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Hopf surfaces: locally conformal Kähler metrics and foliations

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Abstract. In this paper we describe a family of locally conformal Kähler metrics on class 1 Hopf surfaces $H_{\alpha,\beta}$ containing some recent metrics constructed in [GO98]. We study some canonical foliations associated to these metrics, in particular a 2-dimensional foliation $\mathcal{E}_{\alpha,\beta}$ that is shown to be independent of the metric. We prove with elementary tools that $\mathcal{E}_{\alpha,\beta}$ has compact leaves if and only if $\alpha^m = \beta^n$ for some integers m and n, namely in the elliptic case. In this case we prove that the leaves of $\mathcal{E}_{\alpha,\beta}$ give explicitly the elliptic fibration of $H_{\alpha,\beta}$, and we describe the natural orbifold structure on the leaf space.

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1. Introduction

The study of metrics on complex surfaces arose in the sixties out of Kodaira's classification of minimal complex surfaces in seven classes I_0,...,VII_0 (see [Kod64, Kod66, Kod68a, Kod68b]): which complex surfaces, with respect to this classification, admit a Kähler metric? The surfaces in classes I_0, III_0 and V_0 are easily seen to be Kähler, while the surfaces in classes VI_0 and VII_0 are not, due to topological obstructions. (O. Biquard showed that elliptic surfaces in classes VI_0 and VII_0 are also non-symplectic, see [Biq98].) The surfaces in class IV_0 are Kähler as shown by Miyaoka and in 1983, when Todorov and Siu proved that every surface of class II_0 is Kähler, the question was at last settled: only the surfaces of classes VI_0 and VII_0 are not Kähler (see for instance [BPV84]). This theorem has been recently proved by direct methods independently by N. Buchdahl and A. Lamari in [Buc99] and [Lam99] respectively.

Is there a weakened version of the Kähler hypothesis that we can hope to prove for surfaces in classes VI_0 and VII_0? The notion of locally conformal Kähler manifold was introduced in this context by I. Vaisman in [Vai76]; in [Vai79] he thoroughly studied locally conformal Kähler metrics with parallel Lee form; subsequently F. Tricerri in [Tri82] gave an example of a locally conformal Kähler metric with non-parallel Lee form. Further properties of locally conformal Kähler manifolds were proved by B. Y. Chen and P. Piccinni in [CP85]; in particular, the existence on them of some canonical foliations. Until 1998, there were very few examples of locally conformal Kähler manifolds, namely *some* Hopf surfaces,

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some Inoue surfaces and manifolds of type $(G/\Lambda) \times S^1$ where G is a nilpotent or solvable group. Recent results were obtained by P. Gauduchon and L. Ornea in the paper [GO98], where they showed that *every* primary Hopf surface is locally conformal Kähler by finding a (family of) locally conformal Kähler metric (with parallel Lee form) on those of class 1 and then deforming it; and by F. A. Belgun in [Bel00] where he classified the locally conformal Kähler surfaces with parallel Lee form and showed that also secondary Hopf surfaces are locally conformal Kähler.

In this paper we show that the metrics written in [GO98] for Hopf surfaces of class 1 belong to a family of locally conformal Kähler metrics that are parametrized by the smooth positive functions defined on the circle S^1. Among all these locally conformal Kähler metrics, the only ones with parallel Lee form are those of [GO98]. Then we explicitly study the canonical foliations associated to the metrics of this family. Class 1 Hopf surfaces $H_{\alpha,\beta}$ are elliptic if and only if an algebraic condition is satisfied, that is $\alpha^m = \beta^n$ for some integers m and n (see [Kod64, 2]): we find that, whenever this condition is satisfied, one of the canonical foliations gives exactly the elliptic fibration. Finally, we look at the regularity of this foliation and at the natural orbifold structure on the leaf space: we find that this foliation is quasi-regular if and only if $\alpha^m = \beta^n$, and that in this case the leaf space is an orbifold with two conical points of order m and n.

In section 2 we give some basic preliminaries and we state more precisely the theorem of [GO98] we used (see theorem 2). Firstly, we develop some tools we shall need, namely a diffeomorphism between $H_{\alpha,\beta}$ and $S^1 \times S^3$ (see formula (5)), a parallelization on $S^1 \times S^3$ (see formulas (6)) and the explicit description of the induced complex structure on $S^1 \times S^3$ via the diffeomorphism (see formulas (8)). This is the point of view we adopt to study $H_{\alpha,\beta}$. Secondly, we look at the simplest case, that is $|\alpha| = |\beta|$: we deform the classical locally conformal Kähler metric $(dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2)/(z_1\bar{z}_1 + z_2\bar{z}_2)$ on $\mathbb{C}^2 \setminus 0$ by means of a positive function $k\colon S^1 \to \mathbb{R}$ obtaining a family of locally conformal Kähler metrics, with parallel Lee form if and only if k is constant (see theorem 3). Thirdly, we apply our method to the metric of [GO98] to obtain a family of locally conformal Kähler metrics on $H_{\alpha,\beta}$ (see theorem 6) parametrized by the real positive functions on S^1. Then we verify that the only metrics with parallel Lee form in this family are the ones of [GO98] (see theorem 9).

In section 4 we begin by recalling the definitions of three canonical distributions on a locally conformal Kähler manifold, as given in [CP85], then we study each of them in detail, and we define $\mathcal{E}_{\alpha,\beta}$. We remark that they are all integrable and explicitly find the leaves, then we study their properties obtaining necessary and sufficient conditions for compactness (see theorems 13, 14 and 15).

In section 5 we recall the definition of elliptic surface, as given in [Kod64]. Then we show that when the foliation $\mathcal{E}_{\alpha,\beta}$ has all compact leaves -and this happens, according to theorem 15, if and only if $\alpha^m = \beta^n$ for some integers m and n-, we can identify the leaf space with $\mathbb{P}^1\mathbb{C}$ in such a way that the canonical projection is a holomorphic map (see theorem 19). This means that, whenever $H_{\alpha,\beta}$ is elliptic, the ellipticity is explicitly given by the foliation $\mathcal{E}_{\alpha,\beta}$.

In the same section we recall the definitions of regularity and quasi-regularity, and we show that $\mathcal{E}_{\alpha,\beta}$ is quasi-regular if and only if $H_{\alpha,\beta}$ is elliptic, and it is regular if and only if $\alpha = \beta$ (see theorem 20). The quasi-regularity gives the leaf space a natural structure of orbifold with two conical points.

2. Preliminaries

A Hermitian manifold (M^{2n}, J, g) is said to be *locally conformal Kähler*, briefly *l.c.K.*, if there exist an open covering $\{U_i\}_{i\in I}$ of M and a family $\{f_i\}_{i\in I}$ of smooth functions $f_i\colon U_i \to \mathbb{R}$ such that the metrics g_i on U_i given by

$$g_i := \exp(-f_i)\, g_{|_{U_i}}$$

are Kählerian metrics. The following relation holds on U_i between the fundamental forms Ω_i and Ω respectively of g_i and g:

$$\Omega_i = \exp(-f_i)\, \Omega_{|_{U_i}},$$

so the *Lee form* ω locally defined by

$$\omega_{|_{U_i}} := df_i \tag{1}$$

is in fact global, and satisfies $d\Omega = \omega \wedge \Omega$. The manifold (M, J, g) is then l.c.K. if and only if there exists a global closed 1-form ω such that

$$d\Omega = \omega \wedge \Omega$$

(see for instance the recent book [DO98]).

As Kodaira defined in [Kod66, 10], a *Hopf surface* is a complex compact surface H whose universal covering is $\mathbb{C}^2 \setminus 0$. If $\pi_1(H) \simeq \mathbb{Z}$ then we say that H is a *primary* Hopf surface. Kodaira showed that every primary Hopf surface can be obtained as

$$\frac{\mathbb{C}^2 \setminus 0}{< f >}, \qquad f(z_1, z_2) := (\alpha z_1 + \lambda z_2^m, \beta z_2),$$

where m is a positive integer and α, β and λ are complex numbers such that

$$(\alpha - \beta^m)\lambda = 0 \qquad \text{and} \qquad |\alpha| \geq |\beta| > 1.$$

These relations will be assumed throughout this paper.

We write $H_{\alpha,\beta,\lambda,m}$ for the generic primary Hopf surface. If $\lambda \neq 0$ we have

$$f(z_1, z_2) = (\beta^m z_1 + \lambda z_2^m, \beta z_2)$$

and the surface $H_{\beta,\lambda,m} := H_{\beta^m,\beta,\lambda,m}$ is called *of class* 0, while if $\lambda = 0$ we have

$$f(z_1, z_2) = (\alpha z_1, \beta z_2)$$

and the surface $H_{\alpha,\beta} := H_{\alpha,\beta,0,m}$ is called *of class* 1 (this terminology refers to the notion of *Kähler rank* as given in [HL83, § 9]).

A globally conformal Kähler metric on $\mathbb{C}^2 \setminus 0$ (that is, of the form $\exp(-f)\, g$ where g is Kähler and $f\colon \mathbb{C}^2 \setminus 0 \to \mathbb{R}$), which is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1 + \lambda z_2^m, \beta z_2)$, defines a l.c.K. metric on $H_{\alpha,\beta,\lambda,m}$: this is the case for the metric

$$\frac{dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2}{z_1\bar{z}_1 + z_2\bar{z}_2} \tag{2}$$

which is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$ (and so defines a l.c.K. metric on $H_{\alpha,\beta}$) whenever $|\alpha| = |\beta|$. The Lee form of this metric is parallel for the Levi-Civita connection (see [Vai79]).

In [Vai82], I. Vaisman called *generalized Hopf (g.H.)* manifolds those l.c.K. manifolds (M, J, g) with a parallel Lee form. Recently, since F. A. Belgun proved that primary Hopf surfaces of class 0 do not admit any generalized Hopf structure (see [Bel00]), some authors (see for instance [DO98, GO98]) decided to use the term *Vaisman manifold* instead. We shall adher to this terminology and thus give the following

Definition 1. *A* Vaisman manifold *is a l.c.K. manifold* (M, J, g) *with parallel Lee form with respect to the Levi-Civita connection of* g.

Define the operator d^c by $d^c(f)(X) := -\, df(J(X))$ for $f \in C^\infty$ and $X \in \mathfrak{X}(M)$. Let $\mathcal{U}$ be an open set of a complex manifold (M, J). We call a *potential* on $\mathcal{U}$ a map $f\colon \mathcal{U} \to \mathbb{R}$ such that the 2-form on $\mathcal{U}$ of type $(1,1)$ given by $(dd^c f)/2$ is positive: namely, such that the bilinear map g on $\mathfrak{X}(\mathcal{U}) \times \mathfrak{X}(\mathcal{U})$ given by

$$g(X, Y) := -\frac{dd^c f}{2}(J(X), Y)$$

is a (Kählerian) metric on $\mathcal{U}$.

Take the map $\Phi_{\alpha,\beta}\colon \mathbb{C}^2 - 0 \to \mathbb{R}$ given by

$$\Phi_{\alpha,\beta}(z_1, z_2) := \exp\left(\frac{(\log|\alpha| + \log|\beta|)\theta}{2\pi}\right) \tag{3}$$

where θ is given by

$$\frac{|z_1|^2}{\exp\left(\dfrac{\theta \log|\alpha|}{\pi}\right)} + \frac{|z_2|^2}{\exp\left(\dfrac{\theta \log|\beta|}{\pi}\right)} = 1. \tag{4}$$

In [GO98] the following theorem is proved:

Theorem 2 ([GO98, Proposition 1 and Corollary 1]). *The map* $\Phi_{\alpha,\beta}$ *given by (3) is a potential on* $\mathbb{C}^2 \setminus 0$ *with respect to the standard complex structure* J. *The metric* $g_{\alpha,\beta}$ *on* $\mathbb{C}^2 \setminus 0$ *given by*

$$g_{\alpha,\beta}(X, Y) := -\frac{dd^c \Phi_{\alpha,\beta}}{2\Phi_{\alpha,\beta}}(J(X), Y), \qquad X, Y \in \mathfrak{X}(\mathbb{C}^2 \setminus 0)$$

is invariant for the map $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$. *Moreover, the induced metric on* $H_{\alpha,\beta}$ *is Vaisman for every* α *and* β.

3. Metrics on $S^1 \times S^3$

We define the 3-sphere by

$$S^3 := \left\{ (\xi_1, \xi_2) \in \mathbb{C}^2 : |\xi_1|^2 + |\xi_2|^2 = 1 \right\}$$

and S^1 by the quotient of $\mathbb{R}$ by the map $\theta \mapsto \theta + 2\pi$. The manifolds $S^1 \times S^3$ and $H_{\alpha,\beta}$ are diffeomorphic (see [Kat75, theorem 9]) by means of the map $F_{\alpha,\beta}$ given by F in the diagram

$$\begin{array}{ccc} \mathbb{R} \times S^3 & \xrightarrow{F} & \mathbb{C}^2 \setminus 0 \\ h \big\downarrow & & \big\downarrow f \\ \mathbb{R} \times S^3 & \xrightarrow{F} & \mathbb{C}^2 \setminus 0 \end{array}$$

where

$$\begin{aligned} h(\theta, \xi_1, \xi_2) &:= (\theta + 2\pi, \xi_1, \xi_2), \\ f(\xi_1, \xi_2) &:= (\alpha \xi_1, \beta \xi_2), \\ F(\theta, \xi_1, \xi_2) &:= (\exp\left(\frac{\theta \log \alpha}{2\pi}\right) \xi_1, \exp\left(\frac{\theta \log \beta}{2\pi}\right) \xi_2). \end{aligned}$$

Here and henceforth, we fix the arguments of α, β and α/β in such a way that

$$\log(\alpha/\beta) = \log \alpha - \log \beta.$$

If $[z_1, z_2]$ is the element in $H_{\alpha,\beta}$ corresponding to $(z_1, z_2) \in \mathbb{C}^2 \setminus 0$, we have

$$F_{\alpha,\beta}(\theta, \xi_1, \xi_2) := [\exp\left(\frac{\theta \log \alpha}{2\pi}\right) \xi_1, \exp\left(\frac{\theta \log \beta}{2\pi}\right) \xi_2] \tag{5}$$

and the inverse is

$$F^{-1}_{\alpha,\beta}([z_1, z_2]) = (\theta, \exp\left(-\frac{\theta \log \alpha}{2\pi}\right) z_1, \exp\left(-\frac{\theta \log \beta}{2\pi}\right) z_2)$$

where θ is given by (4).

Via this diffeomorphism $H_{\alpha,\beta}$ induces a complex structure on $S^1 \times S^3$, which we denote by $J_{\alpha,\beta}$; in particular, all complex structures of the form $J_{\alpha,\alpha}$ were studied and classified by P. Gauduchon in [Gau81, propositions 2 and 3, pages 138 and 140], by means of the parallelizability of $S^1 \times S^3$.

Let θ be the point in $S^1 \subset \mathbb{C}$ given by the embedding $\theta \mapsto \exp(i\theta)$, and let (ξ_1, ξ_2) be a point in $S^3 \subset \mathbb{C}^2$. The parallelization $\mathcal{E} := (e_1, e_2, e_3, e_4)$ on $S^1 \times S^3$ and its dual $\mathcal{E}^* = (e^1, e^2, e^3, e^4)$ are given by:

$$\begin{aligned} e_1(\theta, \xi_1, \xi_2) &:= i \exp(i\theta) \in T_\theta(S^1), \\ e_2(\theta, \xi_1, \xi_2) &:= (i\xi_1, i\xi_2) \in T_{(\xi_1,\xi_2)}(S^3), \\ e_3(\theta, \xi_1, \xi_2) &:= (-\bar{\xi}_2, \bar{\xi}_1) \in T_{(\xi_1,\xi_2)}(S^3), \\ e_4(\theta, \xi_1, \xi_2) &:= (-i\bar{\xi}_2, i\bar{\xi}_1) \in T_{(\xi_1,\xi_2)}(S^3). \end{aligned} \tag{6}$$

The differential structure of this frame is given by the following formulas:

$$de^1 = 0, \qquad de^2 = 2e^3 \wedge e^4, \qquad de^3 = -2e^2 \wedge e^4, \qquad de^4 = 2e^2 \wedge e^3,$$

and the non-zero brackets are

$$[e_2, e_3] = -2e_4, \qquad [e_2, e_4] = 2e_3, \qquad [e_3, e_4] = -2e_2.$$

One finds that

$$\begin{aligned} dF = &\left(\frac{\log\alpha}{2\pi} \exp\left(\frac{\theta\log\alpha}{2\pi}\right) \xi_1 d\theta + \exp\left(\frac{\theta\log\alpha}{2\pi}\right) d\xi_1 \right) \otimes \partial_{z_1} \\ &+ \left(\frac{\log\beta}{2\pi} \exp\left(\frac{\theta\log\beta}{2\pi}\right) \xi_2 d\theta + \exp\left(\frac{\theta\log\beta}{2\pi}\right) d\xi_2 \right) \otimes \partial_{z_2}. \end{aligned} \tag{7}$$

Letting G be the complex function on $S^1 \times S^3$ given by (see [GO98, formula 45])

$$\begin{aligned} G(\theta, \xi_1, \xi_2) :=& |\xi_1|^2 \log\alpha + |\xi_2|^2 \log\beta \\ =& |\xi_1|^2 \log|\alpha| + |\xi_2|^2 \log|\beta| + i(|\xi_1|^2 \arg\alpha + |\xi_2|^2 \arg\beta), \end{aligned}$$

the complex structure $J_{\alpha,\beta}$ with respect to the basis $\mathcal{E}$ is given by

$$\begin{aligned} J_{\alpha,\beta}(e_1) &= -\frac{\Im m\, G}{\Re e\, G} e_1 \\ &+ \frac{|G|^2}{2\pi\, \Re e\, G} e_2 - \frac{\Re e\left(i\xi_1\xi_2\overline{G}\log(\alpha/\beta)\right)}{2\pi\, \Re e\, G} e_3 - \frac{\Im m\left(i\xi_1\xi_2\overline{G}\log(\alpha/\beta)\right)}{2\pi\, \Re e\, G} e_4, \\ J_{\alpha,\beta}(e_2) &= -\frac{2\pi}{\Re e\, G} e_1 \\ &+ \frac{\Im m\, G}{\Re e\, G} e_2 - \frac{\Re e\left(\xi_1\xi_2\log(\alpha/\beta)\right)}{\Re e\, G} e_3 - \frac{\Im m\left(\xi_1\xi_2\log(\alpha/\beta)\right)}{\Re e\, G} e_4, \\ J_{\alpha,\beta}(e_3) &= e_4, \\ J_{\alpha,\beta}(e_4) &= -e_3, \end{aligned} \tag{8}$$

(see [GO98, formulas 49], where the notations T, Z, E, iE, z_1, z_2 and F are used instead respectively of $2\pi e_1$, e_2, $-e_3$, $-e_4$, ξ_1, ξ_2 and G).

Since $S^1 \times S^3$ is parallelizable, the choice of two $J_{\alpha,\beta}$-independent vector fields gives an isomorphism between $T(S^1 \times S^3)$ and $S^1 \times S^3 \times \mathbb{C}^2$, such that a $J_{\alpha,\beta}$-Hermitian metric on $S^1 \times S^3$ is given by a Hermitian 2×2 matrix. We choose e_2 and e_3 as such $J_{\alpha,\beta}$-independent vector fields.

We now examine the case $|\alpha| = |\beta|$.

Let k be any positive real function on S^1. By a direct computation we obtain:

Theorem 3. *Let $|\alpha| = |\beta|$. The Hermitian matrix*

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

gives a family of l.c.K. metrics on $H_{\alpha,\beta}$ which is parametrized by the positive functions k on S^1. In this family the Vaisman ones are given exactly by constant functions k.

Remark 4. $k = 1$ gives just the classical metric (2).

Remark 5. A family $\{g_t\}_{t>-1}$ of l.c.K. metrics (in the case $|\alpha| = |\beta|$) can be found in [Vai82, formula 2.13]. The metrics of this family coincide (up to coefficients) with the metrics of our family with k constant, where $k = t + 1$. The claim, on page 240 of [Vai82], that only g_0 has parallel Lee form is incorrect. The author uses the Weyl connection with the hypothesis $\omega_t(B_t) = |\omega_t|^2 = 1$, before proving that ω_t is parallel: in such a way, what is in fact proved is that g_0 is the only metric with $\nabla\omega = 0$ and $|\omega_t| = 1$. Actually, by using (2.14) and (2.17), one can check that $|\omega_t| = 1 + t$, hence the same computation proves that all the g_t have parallel Lee form. The author acknowledges a useful conversation and an exchange of e-mail messages with I. Vaisman.

We now go to general case, that is, no restrictions on α and β.

Let $l\colon \mathcal{U} \to \mathbb{R}$ be a real function defined on an open set $\mathcal{U}$ of $\mathbb{R}$, and

$$\Phi_l\colon \frac{\mathcal{U}}{2\pi\mathbb{Z}} \times S^3 \to \mathbb{R}^+$$

the real positive function given by

$$\Phi_l(\theta, \xi_1, \xi_2) := \exp\left(l(\theta)\right). \tag{9}$$

The local 2-form $\Omega := (1/2) dd^c \Phi_l$ gives the Hermitian bilinear form

$$2\Phi_l \pi l' A \tag{10}$$

where

$$A := \begin{pmatrix} \dfrac{\pi}{\Re e^2\, G} \dfrac{l'^2 + l''}{l'} + \dfrac{|\xi_1|^2|\xi_2|^2 \log^2(|\alpha|/|\beta|)}{\Re e^3\, G} & \dfrac{\imath\xi_1\xi_2 \log(|\alpha|/|\beta|)}{\Re e^2\, G} \\ \dfrac{\overline{\imath\xi_1\xi_2} \log(|\alpha|/|\beta|)}{\Re e^2\, G} & \dfrac{1}{\Re e\, G} \end{pmatrix} \tag{11}$$

The condition that Ω be positive translates then into l' and $l'^2 + l''$ both positive. So we have a local generalization of the proposition 1 in [GO98], that is we can take the local function $\exp(l)$ as a potential, where l is increasing and $l'^2 + l'' > 0$ on $\mathcal{U}$.

In the matrix A the dependance on θ is only given by $(l'^2 + l'')/l'$. Consider a family $\{l_\mathcal{U}\}_{\mathcal{U}\in U}$ of local functions, where U is an open covering of $\mathbb{R}$, all satisfying $l' > 0$ and $l'^2 + l'' > 0$ and such that the quantities $(l'^2 + l'')/l'$ paste to a well defined function h on S^1. The matrix (11) then gives a global Hermitian l.c.K. metric on $(S^1 \times S^3, J_{\alpha,\beta})$. In fact such a family can be found, as we show in the following

Theorem 6. *Given any real positive function h with period 2π on $\mathbb{R}$, the metric $g^h_{\alpha,\beta}$ given in the complex basis (e_2, e_3) of $T(S^1 \times S^3)$ by the Hermitian matrix*

$$\begin{pmatrix} \dfrac{\pi h}{\Re e^2\, G} + \dfrac{|\xi_1|^2|\xi_2|^2 \log^2(|\alpha|/|\beta|)}{\Re e^3\, G} & \dfrac{\imath\xi_1\xi_2 \log(|\alpha|/|\beta|)}{\Re e^2\, G} \\ \dfrac{\overline{\imath\xi_1\xi_2} \log(|\alpha|/|\beta|)}{\Re e^2\, G} & \dfrac{1}{\Re e\, G} \end{pmatrix}$$

is (well defined and) l.c.K. on $(S^1 \times S^3, J_{\alpha,\beta})$.

Proof: For fixed h, the Cauchy problem

$$\begin{cases} \dfrac{l'^2 + l''}{l'} = h \\ l'(\theta_0) > 0 \end{cases} \tag{12}$$

satisfies the local existence theorem for any $\theta_0 \in \mathbb{R}$. This means we can find an open covering U of $\mathbb{R}$ and functions $l_{\mathcal{U}} \colon \mathcal{U} \to \mathbb{R}$ which satisfy the equation. Moreover U can be chosen so that $l_{\mathcal{U}}$ is increasing for any $\mathcal{U} \in U$; finally, note that, since h is positive, so is ${l'_{\mathcal{U}}}^2 + l''_{\mathcal{U}}$, and this gives the required family. □

The previous theorem extends the corollary 1 of [GO98].

Remark 7. If $h \colon S^1 \to \mathbb{R}^+$ is constant, a (global) solution of the Cauchy problem (12) is given by $l(\theta) = h\theta$, and the potential of the corresponding $g^h_{\alpha,\beta}$ is given by (see (9)) $\exp(h\theta)$. In [GO98] the potential is $\exp(l(\log|\alpha| + \log|\beta|)\theta/(2\pi))$, where l is any positive real number (see [GO98, after remark 3]): thus, for h constant, the constant l of [GO98] is given by

$$l = \frac{2\pi h}{\log|\alpha| + \log|\beta|}.$$

Remark 8. If $|\alpha| = |\beta|$, we get $\mathfrak{Re}\, G = \log|\alpha|$, $\log(|\alpha|/|\beta|) = 0$ and

$$g^h_{\alpha,\beta} = \frac{1}{\log|\alpha|} \begin{pmatrix} \dfrac{\pi h}{\log|\alpha|} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus in the case $|\alpha| = |\beta|$ the family given by the theorem 6 coincide up to a constant with the family given by theorem 3, where $k = \pi h / \log|\alpha|$.

Using the “six terms formula” ([KN63, proposition 2.3]), we obtain:

Theorem 9. *The metric $g^h_{\alpha,\beta}$ of theorem 6 is Vaisman if and only if h is constant.*

4. Foliations on $S^1 \times S^3$

On any l.c.K. manifold (M, J, g) with a nowhere-vanishing Lee form ω, the following canonical distributions are given (see [CP85] and [Pic90]):

1. the flow of the Lee vector field B, dual via g of ω: that is, defined by

$$g(B, X) := \omega(X) \qquad \text{for every } X \in \mathfrak{X}(M);$$

2. the flow of the vector field $J(B)$;
3. the 2-dimensional distribution spanned by B and $J(B)$: whenever the Lee form is parallel, this distribution is integrable (see e.g. [CP85, theorem 4.3], but this condition is not necessary, as we shall see), and moreover, it defines a Riemannian foliation (see [DO98, Theorem 5.1]).

By (1) and (10) we obtain the Lee form $\omega^h_{\alpha,\beta} = -he^1$ of the metric $g^h_{\alpha,\beta}$ defined in theorem 6, and a direct computation shows that the Lee vector field and its dual do not depend on h:

$$\begin{aligned} B_{\alpha,\beta} &= -4\pi e_1 \\ &\quad + 2\,\Im m\, Ge_2 + 2\,\Im m(\xi_1\xi_2)\arg(\alpha/\beta)e_3 - 2\,\Re e(\xi_1\xi_2)\arg(\alpha/\beta)e_4, \\ J_{\alpha,\beta}(B_{\alpha,\beta}) &= -2\,\Re e\, Ge_2 - 2\,\Im m(\xi_1\xi_2)\log|\alpha/\beta|e_3 + 2\,\Re e(\xi_1\xi_2)\log|\alpha/\beta|e_4. \end{aligned} \tag{13}$$

Thus the following holds:

Theorem 10. *The distribution spanned by $B_{\alpha,\beta}$ and $J_{\alpha,\beta}(B_{\alpha,\beta})$ is integrable.*

Proof: It is well known (see [CP85]) that if the Lee form is parallel then the distribution is integrable: since $B_{\alpha,\beta}$ and $J_{\alpha,\beta}(B_{\alpha,\beta})$ are independent of the function h, we get a unique distribution on $S^1 \times S^3$. This coincides with the one induced by the Vaisman metric given by constant h, and is thus integrable. □

Definition 11. *We call $\mathcal{E}_{\alpha,\beta}$ the unique foliation given by theorem 10.*

We now introduce some notation.

Let us consider the torus $S^1 \times S^1$ with coordinates (t_1, t_2). The following is well known:

Lemma 12. *The curve in $S^1 \times S^1$ given by the linear functions*

$$t_1(t) = \gamma_1 + \delta_1 t \mod 2\pi, \qquad t_2(t) = \gamma_2 + \delta_2 t \mod 2\pi \tag{14}$$

is compact if $\delta_2/\delta_1 \in \mathbb{Q}$, dense in $S^1 \times S^1$ otherwise.

In the compact case of the previous lemma, the curve (14) is called a *toral knot of type* δ_2/δ_1.

For all (Ξ_1, Ξ_2) in S^3 there is a submanifold $T(\Xi_1, \Xi_2)$ in S^3 defined as the product of two circles of radius respectively $|\Xi_1|$ e $|\Xi_2|$:

$$T(\Xi_1, \Xi_2) := S^1_{|\Xi_1|} \times S^1_{|\Xi_2|} \subset \mathbb{C} \times \mathbb{C}$$

and we denote by t_1 and t_2 the coordinates on the torus $T(\Xi_1, \Xi_2)$ given by

$$\xi_1(t_1) = \Xi_1 \exp(it_1), \qquad \xi_2(t_2) = \Xi_2 \exp(it_2). \tag{15}$$

Thus, a curve in the 3-torus $S^1 \times T(\Xi_1, \Xi_2)$ is given by

$$\theta = \theta(t) \mod 2\pi, \qquad t_1 = t_1(t) \mod 2\pi, \qquad t_2 = t_2(t) \mod 2\pi. \tag{16}$$

We can visualize $S^1 \times T(\Xi_1, \Xi_2)$ as a cube with identifications.

Theorem 13. *The 1-dimensional foliation on $S^1 \times S^3$ spanned by $B_{\alpha,\beta}$ has the following properties:*

1. *for every* α *and* β *the leaf through* $(\Theta, \Xi_1, 0)$ *(respectively* $(\Theta, 0, \Xi_2)$*) is a subset of* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*). This leaf is*
   – *compact if* $\arg\alpha \in \mathbb{Q}\pi$ *(respectively* $\arg\beta \in \mathbb{Q}\pi$*);*
   – *dense in* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively in* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*) otherwise;*
2. *for every* α *and* β *the leaf through* (Θ, Ξ_1, Ξ_2)*, where* $\Xi_1\Xi_2 \neq 0$*, is a subset of* $S^1 \times T(\Xi_1, \Xi_2)$*, where* $T(\Xi_1, \Xi_2)$ *is the torus in the factor* S^3 *of* $S^1 \times S^3$ *given by (15). This leaf is*
   – *compact if any two of the relations*

$$\arg\alpha \in \mathbb{Q}\pi; \qquad \arg\beta \in \mathbb{Q}\pi; \qquad \arg\alpha / \arg\beta \in \mathbb{Q}, \tag{17}$$

   hold;
   – *non compact otherwise;*

   if the leaf is not compact, then its projection on $T(\Xi_1, \Xi_2)$ *is*
   – *a toral knot of type* $\arg\alpha / \arg\beta$ *if this ratio is rational;*
   – *dense in* $T(\Xi_1, \Xi_2)$ *otherwise.*

Proof: Fix (Θ, Ξ_1, Ξ_2) in $S^1 \times S^3$, and let $[Z_1, Z_2] := F_{\alpha,\beta}(\Theta, \Xi_1, \Xi_2)$. Using formula (7), the Lee vector field given by (13) becomes (see also [GO98, formula (23)])

$$B_{\alpha,\beta} = -2(z_1 \log|\alpha|, z_2 \log|\beta|). \tag{18}$$

The flow of $B_{\alpha,\beta}$ carries the point $[Z_1, Z_2]$ to

$$[z_1(t), z_2(t)] = [Z_1 \exp(-2t \log|\alpha|), Z_2 \exp(-2t \log|\beta|)] \qquad t \in \mathbb{R}. \tag{19}$$

A computation gives the following equation for $\theta(t)$:

$$|Z_1|^2 \exp\left(-\log|\alpha|(4t + \frac{\theta(t)}{\pi})\right) + |Z_2|^2 \exp\left(-\log|\beta|(4t + \frac{\theta(t)}{\pi})\right) = 1;$$

and finally the equations for the leaf:

$$\theta(t) = \Theta – 4\pi t \mod 2\pi, \xi_1(t) = \Xi_1 \exp(2it \arg\alpha), \xi_2(t) = \Xi_2 \exp(2it \arg\beta). \tag{20}$$

We distinguish two kinds of points in $S^1 \times S^3$. If $\Xi_1\Xi_2 = 0$, say $\Xi_2 = 0$, the leaf given by (20) is contained in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$. According to lemma 12, if $\arg\alpha$ is a rational multiple of π, the leaf is compact; otherwise it is dense in $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$. If $\Xi_1\Xi_2 \neq 0$, from equations (20) we obtain that $\xi_1(t)$ and $\xi_2(t)$ have a constant positive length for every t, so the leaf is contained in the real 3-torus $S^1 \times T(\Xi_1, \Xi_2)$ defined at page 9. According to (16), the equations (20) can be written as

$$\theta(t) = \Theta – 4\pi t \mod 2\pi, t_1(t) = 2t \arg\alpha \mod 2\pi, t_2(t) = 2t \arg\beta \mod 2\pi. \tag{21}$$

In order to study the compactness of the leaves we remark that:

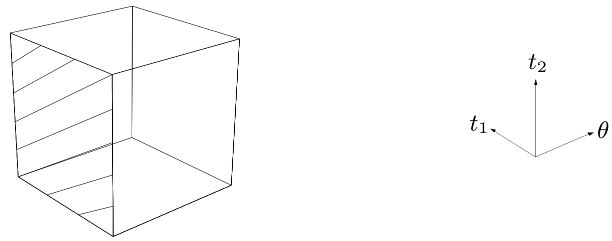


Fig. 1. projection of the leaf of the foliation spanned by $B_{\alpha,\beta}$ to $T(\Xi_1, \Xi_2)$: case $\arg\alpha/\arg\beta \in \mathbb{Q}$.

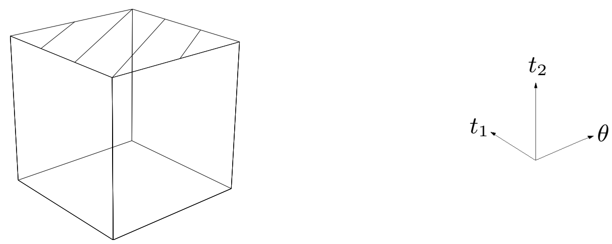


Fig. 2. projection of the leaf of the foliation spanned by $B_{\alpha,\beta}$ to $\{t_2 = 0\}$: case $(\arg\alpha)/\pi \in \mathbb{Q}$.

1. the leaf projected on $T(\Xi_1, \Xi_2)$ is given by
$$t_1(t) = 2t\arg\alpha \mod 2\pi, \qquad t_2(t) = 2t\arg\beta \mod 2\pi, \tag{22}$$
and by lemma 12 this is a compact set if the ratio of $\arg\alpha$ to $\arg\beta$ is rational; otherwise it is dense in $T(\Xi_1, \Xi_2)$. Since the projection from $S^1 \times T(\Xi_1, \Xi_2)$ on $T(\Xi_1, \Xi_2)$ is a closed map, we can infer that if the ratio of $\arg\alpha$ to $\arg\beta$ is not rational then the leaf is not compact. If this ratio is rational, then the projected set is a toral knot of type $\arg\alpha/\arg\beta$ (see figure 1);
2. the projection of the leaf on the face $t_2 = 0$ is given by
$$\theta(t) = \Theta - 4\pi t \mod 2\pi, \qquad t_1(t) = 2t\arg\alpha \mod 2\pi,$$
and lemma 12 gives the condition $(\arg\alpha)/\pi \in \mathbb{Q}$ (see figure 2);
3. in the same way, if we consider the projection on the face $t_1 = 0$, we obtain $(\arg\beta)/\pi \in \mathbb{Q}$ (see figure 3).

We have thus obtained that the three conditions (17) are necessary for the compactness of the leaf. Let us show that they are also sufficient. If the (17) hold, we can choose coprime integers l and k such that
$$\frac{\arg\alpha}{\arg\beta} = \frac{l}{k}.$$
The equations (22) define a closed curve with period $l\pi/\arg\alpha$(=$k\pi/\arg\beta$), and the leaf is closed whenever the $\theta(t)$ given by equations (21) also has a period that is an integer multiple of $l\pi/\arg\alpha$. If we choose integers p and q such that $(\arg\alpha)/\pi = p/q$, it is straightforward to check that $pl\pi/\arg\alpha$ is a period of $\theta(t)$, and the proof is complete. □

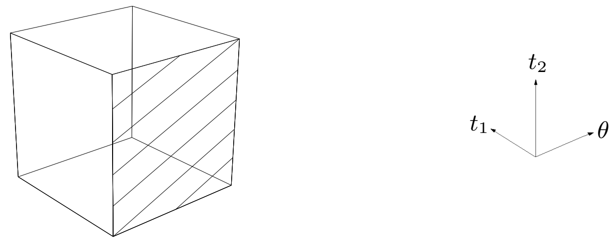


Fig. 3. projection of the leaf of the foliation spanned by $B_{\alpha,\beta}$ to $\{t_1 = 0\}$: case $(\arg\beta)/\pi \in \mathbb{Q}$.

Theorem 14. *The* 1*-dimensional foliation on* $S^1 \times S^3$ *spanned by* $J_{\alpha,\beta}(B_{\alpha,\beta})$ *has the following properties:*

1. *for every* α *and* β *the leaf through* $(\Theta, \Xi_1, 0)$ *(respectively* $(\Theta, 0, \Xi_2)$*) is* $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively* $\{\Theta\} \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*), so it is compact;*
2. *for every* α *and* β *the leaf through* (Θ, Ξ_1, Ξ_2)*, where* $\Xi_1\Xi_2 \neq 0$*, is a subset of* $\{\Theta\} \times T(\Xi_1, \Xi_2)$*, where* $T(\Xi_1, \Xi_2)$ *is the torus in the factor* S^3 *of* $S^1 \times S^3$ *given by (15). This leaf is*
   - *a toral knot of type* $\log|\alpha|/\log|\beta|$ *if this ratio is rational;*
   - *dense in* $\{\Theta\} \times T(\Xi_1, \Xi_2)$ *otherwise.*

Proof: The proof is similar to (and actually easier than) that of previous theorem. □

In the following theorem we look at the most interesting distribution, that is $\mathcal{E}_{\alpha,\beta}$. In the case of compact leaves, we also give a description of the (constant) complex structure on them.

Theorem 15. *The foliation* $\mathcal{E}_{\alpha,\beta}$ *on* $S^1 \times S^3$ *is described by the following properties:*

1. *for every* α *and* β *the leaf through* $(\Theta, \Xi_1, 0)$ *(respectively* $(\Theta, 0, \Xi_2)$*) is* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_2 = 0\}$ *(respectively* $S^1 \times \{(\xi_1, \xi_2) \in S^3 : \xi_1 = 0\}$*), and it is thus compact;*
2. *for every* α *and* β *the leaf through* (Θ, Ξ_1, Ξ_2)*, where* $\Xi_1\Xi_2 \neq 0$*, is a subset of* $S^1 \times T(\Xi_1, \Xi_2)$*, where* $T(\Xi_1, \Xi_2)$ *is the torus in the factor* S^3 *of* $S^1 \times S^3$ *given by (15). This leaf is*
   - *compact if there exist integers* m *and* n *such that* $\alpha^m = \beta^n$*: in this case the leaf is a Riemann surface* $\mathbb{C}/\Lambda$ *of genus one, where* Λ *is the lattice in* $\mathbb{C}$ *generated by the vectors* v *and* w *given by (26);*
   - *non compact otherwise, and in this case it is dense in* $S^1 \times T(\Xi_1, \Xi_2)$.

Proof: Look at the 2-dimensional real distribution as a field of 1-dimensional complex lines generated by B. In the expression (19) substitute with the real parameter t a complex parameter w, then, as for (20), we obtain

$$\begin{aligned}
\theta(w) &= \Theta - 4\pi\, \Re e\, w \mod 2\pi, \\
\xi_1(w) &= \Xi_1 \exp\left(2i \arg\alpha\, \Re e\, w\right) \exp\left(-2i \log|\alpha|\, \Im m\, w\right), \\
\xi_2(w) &= \Xi_2 \exp\left(2i \arg\beta\, \Re e\, w\right) \exp\left(-2i \log|\beta|\, \Im m\, w\right).
\end{aligned} \tag{23}$$

The simplest case $\Xi_1\Xi_2 = 0$ easily follows. So we can suppose $\Xi_1\Xi_2 \neq 0$, and in this case the leaf is a subset of $S^1 \times T(\Xi_1, \Xi_2)$, where $T(\Xi_1, \Xi_2)$ is given by (15). By means of $F_{\alpha,\beta}$, and setting $(t, s){:=}(\Re e\, w, \Im m\, w)$, the equations (23) become

$$\begin{aligned} \theta(t,s) &= \Theta - 4\pi t \mod 2\pi, \\ t_1(t,s) &= 2(\arg\alpha t - \log|\alpha| s) \mod 2\pi, \\ t_2(t,s) &= 2(\arg\beta t - \log|\beta| s) \mod 2\pi. \end{aligned} \tag{24}$$

Call N the leaf given by (24), and consider $N \cap (\{\Theta\} \times T(\Xi_1, \Xi_2))$. We observe that $\theta(t) = \Theta$ is equivalent to $t = j/2$ where j is an integer: call N_j the curve given by the equations

$$\begin{aligned} \theta(\frac{j}{2}, s) &= \Theta \mod 2\pi, \\ t_1(\frac{j}{2}, s) &= 2(\arg\alpha \frac{j}{2} - \log|\alpha| s) \mod 2\pi, \\ t_2(\frac{j}{2}, s) &= 2(\arg\beta \frac{j}{2} - \log|\beta| s) \mod 2\pi. \end{aligned}$$

Clearly $N \cap (\{\Theta\} \times T(\Xi_1, \Xi_2))$ is the union of the curves N_j for $j \in \mathbb{Z}$. By lemma 12 we know that N_j is dense in $\{\Theta\} \times T(\Xi_1, \Xi_2)$ whenever $\log|\alpha|/\log|\beta|$ is irrational: $N \cap (\{\Theta\} \times T(\Xi_1, \Xi_2))$ is then *a fortiori* dense (and properly contained) in $\{\Theta\} \times T(\Xi_1, \Xi_2)$. We can repeat this argument for all θ, so in this case N is dense in $S^1 \times T(\Xi_1, \Xi_2)$. Otherwise if $\log|\alpha|/\log|\beta|$ is rational, the intersection of N with $\{\theta\} \times T(\Xi_1, \Xi_2)$ is the union of toral knots of type $\log|\alpha|/\log|\beta|$.

Let us now consider the intersection of N with the surface given by $t_2 = 0$: after observing that $t_2 = 0$ is equivalent to $s = (t \arg\beta - j\pi)/\log|\beta|$ for j integer, let us call N_j the curve given by

$$\begin{aligned} \theta(t, \frac{t\arg\beta - j\pi}{\log|\beta|}) &= -\pi \log x - 4\pi t \mod 2\pi, \\ t_1(t, \frac{t\arg\beta - j\pi}{\log|\beta|}) &= 2(\arg\alpha t - \log|\alpha| \frac{t\arg\beta - j\pi}{\log|\beta|}) \mod 2\pi, \\ t_2(t, \frac{t\arg\beta - j\pi}{\log|\beta|}) &= 0 \mod 2\pi, \end{aligned}$$

(see figure 2). In this case lemma 12 shows that every N_j is dense in $S^1 \times \{(t_1, 0) \in T(\Xi_1, \Xi_2)\}$ whenever $(\arg\alpha - \arg\beta \log|\alpha|/\log|\beta|)/\pi$ is irrational: the argument used for $t_2 \neq 0$ shows that in this case N is dense in $S^1 \times T(\Xi_1, \Xi_2)$.

We are then left to the case

$$\frac{\arg\alpha - \arg\beta \log|\alpha|/\log|\beta|}{\pi} \in \mathbb{Q}, \qquad \frac{\log|\alpha|}{\log|\beta|} \in \mathbb{Q}$$

namely

$$\frac{k \arg\alpha - l \arg\beta}{\pi} = \frac{p}{q}, \qquad \frac{\log|\alpha|}{\log|\beta|} = \frac{l}{k} \tag{25}$$

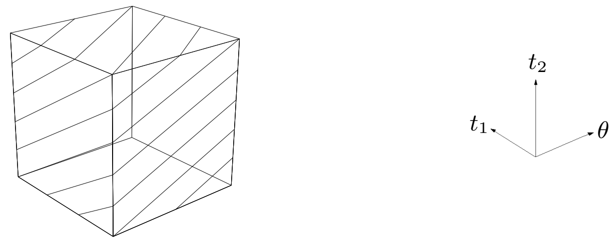


Fig. 4. intersection of the leaf N with the faces of $S^1 \times T(\Xi_1, \Xi_2)$: case $\alpha^m = \beta^n$.

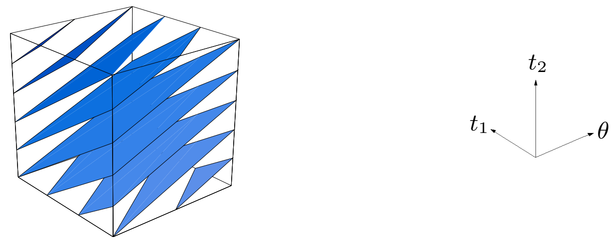


Fig. 5. the compact leaf N in the case $\alpha^m = \beta^n$.

where l, k, p and q are integers and $(p, q) = (l, k) = 1$: by lemma 16 this is equivalent to the existence of integers m and n such that $\alpha^m = \beta^n$. In this case the intersection of N with the faces of the cube is a union of closed curves (see figure 4).

Choose two integers b and c such that $bk - cl = 1$. Set

$$q' := \begin{cases} q & \text{if } p \text{ is odd} \\ q/2 & \text{if } p \text{ is even} \end{cases}, \qquad p' := \begin{cases} p & \text{if } p \text{ is odd} \\ p/2 & \text{if } p \text{ is even} \end{cases}$$

and remark that in this case the map

$$\begin{aligned} F :\ \mathbb{R}^2 &\longrightarrow N \subset S^1 \times T(\Xi_1, \Xi_2) \\ (t, s) &\longmapsto (\theta(t, s), t_1(t, s), t_2(t, s)) \end{aligned}$$

is invariant with respect to the action on $\mathbb{R}^2$ of the lattice $\Lambda := v\mathbb{Z} \oplus w\mathbb{Z}$ (see figure 5) where

$$v = (q', \frac{q' \arg \beta - p' c\pi}{\log |\beta|}), \qquad w = (0, \frac{k\pi}{\log |\beta|}). \tag{26}$$

So we may consider the diagram

$$\begin{array}{ccc} \mathbb{C} & & \\ {\scriptstyle p}\downarrow & \searrow^{F} & \\ \frac{\mathbb{C}}{\Lambda} & \xrightarrow[\bar{F}]{} & N \end{array} \tag{27}$$

where p is the canonical projection of $\mathbb{C}$ onto $\mathbb{C}/\Lambda$ and $\bar{F}$ is the quotient map of F. Obviously $\bar{F}$ is onto, and the leaf $N = \bar{F}(\mathbb{C}/\Lambda)$ is compact. Moreover, since

$F' = B_{\alpha,\beta} \neq 0$, $\bar{F}$ is a local diffeomorphism; this implies that N is a submanifold of $H_{\alpha,\beta}$. Thus N is a compact Riemann surface of genus one. Furthermore $\bar{F}$ is holomorphic, because, with the chosen parametrization, the horizontal and the vertical axes of $\mathbb{C}$ are the integral curves respectively of $B_{\alpha,\beta}$ and $J_{\alpha,\beta}(B_{\alpha,\beta})$; finally it is straightforward to check that $\bar{F}$ is injective, so it is a biholomorphism. □

Lemma 16. *The conditions (25) are equivalent to the existence of integers m and n, where $m/n = k/l$, such that $\alpha^m = \beta^n$.*

Proof: The existence of integers m and n such that $m/n = k/l$ and $\alpha^m = \beta^n$ is equivalent to

$$\frac{\log|\alpha|}{\log|\beta|} = \frac{n}{m} = \frac{l}{k} \qquad \text{and} \qquad \{m \arg\alpha + 2r\pi\}_{r\in\mathbb{Z}} = \{n \arg\beta + 2s\pi\}_{s\in\mathbb{Z}}. \tag{28}$$

These conditions obviously imply (25).

Vice versa, from (25) we obtain

$$2qk \arg\alpha + 2r\pi = 2ql \arg\beta + 2\pi(p+r) \qquad \text{for every integer } r;$$

so, setting $m{:=}2qk$ and $n{:=}2ql$, we get (28). □

The proof of theorem 15 allows us to complete the description of the foliation when the leaves are not compact:

Corollary 17. *When α and β do not satisfy (25), the saturated components of $\mathcal{E}_{\alpha,\beta}$ are of two kinds:*

1. $S^1 \times \{(\xi_1,\xi_2) \in S^3 : \xi_2 = 0\}$ *and* $S^1 \times \{(\xi_1,\xi_2) \in S^3 : \xi_1 = 0\}$;
2. $S^1 \times T(\xi_1,\xi_2)$.

Remark 18. Because of (18), $\mathcal{E}_{\alpha,\beta}$ is linear in the classification recently given by D. Mall in [Mal98].

5. Elliptic fibrations and orbifolds

By the definition of Kodaira in [Kod64, 2], an *elliptic surface* is a complex fibre space of elliptic curves over a non singular algebraic curve, namely a map $\Xi\colon S \to \Delta$ where S is a complex surface, Δ is a non singular algebraic curve, Ψ is a holomorphic map and the generic fibre is a torus. The curve Δ is called the *base space* of S.

In theorem 15 we showed that, if $\alpha^m = \beta^n$ for some integers m and n, then $S^1 \times S^3$ is a fibre space of elliptic curves over a topological space Δ -the leaf space. In this section we show that such a Δ is a non singular algebraic curve (actually $\mathbb{P}^1\mathbb{C}$) and that the projection Ψ is holomorphic with respect to this complex structure.

A *quasi-regular foliation* is a foliation $\mathcal{F}$ on a smooth manifold M such that for each point p of M there is a natural number $N(p)$ and a Frobenius chart U

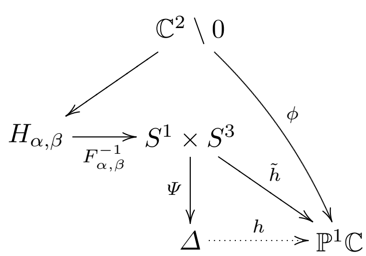
(namely, a $\mathcal{F}$-flat cubical neighborhood) where each leaf of $\mathcal{F}$ intersects U in $N(p)$ slices, if any. If $N(p) = 1$ for all p, then $\mathcal{F}$ is called a *regular foliation* (see for instance [BG98]). For a compact manifold M, the assumption that the foliation is quasi-regular is equivalent to the assumption that all leaves are compact. A Riemannian foliation with compact leaves induces a natural orbifold structure on the leaf space (see [Mol88, Proposition 3.7]). This is the case we are concerned with, since by [DO98, Theorem 5.1] $\mathcal{E}_{\alpha,\beta}$ is Riemannian. In this section we show that $\mathcal{E}_{\alpha,\beta}$ is quasi-regular if and only if $\alpha^m = \beta^n$ for some integers m and n, and this gives an orbifold structure to the leaf space. This does not contradict the structure of complex curve, it simply means that the two structures are not isomorphic in the orbifold category. In fact, any 2-dimensional orbifold with only conical points is homeomorphic to a manifold.

Theorem 19. *If $\alpha^m = \beta^n$ for some integers m and n, the leaf space Δ of the foliation in tori given on $S^1 \times S^3$ by the theorem 15 is homeomorphic to $\mathbb{P}^1\mathbb{C}$, and the projection $\Psi\colon S^1 \times S^3 \to \Delta$ is holomorphic with respect to the induced complex structure.*

Proof: By lemma 16 we can suppose that (25) holds. Choose then the integers m and n minimal with respect to the property $\alpha^m = \beta^n$, and observe that this implies $m \arg \alpha = n \arg \beta + 2\pi c$, where c is an integer such that GCD$(m, n, c) = 1$, and consider the following map:

$$\begin{aligned} \tilde{h}:\ S^1 \times S^3 &\longrightarrow \mathbb{P}^1\mathbb{C} \\ (\theta, \xi_1, \xi_2) &\longmapsto [\exp(\theta i c)\, \xi_1^m : \xi_2^n]. \end{aligned}$$

It is an easy matter to verify that on $H_{\alpha,\beta}$ this map is nothing but the quotient of $\phi(z_1, z_2) := [z_1^m : z_2^n]$, and we obtain the diagram



(29)

We show that $\tilde{h}$ is well defined on the leaf space, and that its quotient h is in fact the homeomorphism we are looking for:

1. h is well defined: if (θ, ξ_1, ξ_2) is on the leaf passing through (Θ, Ξ_1, Ξ_2), then θ, ξ_1 and ξ_2 are of the form (see (23))

$$\begin{aligned} \theta(t,s) &= \Theta - 4\pi t \mod 2\pi, \\ \xi_1(t,s) &= \Xi_1 \exp(2i \arg \alpha t) \exp(-2i \log|\alpha| s), \\ \xi_2(t,s) &= \Xi_2 \exp(2i \arg \beta t) \exp(-2i \log|\beta| s), \end{aligned}$$

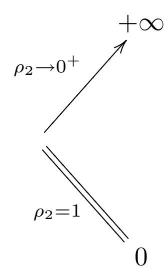
and we obtain that $(\theta(t,s),\xi_1(t,s),\xi_2(t,s))$ is mapped to

$$\begin{aligned}&[\exp\left(i(\Theta-4\pi t)c\right)\varXi_1^m\exp\left(2itm\arg\alpha\right):\varXi_2^n\exp\left(2itn\arg\beta\right)]\\ =&[\exp\left(i(\Theta-4\pi t)c+2it(m\arg\alpha-n\arg\beta)\right)\varXi_1^m:\varXi_2^n]\\ =&[\exp\left(i\Theta c\right)\varXi_1^m:\varXi_2^n],\end{aligned}$$

and the last member does not depend on t and s. Namely, $\tilde{h}$ is constant on every leaf and h is well defined on $\varDelta$;

2. h is onto: $(\theta,1,0)\mapsto[1:0]$ and if we put $h(\theta,\xi_1,\xi_2)=[z_1:z_2]$ where $z_2\neq 0$ we obtain $z_1z_2^{-1}=\exp\left(i\theta c\right)\xi_1^m\xi_2^{-n}$. Using polar coordinates, that is, choosing real numbers ρ_1, ρ_2, θ_1 and θ_2 such that $\xi_1=\rho_1\exp\left(i\theta_1\right)$ and $\xi_2=\rho_2\exp\left(i\theta_2\right)$, the last member becomes $\exp\left(i\theta c+m\theta_1-n\theta_2\right)\rho_1^m\rho_2^{-n}$ where $\rho_1^2+\rho_2^2=1$. The exponent $\theta c+m\theta_1-n\theta_2$ covers all the real numbers, and the map

$$\rho_1^m\rho_2^{-n}\big|_{\rho_1=\sqrt{1-\rho_2^2}}=(1-\rho_2^2)^{\frac{m}{2}}\rho_2^{-n}$$



covers all the positive real numbers, so $\tilde{h}$ -and, consequently, h- is onto;

3. h is injective: suppose that $h(\theta,\xi_1,\xi_2)=h(\Theta,\varXi_1,\varXi_2)$ for (θ,ξ_1,ξ_2) and $(\Theta,\varXi_1,\varXi_2)$ in $S^1\times S^3$. If $\xi_1\varXi_1=0$, then ξ_1 and $\varXi_1$ must both be zero, whence (θ,ξ_1,ξ_2) and $(\Theta,\varXi_1,\varXi_2)$ lie on the same leaf. If $\xi_1\varXi_1\neq 0$, we can write
$$\frac{\xi_2^n}{\exp\left(i\theta c\right)\xi_1^m}=\frac{\varXi_2^n}{\exp\left(i\Theta c\right)\varXi_1^m}. \tag{30}$$
Let $\xi_1=\rho_1\exp\left(i\eta_1\right)$, $\xi_2=\rho_2\exp\left(i\eta_2\right)$, $\varXi_1=P_1\exp\left(iH_1\right)$ and $\varXi_2=P_2\exp\left(iH_2\right)$; the equation (30) becomes
$$\frac{\rho_2^n\exp\left(i\eta_2 n\right)}{\rho_1^m\exp\left(i(\theta c+\eta_1 m)\right)}=\frac{P_2^n\exp\left(iH_2 n\right)}{P_1^m\exp\left(i(\theta c+H_1 m)\right)},$$
that is
$$\begin{cases}\dfrac{\rho_2^n}{\rho_1^m}=\dfrac{P_2^n}{P_1^m},\\ (\theta-\Theta)c+m(\eta_1-H_1)-n(\eta_2-H_2)=0\mod 2\pi.\end{cases} \tag{31}$$
The first equation in (31), together with $\rho_1^2+\rho_2^2=1=P_1^2+P_2^2$, easily gives
$$\rho_1=P_1\qquad\text{and}\qquad\rho_2=P_2. \tag{32}$$

In order to show that (θ, ξ_1, ξ_2) and (Θ, Ξ_1, Ξ_2) lie on the same leaf, we want to find two real numbers t and s such that

$$\begin{aligned} \theta &= \Theta - 4\pi t \mod 2\pi, \\ \xi_1 &= \Xi_1 \exp\left(2(\arg \alpha t - \log|\alpha| s)\right), \\ \xi_2 &= \Xi_2 \exp\left(2(\arg \beta t - \log|\beta| s)\right), \end{aligned} \tag{33}$$

that is, by using (32), we want to find two real numbers t and s satisfying

$$\begin{cases} 4\pi t &= \Theta - \theta \mod 2\pi, \\ 2\arg\alpha t - 2\log|\alpha| s = \eta_1 - H_1 \mod 2\pi, \\ 2\arg\beta t - 2\log|\beta| s = \eta_2 - H_2 \mod 2\pi. \end{cases}$$

The determinant of

$$\begin{pmatrix} 4\pi & 0 & \Theta - \theta \\ 2\arg\alpha & -2\log|\alpha| & \eta_1 - H_1 \\ 2\arg\beta & -2\log|\beta| & \eta_2 - H_2 \end{pmatrix}$$

is zero, because the second equation of (31) gives

$$m(\text{second row}) - n(\text{third row}) = c(\text{first row}),$$

and the injectivity of h is proved.

From 1, 2 and 3 we obtain that $h\colon \Delta \to \mathbb{P}^1\mathbb{C}$ is a bijective continous map, and so is a homeomorphism because of the compactness of Δ. At least, Ψ is holomorphic with respect to the induced complex structure -that is, $\tilde{h}$ is holomorphic- because the map ϕ in the diagram (29) is holomorphic. □

Theorem 20. *The foliation $\mathcal{E}_{\alpha,\beta}$ is quasi-regular if and only if $\alpha^m = \beta^n$ for some integers m and n; in this case, choosing m and n minimal positive such integers, $N(\Theta, \Xi_1, \Xi_2) = 1$ if $\Xi_1\Xi_2 \neq 0$, whereas $N(\Theta, 0, \Xi_2) = m$ and $N(\Theta, \Xi_1, 0) = n$. In particular, the foliation $\mathcal{E}_{\alpha,\beta}$ is regular if and only if $\alpha = \beta$.*

Proof: By theorem 15 we know that all leaves are compact if and only if $\alpha^m = \beta^n$, and for the points (Θ, Ξ_1, Ξ_2) where $\Xi_1\Xi_2 \neq 0$ the thesis is given by the figure 5. We are then left to the points $(\Theta, 0, \Xi_2)$ and $(\Theta, \Xi_1, 0)$, when $\alpha^m = \beta^n$. We look at the points $(\Theta, \Xi_1, 0)$, the study of the other ones being analogous.

We remark that the figure 5 is 3-dimensional, and in order to visualize the 4-dimensional neighborhood of a point of $S^1 \times S^3$ we need another 3-dimensional description of the foliation $\mathcal{E}_{\alpha,\beta}$: consider the stereographic projection

$$\begin{aligned} \phi : S^3 \setminus (0,0,0,1) &\longrightarrow \mathbb{R}^3 \\ (x_1, x_2, x_3, x_4) &\longmapsto \frac{1}{1 - x_4}(x_1, x_2, x_3). \end{aligned}$$

It is easy to check that $\phi(T(\xi_1, \xi_2))$ is generated by the revolution around the y_3-axis of the circle $C(\xi_1, \xi_2)$ in the y_2y_3-plane centered in $(1/|\xi_1|, 0)$ with radius $|\xi_2|/|\xi_1|$. We are thus led to the figure 6.

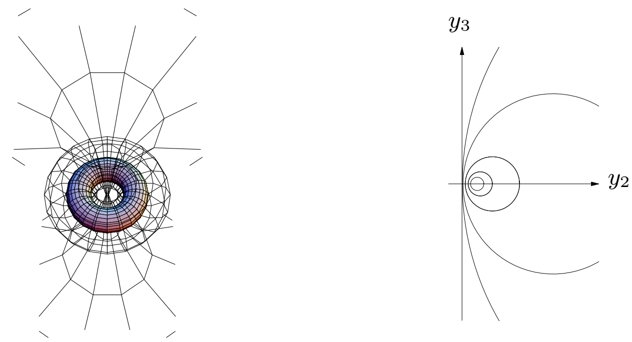


Fig. 6. On the left, the partition of $\mathbb{R}^3$ in tori $T(\xi_1, \xi_2)$; on the right, the circles that generate the tori.

By refining the computation in the proof of theorem 15, we see that any leaf intersects $T(\xi_1, \xi_2)$ along r toral knots of type l/k, r being the greatest common divisor of m and n. This means that each leaf contained in $T(\xi_1, \xi_2)$ intersects $C(\xi_1, \xi_2)$ in exactly $n = rl$ points. Now let

$$D_\rho := \bigcup_{|\xi_2|/|\xi_1| < \rho} C(\xi_1, \xi_2)$$

and let $U_{\delta,\rho}$ the piece of solid torus given by the revolution of angle $(-\delta, \delta)$ of D_ρ. The neighborhoods of $(\Theta, \Xi_1, 0)$ of the form $(\Theta - \varepsilon, \Theta + \varepsilon) \times U_{\delta,\rho}$ contain each leaf in $n = rl$ distinct connected components, and this ends the proof. □

Remark 21. We thus have an orbifold structure on the leaf space Δ, with two conical points of order m and n, respectively (see [Mol88, Proposition 3.7]). In particular, a local chart around the leaf through $(\Theta, \Xi_1, 0)$ is given by D_ρ/Γ_n, Γ_n being the finite group generated by the rotation of angle $2\pi/n$.

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