{2\pi}}\xi_1, e^{\frac{\theta \log \beta}{2\pi}}\xi_2).$$

If $[z_1, z_2]$ is the element in $H_{\alpha,\beta}$ corresponding to $(z_1, z_2) \in \mathbb{C}^2 - 0$, then

$$F_{\alpha,\beta}(\theta, (\xi_1, \xi_2)) \stackrel{\text{def}}{=} [e^{\frac{\theta \log \alpha}{2\pi}}\xi_1, e^{\frac{\theta \log \beta}{2\pi}}\xi_2] \tag{1.5}$$

and the inverse map is

$$F^{-1}_{\alpha,\beta}([z_1, z_2]) = (\theta, (e^{-\frac{\theta \log \alpha}{2\pi}} z_1, e^{-\frac{\theta \log \beta}{2\pi}} z_2))$$

where θ is given by (1.4).

This diffeomorphism induce a complex structure $J_{\alpha,\beta}$ on $S^1 \times S^3$. In particular, $J_{\alpha,\alpha}$ were studied and classified by P. Gauduchon in [Gau81, propositions 2 and 3, pages 138 and 140], using an explicit parallelization of $S^1 \times S^3$.

Let $\mathbb{H}$ be the non-commutative field of quaternions, identified with $\mathbb{C}^2$ by $(\xi_1, \xi_2) \mapsto \xi_1 + j\bar{\xi}_2$. Let $S^1 \subset \mathbb{C}$ by $\theta \mapsto e^{i\theta}$, and let $Q = Q(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in S^3 \subset \mathbb{H}$ given by $\xi_1 = \alpha_1 + i\alpha_2$, $\xi_2 = \alpha_3 + i\alpha_4$. The Lie frame $\mathcal{E} \stackrel{\text{def}}{=} (e_1, e_2, e_3, e_4)$ on $S^1 \times S^3$ is then given by:

$$\begin{aligned} e_1((\theta, Q)) &\stackrel{\text{def}}{=} ie^{i\theta} \in T_\theta(S^1), \\ e_2((\theta, Q)) &\stackrel{\text{def}}{=} iQ = (i\xi_1, i\xi_2) = (-\alpha_2, \alpha_1, -\alpha_4, \alpha_3) \in T_Q(S^3), \\ e_3((\theta, Q)) &\stackrel{\text{def}}{=} jQ = (-\bar{\xi}_2, \bar{\xi}_1) = (-\alpha_3, \alpha_4, \alpha_1, -\alpha_2) \in T_Q(S^3), \\ e_4((\theta, Q)) &\stackrel{\text{def}}{=} kQ = (-i\bar{\xi}_2, i\bar{\xi}_1) = (-\alpha_4, -\alpha_3, \alpha_2, \alpha_1) \in T_Q(S^3). \end{aligned} \tag{1.6}$$

The structure equations are:

$$de^1 = 0, \qquad de^2 = 2e^3 \wedge e^4, \qquad de^3 = -2e^2 \wedge e^4, \qquad de^4 = 2e^2 \wedge e^3,$$

and the non-zero brackets are

$$[e_2, e_3] = -2e_4, \qquad [e_2, e_4] = 2e_3, \qquad [e_3, e_4] = -2e_2. \tag{1.7}$$

One finds that

$$dF = \Big(\frac{\log\alpha}{2\pi} e^{\frac{\theta\log\alpha}{2\pi}} \xi_1 d\theta + e^{\frac{\theta\log\alpha}{2\pi}} d\xi_1\Big) \otimes \partial_{z_1} + \Big(\frac{\log\beta}{2\pi} e^{\frac{\theta\log\beta}{2\pi}} \xi_2 d\theta + e^{\frac{\theta\log\beta}{2\pi}} d\xi_2\Big) \otimes \partial_{z_2}, \tag{1.8}$$

where $d\xi_1, d\xi_2$ and $d\theta$ are given by

$$\begin{aligned} &d\xi_1(e_1) = 0 && d\xi_2(e_1) = 0 && d\theta(e_1) = 1 \\ &d\xi_1(e_2) = -\alpha_2 + i\alpha_1 = i\xi_1 && d\xi_2(e_2) = -\alpha_4 + i\alpha_3 = i\xi_2 && d\theta(e_2) = 0 \\ &d\xi_1(e_3) = -\alpha_3 + i\alpha_4 = -\bar{\xi}_2 && d\xi_2(e_3) = \alpha_1 - i\alpha_2 = \bar{\xi}_1 && d\theta(e_3) = 0 \\ &d\xi_1(e_4) = -\alpha_4 - i\alpha_3 = -i\bar{\xi}_2 && d\xi_2(e_4) = \alpha_2 + i\alpha_1 = i\bar{\xi}_1 && d\theta(e_4) = 0. \end{aligned} \tag{1.9}$$

If G denotes the complex function on $S^1 \times S^3$ given by (see [GO98, formula 45])

$$\begin{aligned} G(\theta, (\xi_1, \xi_2)) &\stackrel{\text{def}}{=} |\xi_1|^2 \log\alpha + |\xi_2|^2 \log\beta \\ &= |\xi_1|^2 \log|\alpha| + |\xi_2|^2 \log|\beta| + i(|\xi_1|^2 \arg\alpha + |\xi_2|^2 \arg\beta), \end{aligned}$$

the complex structure $J_{\alpha,\beta}$ with respect to the basis $\mathcal{E}$ is given by

$$\begin{aligned} J_{\alpha,\beta}(e_1) &= -\frac{\mathfrak{Im}\, G}{\mathfrak{Re}\, G} e_1 + \frac{|G|^2}{2\pi\,\mathfrak{Re}\, G} e_2 - \frac{\mathfrak{Re}\left(i\xi_1\xi_2\overline{G}\log(\alpha/\beta)\right)}{2\pi\,\mathfrak{Re}\, G} e_3 - \frac{\mathfrak{Im}\left(i\xi_1\xi_2\overline{G}\log(\alpha/\beta)\right)}{2\pi\,\mathfrak{Re}\, G} e_4, \\ J_{\alpha,\beta}(e_2) &= -\frac{2\pi}{\mathfrak{Re}\, G} e_1 + \frac{\mathfrak{Im}\, G}{\mathfrak{Re}\, G} e_2 - \frac{\mathfrak{Re}\left(\xi_1\xi_2\log(\alpha/\beta)\right)}{\mathfrak{Re}\, G} e_3 - \frac{\mathfrak{Im}\left(\xi_1\xi_2\log(\alpha/\beta)\right)}{\mathfrak{Re}\, G} e_4, \\ J_{\alpha,\beta}(e_3) &= e_4, \\ J_{\alpha,\beta}(e_4) &= -e_3, \end{aligned} \tag{1.10}$$

(see [GO98, formulas 49], where the notations T, Z, E, iE, z_1, z_2 and F are used instead of $2\pi e_1$, e_2, $-e_3$, $-e_4$, ξ_1, ξ_2 and G).

The real vector bundle $T(S^1 \times S^3)$ of rank 4 becomes a complex vector bundle of rank 2 by means of $J_{\alpha,\beta}$: the vector fields e_2 and e_3 are $\mathbb{C}$-independent with respect to $J_{\alpha,\beta}$. A Hermitian metric on $S^1 \times S^3$ is given then by a Hermitian 2×2 matrix.

1.2.2 Case $|\alpha| = |\beta|$

First, consider the well-known case $\alpha = \beta$.

It can be checked that the pull-back of (1.2) by $F_{\alpha,\alpha}$ is the identity matrix in the Hermitian basis (e_2, e_3) of $T(S^1 \times S^3)$. It is then reasonable to wonder whether there exist other l.c.K. metrics given by Hermitian matrices of the form

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \tag{1.11}$$

where $k\colon S^1 \times S^3 \to \mathbb{R}^+$ is any real positive function; the Lee form is given by

$$\omega = -k\frac{\log|\alpha|}{\pi}e^1$$

and the l.c.K. condition $d\omega = 0$ gives

$$e_2(k) = 0, \qquad \log|\alpha| e_3(k) + \pi e_1\left(\frac{e_3(k)}{k}\right) = 0, \qquad \log|\alpha| e_4(k) + \pi e_1\left(\frac{e_4(k)}{k}\right) = 0. \tag{1.12}$$

This is a differential system of the second order, and it is trivially solved by any function k satisfying $e_2(k) = e_3(k) = e_4(k) = 0$, namely, by any function k which depends only on θ; using $F_{\alpha,\alpha}$ in the opposite direction the following invariant metrics on $\mathbb{C}^2 - 0$ are obtained:

$$\begin{aligned}(|z_1|^2 + |z_2|^2)^{-2}\Big(&(k(\theta)z_1\bar{z}_1 + z_2\bar{z}_2)\, dz_1 \otimes d\bar{z}_1 + (k(\theta) - 1)\, z_2\bar{z}_1 dz_1 \otimes d\bar{z}_2 \\ &+ (k(\theta) - 1)\, z_1\bar{z}_2 dz_2 \otimes d\bar{z}_1 + (z_1\bar{z}_1 + k(\theta)z_2\bar{z}_2)\, dz_2 \otimes d\bar{z}_2\Big)\end{aligned} \tag{1.13}$$

where

$$\theta = \frac{\log(|z_1|^2 + |z_2|^2)}{2\log|\alpha|}$$

and k is a positive function on S^1, i.e. a positive 2π-periodic real variable function.

Remark 1.2.1 If $k \neq 1$, the above metrics are not conformally equivalent to the classical invariant metric (1.2).

Let us call θ_j^k the 1-forms

$$\theta_j^k \stackrel{\text{def}}{=} \sum_{i=1}^4 \Gamma_{ij}^k e^i$$

of the Levi-Civita connection. Using Cartan's structure equations one obtains

$$\begin{aligned}
&\theta_1^1 = \frac{k'(\log^2|\alpha| - \arg^2\alpha)}{2k\log^2|\alpha|}e^1 - \frac{\pi k' \arg\alpha}{k\log^2|\alpha|}e^2, && \theta_2^1 = -\frac{\pi k'\arg\alpha}{k\log^2|\alpha|}e^1 - \frac{2\pi^2 k'}{k\log^2|\alpha|}e^2,\\
&\theta_1^2 = \frac{k'|\log\alpha|^2\arg\alpha}{4\pi k\log^2|\alpha|}e^1 + \frac{k'|\log\alpha|^2}{2k\log^2|\alpha|}e^2, && \theta_2^2 = \frac{k'|\log\alpha|^2}{2k\log^2|\alpha|}e^1 + \frac{\pi k'\arg\alpha}{k\log^2|\alpha|}e^2,\\
&\theta_3^2 = e^4, \quad \theta_2^3 = -ke^4, \quad \theta_2^4 = ke^3, \quad \theta_4^2 = -e^3, && \theta_4^3 = -\frac{k\arg\alpha}{2\pi}e^1 + (2-k)e^2,\\
&\theta_1^3 = -\frac{k\arg\alpha}{2\pi}e^4, \quad \theta_1^4 = \frac{k\arg\alpha}{2\pi}e^3, && \theta_3^4 = \frac{k\arg\alpha}{2\pi}e^1 + (k-2)e^2,\\
&\theta_3^1 = \theta_4^1 = \theta_3^3 = \theta_4^4 = 0.
\end{aligned}$$

A straightforward calculation thus gives

$$\begin{aligned}
&\nabla_{e_1}\omega = -\frac{k'|\log\alpha|^2}{2\pi\log|\alpha|}e^1 - \frac{k'\arg\alpha}{\log|\alpha|}e^2, \qquad \nabla_{e_2}\omega = -\frac{k'\arg\alpha}{\log|\alpha|}e^1 - \frac{2\pi k'}{\log|\alpha|}e^2,\\
&\nabla_{e_3}\omega = \nabla_{e_4}\omega = 0.
\end{aligned}$$

So, in the family of l.c.K. metrics given by (1.13), the Vaisman ones are those in which k is a constant function:

$$\begin{aligned}
(|z_1|^2 + |z_2|^2)^{-2}\Big(&(kz_1\bar z_1 + z_2\bar z_2)\, dz_1 \otimes d\bar z_1 + (k-1)\, z_2\bar z_1 dz_1 \otimes d\bar z_2\\
&+ (k-1)\, z_1\bar z_2 dz_2 \otimes d\bar z_1 + (z_1\bar z_1 + kz_2\bar z_2)\, dz_2 \otimes d\bar z_2\Big).
\end{aligned}$$

Can this method be applied also if only the weaker relation $|\alpha| = |\beta|$ holds? Answer is positive.

Again the pull-back via $F_{\alpha,\beta}$ of the metric (1.2) is the identity matrix in the Hermitian basis (e_2, e_3), and the same construction can be repeated: the Hermitian matrix

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

where $k\colon S^1 \times S^3 \to \mathbb{R}^+$ is a real positive function, is a l.c.K. metric if and only if it is a solution of

$$\begin{aligned} & e_2(k) = 0, \\ \arg\frac{\alpha}{\beta}\left((|\xi_1|^2 - |\xi_2|^2)\frac{e_4(k)}{k} - \Im m(\xi_1\xi_2)e_3\left(\frac{e_3(k)}{k}\right) + \Re e(\xi_1\xi_2)e_3\left(\frac{e_4(k)}{k}\right)\right) & \\ +2\left(\log|\alpha|e_3(k) + \pi e_1\left(\frac{e_3(k)}{k}\right)\right) &= 0, \\ \arg\frac{\alpha}{\beta}\left((|\xi_1|^2 - |\xi_2|^2)\frac{e_3(k)}{k} - \Re e(\xi_1\xi_2)e_4\left(\frac{e_4(k)}{k}\right) + \Im m(\xi_1\xi_2)e_3\left(\frac{e_4(k)}{k}\right)\right) & \\ +\arg\frac{\alpha}{\beta}\Im m(\xi_1\xi_2)\frac{(\log|\alpha|e_4(k)k - 1)e_4(e_3(k))}{k^2} - 2\left(\log|\alpha|e_4(k) + \pi e_1\left(\frac{e_4(k)}{k}\right)\right) &= 0. \end{aligned}$$

Computations are now much harder, due to the factor $\arg(\alpha/\beta)$: nevertheless, any function $k\colon S^1 \subset S^1 \times S^3 \to R^+$ is trivially again a solution, and the covariant derivative of the Lee form of the corresponding l.c.K. metric is given by

$$\nabla_{e_1}\omega = -\frac{k'|G|^2}{2\pi\log|\alpha|}e^1 - \frac{k'\,\Im m\,G}{\log|\alpha|}e^2, \qquad \nabla_{e_2}\omega = -\frac{k'\,\Im m\,G}{\log|\alpha|}e^1 - \frac{2\pi k'}{\log|\alpha|}e^2,$$
$$\nabla_{e_3}\omega = \nabla_{e_4}\omega = 0,$$

that is, the l.c.K. metric given in the complex basis (e_2, e_3) by $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ is a Vaisman metric if and only if k is constant:

Theorem 1.2.2 *The formula (1.11) gives a family of l.c.K. metrics on $H_{\alpha,\beta}$, in the case $|\alpha| = |\beta|$. In this family the Vaisman ones are given exactly by constant functions k.*

Remark 1.2.3 A family $\{g_t\}_{t>-1}$ of l.c.K. metrics (in the case $|\alpha| = |\beta|$) can be found in [Vai82, formula 2.13]. This family coincide (up to coefficients) with the Vaisman metrics of the above family given by $k = t + 1$. The claim, on page 240 of [Vai82], that only g_0 has parallel Lee form is uncorrect. The Weyl connection is used with the hypothesis $\omega_t(B_t) = |\omega_t|^2 = 1$, before proving that ω_t is parallel: in such a way, what is in fact proved is that g_0 is the only metric with $\nabla\omega = 0$ and $|\omega_t| = 1$. Actually, by using (2.14) and (2.17), one can check that $|\omega_t| = 1 + t$, hence the same computation proves that all the g_t have parallel Lee form. I acknowledge a useful conversation and an exchange of e-mail messages with I. Vaisman. □

1.2.3 General case

Unfortunately the same construction does not apply to the general case since the metric (1.2) is not $\langle f\rangle$-invariant, hence is not defined on $H_{\alpha,\beta}$.

The starting point in this case is the l.c.K. metric given by P. Gauduchon and L. Ornea in the recent work [GO98]. At the beginning of their paper they explicitly find a family of Vaisman metrics on

$H_{\alpha,\beta}$ by modifying the potential of (1.2). In what follows, the same techniques used in previous sections are applied to further modify the potential of (1.2).

Let $l\colon \mathcal{U} \to \mathbb{R}$ be a real function defined on an open set $\mathcal{U}$ of $\mathbb{R}$, and

$$\Phi_{\alpha,\beta}\colon \frac{\mathcal{U}}{2\pi\mathbb{Z}} \times S^3 \to \mathbb{R}^+$$

the real positive function given by

$$\Phi_{\alpha,\beta}((\theta,(\xi_1,\xi_2))) \stackrel{\text{def}}{=} e^{l(\theta)}. \tag{1.14}$$

The local 2-form $\Omega \stackrel{\text{def}}{=} \frac{1}{2}dd^c\Phi_{\alpha,\beta}$ is

$$\begin{aligned}\Omega = \frac{\Phi_{\alpha,\beta}\pi l'}{\mathfrak{Re}\, G}\Bigg(&\frac{l'^2+l''}{l'}e^{12} - \frac{\mathfrak{Re}(\xi_1\xi_2)(\log|\alpha|\arg\beta - \log|\beta|\arg\alpha)}{\pi\,\mathfrak{Re}\, G}e^{13}\\ &- \frac{\mathfrak{Im}(\xi_1\xi_2)(\log|\alpha|\arg\beta - \log|\beta|\arg\alpha)}{\pi\,\mathfrak{Re}\, G}e^{14} - \frac{2\,\mathfrak{Re}(\xi_1\xi_2)\log(|\alpha|/|\beta|)}{\mathfrak{Re}\, G}e^{23}\\ &- \frac{2\,\mathfrak{Im}(\xi_1\xi_2)\log(|\alpha|/|\beta|)}{\mathfrak{Re}\, G}e^{24} + 2e^{34}\Bigg)\end{aligned}$$

where e^{ij} denotes the wedge product $e^i \wedge e^j$. The matrix of the Hermitian bilinear form[(1)] in the complex basis (e_2, e_3) of $S^1 \times S^3$ is

$$2\Phi_{\alpha,\beta}\pi l' A \tag{1.15}$$

where

$$A \stackrel{\text{def}}{=} \begin{pmatrix} \dfrac{\pi}{\mathfrak{Re}^2\, G}\dfrac{l'^2+l''}{l'} + \dfrac{|\xi_1|^2|\xi_2|^2\log^2(|\alpha|/|\beta|)}{\mathfrak{Re}^3\, G} & \dfrac{\imath\xi_1\xi_2\log(|\alpha|/|\beta|)}{\mathfrak{Re}^2\, G} \\ \dfrac{\overline{\imath\xi_1\xi_2}\log(|\alpha|/|\beta|)}{\mathfrak{Re}^2\, G} & \dfrac{1}{\mathfrak{Re}\, G}\end{pmatrix}$$

The condition that Ω be positive translates then in l' and $l'^2 + l''$ both positive. This results in a local generalization of the proposition 1 of [GO98], that is, the local function e^l, where l is increasing and $l'^2 + l'' > 0$ on $\mathcal{U}$, define a potential $\Phi_{\alpha,\beta}$.

The matrix A does not depend directly on θ, but only by $(l'^2 + l'')/l'$. Consider a family $\{l_{\mathcal{U}}\}_{\mathcal{U}\in U}$ of local functions, where U is an open covering of $\mathbb{R}$, all satisfying $l' > 0$ and $l'^2 + l'' > 0$ and such that the quantities $(l'^2 + l'')/l'$ paste to a well-defined function h on S^1. The matrix (1.15) then gives a global Hermitian l.c.K. metric on $(S^1 \times S^3, J_{\alpha,\beta})$. In fact such a family can be found, as it is shown in the following theorem:

Theorem 1.2.4 *Given any real positive function h with period 2π on $\mathbb{R}$, the metric $g^h_{\alpha,\beta}$ given in the complex basis (e_2, e_3) of $T(S^1 \times S^3)$ by the Hermitian matrix*

$$\begin{pmatrix} \dfrac{\pi h}{\mathfrak{Re}^2\, G} + \dfrac{|\xi_1|^2|\xi_2|^2\log^2(|\alpha|/|\beta|)}{\mathfrak{Re}^3\, G} & \dfrac{\imath\xi_1\xi_2\log(|\alpha|/|\beta|)}{\mathfrak{Re}^2\, G} \\ \dfrac{\overline{\imath\xi_1\xi_2}\log(|\alpha|/|\beta|)}{\mathfrak{Re}^2\, G} & \dfrac{1}{\mathfrak{Re}\, G}\end{pmatrix}$$

is (well defined and) l.c.K on $(S^1 \times S^3, J_{\alpha,\beta})$.

---

(1) Given by $H(X,Y) \stackrel{\text{def}}{=} -\Omega(JX,Y) - i\Omega(X,Y)$.

Proof: For fixed h, the Cauchy problem

$$\begin{cases} \dfrac{l'^2 + l''}{l'} = h \\ l'(\theta_0) > 0 \end{cases} \tag{1.16}$$

satisfies the local existence theorem for any $\theta_0 \in \mathbb{R}$. This means one can find an open covering U of $\mathbb{R}$ and functions $l_{\mathcal{U}} \colon \mathcal{U} \to \mathbb{R}$ which satisfy the equation. Moreover, U and $\{l_{\mathcal{U}}\}_{\mathcal{U} \in U}$ can be chosen so that h is increasing for any $\mathcal{U} \in U$; finally, note that, since h is positive, so is $l'^2 + l''$, and this gives the required family. ■

The previous theorem extends the corollary 1 of [GO98].

The Lee form of the metric $g^h_{\alpha,\beta}$ associated to a function h is given by (see (1.1) and (1.15))

$$\omega = -d \log\left(2\Phi_{\alpha,\beta}\pi l'\right) = -\frac{l'^2 + l''}{l'} e^1 = -he^1.$$

Remark 1.2.5 If $h \colon S^1 \to \mathbb{R}^+$ is constant, a (global) solution of the Cauchy problem (1.16) is given by $l(\theta) = h\theta$, and the potential of the corresponding $g^h_{\alpha,\beta}$ is given by (see (1.14)) $e^{h\theta}$. In [GO98] the potential is $e^{l(\log|\alpha| + \log|\beta|)\theta/(2\pi)}$, where l is any positive real number (see [GO98, after remark 3]): thus, for h constant, the constant l of [GO98] is given by

$$l = \frac{2\pi h}{\log|\alpha| + \log|\beta|}.$$

□

Remark 1.2.6 If $|\alpha| = |\beta|$ then $\mathfrak{Re}\, G = \log|\alpha|$, $\log(|\alpha|/|\beta|) = 0$ and

$$g^h_{\alpha,\beta} = \frac{1}{\log|\alpha|} \begin{pmatrix} \dfrac{\pi h}{\log|\alpha|} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus in the case $|\alpha| = |\beta|$ the family given by the theorem 1.2.4 coincide up to a constant with the family given by (1.11), where $k = \pi h / \log|\alpha|$. □

For a general $h^{(2)}$ the Lee vector field B of $g^h_{\alpha,\beta}$ is

$$B = -4\pi e_1 + 2\,\mathfrak{Im}\, G e_2 + 2\,\mathfrak{Im}(\xi_1\xi_2)\arg(\alpha/\beta) e_3 - 2\,\mathfrak{Re}(\xi_1\xi_2)\arg(\alpha/\beta) e_4$$

and the six terms formula gives

$$\begin{aligned} &g^h_{\alpha,\beta}(\nabla_{e_1}(B), e_1) = -\frac{h'|G|^2}{2\,\mathfrak{Re}^2\, G}, \qquad g^h_{\alpha,\beta}(\nabla_{e_2}(B), e_2) = -\frac{2h'\pi^2}{\mathfrak{Re}^2\, G}, \\ &g^h_{\alpha,\beta}(\nabla_{e_1}(B), e_2) = g^h_{\alpha,\beta}(\nabla_{e_2}(B), e_1) = -\frac{h'\,\mathfrak{Im}\, G\pi}{\mathfrak{Re}^2\, G}, \\ &g^h_{\alpha,\beta}(\nabla_{e_i}(B), e_j) = 0 \qquad \text{otherwise}. \end{aligned}$$

Then the following holds:

(2) If h is not constant the metric $g^h_{\alpha,\beta}$ restricted to the fibre S^3 of the projection $S^1 \times S^3 \to S^1$ does depend on θ, so the argument of [GO98, proposition 3 and corollary 2] doesn't apply.

Theorem 1.2.7 *The metric $g^h_{\alpha,\beta}$ of theorem 1.2.4 is Vaisman if and only if h is constant.*

1.3 Some foliations on $S^1 \times S^3$

On any l.c.K. manifold (M, J, g) with a never-vanishing Lee form ω, the following canonical distributions are given:

i) The kernel of the Lee form: since $d\omega = 0$, and

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) \qquad X, Y \in \mathfrak{X}(M)$$

such a distribution is integrable. This codimension 1 foliation is denoted by $\mathcal{F}$;

ii) the flow of the Lee vector field B, dual via g of ω: since

$$g(B, X) = \omega(X) = 0 \qquad \text{for every } X \in \ker\omega$$

this foliation is in fact $\mathcal{F}^\perp$;

iii) the flow of the vector field JB: this foliation is denoted by $J\mathcal{F}^\perp$;

iv) the 2-dimensional distribution spanned by B and JB is $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$: whenever the Lee form is parallel, this distribution is integrable (see e.g. [CP85, theorem 4.3], but this condition is not necessary, see theorem 1.3.4), and moreover, it defines a Riemannian foliation (see [DO98, Theorem 5.1]).

The notation is taken from [CP85] and [Pic90], where these and other related distributions are studied.

Referring to $g^h_{\alpha,\beta}$, remark that $\omega = he^1$, where h is strictly positive, implies that ω is never-vanishing.

1.3.1 The foliation $\mathcal{F}$

The foliation $\mathcal{F}$ is simply the S^3 spheres foliation given by the diffeomorphism $F_{\alpha,\beta}$: so in the parallel case -namely, for h constant- these S^3 are totally geodesic submanifolds of $(S^1 \times S^3, g^h_{\alpha,\beta})$ (see [CP85, lemma 4.1]).

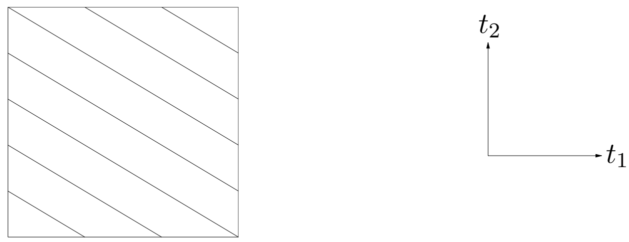


Figure 1.1: toral knot of type $-\frac{3}{5}$.

1.3.2 The foliations $\mathcal{F}^\perp$, $J\mathcal{F}^\perp$ and $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$

Consider the torus $S^1 \times S^1$ with coordinates (t_1, t_2). The following is well-known:

Lemma 1.3.1 *The curve in $S^1 \times S^1$ given by the linear functions*

$$t_1(t) = \gamma_1 + \delta_1 t \mod 2\pi, \qquad t_2(t) = \gamma_2 + \delta_2 t \mod 2\pi \tag{1.17}$$

is

i) compact if $\delta_2/\delta_1 \in \mathbb{Q}$;

ii) dense in $S^1 \times S^1$ otherwise.

In the case i) of the previous lemma, the curve (1.17) is called a *toral knot of type* δ_2/δ_1 (see figure 1.1).

Let $(\Theta, \Xi_1, \Xi_2) \in S^1 \times S^3$, and suppose $\Xi_1\Xi_2 \neq 0$. To study the leaves of $\mathcal{F}^\perp$, $J\mathcal{F}^\perp$ passing through (Θ, Ξ_1, Ξ_2), define the submanifold $T \subset S^3$ product of two circles of radius $|\Xi_1|$, $|\Xi_2|$:

$$T \overset{\text{def}}{=} T(\Xi_1, \Xi_2) \overset{\text{def}}{=} S^1_{|\Xi_1|} \times S^1_{|\Xi_2|} \subset \mathbb{C} \times \mathbb{C},$$

and denote by t_1, t_2 the coordinates on the torus T given by

$$\xi_1(t_1) = \Xi_1 e^{it_1}, \qquad \xi_2(t_2) = \Xi_2 e^{it_2}. \tag{1.18}$$

Consider in $S^1 \times S^3$ the real 3-dimensional torus $S^1 \times T$, containing the point (Θ, Ξ_1, Ξ_2); a curve in this 3-torus is given by

$$\theta = \theta(t) \mod 2\pi, \qquad t_1 = t_1(t) \mod 2\pi, \qquad t_2 = t_2(t) \mod 2\pi.$$

The 3-torus $S^1 \times T$ can be visualized as a cube with identifications (see figure 1.2).

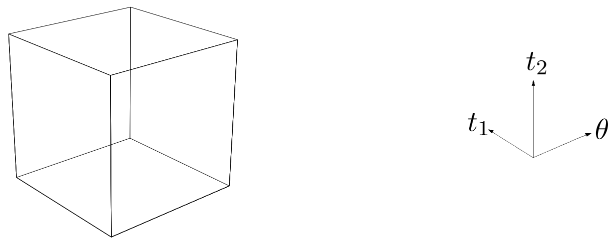


Figure 1.2: the 3-torus $S^1 \times T$.

The foliation $\mathcal{F}^{\perp}$

The Lee vector field of $g^h_{\alpha,\beta}$ is

$$B = -4\pi e_1 + 2\,\Im m\, G e_2 + 2\,\Im m(\xi_1\xi_2)\arg(\alpha/\beta)e_3 - 2\,\Re e(\xi_1\xi_2)\arg(\alpha/\beta)e_4$$

-it does not depend on h- and using (1.6) one obtains

$$B = -4\pi e_1 + 2i(\xi_1 \arg\alpha, \xi_2 \arg\beta).$$

By means of $F_{\alpha,\beta}$ (formula (1.8)) the Lee vector field induces a vector field in $\mathbb{C}^2 - 0$, where it becomes (see also [GO98, formula (23)])

$$B = -2(z_1 \log|\alpha|, z_2 \log|\beta|). \tag{1.19}$$

The flow of B is then

$$(z_1(t), z_2(t)) = (z_1(0)e^{-2t\log|\alpha|}, z_2(0)e^{-2t\log|\beta|}) \qquad t \in \mathbb{R}, \tag{1.20}$$

where $(z_1(0), z_2(0))$ and (Θ, Ξ_1, Ξ_2) are related by

$$\Xi_1 e^{\frac{\Theta\log\alpha}{2\pi}} = z_1(0), \qquad \Xi_2 e^{\frac{\Theta\log\beta}{2\pi}} = z_2(0). \tag{1.21}$$

Pull now the integral curve back to $S^1 \times S^3$ via $F_{\alpha,\beta}$: setting

$$\xi_1(t)e^{\frac{\theta(t)\log\alpha}{2\pi}} = z_1(0)e^{-2t\log|\alpha|}, \qquad \xi_2(t)e^{\frac{\theta(t)\log\beta}{2\pi}} = z_2(0)e^{-2t\log|\beta|}, \tag{1.22}$$

one obtains the following implicit expression of $\theta(t)$:

$$|z_1(0)|^2 e^{-\log|\alpha|(4t+\frac{\theta(t)}{\pi})} + |z_2(0)|^2 e^{-\log|\beta|(4t+\frac{\theta(t)}{\pi})} = 1;$$

if $x = x(\Theta, \Xi_1, \Xi_2)$ denotes the unique solution of the equation

$$|z_1(0)|^2 x^{\log|\alpha|} + |z_2(0)|^2 x^{\log|\beta|} = 1, \tag{1.23}$$

one gets

$$\theta(t) = -\pi(\log x + 4t), \tag{1.24}$$

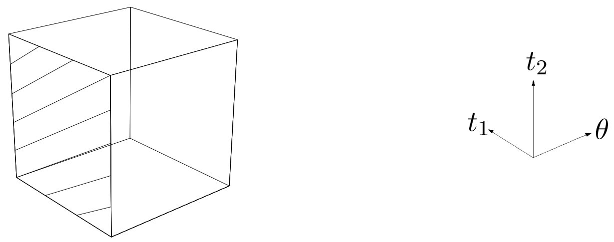


Figure 1.3: projection of the leaf of $J\mathcal{F}^\perp$ to T: case $\arg\alpha/\arg\beta \in \mathbb{Q}$.

that, together with (1.22) and (1.21), gives the following parametric equations for the integral curve of $\mathcal{F}^\perp$ through x:

$$\xi_1(t) = z_1(0)e^{\frac{\log x \log\alpha}{2}} e^{2it\arg\alpha} = \Xi_1 e^{2it\arg\alpha}, \quad \xi_2(t) = z_2(0)e^{\frac{\log x \log\beta}{2}} e^{2it\arg\beta} = \Xi_2 e^{2it\arg\beta}. \quad (1.25)$$

There are two types of points in $S^1 \times S^3$. If $\Xi_1\Xi_2 = 0$, say $\Xi_2 = 0$, the leaf given by (1.24) and (1.25) is contained in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$. According to lemma 1.3.1, if $\arg\alpha$ is a rational multiple of π, the leaf is compact; otherwise it is dense in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$. If $\Xi_1\Xi_2 \neq 0$, equations (1.25) implies that $\xi_1(t)$ and $\xi_2(t)$ have a constant positive length for every t, therefore the leaf is contained in the real 3-torus $S^1 \times T$ defined at page 11. Once observed that $\Theta = -\pi \log x \mod 2\pi$, the equations (1.24) and (1.25) can be written as

$$\theta(t) = \Theta - 4\pi t \mod 2\pi, \quad t_1(t) = 2t\arg\alpha \mod 2\pi, \quad t_2(t) = 2t\arg\beta \mod 2\pi. \quad (1.26)$$

In order to study the compactness of the leaves it should be remarked that:

i) the leaf projected on T is given by

$$t_1(t) = 2t\arg\alpha \mod 2\pi, \qquad t_2(t) = 2t\arg\beta \mod 2\pi, \quad (1.27)$$

and by lemma 1.3.1 this is a compact set if the ratio of $\arg\alpha$ to $\arg\beta$ is rational; otherwise it is dense in T. Since the projection from $S^1 \times T$ on T is a closed map, it can be inferred that if the ratio of $\arg\alpha$ to $\arg\beta$ is not rational then the leaf is not compact. If this ratio is rational, then the projected set is a toral knot of type $\arg\alpha/\arg\beta$ (see figure 1.3);

ii) the projection of the leaf on the face $t_2 = 0$ of the cube in figure 1.2 is given by

$$\theta(t) = \Theta - 4\pi t \mod 2\pi, \qquad t_1(t) = 2t\arg\alpha \mod 2\pi,$$

and lemma 1.3.1 gives the condition $(\arg\alpha)/\pi \in \mathbb{Q}$ (see figure 1.4);

iii) in the same way, consider the projection on the face $t_1 = 0$ to obtain $(\arg\beta)/\pi \in \mathbb{Q}$ (see figure 1.5).

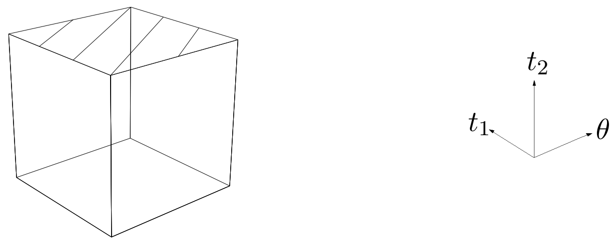


Figure 1.4: projection of the leaf of $J\mathcal{F}^\perp$ to $\{t_2 = 0\}$: case $(\arg\alpha)/\pi \in \mathbb{Q}$.

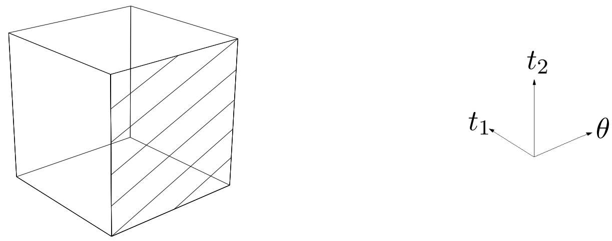


Figure 1.5: projection of the leaf of $J\mathcal{F}^\perp$ to $\{t_1 = 0\}$: case $(\arg\beta)/\pi \in \mathbb{Q}$.

Then, the following three conditions are *necessary* for the compactness of the leaf:

$$\arg\alpha \in \mathbb{Q}\pi; \qquad \arg\beta \in \mathbb{Q}\pi; \qquad \arg\alpha/\arg\beta \in \mathbb{Q}, \tag{1.28}$$

and any two of them obviously imply the third. The conditions (1.28) are also *sufficient*: if (1.28) hold, one can choose coprime integers l and k such that

$$\frac{\arg\alpha}{\arg\beta} = \frac{l}{k}.$$

The equations (1.27) define a closed curve with period $l\pi/\arg\alpha(=k\pi/\arg\beta)$, and the leaf is closed whenever $\theta(t)$ given by equations (1.26) has a period that is an integer multiple of $l\pi/\arg\alpha$. Choosing integers p and q such that $(\arg\alpha)/\pi = p/q$, it is straightforward to check that $pl\pi/\arg\alpha$ is a period of $\theta(t)$, and the proof is complete. To summarize:

Theorem 1.3.2 *Given the* 1*-dimensional foliation* $\mathcal{F}^\perp$ *on* $(S^1 \times S^3, J_{\alpha,\beta}, g^h_{\alpha,\beta})$ *the following holds:*

i) for every α *and* β *the leaf through the point* $(\Theta, \Xi_1, 0)$ *(respectively* $(\Theta, 0, \Xi_2)$*) is a subset of* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*). This leaf is*

- *compact if* $\arg\alpha \in \mathbb{Q}\pi$ *(respectively* $\arg\beta \in \mathbb{Q}\pi$*);*
- *dense in* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively in* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*) otherwise;*

ii) for every α *and* β *the leaf through the point* (Θ, Ξ_1, Ξ_2)*, where* $\Xi_1\Xi_2 \neq 0$*, is a subset of* $S^1 \times T$*, where* T *is the torus in the factor* S^3 *of* $S^1 \times S^3$ *given by (1.18). This leaf is*

- *compact if any two of (1.28) hold;*

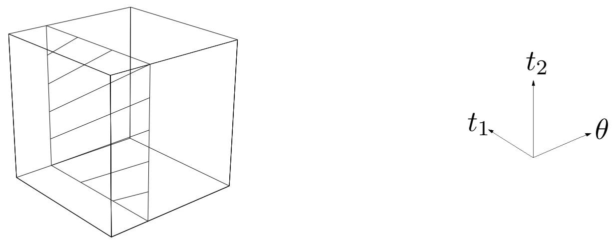


Figure 1.6: the leaf of $J\mathcal{F}^\perp$, case $\log|\alpha|/\log|\beta| \in \mathbb{Q}$.

- *non compact otherwise;*

if the leaf is not compact, then its projection on T is

- *a toral knot of type* $\arg\alpha/\arg\beta$ *if this ratio is rational;*
- *dense in T otherwise.*

The foliation $J\mathcal{F}^\perp$

The anti Lee vector field JB is given by

$$JB = -2\,\Re e\, G e_2 - 2\,\Im m(\xi_1\xi_2)\log|\alpha/\beta| e_3 + 2\,\Re e(\xi_1\xi_2)\log|\alpha/\beta| e_4$$

-it is independent of h- and again by (1.6) and (1.8) one obtains

$$JB = -2i(\xi_1 \log|\alpha|, \xi_2 \log|\beta|) = -2i(z_1 \log|\alpha|, z_2 \log|\beta|),$$

thus the integral curves are

$$(z_1(s), z_2(s)) = (z_1(0)e^{-2is\log|\alpha|}, z_2(0)e^{-2is\log|\beta|}).$$

These formulas are profoundly different from the previous ones, because of the complex exponent: in fact

$$\theta(s) = -\pi \log x,$$

where x is a solution of (1.23), and

$$\xi_1(s) = z_1(0)e^{\frac{\log x \log\alpha}{2}} e^{-2is\log|\alpha|} = \Xi_1 e^{-2is\log|\alpha|},$$
$$\xi_2(s) = z_2(0)e^{\frac{\log x \log\beta}{2}} e^{-2is\log|\beta|} = \Xi_2 e^{-2is\log|\beta|}.$$

If $\Xi_1\Xi_2 = 0$, say $\Xi_2 = 0$, the leaf through $(\Theta, \Xi_1, 0)$ is $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$, so it is closed. If $\Xi_1\Xi_2 \neq 0$, then $\xi_1(s)$ and $\xi_2(s)$ have constant positive length, so the leaf through (Θ, Ξ_1, Ξ_2) is a subset of $\{\Theta\} \times T$, where T is given by (1.18) (see figure 1.6):

Theorem 1.3.3 *Given the* 1*-dimensional foliation* $J\mathcal{F}^\perp$ *on* $(S^1 \times S^3, J_{\alpha,\beta}, g^h_{\alpha,\beta})$*, the following holds:*

i) for every α *and* β *the leaf through the point* $(\Theta, \Xi_1, 0)$ *(respectively* $(\Theta, 0, \Xi_2)$*) is* $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively* $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*), so it is compact;*

ii) for every α *and* β *the leaf through the point* (Θ, Ξ_1, Ξ_2)*, where* $\Xi_1\Xi_2 \neq 0$*, is a subset of* $\{\Theta\} \times T$*, where* T *is the torus in the factor* S^3 *of* $S^1 \times S^3$ *given by (1.18). This leaf is*

- *a toral knot of type* $\log|\alpha|/\log|\beta|$ *if this ratio is rational;*
- *dense in* $\{\Theta\} \times T$ *otherwise.*

The foliation $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$

The most interesting distribution is the one generated by both the Lee and the anti Lee vector fields: these planes are closed with respect to J, thus if the distribution is integrable then the integral surfaces are complex curves with a never-vanishing vector field:

Theorem 1.3.4 *The distribution* $\mathcal{F}^\perp \oplus J\mathcal{F}^\perp$ *is integrable. Moreover this distribution only depends on* α *and* β*.*

Proof: It is well known (see [CP85]) that if the Lee form is parallel then the distribution is integrable: now recall that

$$B = -2(z_1 \log|\alpha|, z_2 \log|\beta|), \qquad JB = -2i(z_1 \log|\alpha|, z_2 \log|\beta|),$$

and these expressions are independent of the function h. Then, fixing α and β, one obtains a unique distribution on $S^1 \times S^3$, that coincides with the distribution induced by any constant h, and is thus integrable. ∎

Definition 1.3.5 Call $\mathcal{E}_{\alpha,\beta}$ the unique foliation given by theorem 1.3.4.

The following theorem gives an explicit description of the leaves of $\mathcal{E}_{\alpha,\beta}$:

Theorem 1.3.6 *The foliation* $\mathcal{E}_{\alpha,\beta}$ *on* $S^1 \times S^3$ *is described by the following properties:*

i) for every α *and* β *the leaf through the point* $(\Theta, \Xi_1, 0)$ *(respectively* $(\Theta, 0, \Xi_2)$*) is* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*), and it is thus compact;*

ii) for every α *and* β *the leaf through the point* (Θ, Ξ_1, Ξ_2)*, where* $\Xi_1\Xi_2 \neq 0$*, is a subset of* $S^1 \times T$*, where* T *is the torus in the factor* S^3 *of* $S^1 \times S^3$ *given by (1.18). This leaf is*

- *compact if there exist integers m and n such that $\alpha^m = \beta^n$: in this case the leaf is a Riemann surface of genus one $\mathbb{C}/\Lambda$, where Λ is the lattice in $\mathbb{C}$ generated by the vectors v and w given by (1.33);*
- *non compact otherwise, and in this case it is dense in $S^1 \times T$.*

Proof: The 2-dimensional real distribution is a 1-dimensional complex distribution generated by B. By a formal substitution of $t \in \mathbb{R}$ with $w \in \mathbb{C}$ in (1.20), one obtains

$$(z_1(w), z_2(w)) = (z_1(0)e^{-2w\log|\alpha|}, z_2(0)e^{-2w\log|\beta|}), \tag{1.29}$$

which results in a complex parametrization of the integral surface of $\mathcal{E}_{\alpha,\beta}$ that passes through (Θ, Ξ_1, Ξ_2), in the coordinates $[z_1, z_2]$. As in the proof of theorem 1.3.2, one obtains the following parametrization:

$$\begin{aligned} \theta(w) &= \Theta - 4\pi\,\mathfrak{Re}\,w \mod 2\pi, \\ \xi_1(w) &= \Xi_1 e^{2i\arg\alpha\,\mathfrak{Re}\,w} e^{-2i\log|\alpha|\,\mathfrak{Im}\,w}, \\ \xi_2(w) &= \Xi_2 e^{2i\arg\beta\,\mathfrak{Re}\,w} e^{-2i\log|\beta|\,\mathfrak{Im}\,w}. \end{aligned} \tag{1.30}$$

The simplest case $\Xi_1\Xi_2 = 0$ follows from equations (1.30). Suppose $\Xi_1\Xi_2 \neq 0$. In this case the leaf is a subset of $S^1 \times T$, where T is the 2-torus given by (1.18). Let $(t,s) \stackrel{\text{def}}{=} (\mathfrak{Re}\,w, \mathfrak{Im}\,w)$. Then equations (1.30) become

$$\begin{aligned} \theta(t,s) &= \Theta - 4\pi t \mod 2\pi, \\ t_1(t,s) &= 2(\arg\alpha t - \log|\alpha|s) \mod 2\pi, \\ t_2(t,s) &= 2(\arg\beta t - \log|\beta|s) \mod 2\pi. \end{aligned} \tag{1.31}$$

Call N this leaf, and consider $N \cap (\{\Theta\} \times T)$. Observe that $\theta(t) = \Theta$ is equivalent to $t = m/2$ where m is an integer, and call N_m the curve given by the equations

$$\begin{aligned} \theta(\frac{m}{2}, s) &= \Theta \mod 2\pi, \\ t_1(\frac{m}{2}, s) &= 2(\arg\alpha\frac{m}{2} - \log|\alpha|s) \mod 2\pi, \\ t_2(\frac{m}{2}, s) &= 2(\arg\beta\frac{m}{2} - \log|\beta|s) \mod 2\pi. \end{aligned}$$

Clearly $N \cap (\{\Theta\} \times T)$ is the union of the curves N_m for $m \in \mathbb{Z}$. Lemma 1.3.1 says that N_m is dense in $\{\Theta\} \times T$ whenever $\log|\alpha|/\log|\beta|$ is irrational: $N \cap (\{\Theta\} \times T)$ is then *a fortiori* dense in $\{\Theta\} \times T$, and it is not $\{\Theta\} \times T$ since it does not contain for instance the points

$$\begin{aligned} \theta &= \Theta \mod 2\pi, \\ t_1(s) &= 2(\arg\alpha\frac{2m+1}{4} - \log|\alpha|s) \mod 2\pi, \\ t_2(s) &= 2(\arg\beta\frac{2m+1}{4} - \log|\beta|s) \mod 2\pi. \end{aligned}$$

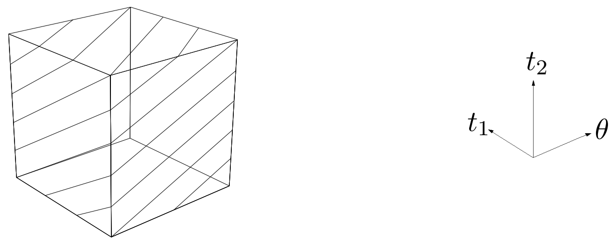


Figure 1.7: intersection of the leaf with the faces of $S^1 \times T$: case $\arg\alpha - \arg\beta \log|\alpha|/\log|\beta| \in \mathbb{Q}\pi$ and $\log|\alpha|/\log|\beta| \in \mathbb{Q}$.

One can use this argument for all θ, so in this case N is dense in $S^1 \times T$. Otherwise if $\log|\alpha|/\log|\beta|$ is rational, the intersection of N with $\{\theta\} \times T$ is the union of toral knots of type $\log|\alpha|/\log|\beta|$.

Consider now the intersection of N with the surface $t_2 = 0$: recall that $t_2 = 0$ is equivalent to $s = (t\arg\beta - m\pi)/\log|\beta|$ for m integer, and call N_m the curve given by

$$\begin{aligned}
\theta(t, \frac{t\arg\beta - m\pi}{\log|\beta|}) &= -\pi\log x - 4\pi t \mod 2\pi,\\
t_1(t, \frac{t\arg\beta - m\pi}{\log|\beta|}) &= 2(\arg\alpha t - \log|\alpha|\frac{t\arg\beta - m\pi}{\log|\beta|}) \mod 2\pi,\\
t_2(t, \frac{t\arg\beta - m\pi}{\log|\beta|}) &= 0 \mod 2\pi,
\end{aligned}$$

(see figure 1.4). In this case lemma 1.3.1 shows that every N_m is dense in $S^1 \times \{(t_1, 0) \in T\}$ whenever $(\arg\alpha - \arg\beta\log|\alpha|/\log|\beta|)/\pi$ is irrational: the same argument for $t_2 \neq 0$ shows that in this case N is dense in $S^1 \times T$.

One is then left with the case

$$\frac{\arg\alpha - \arg\beta\log|\alpha|/\log|\beta|}{\pi} \in \mathbb{Q}, \qquad \frac{\log|\alpha|}{\log|\beta|} \in \mathbb{Q}$$

namely

$$\frac{k\arg\alpha - l\arg\beta}{\pi} = \frac{p}{q}, \qquad \frac{\log|\alpha|}{\log|\beta|} = \frac{l}{k} \tag{1.32}$$

where l, k, p and q are integers and $(p,q) = (l,k) = 1$: in this case the intersection of N with the faces of the figure 1.2 is a union of closed curves (see figure 1.7).

Choose two integers b and c such that $bk - cl = 1$. Set

$$q' \overset{\text{def}}{=} \begin{cases} q & \text{if } p \text{ is odd} \\ q/2 & \text{if } p \text{ is even} \end{cases}, \qquad p' \overset{\text{def}}{=} \begin{cases} p & \text{if } p \text{ is odd} \\ p/2 & \text{if } p \text{ is even} \end{cases}$$

and remark that in this case the map

$$\begin{aligned}
F: \quad \mathbb{R}^2 &\longrightarrow N \subset S^1 \times T\\
(t,s) &\longmapsto (\theta(t,s), t_1(t,s), t_2(t,s))
\end{aligned}$$

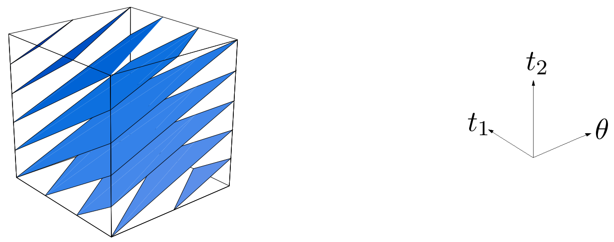


Figure 1.8: the compact leaf in the case $\arg\alpha - \arg\beta \log|\alpha|/\log|\beta| \in \mathbb{Q}\pi$ and $\log|\alpha|/\log|\beta| \in \mathbb{Q}$.

is invariant with respect to the action on $\mathbb{R}^2$ of the lattice $\Lambda \stackrel{\text{def}}{=} v\mathbb{Z} \oplus w\mathbb{Z}$ (see figure 1.8) where

$$v = (q', \frac{q' \arg\beta - p'c\pi}{\log|\beta|}), \qquad w = (0, \frac{k\pi}{\log|\beta|}). \tag{1.33}$$

Consider the diagram

$$\begin{array}{ccc} \mathbb{C} & & \\ {\scriptstyle p}\downarrow & \searrow^{F} & \\ \frac{\mathbb{C}}{\Lambda} & \xrightarrow[\bar F]{} & N \end{array} \tag{1.34}$$

where p is the canonical projection of $\mathbb{C}$ on $\mathbb{C}/\Lambda$ and $\bar F$ is the quotient map of F. The map $\bar F$ is onto, and the leaf $N = \bar F(\mathbb{C}/\Lambda)$ is compact. Moreover, since $F' = B \neq 0$, $\bar F$ is a local diffeomorphism; this implies that N, being the image of a compact manifold via a local diffeomorphism, is a submanifold of $H_{\alpha,\beta}$. Thus N, being closed with respect to $J_{\alpha,\beta}$, is a compact Riemann surface and its genus is one, since it supports a non-vanishing vector field. Furthermore $\bar F$ is holomorphic, because, with the chosen parameterization, the horizontal and the vertical axes of $\mathbb{C}$ are just the integral curves respectively of B and JB. It follows that $\bar F$ is a non ramified covering. But it is straightforward to check that $\bar F$ is injective also, so it is a biholomorphism.

Lemma 1.3.7 shows that the conditions (1.32) coincide with the condition $\alpha^m = \beta^n$ and the theorem is proved. ■

Lemma 1.3.7 *The conditions (1.32) are equivalent to the existence of integers m and n, where $m/n = k/l$, such that $\alpha^m = \beta^n$.*

Proof: The existence of integers m and n such that $m/n = k/l$ and $\alpha^m = \beta^n$ is equivalent to

$$\frac{\log|\alpha|}{\log|\beta|} = \frac{n}{m} = \frac{l}{k} \qquad \text{and} \qquad \{m\arg\alpha + 2r\pi\}_{r\in\mathbb{Z}} = \{n\arg\beta + 2s\pi\}_{s\in\mathbb{Z}}, \tag{1.35}$$

and these conditions imply (1.32).

Vice versa, from (1.32) one obtains that

$$2qk\arg\alpha + 2r\pi = 2ql\arg\beta + 2\pi(p+r) \qquad \text{for every integer } r;$$

set $m \stackrel{\text{def}}{=} 2qk$, $n \stackrel{\text{def}}{=} 2ql$ to obtain (1.35) and complete the proof. ■

The proof of theorem 1.3.6 complete the description of the foliation when the leaves are not compact:

Corollary 1.3.8 *When α and β do not satisfy (1.32), the saturated components of $\mathcal{E}_{\alpha,\beta}$ are of two kinds:*

i) $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *and* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*;*

ii) $S^1 \times T(\xi_1, \xi_2)$.

Remark 1.3.9 Because of (1.19), $\mathcal{E}_{\alpha,\beta}$ is linear in the classification recently given by D. Mall in [Mal98]. □

1.4 Elliptic fibrations on $S^1 \times S^3$

By the definition of Kodaira in [Kod64, 2], an *elliptic surface* is a complex fibre space of elliptic curves over a non singular algebraic curve, namely a map $\Xi\colon S \to \Delta$ where S is a complex surface, Δ is a non singular algebraic curve, Ψ is a holomorphic map and the generic fibre is a torus. The curve Δ is called the *base space* of S.

In theorem 1.3.6 it is showed that, if $\alpha^m = \beta^n$ for some integers m and n, then $S^1 \times S^3$ is a fibre space of elliptic curves over a topological space Δ -the leaf space. In this section it is shown that such a Δ is a non singular algebraic curve (actually $\mathbb{CP}^1$) and that the projection Ψ is holomorphic with respect to the induced complex structure.

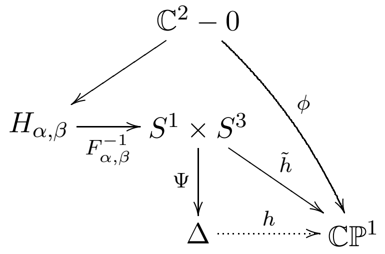
Theorem 1.4.1 *If $\alpha^m = \beta^n$ for some integers m and n, the leaf space Δ of the foliation in tori given on $S^1 \times S^3$ by the theorem 1.3.6 is homeomorphic to $\mathbb{CP}^1$, and the projection $\Psi\colon S^1 \times S^3 \to \Delta$ is holomorphic with respect to the induced complex structure.*

Proof: By lemma 1.3.7 the hypothesis is equivalent to (1.32). Choose then the integers m and n minimal with respect to the property $\alpha^m = \beta^n$, and observe that this implies $m \arg\alpha = n \arg\beta + 2\pi c$, where c is an integer such that $\mathrm{MCD}(m, n, c) = 1$, and consider the following map:

$$\begin{array}{rccc} \tilde{h}: & S^1 \times S^3 & \longrightarrow & \mathbb{CP}^1 \\ & (\theta, \xi_1, \xi_2) & \longmapsto & [e^{\theta ic}\xi_1^m : \xi_2^n]. \end{array}$$

It is an easy matter to verify that on $H_{\alpha,\beta}$ this map is nothing but the quotient of $\phi(z_1, z_2) \stackrel{\text{def}}{=}$

$[z_1^m : z_2^n]$, and one gets the diagram



(1.36)

i) h is well defined: if (θ, ξ_1, ξ_2) belongs to the leaf passing through (Θ, Ξ_1, Ξ_2), then θ, ξ_1 and ξ_2 satisfy (see (1.30))

$$\theta(t,s) = \Theta - 4\pi t \mod 2\pi,$$
$$\xi_1(t,s) = \Xi_1 e^{2i \arg \alpha t} e^{-2i \log |\alpha| s},$$
$$\xi_2(t,s) = \Xi_2 e^{2i \arg \beta t} e^{-2i \log |\beta| s},$$

and one obtains

$$(\theta(t,s), \xi_1(t,s), \xi_2(t,s)) \mapsto [e^{i(\Theta - 4\pi t)c} \Xi_1^m e^{2itm \arg \alpha} : \Xi_2^n e^{2itn \arg \beta}]$$

that is

$$(\theta(t,s), \xi_1(t,s), \xi_2(t,s)) \mapsto [e^{i(\Theta-4\pi t)c + 2it(m \arg \alpha - n \arg \beta)} \Xi_1^m : \Xi_2^n] = [e^{i\Theta c} \Xi_1^m : \Xi_2^n],$$

where the last member does not depend on t and s. Namely, $\tilde{h}$ is constant on every leaf and h is well defined on Δ;

ii) h is onto: $(\theta, 1, 0) \mapsto [1:0]$ and if $h(\theta, \xi_1, \xi_2) = [z_1 : z_2]$, where $z_2 \neq 0$, then

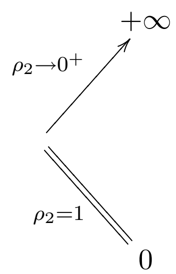
$$z_1 z_2^{-1} = e^{i\theta c} \xi_1^m \xi_2^{-n}.$$

Using polar coordinates, that is, choosing real numbers ρ_1, ρ_2, θ_1 and θ_2 such that $\xi_1 = \rho_1 e^{i\theta_1}$ and $\xi_2 = \rho_2 e^{i\theta_2}$, the last member becomes

$$e^{(i\theta c + m\theta_1 - n\theta_2)} \rho_1^m \rho_2^{-n} \qquad \text{where} \qquad \rho_1^2 + \rho_2^2 = 1.$$

The exponent $\theta c + m\theta_1 - n\theta_2$ covers all the real numbers, and the map

$$\rho_1^m \rho_2^{-n}\Big|_{\rho_1 = \sqrt{1-\rho_2^2}} = (1-\rho_2^2)^{\frac{m}{2}} \rho_2^{-n}$$



covers all the positive real numbers, so $\tilde{h}$ -and consequently h- is onto;

iii) h is injective: suppose that $h(\theta, \xi_1, \xi_2) = h(\Theta, \Xi_1, \Xi_2)$ for (θ, ξ_1, ξ_2) and (Θ, Ξ_1, Ξ_2) on $S^1 \times S^3$. If $\xi_1 \Xi_1 = 0$, then ξ_1 and Ξ_1 must both of them be zero, hence (θ, ξ_1, ξ_2) and (Θ, Ξ_1, Ξ_2) lie on the same leaf. If $\xi_1 \Xi_1 \neq 0$, it can be written

$$\frac{\xi_2^n}{e^{i\theta c} \xi_1^m} = \frac{\Xi_2^n}{e^{i\Theta c} \Xi_1^m}. \tag{1.37}$$

Let $\xi_1 = \rho_1 e^{i\eta_1}$, $\xi_2 = \rho_2 e^{i\eta_2}$, $\Xi_1 = P_1 e^{iH_1}$ and $\Xi_2 = P_2 e^{iH_2}$; the equation (1.37) becomes

$$\frac{\rho_2^n e^{i\eta_2 n}}{\rho_1^m e^{i(\theta c + \eta_1 m)}} = \frac{P_2^n e^{iH_2 n}}{P_1^m e^{i(\theta c + H_1 m)}},$$

that is

$$\begin{cases} \dfrac{\rho_2^n}{\rho_1^m} = \dfrac{P_2^n}{P_1^m}, \\ (\theta - \Theta)c + m(\eta_1 - H_1) - n(\eta_2 - H_2) = 0 \mod 2\pi. \end{cases} \tag{1.38}$$

The first equation in (1.38), together with $\rho_1^2 + \rho_2^2 = 1 = P_1^2 + P_2^2$, easily gives

$$\rho_1 = P_1 \qquad \text{and} \qquad \rho_2 = P_2. \tag{1.39}$$

In order to show that (θ, ξ_1, ξ_2) and (Θ, Ξ_1, Ξ_2) lie on the same leaf, find two real numbers t and s such that

$$\begin{aligned} \theta &= \Theta - 4\pi t \mod 2\pi, \\ \xi_1 &= \Xi_1 e^{2(\arg \alpha t - \log|\alpha| s)}, \\ \xi_2 &= \Xi_2 e^{2(\arg \beta t - \log|\beta| s)}, \end{aligned} \tag{1.40}$$

that is, by using (1.39), find two real numbers t and s satisfying

$$\begin{cases} 4\pi t & = \Theta - \theta \mod 2\pi, \\ 2\arg \alpha t - 2\log|\alpha| s & = \eta_1 - H_1 \mod 2\pi, \\ 2\arg \beta t - 2\log|\beta| s & = \eta_2 - H_2 \mod 2\pi. \end{cases}$$

The determinant of

$$\begin{pmatrix} 4\pi & 0 & \Theta - \theta \\ 2\arg\alpha & -2\log|\alpha| & \eta_1 - H_1 \\ 2\arg\beta & -2\log|\beta| & \eta_2 - H_2 \end{pmatrix}$$

is zero, because the second equation of (1.38) gives

$$m(\text{second row}) - n(\text{third row}) = c(\text{first row}),$$

and the injectivity of h is proved.

From i), ii) and iii) one obtains that $h\colon \Delta \to \mathbb{CP}^1$ is a bijective continous map, and so is a homeomorphism because of the compactness of Δ. At least, Ψ is holomorphic with respect to the induced complex structure -that is, $\tilde{h}$ is holomorphic- because the map ϕ in the diagram (1.36) is holomorphic. ■

1.5 Regularity of $\mathcal{E}_{\alpha,\beta}$ and orbifold structure on Δ

A *quasi-regular foliation* is a foliation $\mathcal{F}$ on a smooth manifold M such that for each point p of M there is a natural number $N(p)$ and a Frobenius chart U (namely, a $\mathcal{F}$-flat cubical neighborhood) where each leaf of $\mathcal{F}$ intersects U in $N(p)$ slices, if any. If $N(p) = 1$ for all p, then $\mathcal{F}$ is called a *regular foliation* (see for instance [BG98]). For a compact manifold M, the assumption that the foliation is quasi-regular is equivalent to the assumption that all leaves are compact. A Riemannian foliation with compact leaves induces a natural orbifold structure on the leaf space (see [Mol88, Proposition 3.7]). Since by [DO98, Theorem 5.1] $\mathcal{E}_{\alpha,\beta}$ is Riemannian, this is the case.

Theorem 1.5.1 *The foliation $\mathcal{E}_{\alpha,\beta}$ is quasi-regular if and only if $\alpha^m = \beta^n$ for some integers m and n; in this case $N(\Theta, \Xi_1, \Xi_2) = 1$ if $\Xi_1\Xi_2 \neq 0$, whereas $N(\Theta, 0, \Xi_2) = m$ and $N(\Theta, \Xi_1, 0) = n$. In particular, the foliation $\mathcal{E}_{\alpha,\beta}$ is regular if and only if $\alpha = \beta$.*

Proof: By theorem 1.3.6, all the leaves are compact if and only if $\alpha^m = \beta^n$, and for (Θ, Ξ_1, Ξ_2) where $\Xi_1\Xi_2 \neq 0$ the thesis follows by figure 1.8. One is then left with $(\Theta, 0, \Xi_2)$ and $(\Theta, \Xi_1, 0)$, when $\alpha^m = \beta^n$. It is now described the case $(\Theta, \Xi_1, 0)$, the other case being analogous.

To visualize the 4-dimensional neighborhood of a point of $S^1 \times S^3$, another 3-dimensional description of the foliation $\mathcal{E}_{\alpha,\beta}$ is needed: consider the stereographic projection

$$\begin{array}{rccc} \phi: & S^3 - (0,0,0,1) & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, x_4) & \longmapsto & \dfrac{1}{1-x_4}(x_1, x_2, x_3). \end{array}$$

It is easy to check that $\phi(T(\xi_1, \xi_2))$ is generated by the revolution around the y_3-axis of the circle $C(\xi_1, \xi_2)$ in the y_2y_3-plane, centered in $(1/|\xi_1|, 0)$ with radius $|\xi_2|/|\xi_1|$. One is thus led to figure 1.9.

Refining the computation in the proof of theorem 1.3.6, one sees that any leaf intersects $T(\xi_1, \xi_2)$ along r toral knots of type l/k, r being the greatest common divisor of m and n. This means that each leaf contained in $T(\xi_1, \xi_2)$ intersects $C(\xi_1, \xi_2)$ in exactly $n = rl$ points. Now let

$$D_\rho \stackrel{\text{def}}{=} \bigcup_{|\xi_2|/|\xi_1| < \rho} C(\xi_1, \xi_2)$$

and let $U_{\delta,\rho}$ the piece of solid torus given by the revolution of angle $(-\delta, \delta)$ of D_ρ. The neighborhoods of $(\Theta, \Xi_1, 0)$ of the form $(\Theta - \varepsilon, \Theta + \varepsilon) \times U_{\delta,\rho}$ contain each leaf in $n = rl$ distinct connected components, and this ends the proof. ■

Remark 1.5.2 The previous theorem defines an orbifold structure on the leaf space Δ, with two conical points of order m and n, respectively (see [Mol88, Proposition 3.7]). In particular, a local

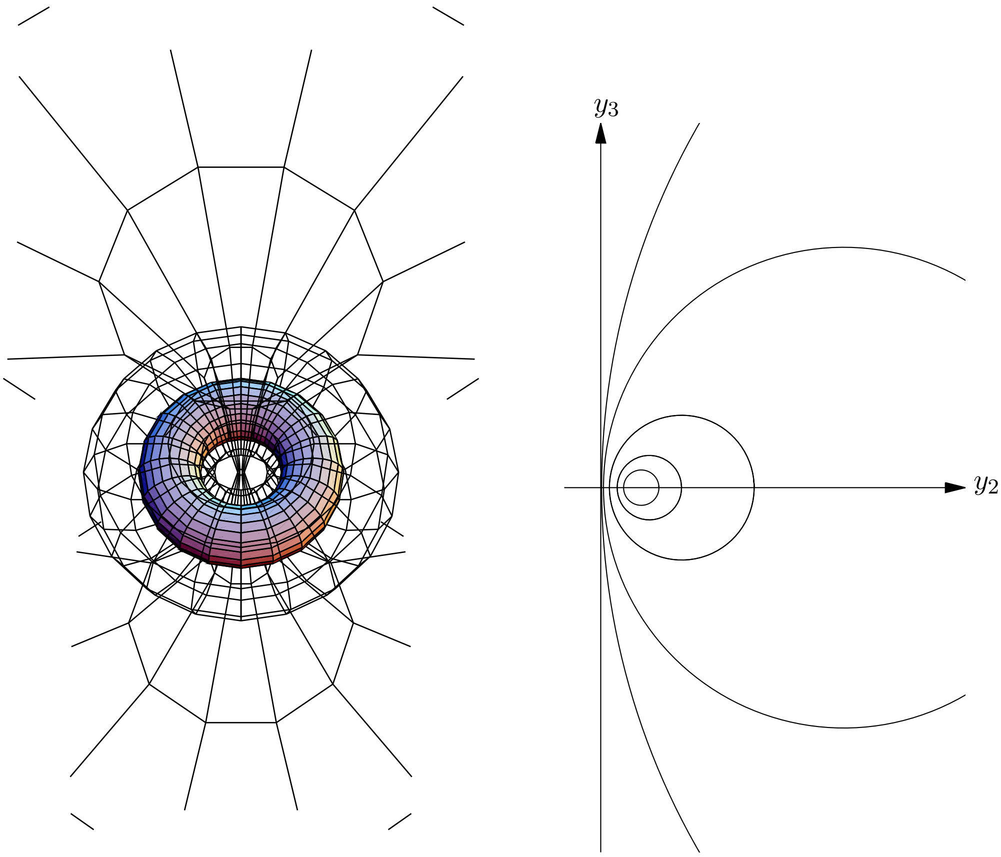


Figure 1.9: On the left, the partition of $\mathbb{R}^3$ in tori $T(\xi_1, \xi_2)$; on the right, the circles that generate the tori.

chart around the leaf through $(\Theta, \Xi_1, 0)$ is given by D_ρ/Γ_n, Γ_n being the finite group generated by the rotation of angle $2\pi/n$. □

Remark 1.5.3 In the preceding section Δ was equipped with a structure of complex curve; this does not contradict the orbifold structure, it simply means that the two structures are not isomorphic in the orbifold category. In fact, any 2-dimensional orbifold with only conical points is homeomorphic to a manifold. □

Chapter 2

Explicit parallelizations on products of spheres

2.1 An explicit parallelization $\mathcal{B}$ on $S^m \times S^1$

Denote by $x = (x_i)$ the coordinates on $\mathbb{R}^{m+1}$, and let $S^m \subset \mathbb{R}^{m+1}$ be given by

$$S^m \stackrel{\text{def}}{=} \{x = (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \text{ such that } |x|^2 = x_1^2 + \cdots + x_{m+1}^2 = 1\}.$$

The orthogonal projection of the standard coordinate frame $\{\partial_{x_i}\}_{i=1,\ldots,m+1}$ to the sphere plays an important role in the game, and deserves its own definition:

Definition 2.1.1 The $\imath^{th}$ *meridian vector field* M_i on S^m is

$$M_i \stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{x_i} \text{ on } S^m \qquad i = 1, \ldots, m+1.$$

□

Let M be the normal versor field of $S^m \subset \mathbb{R}^{m+1}$, that is,

$$M \stackrel{\text{def}}{=} \sum_{i=1}^{m+1} x_i \partial_{x_i}.$$

Since

$$\langle \partial_{x_i}, M \rangle = x_i \qquad i = 1, \ldots, m+1,$$

one obtains the following expression for M_i:

$$M_i = \partial_{x_i} - x_i M \qquad i = 1, \ldots, m+1, \tag{2.1}$$

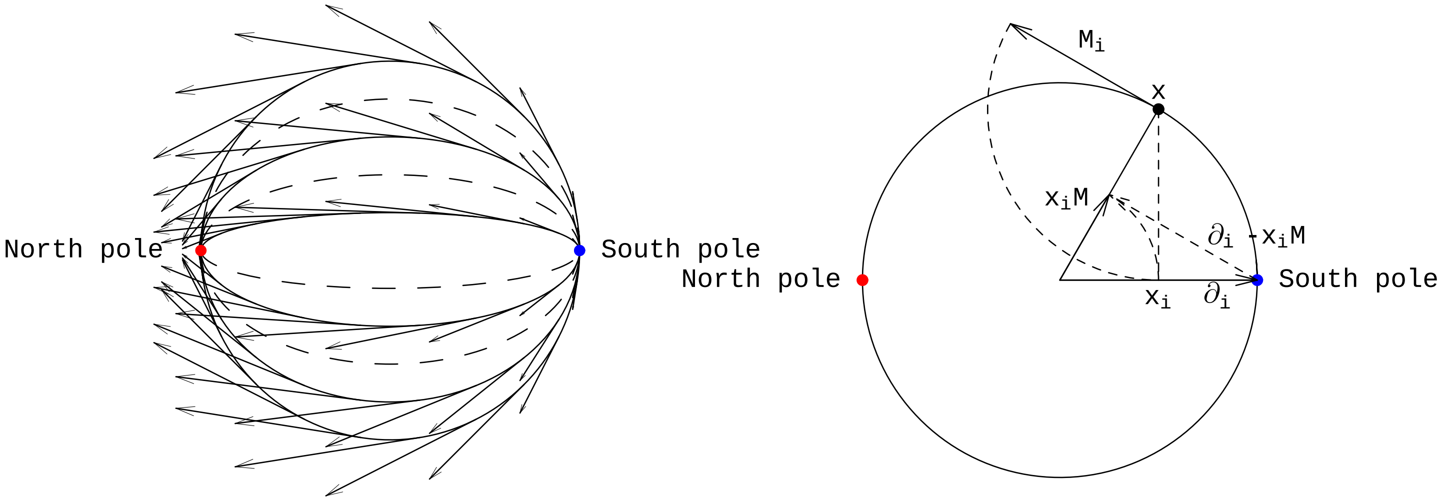


Figure 2.1: Meridian vector field M_i.

and thus

$$\langle M_i, M_j \rangle = \delta_{ij} - x_i x_j \qquad i, j = 1, \ldots, m+1. \tag{2.2}$$

Let Γ be the cyclic infinite group of transformations of $\mathbb{R}^{m+1} - 0$ generated by the map $x \mapsto e^{2\pi} x$. Denote by H the corresponding diagonal real Hopf manifold, that is, the quotient manifold $(\mathbb{R}^{m+1} - 0)/\Gamma$: H turns out to be diffeomorphic to $S^m \times S^1$ by means of the map induced by the projection p:

$$\begin{array}{ccc} \mathbb{R}^{m+1} - 0 & \xrightarrow{p} & S^m \times S^1 \\ x & \longmapsto & (x/|x|, \log|x| \mod 2\pi). \end{array}$$

The standard coordinate frame $\{\partial_{x_i}\}_{i=1,\ldots,m+1}$ on $\mathbb{R}^{m+1} - 0$ becomes Γ-equivariant when multiplied by the function $|x|$, hence it defines a parallelization on $S^m \times S^1$. This proves the following proposition. . .

Proposition 2.1.2 *$S^m \times S^1$ is parallelizable.*

. . . and suggests to give the following definition:

Definition 2.1.3 Denote by $\mathcal{B} = \{b_i\}_{i=1,\ldots,m+1}$ the frame on $S^m \times S^1$ induced by the Γ-equivariant frame $\{|x|\partial_{x_i}\}_{i=1,\ldots,m+1}$ on the universal covering $\mathbb{R}^{m+1}$ of $S^m \times S^1$:

$$b_i \stackrel{\text{def}}{=} p_*(|x|\partial_{x_i}) \qquad i = 1, \ldots, m+1.$$

□

The following theorem explicitly describes the frame $\mathcal{B}$:

Theorem 2.1.4 *Let M_i be the i^{th} meridian vector field on $S^m \subset \mathbb{R}^{m+1}$. Then*

$$b_i = M_i + x_i \partial_\theta \qquad i = 1, \ldots, m+1. \tag{2.3}$$

Proof: Look at $S^m \times S^1$ as a Riemannian submanifold of $R^{m+1} \times S^1$, and in particular look at $T(S^m \times S^1) = TS^m \times TS^1$ as a Riemannian subbundle of $TR^{m+1}_{|S^m} \times TS^1$; this last is a trivial vector bundle and an orthonormal frame is $\{\partial_{x_1}, \ldots, \partial_{x_{m+1}}, \partial_\theta\}$. A computation then shows that

$$p_* = \frac{1}{|x|}\left((dx_1 - x_1\omega) \otimes \partial_{x_1} + \cdots + (dx_{m+1} - x_{m+1}\omega) \otimes \partial_{x_{m+1}} + |x|\omega \otimes \partial_\theta\right),$$

where ω is the 1-form given by

$$\omega \stackrel{\text{def}}{=} -d\left(\frac{1}{|x|}\right) = \frac{1}{|x|^2}\left(x_1 dx_1 + \cdots + x_{m+1} dx_{m+1}\right).$$

Hence, the frame $\mathcal{B}$ in the point $p(x) = (x/|x|, \log|x| \mod 2\pi)$ is given by

$$\frac{1}{|x|^2}\left(-x_1 x_i \partial_{x_1} + \cdots + (|x|^2 - x_i^2)\partial_{x_i} + \cdots - x_{m+1} x_i \partial_{x_{m+1}} + |x| x_i \partial_\theta\right) \qquad i = 1, \ldots, m+1,$$

that is, the frame $\mathcal{B}$ in the point $(x, \theta) \in S^m \times S^1$ is given by

$$\begin{aligned} b_i &= \left(-x_1 x_i \partial_{x_1} + \cdots + (1 - x_i^2)\partial_{x_i} + \cdots - x_{m+1} x_i \partial_{x_{m+1}} + x_i \partial_\theta\right) \\ &= \partial_{x_i} - x_i(x_1 \partial_{x_1} + \cdots + x_{m+1}\partial_{x_{m+1}}) + x_i \partial_\theta \stackrel{(2.1)}{=} M_i + x_i \partial_\theta. \end{aligned} \qquad i = 1, \ldots, m+1. \tag{2.4}$$

■

Remark 2.1.5 The notion of meridian vector field was given in [Bru92]: it was used to describe a parallelization on products of spheres by parallelizable manifolds. In this context, theorem 2.1.4 shows that the frame $\mathcal{B}$ given by definition 2.1.3 coincide with that of [Bru92]. □

Remark 2.1.6 The frame $\mathcal{B}$ is orthonormal with respect to the product metric on $S^m \times S^1$ (use theorem 2.1.4 and formula (2.2)). □

The brackets of $\mathcal{B}$ are

$$[b_i, b_j] = x_i b_j - x_j b_i \qquad i, j = 1, \ldots, m+1. \tag{2.5}$$

Since $\mathcal{B}$ is orthonormal, the coframe $\mathcal{B}^* \stackrel{\text{def}}{=} \{b^i\}_{i=1,\ldots,m+1}$ dual to $\mathcal{B}$ on $S^m \times S^1$ is given by

$$b^i = dx_i + x_i d\theta \qquad i = 1, \ldots, m+1. \tag{2.6}$$

Remark 2.1.7 Since

$$b_i = p_*(|x|\partial_{x_i}) \qquad i = 1, \dots, m+1,$$

the coframe $\mathcal{B}^*$ can be also described as the quotient of the Γ-invariant coframe on $\mathbb{R}^{m+1} - 0$ given by

$$\{|x|^{-1} dx_i\}_{i=1,\dots,m+1}.$$

□

A straightforward computation gives the structure equations for $\mathcal{B}$:

$$db^i = dx_i \wedge d\theta \overset{(2.6)}{=} b^i \wedge d\theta \qquad i = 1, \dots, m+1, \tag{2.7}$$

where the 1-form $d\theta$ is related to $\mathcal{B}^*$ by

$$d\theta = \sum_{i=1}^{m+1} x_i b^i.$$

The following lemma is trivial to prove, but will be useful:

Lemma 2.1.8 *For each permutation π of $\{1, \dots, m+1\}$, the automorphism of $\mathbb{R}^{m+1} - 0$ given by $(x_1, \dots, x_{m+1}) \mapsto (x_{\pi(1)}, \dots, x_{\pi(m+1)})$ is Γ-equivariant. The induced diffeomorphism is*

$$\begin{array}{rccc} f_\pi : & S^m \times S^1 & \longrightarrow & S^m \times S^1 \\ & (x_1, \dots, x_{m+1}, \theta) & \longmapsto & (x_{\pi(1)}, \dots, x_{\pi(m+1)}, \theta), \end{array}$$

and $df_\pi(b_{\pi(i)}) = b_i$.

2.2 An explicit parallelization $\mathcal{B}$ on $S^m \times S^3$

Denote by $y = (y_j)$ the coordinates on $\mathbb{R}^4$, and let $S^3 \subset \mathbb{R}^4$ be given by

$$S^3 \overset{\text{def}}{=} \{y = (y_1, \dots, y_4) \in \mathbb{R}^4 \text{ such that } |y|^2 = y_1^2 + \dots + y_4^2 = 1\}.$$

Let $T = T_1$, T_2, T_3 be the vector fields on S^3 given by multiplication by i, j, $k \in \mathbb{H} = \mathbb{R}^4$ respectively, that is,

$$\begin{aligned} T = T_1 &= -y_2\partial_{y_1} + y_1\partial_{y_2} - y_4\partial_{y_3} + y_3\partial_{y_4}, \\ T_2 &= -y_3\partial_{y_1} + y_4\partial_{y_2} + y_1\partial_{y_3} - y_2\partial_{y_4}, \\ T_3 &= -y_4\partial_{y_1} - y_3\partial_{y_2} + y_2\partial_{y_3} + y_1\partial_{y_4}. \end{aligned} \tag{2.8}$$

The Hopf fibration $S^3 \to S^2$ defines a foliation of $S^m \times S^3$ in $S^m \times S^1$'s, and section 2.1 gives $m+1$ vector fields tangent to the leaves: they can be completed to a parallelization of $S^m \times S^3$ by means of a suitable parallelization of S^3, as it is now going to be shown in the following proposition:

Theorem 2.2.1 ([Bru92]) *$S^m \times S^3$ is parallelizable.*

Proof: The proof of theorem 2.1.4 needs a unitary and tangent to fibers vector field on S^3: this is just what T is. Hence, define $\mathcal{B} \stackrel{\text{def}}{=} \{b_i\}_{i=1,\dots,m+3}$ by

$$\begin{aligned} b_i &\stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \dots, m+1, \\ b_{m+j} &\stackrel{\text{def}}{=} T_j \qquad j = 2, 3, \end{aligned} \tag{2.9}$$

where M_i is the i^{th} meridian vector field on S^m, to obtain the wished frame on $S^m \times S^3$. ■

Remark 2.2.2 The frame $\mathcal{B}$ is orthonormal with respect to the product metric on $S^m \times S^3$ (use formulas (2.9) and formula (2.2)). □

The same argument used in section 2.1 gives the brackets of $\mathcal{B}$:

$$\begin{aligned} [b_i, b_j] &= x_i b_j - x_j b_i \qquad i, j = 1, \dots, m+1, \\ [b_i, b_{m+2}] &= -2x_i b_{m+3} \qquad i = 1, \dots, m+1, \\ [b_i, b_{m+3}] &= 2x_i b_{m+2} \qquad i = 1, \dots, m+1, \\ [b_{m+2}, b_{m+3}] &= -2T = -2 \sum_{i=1}^{m+1} x_i b_i. \end{aligned} \tag{2.10}$$

Let $\tau = \tau_1$, τ_2, τ_3 be the 1-forms on $S^m \times S^3$ dual to $T = T_1$, T_2, T_3 respectively. The coframe $\mathcal{B}^* \stackrel{\text{def}}{=} \{b^i\}_{i=1,\dots,m+3}$ is given by

$$\begin{aligned} b^i &= x_i \tau + dx_i \qquad i = 1, \dots, m+1, \\ b^{m+j} &= \tau_j \qquad j = 2, 3. \end{aligned} \tag{2.11}$$

Differently from $S^m \times S^1$, the 1-form τ is not closed, so structure equations are a bit more complicated:

$$\begin{aligned} db^i &= b^i \wedge \tau + 2x_i b^{m+2} \wedge b^{m+3} \qquad i = 1, \dots, m+1, \\ db^{m+2} &= 2b^{m+3} \wedge \tau, \\ db^{m+3} &= -2b^{m+2} \wedge \tau, \end{aligned} \tag{2.12}$$

where the 1-form τ is related to $\mathcal{B}^*$ by

$$\tau = \sum_{i=1}^{m+1} x_i b^i.$$

Remark 2.2.3 The same argument used above for $S^m \times S^3$ can be applied to the Hopf fibration $S^7 \to \mathbb{CP}^3$ to obtain a frame on $S^m \times S^7$. Nevertheless, formulas in this case are much more complicated. □

Theorem 2.2.1 and the previous remark can be easily generalized:

Theorem 2.2.4 ([Bru92]) *Let Y^n be any parallelizable n-dimensional manifold. Then $S^m \times Y$ is parallelizable.*

Proof: Let $T = T_1, T_2, \ldots, T_n$ be a frame on Y. The required parallelization is thus given by

$$b_i \stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \ldots, m+1,$$
$$b_{m+j} \stackrel{\text{def}}{=} T_j \qquad j = 2, \ldots, n,$$

where M_i is the $\imath^{\text{th}}$ meridian vector field on S^m. ■

2.3 The general problem: when is a product of spheres parallelizable?

The proof of the theorem of Kervaire cited in the introduction is here sketched:

Sketch of proof:

i) Show by induction there exists an embedding of $S^{n_1} \times \cdots \times S^{n_r}$ in $\mathbb{R}^{n_1+\cdots+n_r+1}$. This is true for $r = 1$. Let

$$f = (f_1, \ldots, f_{n_1+\cdots+n_{r-1}+1})\colon S^{n_1} \times \cdots \times S^{n_{r-1}} \to \mathbb{R}^{n_1+\cdots+n_{r-1}+1}$$

be the embedding given by the inductive hypothesis, where f is chosen in such a way that $f_1 \geq 0$. Let $u \in S^{n_1} \times \cdots \times S^{n_{r-1}}$, and let $(\xi_1, \ldots, \xi_{n_r+1}) \in S^{n_r}$: the embedding f is thus given by

$$\begin{array}{ccc} S^{n_1} \times \cdots \times S^{n_r} & \xrightarrow{f} & \mathbb{R}^{n_1+\cdots+n_r+1} \\ (u, (\xi_1, \ldots, \xi_{n_r+1})) & \longmapsto & (f_2(u), \ldots, f_{n_1+\cdots+n_{r-1}+1}(u), \xi_1\sqrt{f_1(u)}, \ldots, \xi_{n_r+1}\sqrt{f_1(u)}); \end{array}$$

ii) suppose without any loss of generality that the odd dimension is not n_1, and observe that the degree of the Gauss map of the embedding f built in i) is given by

$$\chi(D^{n_1+1} \times S^{n_2} \times \cdots \times S^{n_r}) = \chi(D^{n_1+1})\chi(S^{n_2}) \ldots \chi(S^{n_r}) = 0,$$

where D^{n_1+1} denotes a topological disk of dimension $n_1 + 1$;

iii) denote by $G_{k,n}$ and $V_{k,n}$ the Grassmannian and the Stiefel-Whitney manifold of oriented k-planes and oriented orthonormal frames in $\mathbb{R}^{k+n}$, respectively. The tangential map

$$S^{n_1} \times \cdots \times S^{n_r} \longrightarrow G_{n_1+\cdots+n_r,1}$$

is null-homotopic, since by ii) the Gauss map is;

iv) last, denote by $P(S^{n_1} \times \cdots \times S^{n_r})$ the principal bundle of $S^{n_1} \times \cdots \times S^{n_r}$, and look at the following diagram to end the proof:

$$\begin{array}{ccc} P(S^{n_1} \times \cdots \times S^{n_r}) & \dashrightarrow & V_{n_1+\cdots+n_r,1} \\ \vdots & & \downarrow \\ S^{n_1} \times \cdots \times S^{n_r} & \longrightarrow & G_{n_1+\cdots+n_r,1} \end{array}$$

■

Note that, due to the homotopy theory considerations, the above proof is not very suitable to write down explicit parallelizations on products of spheres.

Another proof of Kervaire's theorem can be developed using a series of hints contained in the book [Hir88, exercises 3,4,5 and 6 of section 4.2]. Details of such a proof, as developed by the author, are given in the following.

In what follows, ε^k_B denotes the trivial vector bundle of rank k with base space B; moreover, whenever α is a vector bundle, $E(\alpha)$, p_α, $B(\alpha)$ denote the total space, the projection and the base space of α respectively.

Lemma 2.3.1 *Let α be a vector bundle. The Whitney sum $\alpha \oplus \varepsilon^k_{B(\alpha)}$ is described by*

$$E(\alpha \oplus \varepsilon^k_{B(\alpha)}) \simeq E(\alpha) \times \mathbb{R}^k,$$
$$p_{\alpha\oplus\varepsilon^k_{B(\alpha)}}(e, v) = p_\alpha(e),$$
$$B(\alpha \oplus \varepsilon^k_{B(\alpha)}) = B(\alpha).$$

Proof: The Whitney sum $\alpha \oplus \varepsilon^k_{B(\alpha)}$ is given by the pull-back of $\alpha \times \varepsilon^k_{B(\alpha)}$ by means of the diagonal map $B(\alpha) \to B(\alpha) \times B(\alpha)$ (see for instance [MS74, page 27]). Then

$$E(\alpha \oplus \varepsilon^k_{B(\alpha)}) = \{(e, b, v, b) \in E(\alpha) \times B(\alpha) \times \mathbb{R}^k \times B(\alpha) \text{ such that } p_\alpha(e) = b\}$$

and the thesis follows. ■

Corollary 2.3.2 *Let α, β be vector bundles. Then, for any $k \geq 0$,*

$$\alpha \times (\beta \oplus \varepsilon^k_{B(\beta)}) \simeq (\alpha \oplus \varepsilon^k_{B(\alpha)}) \times \beta.$$

Proof: Observe that

$$E(\alpha \times (\beta \oplus \varepsilon^k_{B(\beta)})) \simeq E(\alpha) \times E(\beta \oplus \varepsilon^k_{B(\beta)}) \overset{2.3.1}{\simeq} E(\alpha) \times E(\beta) \times \mathbb{R}^k,$$
$$E(\alpha \oplus \varepsilon^k_{B(\alpha)}) \times \beta \simeq E(\alpha \oplus \varepsilon^k_{B(\alpha)}) \times E(\beta) \overset{2.3.1}{\simeq} E(\alpha) \times \mathbb{R}^k \times E(\beta),$$

and use the obvious isomorphism. ■

Theorem 2.3.3 *Suppose X^m and Y^n satisfy the following properties:*

i) $T(X) \oplus \varepsilon^1_X$ *is trivial;*

ii) $T(Y) \oplus \varepsilon^1_Y$ *is trivial;*

iii) there is a non-vanishing vector field on Y.

Then $X \times Y$ is parallelizable.

Proof: Let ν be a complement in $T(Y)$ of the non-vanishing vector field on Y, that is,

$$T(Y) \simeq \nu \oplus \varepsilon^1_Y. \tag{2.13}$$

Then

$$\begin{aligned} T(X \times Y) &\simeq T(X) \times T(Y) \overset{(2.13)}{\simeq} T(X) \times (\nu \oplus \varepsilon^1_Y) \\ &\overset{2.3.2}{\simeq} (T(X) \oplus \varepsilon^1_X) \times \nu \overset{\text{i)}}{\simeq} \varepsilon^{m+1}_X \times \nu \\ &\overset{2.3.2}{\simeq} \varepsilon^{m-1}_X \times (\nu \oplus \varepsilon^2_Y) \overset{ii)}{\simeq} \varepsilon^{m-1}_X \times \varepsilon^{n+1}_Y \end{aligned} \tag{2.14}$$

■

Remark 2.3.4 Theorem 2.3.3 was proven in the same way by E. B. Staples in [Sta67].

Remark 2.3.5 Whenever Y is itself parallelizable, formula (2.14) can be shortened:

$$\begin{aligned} T(X \times Y) &\simeq T(X) \times T(Y) \simeq T(X) \times \varepsilon^n_Y \\ &\simeq (T(X) \oplus \varepsilon^1_X) \times \varepsilon^{n-1}_Y \simeq \varepsilon^{m+1}_X \times \varepsilon^{n-1}_Y. \end{aligned} \tag{2.15}$$

□

The embedding $S^n \subset \mathbb{R}^{n+1}$ gives the triviality of $T(S^n) \oplus \varepsilon^1_{S^n}$; whenever n is odd, a non-vanishing vector field on $S^n \subset \mathbb{C}^{(n+1)/2}$ is given by the complex multiplication. Thus, the following:

Corollary 2.3.6 *Let n be any positive odd integer. Then the manifold $S^m \times S^n$ is parallelizable.*

And finally:

Second proof of Kervaire's theorem: Apply $r-1$ times the corollary 2.3.2 to show that $T(S^{n_2} \times \cdots \times S^{n_r}) \oplus \varepsilon^1_{S^{n_2} \times \cdots \times S^{n_r}}$ is a trivial vector bundle, and use theorem 2.3.3. ■

2.4 An explicit parallelization $\mathcal{P}$ for products of 2 spheres

An explicit parallelization $\mathcal{B}$ has already been found on $S^m \times S^n$, for $n = 1, 3, 7$ in the previous sections. Can one use theorem 2.3.3 to explicitly find a parallelization on any parallelizable $S^m \times S^n$? Answer is positive.

The trick in theorem 2.3.3 is simple: split TY by means of the never-vanishing vector field, then use the trivial summand to parallelize TX, and last detach a rank 2 trivial summand to parallelize the remaining part of TY. Remark 2.3.5 simply says that if Y is itself parallelizable, one can avoid to detach the rank 2 trivial summand from X, using the parallelization of Y instead.

Here and henceforth, n is supposed to be the odd dimension in $S^m \times S^n$.

Denote by $y = (y_j)$ the coordinates on $\mathbb{R}^{n+1}$, and let $S^n \subset \mathbb{R}^{n+1}$ be given by

$$S^n \stackrel{\text{def}}{=} \{y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} \text{ such that } y_1^2 + \cdots + y_{n+1}^2 = 1\}.$$

Being n odd, a never-vanishing vector field, and hence a versor field, is defined on S^n: *here and henceforth, T denotes the versor field on S^n given by multiplication by i in $\mathbb{C}^{(n+1)/2}$, namely,*

$$T \stackrel{\text{def}}{=} -y_2\partial_{y_1} + y_1\partial_{y_2} + \cdots - y_{n+1}\partial_{y_n} + y_n\partial_{y_{n+1}}. \tag{2.16}$$

When a shorter form of T is needed, t_j denotes the coordinates of T, that is,

$$T = \sum_{j=1}^{n+1} t_j \partial_{y_j} \tag{2.17}$$

where t_j is given by

$$t_j = \begin{cases} -y_{j+1} & \text{if } j \text{ is odd,} \\ y_{j-1} & \text{if } j \text{ is even.} \end{cases} \tag{2.18}$$

Moreover, denote by N the normal versor field of $S^n \subset \mathbb{R}^{n+1}$ (recall that M denotes the normal versor field of $S^m \subset \mathbb{R}^{m+1}$):

$$N \stackrel{\text{def}}{=} \sum_{j=1}^{n+1} y_j \partial_{y_j}. \tag{2.19}$$

It is convenient to think of $T(S^m \times S^n) = TS^m \times TS^n$ as a Riemannian subbundle of $T\mathbb{R}^{m+1}_{|S^m} \times T\mathbb{R}^{n+1}_{|S^n}$; this last is trivial, and an orthonormal frame is $\{\partial_{x_1}, \ldots, \partial_{x_{m+1}}, \partial_{y_1}, \ldots, \partial_{y_{n+1}}\}$.

Denote by N_j the j^{th} meridian vector field on S^n (recall that M_i denotes the i^{th} meridian vector field on S^m):

$$N_j \stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{y_j} \text{ on } S^n \qquad j = 1, \ldots, n+1.$$

The tangent space in a point $(x, y) \in S^m \times S^n$ is thus given by an Euclidean vector subspace

$$T_x S^m \oplus T_y S^n \subset \mathbb{R}^{m+1} \oplus \mathbb{R}^{n+1},$$

which is generated by the $m + n + 2$ vectors $\{M_1(x), \ldots, M_{m+1}(x), N_1(y), \ldots, N_{n+1}(y)\}$.

One also has

$$T_x S^m \oplus \langle M(x)\rangle_{\mathbb{R}} = \mathbb{R}^{m+1} \qquad \text{and} \qquad T_y S^n \oplus \langle N(y)\rangle_{\mathbb{R}} = \mathbb{R}^{n+1}. \tag{2.20}$$

As in formula (2.1), one obtains

$$\begin{aligned} \partial_{x_i} &= M_i + x_i M \qquad i = 1, \ldots, m+1, \\ \partial_{y_j} &= N_j + y_j N \qquad j = 1, \ldots, n+1. \end{aligned} \tag{2.21}$$

Moreover, denote by $T(y)^{\perp}$ the vector subspace of $T_y(S^n)$ which is orthogonal to $T(y)$:

$$\langle T(y)\rangle_{\mathbb{R}} \oplus T(y)^{\perp} = T_y S^n. \tag{2.22}$$

In what follows, some computation on the vector space $T_x(S^m) \oplus T_y(S^n)$ is done. For the sake of simplicity, the argument of vector fields is omitted, that is, T stands for $T(y)$, M stands for $M(x)$ etc...

Formula (2.14) in theorem 2.3.3 gives the following chain of pointwise isomorphisms:

$$\begin{aligned} T_x(S^m) \oplus T_y(S^n) &\overset{(2.22)}{=} T_x(S^m) \oplus \langle T\rangle_{\mathbb{R}} \oplus T^{\perp} \\ &\overset{\alpha}{\simeq} T_x(S^m) \oplus \langle M\rangle_{\mathbb{R}} \oplus T^{\perp} \\ &\overset{(2.20)}{=} \mathbb{R}^{m+1} \oplus T^{\perp} \\ &\overset{\beta}{\simeq} \mathbb{R}^{m-1} \oplus \langle N\rangle_{\mathbb{R}} \oplus \langle T\rangle_{\mathbb{R}} \oplus T^{\perp} \\ &\overset{(2.22)}{=} \mathbb{R}^{m-1} \oplus \langle N\rangle_{\mathbb{R}} \oplus T_y S^n \\ &\overset{(2.20)}{=} \mathbb{R}^{m-1} \oplus \mathbb{R}^{n+1}, \end{aligned} \tag{2.23}$$

where α and β are defined by

$$\alpha(T) \overset{\text{def}}{=} M, \qquad \beta(\partial_{x_m}) \overset{\text{def}}{=} N, \qquad \beta(\partial_{x_{m+1}}) \overset{\text{def}}{=} T.$$

Pulling back to $T_x(S^m) \oplus T_y(S^n)$ the $m-1$ generators $\{\partial_{x_1}, \ldots, \partial_{x_{m-1}}\}$ of $\mathbb{R}^{m-1}$ one obtains

$$\begin{aligned} \partial_{x_i} &\overset{(2.1)}{=} M_i + x_i M \\ &\overset{\alpha^{-1}}{\longmapsto} M_i + x_i T \end{aligned} \qquad i = 1, \ldots, m-1, \tag{2.24}$$

whereas pulling back to $T_x(S^m) \oplus T_y(S^n)$ the $n+1$ generators $\{\partial_{y_1}, \dots, \partial_{y_{n+1}}\}$ of $\mathbb{R}^{n+1}$ one obtains the more complicated formulas

$$\begin{aligned}
\partial_{y_j} &\overset{(2.1)}{=} N_j + y_j N \\
&= \langle N_j, T\rangle T + (N_j - \langle N_j, T\rangle T) + y_j N \\
&\overset{\beta^{-1}}{\longmapsto} \langle N_j, T\rangle \partial_{x_{m+1}} + (N_j - \langle N_j, T\rangle T) + y_j \partial_{x_m} \qquad j = 1, \dots, n+1. \quad (2.25)\\
&\overset{(2.1)}{=} \langle N_j, T\rangle (M_{m+1} + x_{m+1} M) + (N_j - \langle N_j, T\rangle T) + y_j (M_m + x_m M) \\
&\overset{\alpha^{-1}}{\longmapsto} \langle N_j, T\rangle (M_{m+1} + x_{m+1} T) + (N_j - \langle N_j, T\rangle T) + y_j (M_m + x_m T)
\end{aligned}$$

The following theorem applies the above argument to $S^m \times S^n$, odd n, in order to obtain an explicit frame on it:

Theorem 2.4.1 *Let n be odd, and let $T = \sum_{j=1}^{n+1} t_j \partial_{y_j}$ be the tangent versor field on S^n given by formula (2.18). Also, let $\{M_i\}_{i=1,\dots,m+1}$ and $\{N_j\}_{j=1,\dots,n+1}$ be the meridian vector fields on S^m and S^n respectively. Last, let M and N be the normal versor fields of $S^m \subset \mathbb{R}^{m+1}$ and $S^n \subset \mathbb{R}^{n+1}$ respectively. The product $S^m \times S^n$ is parallelized by the frame $\mathcal{P} \overset{\text{def}}{=} \{p_1, \dots, p_{m+n}\}$ given by*

$$\begin{aligned}
p_i &\overset{\text{def}}{=} M_i + x_i T \qquad i = 1, \dots, m-1, \\
p_{m-1+j} &\overset{\text{def}}{=} y_j M_m + t_j M_{m+1} + (t_j x_{m+1} + y_j x_m - t_j) T + N_j \qquad j = 1, \dots, n+1.
\end{aligned} \tag{2.26}$$

Moreover, $\mathcal{P}$ is orthonormal with respect to the standard metric on $S^m \times S^n$.

Proof: Observe that

$$\langle N_j, T\rangle \overset{(2.1)}{=} \langle \partial_{y_j} - y_j N, T\rangle = \langle \partial_{y_j}, T\rangle = t_j \qquad j = 1, \dots, n+1$$

and use formulas (2.24) and (2.25) to obtain (2.26). The orthonormality can be proved by observing that both α and β in (2.23) are isometries. But one can also directly check the p_i's, taking into account formula (2.2). ■

Remark 2.4.2 To obtain a parallelization in the general case, use induction in the following way: suppose that $S^{n_2} \times \cdots \times S^{n_r}$, $r \geq 2$, has at least one odd-dimensional factor, hence it is parallelizable; then

$$\begin{aligned}
T(S^{n_1} \times \cdots \times S^{n_r}) &= T(S^{n_1}) \times \varepsilon^{n_2 + \cdots + n_r} \\
&= (T(S^{n_1}) \oplus \varepsilon^1) \times \varepsilon^{n_2 + \cdots + n_r - 1} = \varepsilon^{n_1 + 1} \times \varepsilon^{n_2 + \cdots + n_r - 1}.
\end{aligned}$$

□

2.5 The frames $\mathcal{P}$ and $\mathcal{B}$ on $S^m \times S^1$ and $S^m \times S^3$

If $n = 1$, 3 or 7, remark 2.3.5 can be used to obtain a parallelization simpler than $\mathcal{P}$ on $S^m \times S^n$. If $n = 1$, 3 this parallelization is just the one given in sections 2.1, 2.2 respectively, which was called $\mathcal{B}$. In this section relations between $\mathcal{B}$ and $\mathcal{P}$ are exploited.

Let $n = 1$. Formula (2.26) gives the frame $\mathcal{P} = \{p_1, \dots, p_{m+1}\}$ on $S^m \times S^1$, whereas the frame $\mathcal{B}$ is given by formula (2.3). Clearly,

$$p_i = b_i \qquad i = 1, \dots, m-1.$$

Since $\partial_\theta = -y_2\partial_{y_1} + y_1\partial_{y_2} = T$, one obtains

$$\begin{aligned}\langle N_1, \partial_\theta\rangle &= \langle \partial_{y_1} - y_1 N, -y_2\partial_{y_1} + y_1\partial_{y_2}\rangle = -y_2,\\ \langle N_2, \partial_\theta\rangle &= \langle \partial_{y_2} - y_2 N, -y_2\partial_{y_1} + y_1\partial_{y_2}\rangle = y_1,\end{aligned}$$

and thus

$$\begin{aligned}N_1 &= -y_2 T,\\ N_2 &= y_1 T.\end{aligned}$$

Whence

$$\begin{aligned}p_m &= y_1(M_m + x_m T) - y_2(M_{m+1} + x_{m+1}T) + y_2 T - y_2 T = y_1 b_m - y_2 b_{m+1},\\ p_{m+1} &= y_2(M_m + x_m T) + y_1(M_{m+1} + x_{m+1}T) - y_1 T + y_1 T = y_2 b_m + y_1 b_{m+1},\end{aligned}$$

and one gets

$$\mathcal{P} = \mathcal{B}\left(\begin{array}{ccc|cc} & & & 0 & 0\\ & I_{m-1} & & \vdots & \vdots\\ & & & 0 & 0\\ \hline 0 & \cdots & 0 & y_1 & y_2\\ 0 & \cdots & 0 & -y_2 & y_1\end{array}\right) \tag{2.27}$$

Brackets of $\mathcal{P}$ are thus easily obtained by means of formulas (2.27), (2.5):

$$\begin{aligned}[p_i, p_j] &= x_i p_j - x_j p_i \qquad i, j = 1, \dots, m-1\\ [p_i, p_m] &= (-x_m y_1 + x_{m+1} y_2) p_i + x_i p_m - x_i p_{m+1} \qquad i = 1, \dots, m-1\\ [p_i, p_{m+1}] &= (-x_m y_2 - x_{m+1} y_1) p_i + x_i p_m + x_i p_{m+1} \qquad i = 1, \dots, m-1\\ [p_m, p_{m+1}] &= (x_m(y_1 - y_2) - x_{m+1}(y_1 + y_2)) p_m + (x_m(y_1 + y_2) + x_{m+1}(y_1 - y_2)) p_{m+1}\end{aligned} \tag{2.28}$$

Formula (2.27) gives the frame $\mathcal{P}^*$ dual to $\mathcal{P}$:

$$\begin{aligned}p^i &= b^i \qquad i = 1, \dots, m-1,\\ p^m &= y_1 b^m - y_2 b^{m+1},\\ p^{m+1} &= y_2 b^m + y_1 b^{m+1}.\end{aligned}$$

The structure equations for $\mathcal{P}$ are thus obtained by a straightforward computation:

$$\begin{aligned} dp^i &= dx_i \wedge \tau = p^i \wedge \tau \qquad i = 1, \dots, m-1, \\ dp^m &= p^m \wedge \tau + p^{m+1} \wedge \tau, \\ dp^{m+1} &= p^{m+1} \wedge \tau - p^m \wedge \tau, \end{aligned} \tag{2.29}$$

where τ is given by

$$\tau = \sum_{i=1}^{m+1} x_i b^i = \sum_{i=1}^{m-1} x_i p^i + (x_m y_1 - x_{m+1} y_2) p^m + (x_m y_2 + x_{m+1} y_1) p^{m+1}.$$

Let $n = 3$. Formula (2.26) gives the frame $\mathcal{P} = \{p_1, \dots, p_{m+3}\}$ on $S^m \times S^3$, whereas the frame $\mathcal{B}$ is given by formula (2.9). Clearly,

$$p_i = b_i \qquad i = 1, \dots, m-1.$$

Denote by "$(*)_{j^{\text{th}}}$" the j^{th} coordinate of $*$. Since

$$\begin{aligned} \langle N_j - t_j T, T \rangle &= 0 \\ \langle N_j - t_j T, b_{m+2} \rangle &= (b_{m+2})_{j^{\text{th}}}, \qquad j = 1, \dots, 4, \\ \langle N_j - t_j T, b_{m+3} \rangle &= (b_{m+3})_{j^{\text{th}}}, \end{aligned}$$

one gets

$$p_{m-1+j} = y_j b_m + t_j b_{m+1} + (b_{m+2})_{j^{\text{th}}} b_{m+2} + (b_{m+3})_{j^{\text{th}}} b_{m+3} \qquad j = 1, \dots, 4.$$

Whence

$$\mathcal{P} = \mathcal{B} \left(\begin{array}{ccc|cccc} & & & 0 & 0 & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & \cdots & 0 & -y_2 & y_1 & -y_4 & y_3 \\ 0 & \cdots & 0 & -y_3 & y_4 & y_1 & -y_2 \\ 0 & \cdots & 0 & -y_4 & -y_3 & y_2 & y_1 \end{array} \right) \tag{2.30}$$

Brackets of $\mathcal{P}$ can be obtained by means of a not straightforward computation using formulas (2.30), (2.10).One can also refer to the next section, where general formulas for $\mathcal{P}$ are given.

Formula (2.30) gives the frame $\mathcal{P}^*$ dual to $\mathcal{P}$:

$$\mathcal{P}^* = \mathcal{B}^* \left(\begin{array}{ccc|cccc} & & & 0 & 0 & 0 & 0 \\ & I_{m-1} & & \vdots & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & \cdots & 0 & -y_2 & y_1 & -y_4 & y_3 \\ 0 & \cdots & 0 & -y_3 & y_4 & y_1 & -y_2 \\ 0 & \cdots & 0 & -y_4 & -y_3 & y_2 & y_1 \end{array}\right)$$

2.6 General formulas for $\mathcal{P}$

Recall that $T = \sum_{j=1}^{n+1} t_j \partial_{y_j}$. Set

$$\begin{aligned} X_m &\stackrel{\text{def}}{=} M_m + x_m T, \\ X_{m+1} &\stackrel{\text{def}}{=} M_{m+1} + x_{m+1} T, \\ C_{j,k} &\stackrel{\text{def}}{=} y_j t_k - y_k t_j \qquad j,k = 1,\ldots,n+1, \\ D_{j,k} &\stackrel{\text{def}}{=} 2C_{j,k} \underbrace{\mp \delta_{k,j\pm 1}}_{j\ \frac{\text{odd}}{\text{even}}} \underbrace{\pm \delta_{j,k\pm 1}}_{k\ \frac{\text{odd}}{\text{even}}} \qquad j,k = 1,\ldots,n+1. \end{aligned}$$

Formulas (2.26) easily give

$$\begin{aligned} \sum_{j=1}^{n+1} y_j p_{m-1+j} &= M_m + x_m T = X_m, \\ \sum_{j=1}^{n+1} t_j p_{m-1+j} &= M_{m+1} + x_{m+1} T = X_{m+1}. \end{aligned}$$

A hard calculation then gives

$$\begin{aligned}
[p_i, p_j] &= x_i p_j - x_j p_i \qquad i, j = 1, \ldots, m-1, \\
[p_i, p_{m-1+j}] &= -(y_j x_m + t_j x_{m+1}) p_i \\
&\underbrace{\mp x_i p_{m-1+j\pm 1}}_{j\ \frac{\text{odd}}{\text{even}}} + x_i y_j X_m + x_i t_j X_{m+1} \qquad i = 1, \ldots, m-1, j = 1, \ldots, n+1, \\
[p_{m-1+j}, p_{m-1+k}] &= D_{j,k} \sum_{i=1}^{m-1} x_i p_i + y_j p_{m-1+k} - y_k p_{m-1+j} \\
&+ (x_m D_{j,k} - x_{m+1} C_{j,k}) X_m + ((x_{m+1} - 1) D_{j,k} + x_m C_{j,k}) X_{m+1} \\
&+ \underbrace{(\mp y_j x_m \mp t_j x_{m+1} \pm t_j) p_{m-1+k\pm 1}}_{k\ \frac{\text{odd}}{\text{even}}} \\
&+ \underbrace{(\pm y_k x_m \pm t_k x_{m+1} \mp t_k) p_{m-1+j\pm 1}}_{j\ \frac{\text{odd}}{\text{even}}} \qquad j, k = 1, \ldots, n+1.
\end{aligned} \tag{2.31}$$

Chapter 3

Special structures on products of spheres

3.1 A motivating example

Let Γ be the cyclic infinite group of transformations of $\mathbb{R}^4 - 0$ generated by the map $x \mapsto e^{2\pi}x$. Denote by $\mathcal{B} = \{b_1, \dots, b_4\}$ the frame on $S^3 \times S^1$ given by the Γ-equivariant vector fields $\{|x|\partial_{x_1}, \dots, |x|\partial_{x_4}\}$ on $\mathbb{R}^4 - 0$ by means of the map

$$\begin{array}{ccc} \mathbb{R}^4 - 0 & \longrightarrow & S^3 \times S^1 \\ x & \longmapsto & (x/|x|, \log|x| \mod 2\pi). \end{array}$$

Define the almost-Hermitian structures $I_{\mathcal{B}} = I$, $J_{\mathcal{B}} = J$ and $K_{\mathcal{B}} = K$ on $S^3 \times S^1$ by

$$I(b_1) \stackrel{\text{def}}{=} b_2, \quad I(b_3) \stackrel{\text{def}}{=} b_4, \quad J(b_1) \stackrel{\text{def}}{=} b_3, \quad J(b_2) \stackrel{\text{def}}{=} -b_4, \quad K(b_1) \stackrel{\text{def}}{=} b_4, \quad K(b_2) \stackrel{\text{def}}{=} b_3.$$

Then $I_{\mathcal{B}}$ coincide with the integrable Hermitian structure of diagonal Hermitian Hopf surface $H_{e^{2\pi}, e^{2\pi}}$. Moreover, the almost-hyperhermitian structure $(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ on $S^3 \times S^1$ coincide with the integrable hyperhermitian structure of hyperhermitian Hopf manifold $(\mathbb{H} - 0)/\Gamma$.

One can summarize:

Proposition 3.1.1 *The almost-Hermitian structure $I_{\mathcal{B}}$ on $S^3 \times S^1$ is integrable and the associated Hermitian structure coincide with that of standard Hermitian Hopf surface $H_{e^{2\pi}, e^{2\pi}}$. The same is true for $(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$, whose integrability allows to identify $S^3 \times S^1$ with $(\mathbb{H} - 0)/\Gamma$, where Γ is the infinite cyclic group generated by $h \mapsto e^{2\pi}h$.*

3.2 Preliminaries

Recall now the definitions of the frames $\mathcal{B}$ and $\mathcal{P}$ given in chapter 2. Denote by $x = (x_i)$, $y = (y_j)$ the coordinates on $\mathbb{R}^{m+1}$, $\mathbb{R}^{n+1}$ and let S^m, S^n be the unit spheres in $\mathbb{R}^{m+1}$, $\mathbb{R}^{n+1}$ respectively. Look first to the case $n = 1$. Let Γ be the cyclic infinite group of transformations of $\mathbb{R}^{m+1} - 0$ generated by $x \mapsto e^{2\pi}x$. The corresponding diagonal real Hopf manifold, that is, the quotient manifold $(\mathbb{R}^{m+1} - 0)/\Gamma$, is diffeomorphic to $S^m \times S^1$. The frame $\{|x|\partial_{x_i}\}_{i=1,\dots,m+1}$ on $\mathbb{R}^{m+1} - 0$ is Γ-equivariant, and hence it defines a parallelization $\mathcal{B}$ on $S^m \times S^1$.

If $n = 3$, the Hopf fibration $S^3 \to S^2$ defines a family of $S^m \times S^1$'s into $S^m \times S^3$, and the frame $\mathcal{B}$ on each $S^m \times S^1$ can be completed using the Lie frame $\{T, T_2, T_3\}$ of S^3. The resulting parallelization on $S^m \times S^3$ is denoted again by $\mathcal{B}$.

If n is odd, the complex multiplication in $\mathbb{C}^{(n+1)/2} = \mathbb{R}^{n+1}$ induces a tangent unit vector field T on S^n:

$$T \stackrel{\text{def}}{=} \sum_{j=1}^{n+1} t_j \partial_{y_j} \stackrel{\text{def}}{=} -y_2\partial_{y_1} + y_1\partial_{y_2} + \cdots - y_{n+1}\partial_{y_n} + y_n\partial_{y_{n+1}}.$$

Also, let $\{M_i\}_{i=1,\dots,m+1}$ and $\{N_j\}_{j=1,\dots,n+1}$ be the meridian vector fields on S^m and S^n respectively, i. e.

$$\begin{aligned} M_i &\stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{x_i} \text{ on } S^m \qquad i = 1, \dots, m+1, \\ N_j &\stackrel{\text{def}}{=} \text{orthogonal projection of } \partial_{y_j} \text{ on } S^n \qquad j = 1, \dots, n+1. \end{aligned}$$

Denote by $\mathcal{P}$ the parallelization on $S^m \times S^n$ given by the vector fields

$$\begin{aligned} p_i &\stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \dots, m-1, \\ p_{m-1+j} &\stackrel{\text{def}}{=} y_j M_m + t_j M_{m+1} + (t_j x_{m+1} + y_j x_m - t_j)T + N_j \qquad j = 1, \dots, n+1. \end{aligned} \tag{3.1}$$

Of course, the frames $\mathcal{B}$ and $\mathcal{P}$ (the first defined only for $n = 1, 3$) are orthonormal with respect to the product metric.

3.3 Almost-Hermitian structures on $S^{2n-1} \times S^1$

Let $\mathcal{C} \stackrel{\text{def}}{=} \{c_1, \dots, c_{2n}\}$ be an *ordered* orthonormal basis of an Euclidean vector space V^{2n}. The *Hermitian structure* $I_\mathcal{C}$ *on* V *canonically associated to* $\mathcal{C}$ is given by

$$I_\mathcal{C}(c_{2i-1}) \stackrel{\text{def}}{=} c_{2i} \quad i = 1, \dots, n.$$

Let $z_i \overset{\text{def}}{=} x_{2i-1} + ix_{2i}$, for $i = 1, \dots, n$, be complex coordinates in $\mathbb{C}^n = \mathbb{R}^{2n}$. The cyclic infinite group Γ of transformations of $\mathbb{C}^n - 0$ generated by $z \mapsto e^{2\pi} z$ defines the diagonal Hermitian Hopf manifold $H = (\mathbb{C}^n - 0)/\Gamma$. The map

$$\begin{array}{ccc} \mathbb{C}^n - 0 & \longrightarrow & S^{2n-1} \times S^1 \\ z & \longmapsto & (z/|z|, \log|z| \mod 2\pi) \end{array} \tag{3.2}$$

is Γ-invariant, and induces a Hermitian structure $I_{e^{2\pi}}$ on its diffeomorphic product $S^{2n-1} \times S^1$.

Consider on $S^{2n-1} \times S^1$ the frames $\mathcal{B} = \{b_1, \dots, b_{2n}\}$ and $\mathcal{P} = \{p_1, \dots, p_{2n}\}$ given by

$$\begin{aligned} b_i &= p_*(|x|\partial_{x_i}) \qquad i = 1, \dots, 2n, \\ p_i &= M_i + x_i T \qquad i = 1, \dots, 2n-2, \\ p_{2n-1} &= y_1 M_{2n-1} - y_2 M_{2n} + (-y_2 x_{2n} + y_1 x_{2n-1} + y_2)T + N_1, \\ p_{2n} &= y_2 M_{2n-1} + y_1 M_{2n} + (y_1 x_{2n} + y_2 x_{2n-1} - y_1)T + N_2, \end{aligned}$$

where p is the map given by formula (3.2). The almost-Hermitian structures $I_{\mathcal{B}}$ and $I_{\mathcal{P}}$ on $S^{2n-1} \times S^1$ canonically associated to $\mathcal{B}$ and $\mathcal{P}$ respectively, are then defined.

Remark 3.3.1 Note that $I_{\mathcal{B}}$ coincides with $I_{e^{2\pi}}$, and it is therefore integrable. Moreover, since the change of basis from $\mathcal{B}$ to $\mathcal{P}$ is given by a unitary matrix (see formula (2.27)), $I_{\mathcal{P}} = I_{\mathcal{B}}$. □

3.4 Almost-Hermitian structures on $S^{2n-3} \times S^3$

Consider now on products $S^{2n-3} \times S^3$ the frames $\mathcal{B} = \{b_1, \dots, b_{2n}\}$ and $\mathcal{P} = \{p_1, \dots, p_{2n}\}$ given in section 2.2. More explicitly, using the meridian vector fields M_i, N_j on S^{2n-3}, S^3 respectively and the Lie frame $\{T, T_2, T_3\}$ on S^3 given by the quaternionic multiplication, one gets

$$\begin{aligned} b_i &= M_i + x_i T \qquad i = 1, \dots, 2n-2, \\ b_{2n-1} &= T_2, \\ b_{2n} &= T_3, \\ p_i &= M_i + x_i T \qquad i = 1, \dots, 2n-4, \\ p_{2n-3} &= y_1 M_{2n-3} - y_2 M_{2n-2} + (-y_2 x_{2n-2} + y_1 x_{2n-3} + y_2)T + N_1, \\ p_{2n-2} &= y_2 M_{2n-3} + y_1 M_{2n-2} + (y_1 x_{2n-2} + y_2 x_{2n-3} - y_1)T + N_2, \\ p_{2n-1} &= y_3 M_{2n-3} - y_4 M_{2n-2} + (-y_4 x_{2n-2} + y_3 x_{2n-3} + y_4)T + N_3, \\ p_{2n} &= y_4 M_{2n-3} + y_3 M_{2n-2} + (y_3 x_{2n-2} + y_4 x_{2n-3} - y_3)T + N_4. \end{aligned}$$

The almost-Hermitian structures $I_{\mathcal{B}}$ and $I_{\mathcal{P}}$ on $S^{2n-3} \times S^3$ canonically associated to $\mathcal{B}$ and $\mathcal{P}$ respectively, are then defined.

Theorem 3.4.1 *The almost-Hermitian structure $I_{\mathcal{B}}$ on $S^{2n-3} \times S^3$ is integrable.*

Proof: The differentials of the $(1,0)$-type forms are (use formulas (2.12))

$$\begin{aligned} d(b^{2n-1} + ib^{2n}) &= -2i(b^{2n-1} + ib^{2n}) \wedge \tau, \\ d(b^i + ib^j) &= i(x_i + ix_j)(b^{2n-1} + ib^{2n}) \wedge (b^{2n-1} - ib^{2n}) \\ &\quad + (b^i + ib^j) \wedge \tau \qquad i, j = 1, \ldots, 2n-2. \end{aligned}$$

This shows that

$$d(\Omega^{(1,0)}) \subset \Omega^{(2,0)} \oplus \Omega^{(1,1)},$$

hence $I_{\mathcal{B}}$ is integrable. ■

On each product $S^m \times S^n$ of two odd-dimensional spheres is defined a family of *Calabi-Eckmann complex structures*, parametrized by the moduli space of the torus $S^1 \times S^1$ (see [CE53]). The Calabi-Eckmann complex structure on $S^m \times S^n$ given by the non-real complex number τ is defined as follows: denote by S, T the unit vector field given by the complex multiplication on S^m, S^n respectively, and remark that the complex Hopf fibration induces a complex structure on their orthogonal complement (with respect to the product metric); then map S into $\mathfrak{Re}\,\tau S + \mathfrak{Im}\,\tau T$.

Only $\tau = \pm i$ gives thus Calabi-Eckmann Hermitian structures: here and henceforth, *denote by $I^{m,n}$ the Calabi-Eckmann Hermitian structure on $S^m \times S^n$ given by $\tau = -i$.* Therefore, $I^{m,n}(T) = S$. It is well-known that Calabi-Eckmann complex structures are a generalization of Hopf complex structures: in particular, using our notation,

$$I^{m,1} = I_{e^{2\pi}}.$$

One is thus lead to the following question regarding $S^{2n-3} \times S^3$: the Hermitian structure $I_{\mathcal{B}}$ is one of the Calabi-Eckmann Hermitian structures? The answer is given by the following:

Theorem 3.4.2 *The Calabi-Eckmann Hermitian structure $I^{2n-3,3}$ on $S^{2n-3} \times S^3$ coincide with the almost-Hermitian structure $I_{\mathcal{B}}$ on $S^{2n-3} \times S^3$ canonically associated to $\mathcal{B}$.*

Proof: Remark that, since $\{T, b_{2n-1}, b_{2n}\}$ is the Lie Frame on S^3 (b_{2n-1}, b_{2n} are the multiplication by j, $k \in \mathbb{H}$ respectively), b_{2n-1}, b_{2n} span the horizontal bundle of the Hopf fibration $S^3 \to \mathbb{CP}^1$, hence

$$I^{2n-3,3}(b_{2n-1}) = b_{2n}.$$

The vector fields $\{b_1, \ldots, b_{2n-2}\}$ span the tangent bundle of $S^{2n-3} \times S^1$, where S^1 denotes the fiber of the above Hopf fibration. Then

$$I^{2n-3,3}(b_i) = I^{2n-3,1}(b_i) = I_{e^{2\pi}}(b_i) = I_{\mathcal{B}}(b_i) = b_{i+1} \qquad \text{for odd } i = 1, \ldots, 2n-2,$$

and this ends the proof. ■

Remark 3.4.3 Since the change of basis from $\mathcal{B}$ to $\mathcal{P}$ is given by a unitary matrix (see formula (2.27)), then $I_{\mathcal{P}} = I_{\mathcal{B}}$. □

3.5 Calabi-Eckmann revisited

In the general case of products $S^m \times S^n$, n odd, only the parallelization $\mathcal{P}$ is defined:

$$\begin{aligned} p_i &\stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \dots, m-1, \\ p_{m-1+j} &\stackrel{\text{def}}{=} y_j M_m + t_j M_{m+1} + (t_j x_{m+1} + y_j x_m - t_j)T + N_j \qquad j = 1, \dots, n+1. \end{aligned}$$

If both m, n are odd, then the almost-Hermitian structure $I_{\mathcal{P}}$ on $S^m \times S^n$ is defined.

Theorem 3.5.1 *Let m, $n \geq 1$ be odd. Then the Calabi-Eckmann Hermitian structure $I^{m,n}$ on $S^m \times S^n$ coincide with the almost-Hermitian structure $I_{\mathcal{P}}$ on $S^m \times S^n$ canonically associated to $\mathcal{P}$.*

Proof: Let S^1 be the fiber of the Hopf fibration of S^n. Define the frame $\mathcal{B} = \{b_1, \dots, b_{m+1}\}$ on $S^m \times S^1$ by

$$b_i \stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \dots, m+1,$$

and write $\mathcal{B}$ in the basis $\mathcal{P}$:

$$\begin{aligned} b_i &= p_i \qquad i = 1, \dots, m-1, \\ b_m &= \sum_{j=1}^{n+1} y_j p_{m-1+j}, \\ b_{m+1} &= \sum_{j=1}^{n+1} t_j p_{m-1+j}. \end{aligned} \tag{3.3}$$

Then

$$I_{\mathcal{P}}(b_i) = I_{\mathcal{B}}(b_i) = I_{e^{2\pi}}(b_i) = I^{m,1}(b_i) = I^{m,n}(b_i) \qquad i = 1, \dots, m+1,$$

The same way, denoting by S^1 the fiber of the Hopf fibration of S^m, and using the frame $\tilde{\mathcal{B}} = \{\tilde{b}_1, \dots, \tilde{b}_{n+1}\}$ on $S^1 \times S^n$ given by

$$\tilde{b}_j \stackrel{\text{def}}{=} N_j - y_j S \qquad j = 1, \dots, n+1,$$

one obtains

$$I_{\mathcal{P}}(\tilde{b}_j) = I_{\tilde{\mathcal{B}}}(\tilde{b}_j) = I_{e^{2\pi}}(\tilde{b}_j) = I^{1,n}(\tilde{b}_j) = I^{m,n}(\tilde{b}_j) \qquad j = 1, \dots, n+1,$$

and this completes the proof, since $\mathcal{B} \cup \tilde{\mathcal{B}}$ spans $T(S^m \times S^n)$. ■

3.6 Almost-hyperhermitian structures

Let $\mathcal{C} \stackrel{\text{def}}{=} \{c_1, \dots, c_{4n}\}$ be an *ordered* orthonormal basis of an Euclidean vector space V^{4n}. There are, besides $I_\mathcal{C}$, the Hermitian structures $J_\mathcal{C}$, $K_\mathcal{C}$ given by

$$\begin{array}{lll} J_\mathcal{C}(c_{4i-3}) \stackrel{\text{def}}{=} c_{4i-1} & K_\mathcal{C}(c_{4i-3}) \stackrel{\text{def}}{=} c_{4i} & \\ & & i = 1, \dots, n. \\ J_\mathcal{C}(c_{4i-2}) \stackrel{\text{def}}{=} -c_{4i} & K_\mathcal{C}(c_{4i-2}) \stackrel{\text{def}}{=} c_{4i-1} & \end{array}$$

The identity $I_\mathcal{C} J_\mathcal{C} = -J_\mathcal{C} I_\mathcal{C}$ shows that $(I_\mathcal{C}, J_\mathcal{C}, K_\mathcal{C})$ is a hyperhermitian structure on V, that is referred to as *the hyperhermitian structure canonically associated to* $\mathcal{C}$.

Let $h_i \stackrel{\text{def}}{=} x_{4i-3} + i x_{4i-2} + j x_{4i-1} + k x_{4i}$, for $i = 1, \dots, n$, be quaternionic coordinates in $\mathbb{H}^n = \mathbb{R}^{4n}$. The cyclic infinite group Γ of transformations of $\mathbb{H}^n - 0$ generated by $h \mapsto e^{2\pi} h$ defines the diagonal hyperhermitian Hopf manifold $H = (\mathbb{H}^n - 0)/\Gamma$. The map

$$\begin{array}{ccc} \mathbb{H}^n - 0 & \longrightarrow & S^{4n-1} \times S^1 \\ h & \longmapsto & (h/|h|, \log|h| \mod 2\pi) \end{array} \tag{3.4}$$

is Γ-invariant, and induces a hyperhermitian structure $(I_{e^{2\pi}}, J_{e^{2\pi}}, K_{e^{2\pi}})$ on its diffeomorphic product $S^{4n-1} \times S^1$.

Remark 3.6.1 On $S^{4n-1} \times S^1$, the almost-hyperhermitian structure $(I_\mathcal{B}, J_\mathcal{B}, K_\mathcal{B})$ coincides with $(I_{e^{2\pi}}, J_{e^{2\pi}}, K_{e^{2\pi}})$, and it is therefore integrable. On $S^{4n-3} \times S^3$, since the change of basis from $\mathcal{B}$ to $\mathcal{P}$ is given by a symplectic matrix (see formula (2.27)), $(I_\mathcal{B}, J_\mathcal{B}, K_\mathcal{B}) = (I_\mathcal{P}, J_\mathcal{P}, K_\mathcal{P})$. □

The following theorem is a consequence of the integrability theorems for the Hermitian symmetric orbits 4.1.2, 4.1.4, 4.1.7, 4.1.9 and is stated here for completeness:

Theorem 3.6.2 *On $S^{4n-1} \times S^1$, the almost-hyperhermitian structure $(I_\mathcal{P}, J_\mathcal{P}, K_\mathcal{P})$ is non-integrable. On $S^{4n-3} \times S^3$, the almost-hyperhermitian structure $(I_\mathcal{B}, J_\mathcal{B}, K_\mathcal{B}) = (I_\mathcal{P}, J_\mathcal{P}, K_\mathcal{P})$ is non-integrable. On $S^m \times S^n$, for m, n odd and $m + n = 0 \mod 4$, the almost-hyperhermitian structure $(I_\mathcal{P}, J_\mathcal{P}, K_\mathcal{P})$ is non-integrable.*

Remark 3.6.3 Almost-hypercomplex structures on products of spheres were considered in [Bon65, Bon67].

3.7 Algebraic preliminaries: structures related to the octonions

Call $\{e_1, \dots, e_7\}$ the standard basis of $\mathbb{R}^7$, and $\{e^1, \dots, e^7\}$ the corresponding dual basis. Let $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$. Then the standard basis of $\mathbb{R}^8$ is $\{1, e_1, \dots, e_7\}$. Call $\{\lambda, e^1, \dots, e^7\}$ the corresponding dual basis, with an obvious misuse of notation.

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Figure 3.1: Multiplication table of the Cayley numbers, as given by formulas (3.5).

Let $\mathbb{O}$ be the non-associative normed algebra of Cayley numbers, that is, $\mathbb{R}^8$ equipped with the standard scalar product $\langle \cdot, \cdot \rangle$, and with multiplicative structure defined by the following relations (see [BG72]):

$$e_i^2 = -1, \quad e_i e_{i+1} = e_{i+3}, \quad e_{i+1} e_{i+3} = e_i, \quad e_{i+3} e_i = e_{i+1}, \quad e_i e_j = -e_j e_i \text{ for } i, j \in \mathbb{Z}_7, j \neq i. \tag{3.5}$$

The multiplication table of $\mathbb{O}$ is given in figure 3.1.

Remark 3.7.1 The standard quaternion subalgebra $\mathbb{H}$ of $\mathbb{O}$ is generated by $1, e_1, e_2$ and e_4. This choice is made (following for instance [BG72], [Gra77], [Mar81a], [FG82] or [Cab97]) in order to have a simpler definition of the forms associated to the G_2 and *Spin*(7) structures, to be considered on our products of spheres. An orthonormal basis $\{1, e_1, \ldots, e_7\}$ of $\mathbb{O}$ satisfying (3.5) is called in many different ways: a *Cayley basis* (see [FG82], [Cab95a], [Cab95b], [Cab96] or [Cab97]), or also an *adapted basis* (see [Mar81a] or [Mar81b]) or again a *canonical basis* (see [BG72]). Last, some authors use the more classical (though more asymmetric) multiplication table given by choosing $\{1, i, j, k, e, ie, je, ke\}$ in place of $\{1, e_1, \ldots, e_7\}$ (see [Mur89], [Mur92], [CMS96] or [FKMS97]). $\square$

Definition 3.7.2 Given $x = x_0 + x_1 e_1 + \cdots + x_7 e_7 \in \mathbb{O}$, define the *imaginary* and the *real* parts as

$$\mathfrak{Im}(x) \stackrel{\text{def}}{=} x_1 e_1 + \cdots + x_7 e_7, \qquad \mathfrak{Re}(x) \stackrel{\text{def}}{=} x_0,$$

and the *conjugate of* x as

$$\bar{x} \stackrel{\text{def}}{=} \mathfrak{Re}(x) - \mathfrak{Im}(x).$$

$\square$

As usual, the scalar product and the multiplicative structure are related by

$$\langle x, x \rangle = x\bar{x}. \tag{3.6}$$

Definition 3.7.3 The exceptional Lie group G_2 is the automorphism group of $\mathbb{O}$, that is,

$$\mathrm{G}_2 \stackrel{\text{def}}{=} \{\alpha \in \mathrm{GL}(\mathbb{R}^8) \text{ such that } \alpha(xy) = \alpha(x)\alpha(y)\}.$$

□

Proposition 3.7.4 *Denote by* $\mathrm{SO}(7)$ *the Lie group of orthogonal orientation-preserving transformations of* $\mathbb{O}$ *that fix the real part* $\mathfrak{Re}(\mathbb{O})$*: then*

$$\mathrm{G}_2 \subset \mathrm{SO}(7).$$

Proof: Clearly, any $\alpha \in \mathrm{G}_2$ fixes 1. Let $x \in \mathfrak{Im}(\mathbb{O})$. Since

$$x^2 = -x(-x) = -x\bar{x} \stackrel{(3.6)}{=} -\langle x, x\rangle,$$

for all $\alpha \in \mathrm{G}_2$ one obtains

$$\alpha(x)^2 = -\langle x, x\rangle. \tag{3.7}$$

From the other side,

$$\begin{aligned} \alpha(x)^2 &= (\mathfrak{Re}(\alpha(x)) + \mathfrak{Im}(\alpha(x)))^2 \\ &= \mathfrak{Re}^2(\alpha(x)) - \langle \mathfrak{Im}(\alpha(x)), \mathfrak{Im}(\alpha(x))\rangle^2 + 2\,\mathfrak{Re}(\alpha(x))\,\mathfrak{Im}(\alpha(x)). \end{aligned} \tag{3.8}$$

Comparing equations (3.7) and (3.8), one obtains $\mathfrak{Re}(\alpha(x)) = 0$, hence

$$\langle \alpha(x), \alpha(x)\rangle = \langle x, x\rangle, \qquad \text{for any } x \in \mathfrak{Im}(\mathbb{O}).$$

A standard polarization argument gives

$$\langle \alpha(x), \alpha(y)\rangle = \langle x, y\rangle, \qquad \text{for any } x, y \in \mathfrak{Im}(\mathbb{O}),$$

that is, α is an orthogonal transformation. The action of G_2 on $S^6 \subset \mathfrak{Im}(\mathbb{O})$ is transitive: let $y_1 \in S^6$, extend $\{1, y_1\}$ to any Cayley basis $\{1, y_1, \ldots, y_7\}$ of $\mathbb{O}$, and remark that the map sending e_i to y_i, $i = 1, \ldots, 7$, belongs to G_2. Hence, G_2 is a connected subgroup of the orthogonal transformations of $\mathfrak{Im}(\mathbb{O})$, and this completes the proof. ■

Remark 3.7.5 The trilinear map φ defined by

$$\varphi(x, y, z) \stackrel{\text{def}}{=} \langle x, yz\rangle \qquad x, y, z \in \mathfrak{Im}(\mathbb{O})$$

is actually alternating, thus φ is a 3-form, and one can prove that (see [Mur89])

$$\mathrm{G}_2 = \{\alpha \in \mathrm{GL}(\mathbb{R}^7) \text{ such that } \alpha^*\varphi = \varphi\}. \tag{3.9}$$

□

The 3-form of remark 3.7.5 can be easily computed using formulas (3.5) and the multiplication table of figure 3.1:

$$\varphi = \sum_{i \in \mathbb{Z}_7} e^i \wedge e^{i+1} \wedge e^{i+3}. \tag{3.10}$$

The 3-form φ is called *the standard* G_2*-form* on $\mathbb{R}^7$. If $\mathcal{C}$ is any *ordered* orthonormal basis on an Euclidean vector space V of dimension 7, the above equation defines a G_2*-structure* $\varphi_{\mathcal{C}}$ *on* V *canonically associated to* $\mathcal{C}$.

Denote by $*$ the Hodge star operator on $(\mathbb{R}^8, \langle \cdot, \cdot \rangle)$, where the positive orientation is given by $\{1, e_1, \ldots, e_7\}$. Since $\{1, e_1, \ldots, e_7\}$ is orthonormal, one obtains

$$*(\lambda \wedge \varphi) = -\sum_{i \in \mathbb{Z}_7} e^{i+2} \wedge e^{i+4} \wedge e^{i+5} \wedge e^{i+6} = -\sum_{i \in \mathbb{Z}_7} e^i \wedge e^{i+2} \wedge e^{i+3} \wedge e^{i+4}. \tag{3.11}$$

Define the 4-forms ϕ^+ and ϕ^- on $\mathbb{R}^8$ by

$$\phi^+ \stackrel{\text{def}}{=} \lambda \wedge \varphi + *(\lambda \wedge \varphi), \qquad \phi^- \stackrel{\text{def}}{=} \lambda \wedge \varphi - *(\lambda \wedge \varphi).$$

Using (3.10) and (3.11) one gets

$$\begin{aligned} \phi^+ &= \lambda \wedge \sum_{i \in \mathbb{Z}_7} e^i \wedge e^{i+1} \wedge e^{i+3} - \sum_{i \in \mathbb{Z}_7} e^i \wedge e^{i+2} \wedge e^{i+3} \wedge e^{i+4}, \\ \phi^- &= \lambda \wedge \sum_{i \in \mathbb{Z}_7} e^i \wedge e^{i+1} \wedge e^{i+3} + \sum_{i \in \mathbb{Z}_7} e^i \wedge e^{i+2} \wedge e^{i+3} \wedge e^{i+4}. \end{aligned} \tag{3.12}$$

The following definition is equivalent to the classical one, but more useful to us:

Definition 3.7.6

$$\mathrm{Spin}(7) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{GL}(\mathbb{R}^8) \text{ such that } \alpha^* \phi^+ = \phi^+\} (\simeq \{\alpha \in \mathrm{GL}(\mathbb{R}^8) \text{ such that } \alpha^* \phi^- = \phi^-\}).$$

□

The 4-form ϕ^+ is called *the standard positive* Spin(7)*-form* on $\mathbb{R}^8$, and the 4-form ϕ^- is called *the standard negative* Spin(7)*-form* on $\mathbb{R}^8$. If $\mathcal{C}$ is any *ordered* orthonormal basis on an Euclidean vector space V of dimension 8, the above equations define a *positive* Spin(7)*-structure* $\phi_{\mathcal{C}} = \phi^+_{\mathcal{C}}$ *on* V *canonically associated to* $\mathcal{C}$ and a *negative* Spin(7)*-structure* $\phi^-_{\mathcal{C}}$ *on* V *canonically associated to* $\mathcal{C}$.

3.8 G_2-structures on products of spheres

A G_2-structure on a seven-dimensional manifold M is a reduction of the structure group GL(7) to G_2. Since $\mathrm{G}_2 \subset \mathrm{SO}(7)$ (proposition 3.7.4), a G_2-structure canonically defines a metric.

A G_2-structure gives a canonical identification of each tangent space with $\mathbb{R}^7$, in such a way that the local 3-form defined by (3.10) is actually global, because of (3.9). Vice versa, if there exists on M a global 3-form that can be locally written as in (3.10), then M admits a G_2-structure. Hence, the G_2-structure is often identified with the 3-form.

Definition 3.8.1 Let M be a seven-dimensional manifold with a G_2-structure. Let φ be its 3-form and ∇ the Levi-Civita connection of the metric defined by φ. The G_2-structure is then said to be

- *parallel* if $\nabla\varphi = 0$;
- *locally conformal parallel* if φ is locally conformal to local G_2-structures φ_α, which are parallel with respect to the local Levi-Civita connections they define.

□

A G_2-structure is parallel if and only if $d\varphi = d * \varphi = 0$ ([Sal89]). This fact can be used to characterize locally conformal parallel G_2-structures:

Proposition 3.8.2 *A G_2-structure φ on M^7 is locally conformal parallel if and only if there exists a closed $\tau \in \Omega^1(M)$ such that $d\varphi = 3\tau \wedge \varphi$, $d * \varphi = 4\tau \wedge *\varphi$.*

Proof: Let φ be locally conformal parallel. Then for each $x \in M$, there exist a neighborhood U of x and a map $\sigma\colon U \to \mathbb{R}$ such that the local G_2-structure $\varphi_U \stackrel{\text{def}}{=} e^{-3\sigma}\varphi_{|_U}$ is parallel with respect to its local Levi-Civita connection. One then obtains $d\varphi_U = d *_U \varphi_U = 0$, where $*_U$ is the local Hodge star-operator associated to φ_U, and using these relations together with $e^{4\sigma} *_U = e^{3\sigma} *$, one obtains

$$d\varphi_{|_U} = 3d\sigma \wedge \varphi_{|_U}, \qquad d * \varphi_{|_U} = 4d\sigma \wedge *\varphi_{|_U}.$$

The closed 1-form τ locally defined by $d\sigma$ is easily seen to be global. The reverse implication is obtained the same way, once observed that a closed 1-form τ is locally exact, that is, for each $x \in M$ there exist a neighborhood U of x and a map $\sigma\colon U \to \mathbb{R}$ such that $\tau_{|_U} = d\sigma$. ■

Remark 3.8.3 Using the local expression of φ, it can be shown that $\alpha \wedge \varphi = 0$ if and only if $\alpha = 0$, for any 2-form α on the G_2-manifold (M,φ). This means that the requirement of τ to be closed in the previous proposition can be dropped. Moreover, one can also modify the statement in "[...] if and only if there exist $\alpha, \beta \in \Omega^1(M)$ such that $d\varphi = \alpha \wedge \varphi$, $d * \varphi = \beta \wedge *\varphi$", and then prove that $-4\alpha = -3\beta = *(*d\varphi \wedge \varphi)$. □

Remark 3.8.4 The 3-form φ of a parallel G_2-structure on a compact M represents a non trivial element in 3-dimensional cohomology (see [Bon66]). □

Any parallelizable seven-dimensional manifold trivially admits a G_2-structure. As usual, denote by Γ the cyclic infinite group of transformations of $\mathbb{R}^7 - 0$ generated by $x \mapsto e^{2\pi}x$, and consider the induced frame $\mathcal{B}$ on $S^6 \times S^1$. Let $\varphi_{\mathcal{B}}$ be the G_2-structure on $S^6 \times S^1$ canonically associated to $\mathcal{B}$, that is,

$$\varphi_{\mathcal{B}} \stackrel{\text{def}}{=} \sum_{i \in \mathbb{Z}_7} b^i \wedge b^{i+1} \wedge b^{i+3}.$$

Theorem 3.8.5 *The* G_2*-structure on* $S^6 \times S^1$ *given by* $\varphi_{\mathcal{B}}$ *is locally conformal parallel. The local parallel* G_2*-structures are induced by the standard* G_2*-structure on* $\mathbb{R}^7$ *by means of the canonical projection* $\mathbb{R}^7 - 0 \to (\mathbb{R}^7 - 0)/\Gamma$.

Proof: The standard G_2-structure on $\mathbb{R}^7$ is given by

$$\varphi = \sum_{i \in \mathbb{Z}_7} dx_i \wedge dx_{i+1} \wedge dx_{i+3}.$$

It is parallel, and on $\mathbb{R}^7 - 0$ it is globally conformal to the Γ-invariant 3-form

$$\varphi' = \frac{1}{|x|^3} \sum_{i \in \mathbb{Z}_7} dx_i \wedge dx_{i+1} \wedge dx_{i+3}.$$

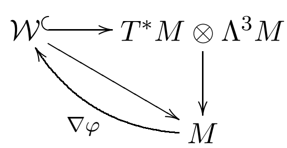
Observe that $\mathbb{R}^7 - 0$ is locally diffeomorphic to $S^6 \times S^1$, and that φ' induces just $\varphi_{\mathcal{B}}$, to end the proof. ■

Remark 3.8.6 By remark 3.8.4, $S^6 \times S^1$ has no parallel G_2-structure. □

Remark 3.8.7 Since $\mathcal{B}$ is orthonormal, the metric induced on $S^6 \times S^1$ by means of $\varphi_{\mathcal{B}}$ is the product metric. □

The same construction can be done on $S^6 \times S^1$, $S^4 \times S^3$ and $S^2 \times S^5$ for the frame $\mathcal{P}$. On $S^4 \times S^3$ also the frame $\mathcal{B}$ is available. One obtains G_2-structures of *general type*. The rest of the section is devoted to explain what is a G_2-structure of general type on a G_2-manifold (M, φ).

Look at $\nabla\varphi$ as belonging to $\Omega^1(\Lambda^3 M) = \Gamma(T^*M \otimes \Lambda^3 M)$. The G_2-structure allows one to identify each tangent space with the standard 7-dimensional orthogonal representation of G_2 given by proposition 3.7.4. The induced action of G_2 splits each fiber of $T^*M \otimes \Lambda^3 M$ into irreducible components, giving rise to a splitting of $T^*M \otimes \Lambda^3 M$, and if $\nabla\varphi$ lifts to a particular component of this splitting, one says that φ belongs to the corresponding particular class. Actually, due to special properties of φ, it can be shown that $\nabla\varphi$ lifts always to a G_2-invariant subbundle $\mathcal{W}$ of $T^*M \otimes \Lambda^3 M$:



As a consequence, the above splitting must be done on fibers of $\mathcal{W}$. The irreducible components of $\mathcal{W}$ turn out to be four, and they are classically denoted by $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$ for the components of rank 1, 14, 27, 7 respectively. The G_2-structure φ is then said *of type* $\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l}$ if $\nabla\varphi$ lifts to $\mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l}$:

$$\begin{array}{ccc} \mathcal{W}_{i_1} \oplus \cdots \oplus \mathcal{W}_{i_l} & \hookrightarrow & \mathcal{W} \\ & \nwarrow & \nabla\varphi \uparrow\downarrow \\ & & M \end{array}$$

For more details, the standard reference is [FG82].

Definition 3.8.8 If $\nabla\varphi$ does not lift to any of the spaces $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$, $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ then φ is said to be *of general type.*

In [FG82] the irreducible components of $\mathcal{W}$ are explicitly given, but the defining relations are rather complicated. These relations can be simplified by looking at the G_2-equivariant maps

$$\begin{array}{ccc} T^*M \otimes \Lambda^3 M & \longrightarrow & \Lambda^4 M \\ \alpha \otimes \beta \wedge \gamma \wedge \delta & \longmapsto & \alpha \wedge \beta \wedge \gamma \wedge \delta \end{array}$$

and

$$\begin{array}{ccc} T^*M \otimes \Lambda^3 M & \longrightarrow & \Lambda^5 M \\ \alpha \otimes \beta \wedge \gamma \wedge \delta & \longmapsto & *(\langle \alpha, \beta \rangle \gamma \wedge \delta + \langle \alpha, \gamma \rangle \delta \wedge \beta + \langle \alpha, \delta \rangle \beta \wedge \gamma). \end{array}$$

This is done in [Cab96]. Here is a list of the resulting simplified relations restricted to the ones that will be useful in the following:

Theorem 3.8.9 *A* G_2*-structure* φ *on a manifold* M *is of type:*

- $\mathcal{W}_4$ *if and only if there exist* α, $\beta \in \Omega^1(M)$ *such that* $d\varphi = \alpha \wedge \varphi$ *and* $d * \varphi = \beta \wedge *\varphi$ *(this class is needed in section 3.10);*
- $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ *if and only if* $(*d\varphi) \wedge \varphi = 0$*;*
- $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ *if and only if there exists* $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$ *such that* $d\varphi = \alpha \wedge \varphi + f * \varphi$*;*
- $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if there exists* $\beta \in \Omega^1(M)$ *such that* $d * \varphi = \beta \wedge *\varphi$*;*
- $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ *if and only if* $d\varphi \wedge \varphi = 0$.

Therefore, to check that a G_2-manifold is of general type, one must verify that none of the above relations is satisfied.

Theorem 3.8.10 *The* G_2*-structure* $\varphi_{\mathcal{P}}$ *canonically associated to the frame* $\mathcal{P}$ *on* $S^6 \times S^1$ *is of general type.*

Proof: The 3-form $\varphi_{\mathcal{P}}$ and the 4-form $*\varphi_{\mathcal{P}}$ are given by

$$\begin{aligned}\varphi_{\mathcal{P}} &= \sum_{i\in\mathbb{Z}/(7)} p^i \wedge p^{i+1} \wedge p^{i+3},\\ *\varphi_{\mathcal{P}} &= -\sum_{i\in\mathbb{Z}/(7)} p^i \wedge p^{i+2} \wedge p^{i+3} \wedge p^{i+4}.\end{aligned}$$

Using formulas (2.29) one obtains

$$\begin{aligned}d\varphi_{\mathcal{P}} &= 3\varphi_{\mathcal{P}} \wedge \tau - (p^{6,1,3} + p^{4,5,6} - p^{3,4,7} - p^{5,7,1}) \wedge \tau,\\ d * \varphi_{\mathcal{P}} &= -4 * \varphi_{\mathcal{P}} \wedge \tau - (p^{7,1,3} + p^{4,5,7} + p^{3,4,6} + p^{5,6,1}) \wedge p^2 \wedge \tau.\end{aligned}$$

A hard computation then shows that no relation of the previous theorem is satisfied, and $\varphi_{\mathcal{P}}$ is of general type. ■

The same result holds for $S^4 \times S^3$ and $S^2 \times S^5$, but computation, being based on the general formulas (2.31), is much harder than before. Therefore, the following theorem was proved by a computer calculation:

Theorem 3.8.11 *The* G_2*-structures canonically associated to the frames* $\mathcal{B}$ *and* $\mathcal{P}$ *on* $S^4 \times S^3$ *are both of general type. The* G_2*-structure canonically associated to the frame* $\mathcal{P}$ *on* $S^2 \times S^5$ *is of general type.*

3.9 Spin(7)-structures on products of spheres

A Spin(7)-structure on an eight-dimensional manifold M is a reduction of the structure group GL(8) to Spin(7). Since Spin(7) $\subset$ SO(8) (see [Mur89]), a Spin(7)-structure canonically defines a metric.

A Spin(7)-structure gives a canonical identification of each tangent space with $\mathbb{R}^8$, in such a way that on each connected component of M one (and only one) of the two local 4-forms defined by (3.12) is actually global, because of definition (3.7.6). Vice versa, if there exists on M a global 4-form that can be locally written as in (3.12), the sign being fixed on each connected component, then M admits a Spin(7)-structure (see [Gra69, theorem 2.4] and [Cab97, page 238]). Hence, the Spin(7)-structure is often identified with its 4-form.

Definition 3.9.1 Let M be an eight-dimensional manifold with a Spin(7)-structure. Let ϕ be its 4-form and ∇ the Levi-Civita connection of the metric defined by ϕ. The Spin(7)-structure is then said to be

- *parallel* if $\nabla\phi = 0$;

- *locally conformal parallel* if ϕ is locally conformal to local Spin(7)-structures ϕ_α, which are parallel with respect to the local Levi-Civita connections they define.

A Spin(7)-structure is parallel if and only if $d\phi = 0$ ([Sal89]). This fact can be used to characterize locally conformal parallel Spin(7)-structures:

Proposition 3.9.2 *A* Spin(7)*-structure* ϕ *on* M^8 *is locally conformal parallel if and only if there exists a closed* $\tau \in \Omega^1(M)$ *such that* $d\phi = \tau \wedge \phi$.

Proof: The same of proposition 3.8.2. ■

Remark 3.9.3 Using the local expression of ϕ, it can be shown that $\alpha\wedge\phi = 0$ if and only if $\alpha = 0$, for any 2-form α on the Spin(7)-manifold (M,ϕ). This means that the requirement of τ to be closed in the previous proposition can be dropped. Moreover, one can prove that $-7\alpha = *(*d\phi \wedge \phi)$. □

Remark 3.9.4 The 4-form ϕ of a parallel Spin(7)-structure on a compact M represents a non trivial element in 4-dimensional cohomology (see [Bon66]).

Any parallelizable eight-dimensional manifold trivially admits a Spin(7)-structure. As usual, denote by Γ the cyclic infinite group of transformations of $\mathbb{R}^8 - 0$ generated by $x \mapsto e^{2\pi}x$, and consider the induced frame $\mathcal{B}$ on $S^7 \times S^1$. Let $\phi_\mathcal{B}$ be the Spin(7)-structure on $S^7 \times S^1$ canonically associated to $\mathcal{B}$, that is,

$$\phi_\mathcal{B} \overset{\text{def}}{=} \lambda \wedge \sum_{i\in\mathbb{Z}_7} b^i \wedge b^{i+1} \wedge b^{i+3} - \sum_{i\in\mathbb{Z}_7} b^i \wedge b^{i+2} \wedge b^{i+3} \wedge b^{i+4}.$$

Theorem 3.9.5 *The* Spin(7)*-structure on* $S^7 \times S^1$ *given by* $\phi_\mathcal{B}$ *is locally conformal parallel. The local parallel* Spin(7)*-structures are induced by the standard positive* Spin(7)*-structure on* $\mathbb{R}^8$*, by means of the canonical projection* $\mathbb{R}^8 - 0 \to (\mathbb{R}^8 - 0)/\Gamma$.

Proof: The same of proposition 3.8.5. ■

Remark 3.9.6 By remark 3.9.4, $S^7 \times S^1$ has no parallel Spin(7)-structure.

Remark 3.9.7 Since $\mathcal{B}$ is orthonormal, the metric induced on $S^7 \times S^1$ by means of $\phi_\mathcal{B}$ is the product metric.

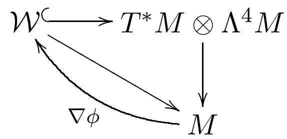
Remark 3.9.8 All the section can be repeated using

$$\phi_\mathcal{B} \overset{\text{def}}{=} \lambda \wedge \sum_{i\in\mathbb{Z}_7} b^i \wedge b^{i+1} \wedge b^{i+3} + \sum_{i\in\mathbb{Z}_7} b^i \wedge b^{i+2} \wedge b^{i+3} \wedge b^{i+4}$$

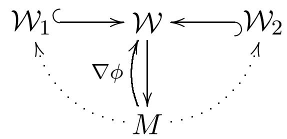
to obtain a Spin(7)-structure on $S^7 \times S^1$ that is locally conformal to the negative standard Spin(7)-structure on $\mathbb{R}^8 - 0$.

The same construction can be done on $S^7 \times S^1$, $S^5 \times S^3$ and $S^3 \times S^5$ for the frame $\mathcal{P}$. On $S^5 \times S^3$ also the frame $\mathcal{B}$ is available. As in the case G_2, one obtains Spin(7)-structures of *general type.* The rest of the section is devoted to explain what is a Spin(7)-structure of general type on a Spin(7)-manifold (M, ϕ). The discussion is formally identical to the case G_2.

Look at $\nabla\phi$ as belonging to $\Omega^1(\Lambda^4 M) = \Gamma(T^*M \otimes \Lambda^4 M)$. The Spin(7)-structure allows one to identify each tangent space with the standard 8-dimensional orthogonal representation of Spin(7) given by the inclusion Spin(7) $\subset$ SO(8). The induced action of Spin(7) splits each fiber of $T^*M \otimes \Lambda^4 M$ into irreducible components, giving rise to a splitting of $T^*M \otimes \Lambda^4 M$, and if $\nabla\phi$ lifts to a particular component of this splitting, one says that ϕ belongs to the corresponding particular class. Actually, due to special properties of ϕ, it can be shown that $\nabla\phi$ lifts always to a Spin(7)-invariant subbundle $\mathcal{W}$ of $T^*M \otimes \Lambda^4 M$:



As a consequence, the above splitting must be done on fibers of $\mathcal{W}$. The irreducible components of $\mathcal{W}$ turn out to be two (this is the main difference from the G_2 case), and they are classically denoted by $\mathcal{W}_1$, $\mathcal{W}_2$ for the components of rank 48, 8 respectively. The Spin(7)-structure ϕ is then said *of type* $\mathcal{W}_1$*,* $\mathcal{W}_2$ if $\nabla\varphi$ lifts to $\mathcal{W}_1$, $\mathcal{W}_2$ respectively:



For more details, the standard reference is [Fer86].

Definition 3.9.9 If $\nabla\phi$ does not lift to neither $\mathcal{W}_1$ nor $\mathcal{W}_2$ then ϕ is said to be *of general type.*

In [Fer86] the irreducible components of $\mathcal{W}$ are explicitly given, but the defining relations are rather complicated. These relations can be simplified by looking at the Spin(7)-equivariant map

$$\begin{array}{ccc} T^*M \otimes \Lambda^4 M & \longrightarrow & \Lambda^5 M \\ \alpha \otimes \beta \wedge \gamma \wedge \delta \wedge \epsilon & \longmapsto & \alpha \wedge \beta \wedge \gamma \wedge \delta \wedge \epsilon. \end{array}$$

This is done in [Cab95a]. Here is the list of the resulting simplified relations:

Theorem 3.9.10 *A* Spin(7)*-structure* ϕ *on a manifold* M *is of type:*

- $\mathcal{W}_1$ *if and only if* $(*d\phi) \wedge \phi = 0$*;*
- $\mathcal{W}_2$ *if and only if there exists* $\alpha \in \Omega^1(M)$ *such that* $d\phi = \alpha \wedge \phi$*.*

Therefore, to check that a Spin(7)-manifold is of general type, one must verify that none of the above relations is satisfied.

The following theorem is based on the general formulas (2.31), and it was proven by a computer calculation:

Theorem 3.9.11 *The* Spin(7)*-structures canonically associated to the frame $\mathcal{P}$ on $S^7 \times S^1$ is of general type. The* Spin(7)*-structures canonically associated to the frames $\mathcal{B}$ and $\mathcal{P}$ on $S^5 \times S^3$ are both of general type. The* Spin(7)*-structure canonically associated to the frame $\mathcal{P}$ on $S^3 \times S^5$ is of general type.*

3.10 Relations among the structures

A unified treatment of G_2 and Spin(7)-structures can be done by means of the vector cross product notion. A beautiful reference is [Gra69].

Definition 3.10.1 Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional Euclidean real vector space. An r-linear map $P\colon V^r \to V$ $(1 \le r \le n)$ is said to be an *r-fold vector cross product* on V if

- $\langle P(v_1, \dots, v_r), v_i\rangle = 0$, for any $i = 1, \dots, r$;
- $\langle P(v_1, \dots, v_r), P(v_1, \dots, v_r)\rangle = \det(\langle v_i, v_j\rangle)_{i,j=1,\dots,r}$.

Given two r-fold vector cross product P and P' on V, one says that P and P' are isomorphic if there is an isometry $f\colon V \to V$ such that $f_*P = P'$.

In [BG67] vector cross products together with their automorphism groups are classified. They span four classes:

(I) $r = 1$, and n even: P is a complex structure on V, the automorphism group of P is the corresponding unitary group $U(n/2)$;

(II) $r = n - 1$: P is the Hodge star operator on V^{n-1}, the automorphism group is $SO(n)$;

(III) $r = 2$, and $n = 7$: $V \simeq \mathfrak{Im}(\mathbb{O})$ in such a way that $P(x, y) = \mathfrak{Im}(xy)$, and the automorphism group is G_2;

(IV) $r = 3$, and $n = 8$: $V \simeq \mathbb{O}$ in such a way that

$$P(x, y, z) = P_+(x, y, z) = -x(\bar{y}z) + \langle x, y\rangle z + \langle y, z\rangle x - \langle z, x\rangle y$$

or

$$P(x, y, z) = P_{-}(x, y, z) = -(x\bar{y})z + \langle x, y\rangle z + \langle y, z\rangle x - \langle z, x\rangle y,$$

and the automorphism group is Spin(7).

An r-fold vector cross product P defines an $(r+1)$-form ω on V by the formula

$$\omega(v_1, \ldots, v_{r+1}) \stackrel{\text{def}}{=} \langle P(v_1, \ldots, v_r), v_{r+1}\rangle.$$

The form ω is called the *fundamental form* of P.

Vector cross products of class (I) are nothing else than complex structures, and ω is just the Kähler form. Vector cross products of class (II) are orientations, and ω is the volume form. For classes (III) and (IV):

Proposition 3.10.2 *The 3-form ω defined by a vector cross product P of class (III) is just the standard* G_2*-form φ on $\mathbb{R}^7$. The 4-form $\omega_\pm$ defined by a vector cross product $P_\pm$ of class (IV) is just the standard* Spin(7)*-form $\phi_\pm$ on $\mathbb{R}^8$.*

Proof: As for ω, look at the definition of φ given in remark 3.7.5, to obtain

$$\varphi(x, y, z) = \langle x, yz\rangle = \langle x, \mathfrak{Im}(yz)\rangle = \langle x, P(y, z)\rangle = \omega(x, y, z) \qquad \text{for } x, y, z \in \mathfrak{Im}(\mathbb{O}).$$

In the Spin(7)-case, a straightforward calculation using the multiplication table of figure 3.1 is all is needed. ■

Requiring all objects to be smooth, one obtains the notion of a *differentiable* vector cross product: ω is then a global differential form of degree $r+1$.

The following proposition gives the link between vector cross product and G-structures:

Proposition 3.10.3 ([Gra69, Proposition 2.2]) *Let M be n-dimensional. Then M has a differentiable vector cross product of class I, II, III, IV if and only if M has a G-structure, where* $G = \mathrm{U}(n/2)$, $\mathrm{SO}(n)$, G_2, Spin(7) *respectively.*

Recall now the following characterization of G_2 and Spin(7)-structures:

- A G_2-structure φ on a manifold M is of type $\mathcal{W}_4$ if and only if there exist α, $\beta \in \Omega^1(M)$ such that $d\varphi = \alpha \wedge \varphi$ and $d * \varphi = \beta \wedge *\varphi$;
- a Spin(7)-structure ϕ on a manifold M is of type $\mathcal{W}_2$ if and only if there exists $\alpha \in \Omega^1(M)$ such that $d\phi = \alpha \wedge \phi$.

Using remarks 3.8.3 and 3.9.3, one obtains:

Proposition 3.10.4 *A* G_2*-structure* φ *is of type* $\mathcal{W}_4$ *if and only if* φ *is locally conformal parallel. A* $\mathrm{Spin}(7)$*-structure* ϕ *is of type* $\mathcal{W}_2$ *if and only if* ϕ *is locally conformal parallel.*

The following theorem gives a method of constructing new vector cross products from old:

Theorem 3.10.5 ([Gra69, Theorem 2.6]) *Let* M *be an oriented hypersurface of* $\bar{M}$*, and let* N *be its unit normal vector field. Let* $\bar{P}$ *a differentiable* $(r+1)$*-fold vector cross product on* $\bar{M}$*. Then the map* P *given by*

$$P(X_1, \dots, X_r) \stackrel{\text{def}}{=} \bar{P}(N, X_1, \dots, X_r), \qquad X_1, \dots, X_r \in \mathfrak{X}(M)$$

defines a differentiable cross vector product on M*.*

In this way the standard Spin(7)-structure on $\mathbb{R}^8$ induces a G_2-structure on $S^7 \subset \mathbb{R}^8$, that is described in [FG82]:

Theorem 3.10.6 ([FG82, Theorem 7.5]) *Let* φ_{S^7} *be the* G_2*-structure on* S^7 *given by theorem 3.10.5. Then*

$$d\varphi_{S^7} = k * \varphi_{S^7}, \qquad k \text{ a non zero constant.}$$

Definition 3.10.7 A G_2-structure on M satisfying the thesis of theorem 3.10.6, is called a *nearly parallel* G_2-structure ([Gra69], [FG82] or [FKMS97]), or also, M is said to have *weak holonomy* G_2 ([Gra71]). Last, in the classification of [FG82], it is called of type $\mathcal{W}_1$.

The nearly parallel G_2-structure φ_{S^7} on S^7 is used in [Cab95a] to give a Spin(7)-structure on $S^7 \times S^1$: let $\phi_{S^7\times S^1}$ be the 4-forms given on $S^7 \times S^1$ by

$$\phi_{S^7\times S^1} \stackrel{\text{def}}{=} d\theta \wedge \varphi_{S^7} + *\varphi_{S^7}. \tag{3.13}$$

Proposition 3.10.8 ([Cab95a, first example at page 278]) *The 4-form* $\phi_{S^7\times S^1}$ *given by formula (3.13) is a locally conformal parallel* $\mathrm{Spin}(7)$*-structure.*

Proof: By theorem 3.10.6,

$$d\phi_{S^7\times S^1} = d\theta \wedge d\varphi_{S^7} + d * \varphi_{S^7} = kd\theta \wedge *\varphi_{S^7} = kd\theta \wedge \phi_{S^7\times S^1},$$

and proposition 3.10.4 ends the proof. ∎

Theorem 3.10.5 gives a canonical G_2-structure on any orientable hypersurface of a Spin(7)-manifold. In [Cab97] the following relation between the Spin(7)-manifold and the induced G_2-structure is proved:

Theorem 3.10.9 ([Cab97, Theorem 4.4]) *Let ϕ be a* Spin(7)*-structure of class $\mathcal{W}_2$ on a manifold $\bar{M}$. Let M be an oriented hypersurface of $\bar{M}$ with unitary normal vector field N and mean curvature H. Then the induced* G_2*-structure on M is of class $\mathcal{W}_4$ if and only if M is totally umbilic in $\bar{M}$ and $*((*d\phi)\wedge\phi)(N) = 28\langle H, N\rangle$.*

Theorem 3.10.9 defines the following G_2-structure on $S^6\times S^1$:

Example 3.10.10 ([Cab97, first example at page 245]) Look at $S^6\times S^1$ as a hypersurface of $S^7\times S^1$. Let $\varphi_{S^6\times S^1}$ be the G_2-structure induced by $\phi_{S^7\times S^1}$. Since $S^6\times S^1$ is totally geodesic in $S^7\times S^1$, and

$$*((*d\phi_{S^7\times S^1})\wedge\phi_{S^7\times S^1}) = -7k d\theta,$$

one obtains $*((*d\phi_{S^7\times S^1})\wedge\phi_{S^7\times S^1})(N) = 0 = 28\langle H, N\rangle$. Then, by theorem 3.10.9 and proposition 3.10.4, $\varphi_{S^6\times S^1}$ is locally conformal parallel.

$S^6\times S^1$ comes then equipped with two locally conformal parallel G_2-structures: $\varphi_{\mathcal{B}}$ given by theorem 3.8.5, and $\varphi_{S^6\times S^1}$ given by example 3.10.10. Also, on $S^7\times S^1$ are defined two locally conformal parallel Spin(7)-structures: $\phi_{\mathcal{B}}$ (theorem 3.9.5 and remark 3.9.8), and $\phi_{S^7\times S^1}$ (formula (3.13)). Are these structures really different each other?

Theorem 3.10.11 *The locally conformal parallel* Spin(7)*-structures $\phi_{\mathcal{B}}$ and $\phi_{S^7\times S^1}$ on $S^7\times S^1$ are the same. Also, the locally conformal parallel* G_2*-structures $\varphi_{\mathcal{B}}$ and $\varphi_{S^6\times S^1}$ on $S^6\times S^1$ are the same.*

Proof: Let p be the projection $\mathbb{R}^8-0\to S^7\times S^1$ given by $x\mapsto (x/|x|, \log|x| \mod 2\pi)$. By theorem 3.9.5, $p^*(\phi_{\mathcal{B}}) = |x|^{-4}\phi$, where ϕ denotes the standard positive Spin(7)-form on $\mathbb{R}^8$. Since p is a local diffeomorphism, it is sufficient to prove that $p^*(\phi_{S^7\times S^1}) = |x|^{-4}\phi$. Define the unitary vector field N on $\mathbb{R}^8-0$ by

$$N \stackrel{\text{def}}{=} \frac{x_1\partial_{x_1}+\cdots+x_8\partial_{x_8}}{|x|} \in \mathfrak{X}(R^8-0),$$

and use the metric to define its dual 1-form $n\in\Omega^1(\mathbb{R}^8-0)$. Then a straightforward computation gives

$$p^*(\phi_{S^7\times S^1}) = \frac{n\wedge i_N\phi + *(n\wedge i_N\phi)}{|x|^4},$$

and using the fact that the action of Spin(7) on S^7 is transitive, one obtains

$$n\wedge i_N\phi + *(n\wedge i_N\phi) = \phi.$$

This completes the proof of the statement about Spin(7). To complete the proof, choose the embedding $S^6\times S^1\subset S^7\times S^1$ given by $x_8=0$. The normal vector field is then $\partial_{x_8} = b_8$, and one obtains

$$\varphi_{S^6\times S^1} = i_{\partial_{x_8}}\phi_{S^7\times S^1} = i_{b_8}\phi_{\mathcal{B}} = \varphi_{\mathcal{B}}.$$

■

3.11 Spin(9)-structures on products of spheres

A Spin(9)-structure on a sixteen-dimensional manifold M is a reduction of the structure group of M to Spin(9). Since Spin(9) $\subset$ SO(16), a Spin(9)-structure canonically defines a metric.

In [BG72] it is shown that Spin(9) is the stabilizer of a Spin(9)-invariant 8-form $\Phi \in \Lambda^8(\mathbb{R}^{16})$, that is called *the standard* Spin(9)*-structure on* $\mathbb{R}^{16}$:

$$\mathrm{Spin}(9) = \{g \in \mathrm{SO}(16) \text{ such that } g^*\Phi = \Phi\}.$$

This allows one to think a Spin(9)-structure on M^{16} as a global 8-form that can be locally written as Φ. In particular, on any parallelizable M^{16}, an explicit parallelization gives such a global 8-form. Therefore one can define Spin(9)-structures on $S^{15} \times S^1$, $S^{13} \times S^3$, $S^{11} \times S^5$, $S^9 \times S^7$, $S^7 \times S^9$, $S^5 \times S^{11}$, $S^3 \times S^{13}$ and $S^1 \times S^{15}$, canonically associated to the frames $\mathcal{P}$ and $\mathcal{B}$. Let as usual Γ denote the cyclic infinite group of transformations of $\mathbb{R}^{16} - 0$ generated by $x \mapsto e^{2\pi}x$.

Theorem 3.11.1 *The* Spin(9)*-structure on* $S^{15} \times S^1$ *given by* $\Phi_{\mathcal{B}}$ *is locally conformal parallel. The local parallel* Spin(9)*-structures are induced by the standard* Spin(9)*-structure on* $\mathbb{R}^{16}$ *by means of the canonical projection* $\mathbb{R}^{16} - 0 \to (\mathbb{R}^{16} - 0)/\Gamma$.

Proof: It follows by the fact that $|x|^{-8}\Phi$ is a Γ-invariant, globally conformal to Φ 8-form that induces just $\Phi_{\mathcal{B}}$. ■

Unfortunately, the 8-form Φ is not easy to handle, and this is probably one of the reasons why a Gray-Hervella-like classification of Spin(9)-structures was lacking until [Fri99].

In what follows, the construction given in [Fri99] is briefly described. Let $\mathcal{R}$ be a Spin(9)-structure on a 16-dimensional Riemannian manifold M^{16}, and denote by $\mathcal{F}(M)$ the principal orthonormal frame bundle. Then $\mathcal{R}$ is a subbundle $\mathcal{F}(M)$:

$$\begin{array}{ccc} \mathcal{R} & \hookrightarrow & \mathcal{F}(M) \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

The Levi-Civita connection

$$Z : T(\mathcal{F}(M)) \longrightarrow \mathfrak{so}(16) = \mathfrak{spin}(9) \oplus \mathfrak{spin}(9)^{\perp}$$

restricted to $T(\mathcal{R})$ decomposes into $Z^* \oplus \Gamma$, where Z^* is a connection in the principal Spin(9)-fibre bundle $\mathcal{R}$, and

$$\Gamma \in \Omega^1(\mathcal{R} \times_{\mathrm{Spin}(9)} \mathfrak{spin}(9)^\perp) = \Omega^1(\Lambda^3(V)), \qquad V^9 \stackrel{\mathrm{def}}{=} \mathcal{R} \times_{\mathrm{Spin}(9)} \mathbb{R}^9.$$

The irreducible components of $\Lambda^1(M) \otimes \Lambda^3(V)$ are described in [Fri99]. In particular, one component is just the standard 16-dimensional representation $\Lambda^1(M)$, and this defines the *nearly parallel* Spin(9)*-structures*. The action of Spin(9) on S^{15} is transitive, with isotropy subgroup Spin(7), and this allows to define the principal Spin(7)-fibre bundle $\mathcal{R}_{S^{15}\times S^1}$

$$\mathrm{Spin}(9) \times S^1 \longrightarrow S^{15} \times S^1,$$

that in [Fri99] is shown to be actually a nearly parallel Spin(7) $\subset$ Spin(9)-structure on $S^{15} \times S^1$.

Theorem 3.11.2 *The nearly parallel* Spin(9)*-structure* $\mathcal{R}_{S^{15}\times S^1}$ *and the locally conformal parallel* Spin(9)*-structure* $\Phi_{\mathcal{B}}$ *on* $S^{15} \times S^1$ *are the same.*

Proof: Consider the following diagram of Spin(7) $\subset$ Spin(9)-structures:

$$\begin{array}{ccccc}
 & & \mathrm{Spin}(9) \times \mathbb{R}^+ & \longrightarrow & \mathrm{Spin}(9) \times S^1 \\
 & & \downarrow \mathcal{R}' & & \downarrow \mathcal{R} \\
\mathbb{R}^{16} - 0 & \xrightarrow{\alpha} & \dfrac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)} \times \mathbb{R}^+ & \xrightarrow{\beta} & \dfrac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)} \times S^1
\end{array}$$

where $\alpha(x) = (x/|x|, |x|)$ and $\beta([g], \rho) = ([g], \log \rho \mod 2\pi)$. The map

$$\beta \circ \alpha : \mathbb{R}^{16} - 0 \longrightarrow S^{15} \times S^1$$

is the canonical projection $\mathbb{R}^{16} - 0 \to S^{15} \times S^1$, and the map

$$\alpha^{-1} \circ \mathcal{R}' : \mathrm{Spin}(9) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^{16} - 0$$

is a Spin(7) $\subset$ Spin(9)-structure on $\mathbb{R}^{16} - 0$. The pull-back $(\beta \circ \alpha)^* \Phi_{\mathcal{B}} \in \Omega^8(\mathbb{R}^{16} - 0)$ gives by definition the admissible frame $\{|x|\partial_{x_1}, \dots, |x|\partial_{x_{16}}\}$. Check that this frame is admissible also for $\alpha^{-1} \circ \mathcal{R}'$ to complete the proof. ■

Chapter 4

Orthogonal and symmetric action

Let $\mathcal{C} \stackrel{\text{def}}{=} \{c_1, \ldots, c_n\}$ be an *ordered* orthonormal basis of the Euclidean vector space V^n. The orbit of $\mathcal{C}$ by the action of $\mathrm{O}(V)$ is a family of ordered orthonormal basis of V. Canonically associated to any of them there is a corresponding structure. One can of course act by any subgroup of $\mathrm{O}(V)$: in particular, the techniques developed in previous chapters are suitable to treat the action of the symmetric group $\mathfrak{S}_n$ of permutations of $\mathcal{C}$.

Define the following families of G-structures on products of spheres $S^m \times S^n$:

- if $m+n$ is even and $G = \mathrm{U}((m+n)/2)$, let $\mathcal{I}_{\mathcal{P}}$ denote the family of almost-Hermitian structures on $S^m \times S^n$ canonically associated to permutations of $\mathcal{P}$. If $n = 1$, 3, define in the similar way the family $\mathcal{I}_{\mathcal{B}}$;
- if $m+n = 0 \mod 4$ and $G = \mathrm{Sp}((m+n)/4)$, let $\mathcal{H}_{\mathcal{P}}$ denote the family of almost-hyperhermitian structures on $S^m \times S^n$ canonically associated to permutations of $\mathcal{P}$. If $n = 1$, 3, define in the similar way the family $\mathcal{H}_{\mathcal{B}}$;
- if $m+n = 7$, 8, 16 and $G = \mathrm{G}_2$, Spin(7), Spin(9), let $\mathcal{G}_{\mathcal{P}}$, $\mathcal{S}_{\mathcal{P}}$, $\mathcal{N}_{\mathcal{P}}$ denote the families of G_2, Spin(7), Spin(9)-structures on $S^m \times S^n$ canonically associated to permutations of $\mathcal{P}$. If $n = 1$, 3, define in the similar way the families $\mathcal{G}_{\mathcal{B}}$, $\mathcal{S}_{\mathcal{B}}$, $\mathcal{N}_{\mathcal{B}}$.

4.1 The symmetric orbit of almost-Hermitian structures

First, consider the simplest case $S^{2n-1} \times S^1$.

Theorem 4.1.1 *On products $S^{2n-1} \times S^1$, all almost-Hermitian structures in $\mathcal{I}_{\mathcal{B}}$ are biholomorphic to the Hopf Hermitian structure $I_{e^{2\pi}}$, and hence they are integrable.*

Proof: Let $I_{\pi(\mathcal{B})} \in \mathcal{I}_{\mathcal{B}}$, where π is a permutation of $\{1, \ldots, 2n\}$. By lemma 2.1.8, the map

$$\begin{array}{rccc} f_\pi : & S^{2n-1} \times S^1 & \longrightarrow & S^{2n-1} \times S^1 \\ & (x_1, \ldots, x_{2n}, \theta) & \longmapsto & (x_{\pi(1)}, \ldots, x_{\pi(2n)}, \theta) \end{array}$$

is a biholomorphism between $I_{\pi(\mathcal{B})}$ and $I_{\mathcal{B}} = I_{e^{2\pi}}$. ■

The following theorem describes the symmetric orbit $\mathcal{I}_{\mathcal{P}}$ of almost-Hermitian structures on $S^{2n-1} \times S^1$.

Theorem 4.1.2 *On products $S^{2n-1} \times S^1$ one gets:*

i) $I \in \mathcal{I}_{\mathcal{P}}$ is integrable if and only if $I(p_{2n-1}) = \pm p_{2n}$;

ii) if $I_{\pi(\mathcal{P})}$ is integrable, then $I_{\pi(\mathcal{P})} = I_{\pi(\mathcal{B})}$.

Proof: Let A be the matrix of the change of basis from $\mathcal{B}$ to $\mathcal{P}$ (see formula (2.27)). To prove sufficiency in i) suppose $I(p_{2n-1}) = \pm p_{2n}$. Then

$$\begin{aligned} d(p^i + ip^j) &= (p^i + ip^j) \wedge \tau \qquad i, j = 1, \ldots, 2n-1, \\ d(p^{2n-1} \pm ip^{2n}) &= (p^{2n-1} \pm ip^{2n}) \wedge (\tau - i\tau), \end{aligned}$$

that is,

$$d(\Omega^{(1,0)}) \subset \Omega^{(2,0)} \oplus \Omega^{(1,1)},$$

where $\Omega^{(a,b)}$ denotes complex (a, b)-type forms, with respect to I. Thus I is integrable. Now suppose that $I(p_{2n-1}) \neq \pm p_{2n}$. Then, taking $-I$ if necessary, there exist $i \neq j \in \{1, \ldots, 2n-2\}$ such that

$$\begin{aligned} I(p_{2n-1}) &= p_i, \\ I(p_{2n}) &= \pm p_j. \end{aligned}$$

The torsion tensor $N(X, Y)$ of I can then be computed for the vector fields $X = p_{2n-1}$, $Y = p_{2n}$. One obtains

$$\langle N(p_{2n-1}, p_{2n}), p_j \rangle = -2x_j \neq 0,$$

that complete the proof of i). To prove ii) one has only to remark that if $\mathcal{P} = A \cdot \mathcal{B}$, then $\pi(\mathcal{P}) = \pi(A) \cdot \pi(\mathcal{B})$, where $\pi(A)$ is just A with rows and columns permuted by means of π. In particular, $I(p_{2n-1}) = \pm p_{2n}$ implies $\pi(A) \in \mathrm{U}(n)$. ■

The following corollary then follows (see also remark 3.3.1):

Corollary 4.1.3 *All integrable almost-Hermitian structures in $\mathcal{I}_{\mathcal{P}}$ are biholomorphic to the Hopf Hermitian structure $I_{e^{2\pi}}$ on $S^{2n-1} \times S^1$. Moreover, $I_{\mathcal{P}}$ coincide with $I_{e^{2\pi}}$.*

Consider now the case $S^{2n-3} \times S^3$.

Theorem 4.1.4 *An almost-Hermitian structure $I \in \mathcal{I}_{\mathcal{B}}$ on $S^{2n-3} \times S^3$ is integrable if and only if $I(b_{2n-1}) = \pm b_{2n}$.*

Proof: Suppose that $I(b_{2n-1}) \neq \pm b_{2n}$. Then, as in the proof of theorem 4.1.2, there exist $i \neq j \in \{1, \ldots, 2n-2\}$ such that

$$\begin{aligned} I(b_{2n-1}) &= b_i, \\ I(b_{2n}) &= \pm b_j. \end{aligned}$$

Then

$$\begin{aligned} N(b_{2n-1}, b_{2n}) &= 2([I(b_{2n-1}), I(b_{2n})] - [b_{2n-1}, b_{2n}] - I([b_{2n-1}, I(b_{2n})]) - I([I(b_{2n-1}), b_{2n}])) \\ &= 2([b_i, \pm b_j] - [b_{2n-1}, b_{2n}] - I([b_{2n-1}, \pm b_j]) - I([b_i, b_{2n}])) \\ &= 2(\pm x_i b_j \mp x_j b_i + 2 \sum_{k=1}^{2n-2} x_k b_k - 2x_j b_j - 2x_i b_i) \neq 0, \end{aligned}$$

showing that I is non-integrable. In order to prove the reverse implication, suppose $I(b_{2n-1}) = \pm b_{2n}$. Since the differentials of the $(1,0)$-type forms are

$$\begin{aligned} d(b^{2n-1} \pm ib^{2n}) &= \mp 2i(b^{2n-1} \pm ib^{2n}) \wedge \tau, \\ d(b^i + ib^j) &= i(x_i + ix_j)(b^{2n-1} \pm ib^{2n}) \wedge (b^{2n-1} \mp ib^{2n}) \\ &\quad + (b^i + ib^j) \wedge \tau \qquad i, j = 1, \ldots, 2n-2, \end{aligned}$$

then

$$d(\Omega^{(1,0)}) \subset \Omega^{(2,0)} \oplus \Omega^{(1,1)},$$

hence I is integrable. ■

The following lemma is similar to the lemma 2.1.8, and can be proven using formulas (2.9).

Lemma 4.1.5 *Let π be a permutation of $\{1, \ldots, 2n-2\}$. The map*

$$\begin{array}{rccc} f_\pi : & S^{2n-3} \times S^3 & \longrightarrow & S^{2n-3} \times S^3 \\ & (x_1, \ldots, x_{2n-2}, y) & \longmapsto & (x_{\pi(1)}, \ldots, x_{\pi(2n-2)}, y) \end{array}$$

is a diffeomorphism and

$$df_\pi(b_{\pi(i)}) = b_i \qquad i = 1, \ldots, 2n-2.$$

Corollary 4.1.6 *All integrable almost-Hermitian structures on $S^{2n-3} \times S^3$ in the family $\mathcal{I}_{\mathcal{B}}$ are biholomorphic to the Calabi-Eckmann Hermitian structure $I^{2n-3,3}$.*

Proof: Let $I_{\pi(\mathcal{B})} \in \mathcal{I}_{\mathcal{B}}$ be integrable. By theorem 4.1.4, taking $-I$ in case, one can suppose $I_{\pi(\mathcal{B})}(b_{2n-1}) = b_{2n}$, that is, there exists an odd $i \in \{1, \ldots, 2n\}$ such that $\pi(i) = b_{2n-1}, \pi(i+1) = b_{2n}$. Define a permutation $\tilde{\pi}$ on $\{1, \ldots, 2n-2\}$ by

$$\tilde{\pi} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & \ldots & i-1 & i & \ldots & 2n-2 \\ \pi(1) & \ldots & \pi(i-1) & \pi(i+1) & \ldots & \pi(2n) \end{pmatrix}$$

The previous lemma implies that

$$df_{\tilde{\pi}} \circ I_{\pi(\mathcal{B})} = I_{\mathcal{B}} \circ df_{\tilde{\pi}},$$

hence $I_{\pi(\mathcal{B})}$ is biholomorphic to $I^{2n-3,3}$ by theorem 3.4.2. ■

The following theorem describes the symmetric orbit $\mathcal{I}_{\mathcal{P}}$ of almost-Hermitian structures on $S^{2n-3} \times S^3$.

Theorem 4.1.7 *On products $S^{2n-3} \times S^3$ one gets:*

i) $I \in \mathcal{I}_{\mathcal{P}}$ is integrable if and only if

$$I(p_{2n-3}) = \pm p_{2n-2} \qquad \text{and} \qquad I(p_{2n-1}) = \pm p_{2n}, \tag{4.1}$$

where the sign is the same in the two equalities;

ii) if $I_{\pi(\mathcal{P})}$ is integrable, then $I_{\pi(\mathcal{P})} = I_{\pi(\mathcal{B})}$.

Proof: Let A be the matrix of the change of basis from $\mathcal{B}$ to $\mathcal{P}$ (see formula (2.30)). To prove the only if part of i) suppose that (4.1) is not satisfied, and for the sake of simplicity suppose that $I(p_{2n-1}) \neq \pm p_{2n}$. Then, taking $-I$ in case,

$$I(p_{2n-1}) = p_i \qquad \text{and} \qquad I(p_{2n}) = \pm p_j \qquad i \neq j \in \{1, \ldots, 2n-2\}.$$

Use formulas (2.31) to check that $N(p_{2n-1}, p_{2n}) \neq 0$ (see also proof of theorem 4.1.9). To prove ii) and the if part of i) remark that, if (4.1) holds, then $\pi(A) \in \mathrm{U}(n)$. Therefore $I_{\pi(\mathcal{P})} = I_{\pi(\mathcal{B})}$, and $I_{\pi(\mathcal{P})}$ is integrable by theorem 4.1.2. ■

Corollary 4.1.6 and theorems 3.4.2, 4.1.7 gives the following corollary:

Corollary 4.1.8 *All integrable almost-Hermitian structures on $S^{2n-3} \times S^3$ in the symmetric orbit $\mathcal{I}_{\mathcal{P}}$ are biholomorphic to the Calabi-Eckmann Hermitian structure $I^{2n-3,3}$. Moreover, $I^{2n-3,3}$ coincide with $I_{\mathcal{P}}$.*

In the general case $S^m \times S^n$, m, n odd, only the symmetric orbit $\mathcal{I}_{\mathcal{P}}$ is defined.

Theorem 4.1.9 *An almost-Hermitian structure $I \in \mathcal{I}_\mathcal{P}$ on $S^m \times S^n$, m, n odd, is integrable if and only if*

$$I(p_{m-1+j}) = \pm p_{m+j} \qquad j \text{ odd}, j = 1, \ldots, n+1, \tag{4.2}$$

where the sign is the same for all j.

Proof: Firstly, the if part. Suppose that $I = I_{\pi(\mathcal{P})}$ for some permutation π of $\{1, \ldots, m+n\}$. Taking $-I$ in case, one can suppose all signs in (4.2) to be positive:

$$I_{\pi(\mathcal{P})}(p_{m-1+j}) = p_{m+j} \qquad j \text{ odd}, j = 1, \ldots, n+1.$$

The same way as in proof of corollary 4.1.6, build a permutation $\tilde{\pi}$ of $\{1, \ldots, m+1\}$, $\tilde{\pi}(m) = m, \tilde{\pi}(m+1) = m+1$. Let S^1 be the fiber of the Hopf fibration of S^n, and let $\mathcal{B} = \{b_1, \ldots, b_{m+1}\}$ be the frame on $S^m \times S^1$ given by

$$b_i \stackrel{\text{def}}{=} M_i + x_i T \qquad i = 1, \ldots, m+1.$$

Use formulas (3.3) to show that

$$I_{\pi(\mathcal{P})}(b_m) = b_{m+1},$$

hence $I_{\pi(\mathcal{P})}$ coincide with $I_{\tilde{\pi}(\mathcal{B})}$ on $S^m \times S^1$. Since theorem 4.1.1 implies that $I_{\tilde{\pi}(\mathcal{B})}$ is integrable, it follows that $I_{\pi(\mathcal{P})}$ is integrable on $S^m \times S^1$. Define the versor field $\tilde{\pi}(S)$ on S^m by

$$\tilde{\pi}(S) \stackrel{\text{def}}{=} -x_{\tilde{\pi}(2)}\partial_{x_{\tilde{\pi}(1)}} + x_{\tilde{\pi}(1)}\partial_{x_{\tilde{\pi}(2)}} + \cdots - x_{\tilde{\pi}(m+1)}\partial_{x_{\tilde{\pi}(m)}} + x_{\tilde{\pi}(m)}\partial_{x_{\tilde{\pi}(m+1)}}.$$

Let now S^1 be the orbit of $\tilde{\pi}(S)$ in S^m, and let $\tilde{\mathcal{B}} = \{\tilde{b}_1, \ldots, \tilde{b}_{n+1}\}$ be the frame on $S^1 \times S^n$ given by

$$\tilde{b}_j \stackrel{\text{def}}{=} N_j - y_j \tilde{\pi}(S) \qquad j = 1, \ldots, n+1.$$

One obtains

$$N_j - y_j\tilde{\pi}(S) = p_{m-1+j} - y_j b_m - t_j b_{m+1} + t_j T - y_j \tilde{\pi}(S).$$

Then, since $I_{\pi(\mathcal{P})}(T) = \tilde{\pi}(S)$,

$$\begin{aligned} I_{\pi(\mathcal{P})}(\tilde{b}_j) &= I_{\pi(\mathcal{P})}(N_j - y_j\tilde{\pi}(S)) \\ &= p_{m+j} - y_j b_{m+1} - y_{j+1} b_m - y_{j+1}\tilde{\pi}(S) + y_j T \qquad \text{odd } j = 1, \ldots, n+1. \\ &= p_{m-1+(j+1)} - y_{j+1} b_m - t_{j+1} b_{m+1} + t_{j+1} T - y_{j+1}\tilde{\pi}(S) = \tilde{b}_{j+1} \end{aligned}$$

Namely, $I_{\pi(\mathcal{P})}$ coincide with $I_{\tilde{\mathcal{B}}} = I_{e^{2\pi}}$ on $S^1 \times S^n$, and it follows that $I_{\pi(\mathcal{P})}$ is integrable on $S^1 \times S^n$. Since $\mathcal{B} \cup \tilde{\mathcal{B}}$ spans $T(S^m \times S^n)$, the proof of the if part is completed. Secondly, the only if part: it is given by a case by case computation, here sketched, which uses formulas (2.31). Suppose that condition (4.2) is not satisfied. Then, taking $-I$ in case, there exist an odd $j \in \{1, \ldots, n+1\}$ such that one of the following conditions holds:

i) there exist $i,\, k \in \{1, \ldots, m-1\}$, $i \neq k$ such that

$$I(p_{m-1+j}) = p_i \qquad \text{and} \qquad I(p_{m+j}) = \pm p_k;$$

ii) there exist $i \in \{1, \ldots, m-1\}$, $k \in \{1, \ldots, n+1\}$, $k \neq j,\, j+1$ such that

$$I(p_{m-1+j}) = p_i \qquad \text{and} \qquad I(p_{m+j}) = \pm p_{m-1+k};$$

iii) there exist $i \in \{1, \ldots, n+1\}$, $k \in \{1, \ldots, m-1\}$, $i \neq j,\, j+1$ such that

$$I(p_{m-1+j}) = p_{m-1+i} \qquad \text{and} \qquad I(p_{m+j}) = \pm p_k;$$

iv) there exist $i,\, k \in \{1, \ldots, n+1\}$, $i,\, k \neq j,\, j+1$, $i \neq k$ such that

$$I(p_{m-1+j}) = p_{m-1+i} \qquad \text{and} \qquad I(p_{m+j}) = \pm p_{m-1+k}.$$

The torsion tensor can then be computed in each case, using formulas (2.31), and in particular one obtains:

i) $\langle N(p_{m-1+j}, p_{m+j}), p_k \rangle = 2(\pm x_i(1 - y_j^2 - y_{j+1}^2) + x_k(1 - 2(y_j^2 + y_{j+1}^2))) \neq 0$

ii) $\langle N(p_{m-1+j}, p_{m+j}), p_i \rangle (y_j = y_{j+1} = y_k = 0, t_k = 1) = 2(x_i \mp x_{m+1}) \neq 0$

iii) $\langle N(p_{m-1+j}, p_{m+j}), p_k \rangle (y_j = y_{j+1} = y_i = 0, t_i = 1) = 2(x_k \pm x_{m+1}) \neq 0$

iv) $\langle N(p_{m-1+j}, p_{m+j}), p_{m-1+i} \rangle (y_j = y_{j+1} = t_i = x_m = 0, y_i = x_{m+1} = 1) = \mp 2t_k \neq 0$

■

The following lemma is a particular case of lemma 2.1.8:

Lemma 4.1.10 *Let π be a permutation of $\{1, \ldots, m+1\}$. The map*

$$\begin{array}{rccc} f_\pi : & S^m \times S^n & \longrightarrow & S^m \times S^n \\ & (x_1, \ldots, x_{m+1}, y) & \longmapsto & (x_{\pi(1)}, \ldots, x_{\pi(m+1)}, y) \end{array}$$

is a diffeomorphism and

$$df_\pi(p_{\pi(i)}) = p_i \qquad i = 1, \ldots, m-1.$$

Theorem 4.1.11 *All integrable almost-Hermitian structures on $S^m \times S^n$, m, n odd, in the symmetric orbit $\mathcal{I}_\mathcal{P}$ are biholomorphic to the Calabi-Eckmann Hermitian structure $I^{m,n}$.*

Proof: Let $I_{\pi(\mathcal{P})} \in \mathcal{I}_\mathcal{P}$ be a Hermitian structure. Let $\tilde{\pi}$ be the permutation of $\{1, \ldots, m+1\}$ built in proof of theorem 4.1.9. Then the above lemma implies $df_{\tilde{\pi}} \circ I_{\pi(\mathcal{P})} = I_\mathcal{P} \circ df_{\tilde{\pi}}$, and theorem 3.5.1 completes the proof. ■

4.2 The symmetric orbit of almost-hyperhermitian structures

The following theorem describes the symmetric orbit $\mathcal{H}_{\mathcal{B}}$ on $S^{4n-1} \times S^1$.

Theorem 4.2.1 *On products $S^{4n-1} \times S^1$, all almost-hyperhermitian structures in $\mathcal{H}_{\mathcal{B}}$ are equivalent with the Hopf hyperhermitian structure $(I_{e^{2\pi}}, J_{e^{2\pi}}, K_{e^{2\pi}})$, and hence they are integrable.*

Proof: The proof is the same of theorem 4.1.1. ■

All remaining dimensions are described in the following theorem:

Theorem 4.2.2 *On $S^{4n-1} \times S^1$, all hyperhermitian structures in $\mathcal{H}_{\mathcal{P}}$ are non-integrable. On $S^{4n-3} \times S^3$, both $\mathcal{H}_{\mathcal{B}}$ and $\mathcal{H}_{\mathcal{P}}$ are families of non-integrable hyperhermitian structures. On $S^m \times S^n$, for m, n odd, $m + n = 0 \mod 4$, all hyperhermitian structures in $\mathcal{H}_{\mathcal{P}}$ are non-integrable.*

Proof: Remark that a 2-dimensional distribution can't be closed for a hyperhermitian structure. Then observe that in theorems 4.1.2, 4.1.4, 4.1.7, 4.1.9, the conditions for an almost-Hermitian structure I in the families $\mathcal{I}_{\mathcal{B}}$ and $\mathcal{I}_{\mathcal{P}}$ to be integrable, implies that a 2-dimensional distribution is closed with respect to I, to end the proof. ■

Remark 4.2.3 Let $\mathcal{B}$ and $\mathcal{P}$ be the frames on $S^{4n-3} \times S^3$, and let A be the matrix of the change of basis from $\mathcal{B}$ to $\mathcal{P}$. One obtains $(I_{\pi(\mathcal{B})}, J_{\pi(\mathcal{B})}, K_{\pi(\mathcal{B})}) = (I_{\pi(\mathcal{P})}, J_{\pi(\mathcal{P})}, K_{\pi(\mathcal{P})})$ for all permutations π of $\{1, \ldots, 4n\}$ such that $\pi(A) \in \mathrm{Sp}(n)$. □

4.3 The symmetric orbit for the special structures

This section describes the symmetric orbits of the G_2 and Spin(7)-structures on products of spheres canonically associated to $\mathcal{B}$, whenever defined, and $\mathcal{P}$. It should be remarked that, since an expression for the Spin(9)-invariant 8-form similar to the G_2 and Spin(7)-case is still lacking, it was not possible to apply these techniques to Spin(9)-structures in the symmetric orbits $\mathcal{N}_{\mathcal{B}}$ and $\mathcal{N}_{\mathcal{P}}$.

The classification problem for structures in the orbits $\mathcal{G}_{\mathcal{B}}$, $\mathcal{G}_{\mathcal{P}}$, $\mathcal{H}_{\mathcal{B}}$ and $\mathcal{H}_{\mathcal{P}}$ can be tackled by a computer using the characterizations of the various classes given in theorems 3.8.9 and 3.9.10, together with the structure equations of $\mathcal{B}$ and $\mathcal{P}$. All statements in this section concerning the symmetric action have been verified by a computer calculation, and proved by a classical argument in a typical case.

Theorem 4.3.1 *On $S^6 \times S^1$, all G_2-structures in $\mathcal{G}_{\mathcal{B}}$ are isomorphic to $\varphi_{\mathcal{B}}$, and hence they are locally conformal parallel. On $S^7 \times S^1$, all $\mathrm{Spin}(7)$-structures in $\mathcal{S}_{\mathcal{B}}$ are isomorphic to $\phi_{\mathcal{B}}$, and hence they are locally conformal parallel. On $S^{15} \times S^1$, all $\mathrm{Spin}(9)$-structures in $\mathcal{N}_{\mathcal{B}}$ are isomorphic to $\Phi_{\mathcal{B}}$, and hence they are locally conformal parallel.*

Proof: The G_2 case. Let $\varphi_{\pi(\mathcal{B})} \in \mathcal{G}_{\mathcal{B}}$. Then the map

$$f_\pi : (S^6 \times S^1, \varphi_{\pi(\mathcal{B})}) \to (S^6 \times S^1, \varphi_{\mathcal{B}})$$

of lemma 2.1.8 is the required isomorphism. The same for the Spin(7) and Spin(9)-cases. ■

Theorem 4.3.2 *On $S^6 \times S^1$, all G_2-structures in $\mathcal{G}_{\mathcal{P}}$ are of general type.*

Proof: See theorem 4.4.4 for the whole orthogonal orbit in the following section. ■

Theorem 4.3.3 *The G_2-structures in $\mathcal{G}_{\mathcal{B}}$ on $S^4 \times S^3$, the G_2-structures in $\mathcal{G}_{\mathcal{P}}$ on $S^4 \times S^3$, $S^2 \times S^5$ are all of general type. The $\mathrm{Spin}(7)$-structures in $\mathcal{S}_{\mathcal{B}}$ on $S^5 \times S^3$, the $\mathrm{Spin}(7)$-structures in $\mathcal{S}_{\mathcal{P}}$ on $S^7 \times S^1$, $S^5 \times S^3$, $S^3 \times S^5$, $S^1 \times S^7$ are all of general type.*

Proof: Calculation. ■

4.4 The orthogonal orbit

The classification problem for structures in the orthogonal orbit by a computer calculation is much harder, and it is not still developed. This section is devoted to prove results about the orthogonal orbits of $I_{\mathcal{B}}$, $(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$, $\varphi_{\mathcal{B}}$, $\phi_{\mathcal{B}}$, $\Phi_{\mathcal{B}}$ on $S^{2n-1} \times S^1$, $S^{4n-1} \times S^1$, $S^7 \times S^1$, $S^8 \times S^1$, $S^{15} \times S^1$ respectively, and about the orthogonal orbit of $\varphi_{\mathcal{P}}$ on $S^6 \times S^1$.

Let Γ be the cyclic infinite group of transformations of $\mathbb{R}^{m+1} - 0$ generated by $x \mapsto e^{2\pi}x$. The following lemma is the natural extension of lemma 2.1.8, and its proof is trivial:

Lemma 4.4.1 *Let $A \in \mathrm{O}(m+1)$. Then $A : \mathbb{R}^{m+1} - 0 \to \mathbb{R}^{m+1} - 0$ is Γ-equivariant, and the induced diffeomorphism is*

$$\begin{array}{rccc} f_A : & S^m \times S^1 & \longrightarrow & S^m \times S^1 \\ & (x, \theta) & \longmapsto & (A(x), \theta). \end{array}$$

Moreover, the matrix of df_A with respect to the basis $\mathcal{B}$ on $S^m \times S^1$ is A.

Theorem 4.4.2 *The almost-Hermitian structures of the orthogonal orbit $\mathrm{O}(2n) \cdot I_{\mathcal{B}}$ on $S^{2n-1} \times S^1$ are biholomorphic to the Hopf Hermitian structure $I_{e^{2\pi}}$.*

Proof: Let $I_{A(\mathcal{B})} \in \mathrm{O}(2n) \cdot I_{\mathcal{B}}$. Then the matrix $[I_{A(\mathcal{B})} \circ df_A]_{\mathcal{B}}$ of $I_{A(\mathcal{B})} \circ df_A$ with respect to the basis $\mathcal{B}$ is

$$[I_{A(\mathcal{B})} \circ df_A]_{\mathcal{B}} = AIA^{-1}A = AI = [df_A \circ I_{\mathcal{B}}]_{\mathcal{B}},$$

and the conclusion follows by remark 3.3.1. ■

Clearly, the proof does not rely on properties of $\mathrm{U}(n)$, and it works for all the other G-structures:

Theorem 4.4.3 *The almost-hyperhermitian structures of the orthogonal orbit* $\mathrm{O}(4n)\cdot(I_{\mathcal{B}}, J_{\mathcal{B}}, K_{\mathcal{B}})$ *on* $S^{4n-1} \times S^1$ *are equivalent to the Hopf hyperhermitian structure* $(I_{e^{2\pi}}, J_{e^{2\pi}}, K_{e^{2\pi}})$. *The* G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$*-structures of the orthogonal orbit* $\mathrm{O}(7) \cdot \varphi_{\mathcal{B}}$, $\mathrm{O}(8) \cdot \phi_{\mathcal{B}}$, $\mathrm{O}(16) \cdot \Phi_{\mathcal{B}}$ *on* $S^6 \times S^1$, $S^7 \times S^1$, $S^{15} \times S^1$ *are isomorphic to* $\varphi_{\mathcal{B}}$, $\phi_{\mathcal{B}}$, $\Phi_{\mathcal{B}}$ *respectively.*

The lemma does not hold for the frame $\mathcal{P}$ on $S^m \times S^1$, because of the twisting of p_m, p_{m+1}. Therefore the following theorem is not trivial:

Theorem 4.4.4 *The* G_2*-structures of the orthogonal orbit* $\mathrm{O}(7) \cdot \varphi_{\mathcal{P}}$ *on* $S^6 \times S^1$ *are of general type.*

Proof: Let $A = (a_{i,j}) \in \mathrm{SO}(7)$, and denote by $\{q^1, \dots, q^7\}$ the coframe on $S^6 \times S^1$ induced by A:

$$q^i \stackrel{\text{def}}{=} \sum_{j=1}^{7} a_{i,j} p^j \qquad i = 1, \dots, 7.$$

Let $\tau = -y_2 dy_1 + y_1 dy_2$ be the usual 1-form on $S^6 \times S^1$, and u_i its coordinates with respect to $\{q^1, \dots, q^7\}$:

$$\tau = u_1 q^1 + \cdots + u_7 q^7.$$

Then

$$\varphi_{A(\mathcal{P})} = \sum_{i \in \mathbb{Z}/7\mathbb{Z}} q^{i,i+1,i+3}, \qquad *\varphi_{A(\mathcal{P})} = -\sum_{i \in \mathbb{Z}/7\mathbb{Z}} q^{i,i+2,i+3,i+4},$$

and using the structure equations one obtains

$$\begin{aligned}
d\varphi_{A(\mathcal{P})} &= 3\varphi_{A(\mathcal{P})} \wedge \tau \\
&\quad + \sum_{i \in \mathbb{Z}/7\mathbb{Z}} ((a_{i,6}p^7 - a_{i,7}p^6) q^{i+1,i+3} - (a_{i+1,6}p^7 - a_{i+1,7}p^6) q^{i,i+3} \\
&\quad + (a_{i+3,6}p^7 - a_{i+3,7}p^6) q^{i,i+1}) \wedge \tau \\
d * \varphi_{A(\mathcal{P})} &= -4 * \varphi_{A(\mathcal{P})} \wedge \tau \\
&\quad + \sum_{i \in \mathbb{Z}/7\mathbb{Z}} ((a_{i,6}p^7 - a_{i,7}p^6) q^{i+2,i+3,i+4} - (a_{i+2,6}p^7 - a_{i+2,7}p^6) q^{i,i+3,i+4} \\
&\quad + (a_{i+3,6}p^7 - a_{i+3,7}p^6) q^{i,i+2,i+4} - (a_{i+4,6}p^7 - a_{i+4,7}p^6) q^{i,i+2,i+3}) \wedge \tau.
\end{aligned}$$

The 3-form $*d\varphi_{A(\mathcal{P})}$ is not easy to write. Let $\alpha_{i,j} \stackrel{\text{def}}{=} a_{i,6}a_{j,7} - a_{i,7}a_{j,6}$. Then by a long calculation one obtains

$$
\begin{aligned}
* d\varphi_{A(\mathcal{P})} = & \sum_{i\in\mathbb{Z}/7\mathbb{Z}} [\\
& (-3u_{i+2} - u_i(-\alpha_{i,i+2} + \alpha_{i+4,i+3} + \alpha_{i+5,i+1}) + u_{i+3}(-\alpha_{i+6,i+1} + \alpha_{i+3,i+2} + \alpha_{i+4,i}) \\
& + u_{i+1}(\alpha_{i+6,i+3} + \alpha_{i+1,i+2} + \alpha_{i+5,i}))q^{i+4,i+5,i+6} \\
& + (3u_{i+4} - u_i(\alpha_{i,i+4} + \alpha_{i+2,i+3} + \alpha_{i+6,i+1}) + u_{i+1}(-\alpha_{i+1,i+4} - \alpha_{i+5,i+3} + \alpha_{i+6,i}) \\
& - u_{i+3}(-\alpha_{i+5,i+1} - \alpha_{i+2,i} + \alpha_{i+3,i+4}))q^{i+2,i+5,i+6} \\
& + (-3u_{i+5} + u_{i+3}(\alpha_{i+4,i+1} + \alpha_{i+6,i} + \alpha_{i+3,i+5}) + u_i(\alpha_{i,i+5} + \alpha_{i+2,i+1} - \alpha_{i+6,i+3}) \\
& - u_{i+1}(\alpha_{i+2,i} + \alpha_{i+4,i+3} - \alpha_{i+1,i+5}))q^{i+2,i+4,i+6} \\
& + (3u_{i+6} - u_{i+3}(-\alpha_{i+5,i} + \alpha_{i+2,i+1} + \alpha_{i+3,i+6}) - u_i(\alpha_{i+5,i+3} + \alpha_{i,i+6} - \alpha_{i+4,i+1}) \\
& + u_{i+1}(-\alpha_{i+4,i} - \alpha_{i+1,i+6} + \alpha_{i+2,i+3}))q^{i+2,i+4,i+5} \\
& + (u_{i+6}(-\alpha_{i,i+2} + \alpha_{i+4,i+3} + \alpha_{i+5,i+1}) - u_{i+3}(\alpha_{i+5,i+2} + \alpha_{i,i+1} + \alpha_{i+4,i+6}) \\
& + u_{i+2}(\alpha_{i+5,i+3} + \alpha_{i,i+6} - \alpha_{i+4,i+1}) - u_{i+1}(-\alpha_{i,i+3} - \alpha_{i+4,i+2} + \alpha_{i+5,i+6}))q^{i,i+4,i+5}].
\end{aligned}
$$

Now use theorem 3.8.9 to check which classes $\varphi_{A(\mathcal{P})}$ belongs to. As for the class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, one obtains

$$0 = d\varphi_{A(\mathcal{P})} \wedge \varphi_{A(\mathcal{P})} = \sigma \wedge \tau$$

where σ is a 6-form on $S^6 \times S^1$ whose coefficients with respect to $\mathcal{P}$ are constant, and this is easily seen to be impossible. The existence of a 1-form β on $S^6 \times S^1$ such that $d * \varphi_{A(\mathcal{P})} = \beta \wedge \varphi_{A(\mathcal{P})}$ implies that

$$\alpha_{i,i+1} + \alpha_{i+5,i+2} - \alpha_{i+6,i+4} = 0 \qquad i \in \mathbb{Z}/7\mathbb{Z}.$$

But this system has no solution, hence $\varphi_{A(\mathcal{P})}$ does not belong to the class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. The above system comes out also requiring the existence of a 1-form α and a function f on $S^6 \times S^1$ such that $d\varphi_{A(\mathcal{P})} = \alpha \wedge \varphi_{A(\mathcal{P})} + f * \varphi_{A(\mathcal{P})}$, hence $\varphi_{A(\mathcal{P})}$ does not belong to $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$. Finally, $*d\varphi_{A(\mathcal{P})} \wedge \varphi_{A(\mathcal{P})} \neq 0$ by a direct computation.

If $\det A = -1$, some signs in formulas are reversed, but the same impossible conditions are obtained.

■

Bibliography

[Ale68] D.V. Alekseevskij. Riemannian spaces with exceptional holonomy groups. *Funkts. Anal. Prilozh.*, 2(2):1–10, 1968.

[Bel99] F. Belgun. *Géometrie conforme et géométrie CR en dimensions* 3 *et* 4. PhD thesis, Centre de Mathématiques, UMR 7640 CNRS, Ecole Polytechnique, 91128 Palaiseau cedex, France, 1999.

[Bel00] F. Belgun. On the metric structure of the non-Kähler complex surfaces. *Math. Ann.*, 317:1–40, 2000.

[Ber55] M. Berger. Sur les groupes d'holonomie homogenes de variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[Bes81] A. Besse. *Géométrie Riemannienne en dimension 4.* Cedic, Paris, 1981.

[BG67] R. B. Brown and A. Gray. Vector cross products. *Comment. Math. Helv.*, 42:222–236, 1967.

[BG72] R. B. Brown and A. Gray. Riemannian Manifolds with Holonomy Group $Spin(9)$. In *Differential Geometry, in honor of K. Yano*, pages 41–59, Kinokuniya, Tokyo, 1972.

[BG98] C. P. Boyer and K. Galicki. On Sasakian-Einstein geometry. electronic preprint, November 1998. math/9811098 in http://xxx.lanl.gov.

[Bon65] E. Bonan. Structure presque quaternale sur une variete differentiable. *C. R. Acad. Sci., Paris*, 260:5445–5448, 1965.

[Bon66] E. Bonan. Sur des varietes riemanniennes a groupe d'holonomie G_2 ou Spin (7). *C. R. Acad. Sci., Paris, Ser. A*, 262:127–129, 1966.

[Bon67] E. Bonan. Sur les G-structures de type quaternionien. *Cah. Topologie Geom. Differ.*, 9:389–463, 1967.

[BPV84] W. Barth, C. Peters, and A. Van de Ven. *Compact complex surfaces.* Springer-Verlag, 1984.

[Bru92] M. Bruni. Sulla parallelizzazione esplicita dei prodotti di sfere. *Rend. di Mat., serie VII*, 12:405–423, 1992.

[Buc99] N. Buchdahl. On compact Kähler surfaces. *Ann. Inst. Fourier*, 49(1):287–302, 1999.

[Cab95a] F. M. Cabrera. On Riemannian manifolds with *Spin*(7)-structure. *Publ. Math. Debrecen*, 46:271–283, 1995.

[Cab95b] F. M. Cabrera. *Spin*(7)-Structures in Principal Fibre Bundles over Riemannian Manifolds with G_2-Structure. *Rend. Circ. Mat. Palermo, serie II*, XLIV:249–272, 1995.

[Cab96] F. M. Cabrera. On Riemannian manifolds with G_2-structure. *Bollettino U.M.I., sezione A*, 7:99–112, 1996.

[Cab97] F. M. Cabrera. Orientable Hypersurfaces of Riemannian Manifolds with *Spin*(7)-Structure. *Acta Math. Hung.*, 76(3):235–247, 1997.

[CE53] E. Calabi and B. Eckmann. A class of compact, complex manifolds wich are not algebraic. *Ann. of Math.*, 58:494–500, 1953.

[CMS96] F. M. Cabrera, M. D. Monar, and A. F. Swann. Classification of G_2-Structures. *J. Lond. Math. Soc., II. Ser.*, 53(2):407–416, 1996.

[CP85] B. Y. Chen and P. Piccinni. The canonical foliations of a locally conformal Kähler manifold. *Ann. di Mat. Pura e Appl.*, 141:289–305, 1985.

[DO98] S. Dragomir and L. Ornea. *Locally Conformal Kähler Geometry*, volume 155 of *Progress in Math.* Birkhäuser, 1998.

[Fer86] M. Fernández. A Classification of Riemannian Manifolds with Structure Group *Spin*(7). *Ann. di Mat. Pura e Appl.*, 148:101–122, 1986.

[FG82] M. Fernández and A. Gray. Riemannian manifolds with structure group G_2. *Ann. Mat. Pura Appl., IV. Ser.*, 132:19–45, 1982.

[FKMS97] Th. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann. On nearly parallel G_2-structures. *J. Geom. Phys.*, 23(3–4):259–286, 1997.

[Fri99] Th. Friedrich. Weak Spin(9)-Structures on 16-dimensional Riemannian Manifolds. pages 1–35, 1999. electronic preprint: math.DG/9912112 15 Dec 1999.

[Gau81] P. Gauduchon. Surfaces de Hopf. Variétés presque complexes de dimension 4. In *Géométrie Riemannienne en dimension 4* [Bes81], pages 134–155.

[GH80] A. Gray and L. M. Hervella. The Sixteen Classes of Almost Hermitian Manifolds and Their Linear Invariants. *Ann. Mat. Pura Appl., IV. Ser.*, 123:35–58, 1980.

[GO98] P. Gauduchon and L. Ornea. Locally conformally Kähler metrics on Hopf surfaces. *Ann. Inst. Fourier*, 48:1107–1127, 1998.

[Gra69] A. Gray. Vector Cross Products on Manifolds. *Trans. Am. Math. Soc.*, 141:465–504, 1969.

[Gra71] A. Gray. Weak holonomy groups. *Math. Z.*, 123:290–300, 1971.

[Gra77] A. Gray. Vector cross products. *Rend. Semin. Mat., Torino*, 35:69–75, 1976–1977.

[Hir88] M. W. Hirsch. *Differential Topology*, volume 33 of *GTM*. Springer-Verlag, third edition, 1988.

[HL83] R. Harvey and H. B. Lawson, Jr. An intrinsic characterization of Kähler manifolds. *Invent. Math.*, 74:169–198, 1983.

[Joy00] D. Joyce. *Compact Manifolds with Special Holonomy.* Oxford University Press, 2000.

[Kat75] M. Kato. Topology of Hopf surfaces. *J. Math. Soc. Japan*, 27(2):223–238, 1975.

[Ker56] M. Kervaire. Courbure intégrale généralisée et homotopie. *Math. Ann.*, 131:219–252, 1956.

[Kod64] K. Kodaira. On the structure of compact complex analytic surfaces, I. *American J. Math.*, 86:751–798, 1964.

[Kod66] K. Kodaira. On the structure of compact complex analytic surfaces, II. *American J. Math.*, 88:682–721, 1966.

[KS00] P. Kobak and A. Swann. *The HyperKähler Potential for an Exceptional Next-to-Minimal Orbit.* http://www.imada.sdu.dk/ swann/g2/index.html, January 2000. 53 pages.

[Lam99] A. Lamari. Kähler currents on compact surfaces. *Ann. Inst. Fourier*, 49:263–285, 1999.

[Mal98] D. Mall. On holomorphic and transversely holomorphic foliations on Hopf surfaces. *J. reine angew. Math.*, 501:41–69, 1998.

[Mar81a] S. Marchiafava. Alcune osservazioni riguardanti i gruppi di Lie G_2 e $Spin(7)$, candidati a gruppi di olonomia. *Ann. Mat. Pura Appl., IV. Ser.*, 129:247–264, 1981.

[Mar81b] S. Marchiafava. Characterization of Riemannian Manifolds with Weak Holonomy Group G_2 (Following A. Gray). *Math. Z.*, 178:157–162, 1981.

[Mol88] P. Molino. *Riemannian Foliations*, volume 73 of *Progress in Math.* Birkhäuser, Basel, 1988.

[MS74] J. W. Milnor and J. D. Stasheff. *Characteristic Classes.* University Press, Princeton, 1974.

[Mur89] S. Murakami. Exceptional simple Lie groups and related topics in recent differential geometry. In *Differential geometry and topology, Proc. Spec. Year, Tianjin/PR China 1986-87*, volume 1369 of *Lect. Notes Math.*, pages 183–221, 1989.

[Mur92] S. Murakami. Exceptional Simple Lie Group F_4 and Spin Representations. In *Algebra, Proc. Int. Conf. Memory A. I. Mal'cev, Novosibirsk/USSR 1989*, volume 131 of *Contemp. Math.*, pages 269–271, 1992.

[Ped99] H. Pedersen. Hypercomplex geometry. In *Proc. Second Meeting on Quaternionic Structures on Mathematics and Physics*, Rome, September 1999. Electronic Library EMS.

[Pic90] P. Piccinni. On some classes of 2-dimensional Hermitian manifolds. *J. Math. Pures et Appl.*, 69:227–237, 1990.

[PPS93] H. Pedersen, Y. S. Poon, and A. Swann. The Einstein-Weyl equations in complex and quaternionic geometry. *Differential Geometry and its Applications*, 3(4):309–322, 1993.

[Sal89] S. Salamon. *Riemannian geometry and holonomy groups.* Longman Scientific & Technical, Essex CM20 2JE, England, 1989.

[Sal00] S. Salamon. *Slides of the conference held in Bilbao at Congress in memory of Alfred Gray.* http://www.maths.ox.ac.uk/ salamon/gray.ps, September 2000. 17 pages.

[Sta67] E. B. Staples. A short and elementary proof that a product of spheres is parallelizable if one of them is odd. *Proc. Am. Math. Soc.*, 18:570–571, 1967.

[Tri82] F. Tricerri. Some examples of locally conformal Kähler manifolds. *Rend. Semin. Mat., Torino*, 40:81–92, 1982.

[Vai76] I. Vaisman. On locally conformal almost Kähler manifolds. *Israel J. Math.*, 24:338–351, 1976.

[Vai79] I. Vaisman. Locally conformal Kähler manifolds with parallel Lee form. *Rend. Mat. Roma*, 12:263–284, 1979.

[Vai82] I. Vaisman. Generalized Hopf manifolds. *Geometriae Dedicata*, 13:231–255, 1982.

[Vai87] I. Vaisman. Non-Kähler metrics on geometric complex surfaces. *Rend. Semin. Mat., Torino*, 45(3):117–123, 1987.