Old and new structures on products of spheres

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ABSTRACT. A classical theorem of Kervaire states that products of spheres are parallelizable if and only if at least one of the factors has odd dimension. In this note an explicit parallelization is given, and it is then used to describe G-structures on products of two spheres, for $G = \mathrm{U}(n)$, $\mathrm{Sp}(n)$, G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$. This approach gives an alternative description of the classical Calabi-Eckmann structures, and of some G_2, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$-structures on $S^6 \times S^1$, $S^7 \times S^1$, $S^{15} \times S^1$ respectively. In other products of spheres some new G-structures are obtained.

1. Introduction

It is a classical result in Algebraic Topology that spheres S^n are parallelizable only in dimension $n = 1, 3, 7$. As for the products of two or more spheres M. Kervaire proved in the fifties the following:

THEOREM 1.1 ([**Ker56**]). *The manifold $S^{n_1} \times \cdots \times S^{n_r}$, $r \geq 2$, is parallelizable if and only if at least one of the n_i is odd.*

The paper [**Bru92**] is the only reference the author knows to provide explicit parallelizations $\mathcal{B}$ on some products of spheres, namely, whenever one of the factors is S^1, S^3, S^5, S^7. In this note the never-vanishing vector field on the odd-dimensional sphere is used to write an explicit isomorphism between $T(S^m \times S^n)$ and $S^m \times S^n \times \mathbb{R}^{m+n}$. Pulling back the standard basis of $\mathbb{R}^{m+n}$ one obtains explicit orthonormal parallelizations $\mathcal{P}$ on $S^m \times S^n$.

These parallelizations are then exploited to define some significant G-structures on products of spheres of suitable dimension. The groups G considered here are: $G = \mathrm{U}((m+n)/2)$, if both the dimensions are odd (almost-Hermitian structures on $S^m \times S^n$); $G = \mathrm{Sp}((m+n)/4)$, if both the dimensions are odd and $m + n = 0$ mod 4 (almost-hyperhermitian structures on $S^m \times S^n$); $G = \mathrm{G}_2$, $\mathrm{Spin}(7)$, $\mathrm{Spin}(9)$ on the 7-dimensional, 8-dimensional, 16-dimensional products $S^m \times S^n$ respectively.

Table 1 summarizes the properties of the orbits of these structures by the standard action on $\mathbb{R}^{m+n}$ of the orthogonal and the symmetric group O_{m+n}, $\mathfrak{S}_{m+n}$. It refers to the frames $\mathcal{B}$ and $\mathcal{P}$ already mentioned, and to their associated G-structures $I_{\mathcal{B}}$, $H_{\mathcal{B}}$, …, $\Phi_{\mathcal{B}}$. In some cases, the orbit contains classical structures,

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TABLE 1. Properties of structures. References are those where the structures were originally introduced.

spheres	G	frame	orbit	type
$S^{2n-1}\times S^1$	$\mathrm{U}(n)$	$\mathcal{B}$	$\mathrm{O}_{2n}I_{\mathcal{B}}$	Hopf Hermitian*
$S^{4n-1}\times S^1$	$\mathrm{Sp}(n)$	$\mathcal{B}$	$\mathrm{O}_{4n}H_{\mathcal{B}}$	Hopf hyperhermitian*
$S^6\times S^1$	G_2	$\mathcal{B}$	$\mathrm{O}_7\varphi_{\mathcal{B}}$	locally conformal parallel*[**Cab97**]
$S^7\times S^1$	$\mathrm{Spin}(7)$	$\mathcal{B}$	$\mathrm{O}_8\phi_{\mathcal{B}}$	locally conformal parallel*[**Cab95**]
$S^{15}\times S^1$	$\mathrm{Spin}(9)$	$\mathcal{B}$	$\mathrm{O}_{15}\Phi_{\mathcal{B}}$	locally conformal parallel*[**Fri99**]
$S^m\times S^n_{m+n\equiv_2 0}$	$\mathrm{U}(\frac{m+n}{2})$	$\mathcal{P}$	$\mathfrak{S}_{m+n}I_{\mathcal{P}}$	integrable if and only if...**
$S^m\times S^n_{m+n\equiv_4 0}$	$\mathrm{Sp}(\frac{m+n}{4})$	$\mathcal{P}$	$\mathfrak{S}_{m+n}H_{\mathcal{P}}$	non-integrable
$S^m\times S^n_{m+n=7}$	G_2	$\mathcal{P}$	$\mathfrak{S}_7\varphi_{\mathcal{P}}$	general type
$S^m\times S^n_{m+n=8}$	$\mathrm{Spin}(7)$	$\mathcal{P}$	$\mathfrak{S}_8\phi_{\mathcal{P}}$	general type
$S^m\times S^n_{m+n=15}$	$\mathrm{Spin}(9)$	$\mathcal{P}$	$\mathfrak{S}_{15}\Phi_{\mathcal{P}}$	?

*all structures in the orbit are isomorphic (see remark 2.1).

**see theorem 3.2. All integrable structures are isomorphic to Calabi-Eckmann.

some of which were originally defined in a non-elementary way: this approach provides then a straightforward encoding of these classical structures. In the other cases, this construction provides new structures on products of spheres. The last part of this note is devoted to explain Table 1, that henceforth will be referred to as the *table*. More details can be found in [**Par00**].

2. Explicit parallelizations on $S^m\times S^n$

Let $n=1$. The vector fields $\{x_i\partial_{x_i}\}_{i=1,\dots,m+1}$ on $\mathbb{R}^{m+1}\setminus 0$ are Γ-equivariant, where Γ is the infinite cyclic group generated by multiplication by $e^{2\pi}$. Hence, they define a parallelization $\mathcal{B}=\{b_1,\dots,b_{m+1}\}$ on $S^m\times S^1=(\mathbb{R}^{m+1}\setminus 0)/\Gamma$. Denoting by $T=-y_2\partial_{y_1}+y_1\partial_{y_2}$ the never-vanishing versor field on S^1, it is easy to see that

$$b_i=(\text{orthogonal projection of } \partial_{x_i} \text{ on } S^m)+x_iT, \qquad i=1,\dots,m+1;$$

hence in particular $\mathcal{B}$ is an orthonormal frame.

REMARK 2.1. Any orthogonal transformation of $\mathbb{R}^{m+1}$ is Γ-equivariant, and the differential of the induced map on $S^m\times S^1$ is given by the same orthogonal transformation with respect to the frame $\mathcal{B}$. This explains the first footnote in the table.

Let now n be any odd integer. Denote by T the never-vanishing vector field on S^n given by the complex multiplication, and by t_j its coordinates:

$$T=\sum_{j=1}^{n+1}t_j\partial_{y_j}=-y_2\partial_{y_1}+y_1\partial_{y_2}+\cdots-y_{n+1}\partial_{y_n}+y_n\partial_{y_{n+1}}.$$

A key role here is played by the *meridian vector fields* (see [**Bru92**]):

$$M_i=\text{orthogonal projection of } \partial_{x_i} \text{ on } S^m\in\mathfrak{X}(S^m), \qquad i=1,\dots,m+1,$$

$$N_j=\text{orthogonal projection of } \partial_{y_j} \text{ on } S^n\in\mathfrak{X}(S^n), \qquad j=1,\dots,n+1,$$

and by the following:

LEMMA 2.2. *Let α, β be vector bundles, and let ε^k be the trivial rank k vector bundle. Then*

$$\alpha \times (\beta \oplus \varepsilon^k) \simeq (\alpha \oplus \varepsilon^k) \times \beta.$$

Split $T(S^n)$ in $\eta \oplus \langle T \rangle$, then use the lemma 2.2 to shift on the left the trivial summand. Since $T(S^m) \oplus \varepsilon^1$ is a trivial vector bundle, a rank 2 trivial summand can be shifted on the right: now remark that $\eta \oplus \varepsilon^2$ is trivial. This argument gives an isomorphism (actually an isometry) between $T(S^m \times S^n)$ and $\varepsilon^{m-1} \times \varepsilon^{n+1}$. Then pull back the sections $\{\partial_{x_1}, \dots, \partial_{x_{m-1}}, \partial_{y_1}, \dots, \partial_{y_{n+1}}\}$ of $\varepsilon^{m-1} \times \varepsilon^{n+1}$ to obtain the explicit parallelization $\mathcal{P}$ given by the following:

THEOREM 2.3. *The vector fields $\{p_1, \dots, p_{m+n}\} \in \mathfrak{X}(S^m \times S^n)$ given by*

$$p_i = M_i + x_i T, \qquad i = 1, \dots, m-1,$$

$$p_{m-1+j} = y_j M_m + t_j M_{m+1} + (t_j x_{m+1} + y_j x_m - t_j) T + N_j, \qquad j = 1, \dots, n+1.$$

define an orthonormal frame $\mathcal{P}$ of $S^m \times S^n$, for any odd n.

PROOF. Once observed that

$$M_i = \partial_{x_i} - x_i M, \qquad i = 1, \dots, m+1,$$
$$N_j = \partial_{y_j} - y_j N, \qquad j = 1, \dots, n+1,$$

where M and N denote the normal versor field of $S^m \subset \mathbb{R}^{m+1}$ and $S^n \subset \mathbb{R}^{n+1}$ respectively, the proof follows from a direct computation. □

REMARK 2.4. To obtain a parallelization for more than two spheres, use induction in the following way: suppose that $S^{n_2} \times \dots \times S^{n_r}$, $r \geq 2$, has at least one odd-dimensional factor, whence it is parallelizable; then

$$\begin{aligned} T(S^{n_1} \times \dots \times S^{n_r}) &= T(S^{n_1}) \times \varepsilon^{n_2 + \dots + n_r} \\ &= (T(S^{n_1}) \oplus \varepsilon^1) \times \varepsilon^{n_2 + \dots + n_r - 1} = \varepsilon^{n_1+1} \times \varepsilon^{n_2 + \dots + n_r - 1}. \end{aligned}$$

3. Hermitian and hyperhermitian structures

The following statement is a consequence of the definition of $\mathcal{B}$. It gives lines 1 and 2 of the table.

THEOREM 3.1. *The almost-Hermitian structure $I_{\mathcal{B}}$ on $S^{2n-1} \times S^1$ coincide with the Hopf Hermitian structure given by the identification $S^{2n-1} \times S^1 = (\mathbb{C}^n \setminus 0)/\Gamma$. The same way, the almost-hyperhermitian structure $H_{\mathcal{B}}$ on $S^{4n-1} \times S^1$ coincide with the Hopf hyperhermitian structure given by the identification $S^{4n-1} \times S^1 = (\mathbb{H}^n \setminus 0)/\Gamma$.*

As for the frame $\mathcal{P}$, the following statement gives line 6 of the table.

THEOREM 3.2. *An almost-Hermitian structure I on $S^m \times S^n$ in the symmetric orbit $\mathfrak{S}_{m+n} I_{\mathcal{P}}$, m, n odd, is integrable if and only if*

$$I(p_{m-1+j}) = \pm p_{m+j}, \qquad j \text{ odd}, 1 \leq j \leq n+1, \tag{3.1}$$

where the sign is the same for all j.

PROOF. The "if" part relies on the following fact: there exists a covering of $S^m \times S^n$ by submanifolds whose tangent spaces span all $T(S^m \times S^n)$ and on these submanifolds I is integrable.

The "only if" part is given by a case by case computation here sketched. Suppose that the condition of the theorem is not satisfied. Then, taking $-I$ in case, there exists an odd $j \in \{1, \dots, n+1\}$ such that one of the following conditions holds true:

(1) there exist i, $k \in \{1, \dots, m-1\}$, $i \neq k$ such that
$$I(p_{m-1+j}) = p_i \qquad \text{and} \qquad I(p_{m+j}) = \pm p_k;$$
(2) there exist $i \in \{1, \dots, m-1\}$, $k \in \{1, \dots, n+1\}$, $k \neq j$, $j+1$ such that
$$I(p_{m-1+j}) = p_i \qquad \text{and} \qquad I(p_{m+j}) = \pm p_{m-1+k};$$
(3) there exist $i \in \{1, \dots, n+1\}$, $k \in \{1, \dots, m-1\}$, $i \neq j$, $j+1$ such that
$$I(p_{m-1+j}) = p_{m-1+i} \qquad \text{and} \qquad I(p_{m+j}) = \pm p_k;$$
(4) there exist i, $k \in \{1, \dots, n+1\}$, i, $k \neq j$, $j+1$, $i \neq k$ such that
$$I(p_{m-1+j}) = p_{m-1+i} \qquad \text{and} \qquad I(p_{m+j}) = \pm p_{m-1+k}.$$

The torsion tensor can then be computed in each case and, in particular, one obtains:

(1) $\langle N(p_{m-1+j}, p_{m+j}), p_k\rangle = 2(\pm x_i(1 - y_j^2 - y_{j+1}^2) + x_k(1 - 2(y_j^2 + y_{j+1}^2))) \neq 0$
(2) $\langle N(p_{m-1+j}, p_{m+j}), p_i\rangle(y_j = y_{j+1} = y_k = 0, t_k = 1) = 2(x_i \mp x_{m+1}) \neq 0$
(3) $\langle N(p_{m-1+j}, p_{m+j}), p_k\rangle(y_j = y_{j+1} = y_i = 0, t_i = 1) = 2(x_k \pm x_{m+1}) \neq 0$
(4) $\langle N(p_{m-1+j}, p_{m+j}), p_{m-1+i}\rangle(y_j = y_{j+1} = t_i = x_m = 0, y_i = x_{m+1} = 1) = \mp 2t_k \neq 0$

□

Since an almost-hyperhermitian structure H cannot fix a 2-dimensional distribution, the previous theorem gives also line 7 of the table.

Let now S be the never-vanishing vector field given by complex multiplication on S^m. Denote by $I^{m,n}$ the Calabi-Eckmann Hermitian structure given by $I^{m,n}(T) = S$.

Theorem 3.3. *The Calabi-Eckmann complex structure $I^{m,n}$ on $S^m \times S^n$ coincides with the almost-Hermitian structure $I_{\mathcal{P}}$, namely*
$$I^{m,n}(p_i) = p_{i+1}, \qquad \textit{if } i \textit{ is odd},$$
$$I^{m,n}(p_i) = -p_{i-1}, \qquad \textit{if } i \textit{ is even}.$$

Proof. Using the Hopf fibrations of S^m and S^n, one is left to show that
$$I_{\mathcal{P}}|_{S^m \times S^1} = I^{m,n}|_{S^m \times S^1},$$
$$I_{\mathcal{P}}|_{S^1 \times S^n} = I^{m,n}|_{S^1 \times S^n},$$
and formulas $b_m = \sum_{j=1}^{n+1} y_j p_{m-1+j}$, $b_{m+1} = \sum_{j=1}^{n+1} t_j p_{m-1+j}$ gives $I_{\mathcal{P}}|_{S^m \times S^1} = I_{\mathcal{B}} = I^{m,1} = I^{m,n}|_{S^m \times S^1}$. The same way, using the frame $\mathcal{B}$ on $S^1 \times S^n$ given by $b_j = N_j - y_j S$, $j = 1, \dots, n+1$, one gets $I_{\mathcal{P}}|_{S^1 \times S^n} = I_{\mathcal{B}} = I^{1,n} = I^{m,n}|_{S^1 \times S^n}$, and this completes the proof. □

The following corollary proves the second footnote in the table.

Corollary 3.4. *All integrable almost-Hermitian structures on $S^m \times S^n$ in the symmetric orbit $\mathfrak{S}_{m+n} I_{\mathcal{P}}$, m, n odd, are biholomorphic to the Calabi-Eckmann Hermitian structure $I^{m,n}$.*

PROOF. Given any permutation π of $\{1,\ldots,m+1\}$, the automorphism f_π of $S^m \times S^n$ given by $(x_1,\ldots,x_{m+1},y) \mapsto (x_{\pi(1)},\ldots,x_{\pi(m+1)},y)$ satisfies

$$df_\pi(p_{\pi(i)}) = p_i, \qquad i = 1,\ldots,m-1.$$

If I is an integrable almost-Hermitian structure in $\mathfrak{S}_{m+n} I_{\mathcal{P}}$, then a permutation $\pi = \pi(I)$ can be defined such that f_π is a biholomorphism between I and $I_{\mathcal{P}}$. □

4. Special structures

THEOREM 4.1. *The* G_2*-structure* $\varphi_{\mathcal{B}}$ *on* $S^6 \times S^1$*, the* $\mathrm{Spin}(7)$*-structure* $\phi_{\mathcal{B}}$ *on* $S^7 \times S^1$ *and the* $\mathrm{Spin}(9)$*-structure* $\Phi_{\mathcal{B}}$ *on* $S^{15} \times S^1$ *are locally conformal parallel.*

PROOF. Denote by $\tilde{\mathcal{B}}$ the frame $\{x_i \partial_{x_i}\}$. The proof follows from the fact that the G_2-structure $\varphi_{\tilde{\mathcal{B}}}$ on $\mathbb{R}^7 \setminus 0$, the $\mathrm{Spin}(7)$-structure $\phi_{\tilde{\mathcal{B}}}$ on $\mathbb{R}^8 \setminus 0$ and the $\mathrm{Spin}(9)$-structure $\Phi_{\tilde{\mathcal{B}}}$ on $\mathbb{R}^{16} \setminus 0$ are globally conformal parallel. □

The previous theorem partially proves lines 3, 4, 5 of the table. That $\varphi_{\mathcal{B}}$, $\phi_{\mathcal{B}}$ and $\Phi_{\mathcal{B}}$ actually coincide with the structures defined in [**Cab97**], [**Cab95**] and [**Fri99**] respectively is a more ticklish question, and it is here just stated.

Only lines 8 and 9 are then left over. Recall that a G-structure of general type, for $G = \mathrm{G}_2$ or $\mathrm{Spin}(7)$, means a structure which does not belong to any proper subclass of $\mathcal{W}$ in the classifications given in [**FG82**] and [**Fer86**]. Using structure equations for $\mathcal{B}$ and $\mathcal{P}$, the problem of checking that a fixed structure belongs to a particular subclass can be translated in a linear algebra problem, and can be solved by a computer calculation. Since the symmetric group is finite, symmetric orbits can be tackled this way, and this is how the following theorem was proved:

THEOREM 4.2. *The* G_2*-structures in* $\mathfrak{S}_7 \varphi_{\mathcal{P}}$ *on* $S^4 \times S^3$ *and* $S^2 \times S^5$ *are all of general type. The* $\mathrm{Spin}(7)$*-structures in* $\mathfrak{S}_8 \phi_{\mathcal{P}}$ *on* $S^7 \times S^1$, $S^5 \times S^3$, $S^3 \times S^5$ *and* $S^1 \times S^7$ *are all of general type.*

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