

Families of strong KT structures in six dimensions

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Abstract. This paper classifies Hermitian structures on 6-dimensional nilmanifolds $M = \Gamma \backslash G$ for which the fundamental 2-form is $\partial\bar{\partial}$ -closed, a condition that is shown to depend only on the underlying complex structure J of M . The space of such J is described when G is the complex Heisenberg group, and explicit solutions are obtained from a limaçon-shaped curve in the complex plane. Related theory is used to provide examples of various types of Ricci-flat structures.

Introduction

Let (M, J, g) be a Hermitian manifold. There is a 1-parameter family of canonical Hermitian connections on M which can be distinguished by properties of the torsion tensor T [10, 27]. In particular, there is a unique connection ∇ satisfying $\nabla g = 0$, $\nabla J = 0$ for which $g(X, T(Y, Z))$ is totally skew-symmetric. The resulting 3-form can then be identified with $Jd\Omega$, where Ω is the fundamental 2-form defined by (1). This connection was used by Bismut [3] to prove a local index formula for the Dolbeault operator when the manifold is non-Kähler, and is the subject of a number of other interesting results [12, 16]. The properties of such a connection give rise to what is loosely called ‘Kähler with torsion geometry’, and if $Jd\Omega$ is closed but non-zero then g is called a *strong KT metric*. Such metrics have applications in type II string theory and in 2-dimensional supersymmetric σ -models [8, 15, 26].

In four real dimensions, a metric satisfying the strong KT condition is ‘standard’ in the terminology of Gauduchon [9]. One can be found in the conformal class of any given Hermitian metric on a compact manifold. But the theory is very different in higher dimensions. Even-dimensional compact Lie groups provide a natural class of strong KT structures [25]. In this case, one may choose J to be a left-invariant complex structure and g to be a compatible bi-invariant metric. Then ∇ is the flat connection with skew-symmetric torsion $g(X, [Y, Z])$ corresponding to an invariant 3-form on the Lie algebra. It is therefore natural to investigate the situation with regard to other groups.

In this paper, we study KT geometry on 6-dimensional nilmanifolds in which J and g arise from corresponding left-invariant tensors. If G is a simply-connected nilpotent Lie group, and if the structure equations of its Lie algebra are rational, then there exists a discrete subgroup Γ of G for which $M = \Gamma \backslash G$ is compact [18, 21]. Any left-invariant complex structure on G will pass to a complex structure J on M but, unless G is abelian, the $\partial\bar{\partial}$ -lemma is not valid for J and in particular there is no compatible Kähler metric [2, 5, 13, 20]. As we explain in §6, there may or may not be invariant pseudo-Kähler metrics on (M, J) .

Eighteen of the thirty-four classes of real 6-dimensional nilpotent Lie algebras \mathfrak{g} admit a complex structure. Exactly four of these classes, all of them 2-step with $b_1 \geq 4$ and including the case in which \mathfrak{g} underlies the complex Heisenberg algebra, give rise to strong KT metrics. Given that compact nilmanifolds with a strong KT structure exist, it is perhaps surprising that there are so few classes. The classification over \mathbb{R} is accomplished in §3, after an analysis of the relevant structure equations over \mathbb{C} in §§1,2. A matrix formalism for describing (1,1)-forms

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is introduced in an attempt to make the calculations of this paper rather more enlightening. A striking feature of our classification is that the existence of a strong KT structure depends only on the complex structure of \mathfrak{g} , and this poses the question of understanding the solutions as a subset of an appropriate moduli space of complex structures.

With this aim, we proceed to a detailed study of the strong KT equations when G is the complex Heisenberg group and $M = \Gamma \backslash G$ is the Iwasawa manifold. It is easy to check that none of the standard complex structures [1] on G are strong KT, so we were intrigued to discover which ones are. According to the third author's joint paper with Ketsetzis [17], essential features of an invariant complex structure J on M depend on $\mathbf{X}\overline{\mathbf{X}}$, where \mathbf{X} is a 2×2 matrix representing the induced action of J on $M/T^2 \cong T^4$. In §4, we prove that the strong KT condition constrains the eigenvalues of $\mathbf{X}\overline{\mathbf{X}}$ to be complex conjugates lying on a curve in the complex plane. We interpret this result in terms of the action of the automorphism group of \mathfrak{g} in §5, and this leads to an explicit description of the solution space. An analogous study can probably be carried out when $G = H_3 \times H_3$ is the product of real Heisenberg groups, using methods from [23].

A Hermitian manifold is called conformally balanced if the Lee 1-form θ (the 'trace' of $d\Omega$) is exact. The study of such structures in connection with the connection ∇ is motivated by work of [22], though there are less subtleties in our context in which θ is exact if and only if it is zero. Having explained in §1 that the vanishing of θ is complementary to the SKT condition, we observe in the final section that θ is also the obstruction to the holonomy of ∇ reducing to $SU(n)$. We list some 6-dimensional Lie algebras giving rise to nilmanifolds admitting such a reduction, and others admitting a pseudo-Riemannian metric with zero Ricci tensor.

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1 Complex structure equations

Let (M, J, g) be a Hermitian manifold of real dimension $2n$. We shall regard the complex structure J as the primary object, so the Riemannian metric g is chosen to render J orthogonal. The fundamental 2-form Ω is then defined by

$$g(X, Y) = \Omega(X, JY) \tag{1}$$

and has type (1,1) relative to J . The Hermitian structure is Kähler if and only if $d\Omega = 0$, which is equivalent to the vanishing of $\partial\Omega = (d\Omega)^{2,1}$.

Somewhat unconventionally, we set

$$\square = \frac{1}{2}idJd.$$

This operator acts as $\partial\bar{\partial}$ on forms of type (p, p) , which it maps to forms of type $(p+1, p+1)$. We shall only be concerned with the case $p = 1$.

Definition 1.1 We shall say that the Hermitian manifold (M, J, g) is 'strong KT' or more briefly 'SKT' if $\square\Omega = 0$ but $d\Omega \neq 0$.

Observe that our definition of SKT excludes the Kähler case.

We wish to combine the notion of SKT with that of an invariant Hermitian structure on a nilmanifold. First recall the definition of nilpotency for a Lie algebra \mathfrak{g} . The descending central series of \mathfrak{g} is the chain of ideals defined inductively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^i = [\mathfrak{g}^{i-1}, \mathfrak{g}]$ for $i \geq 1$. Then \mathfrak{g} is nilpotent if $\mathfrak{g}^s = 0$ for some s . If, in addition, $\mathfrak{g}^{s-1} \neq 0$ then \mathfrak{g} is said to be s -step.

Let \mathfrak{g} be a real $2n$ -dimensional nilpotent Lie algebra. Assigning an almost complex structure $J : \mathfrak{g} \rightarrow \mathfrak{g}$ is equivalent to choosing an n -dimensional subspace Λ of $\mathfrak{g}_\mathbb{C}^*$ such that $\Lambda \cap \bar{\Lambda} = \{0\}$. For the purpose of this paper, we shall call such a subspace of $\mathfrak{g}_\mathbb{C}^*$ ‘maximally complex’. The endomorphism J extends uniquely to a left-invariant almost complex structure (also denoted by J) on any Lie group G with Lie algebra \mathfrak{g} . The subspace Λ generates the space of $(1,0)$ -forms relative to J , and this is a complex structure if and only if $\mathcal{I}(\Lambda)$ is a differential ideal. (We use $\mathcal{I}(S)$ to denote the ideal generated by a subset S of the exterior algebra $\bigwedge^* \mathfrak{g}$.)

By a ‘nilmanifold with an invariant complex structure’ we mean an even-dimensional nilmanifold $\Gamma \backslash G$ endowed with a complex structure J arising from \mathfrak{g} . It is important to note that (G, J) will not in general be a complex Lie group.

Theorem 1.2 *Let $M = \Gamma \backslash G$ be a 6-dimensional nilmanifold with an invariant complex structure J . Then the SKT condition is satisfied by either all invariant Hermitian metrics g or by none. Indeed, it is satisfied if and only if J has a basis (α^i) of $(1,0)$ -forms such that*

$$\begin{cases} d\alpha^1 = 0 \\ d\alpha^2 = 0 \\ d\alpha^3 = A\alpha^{\bar{1}2} + B\alpha^{\bar{2}2} + C\alpha^{1\bar{1}} + D\alpha^{1\bar{2}} + E\alpha^{12} \end{cases} \quad (2)$$

where A, B, C, D, E are complex numbers such that

$$|A|^2 + |D|^2 + |E|^2 + 2\operatorname{Re}(\bar{B}C) = 0. \quad (3)$$

We indicate $\alpha^i \wedge \alpha^{\bar{j}}$ by $\alpha^{i\bar{j}}$ (or $-\alpha^{\bar{j}i}$), and use similar notation for forms of arbitrary degree. Thus, the symbol α stands more for the choice of basis than for an individual element.

The third equation in (2) means that

$$d\alpha^3 \in \bigwedge^2 \langle \alpha^1, \alpha^{\bar{1}}, \alpha^2, \alpha^{\bar{2}} \rangle.$$

and the resulting complex structure J is of ‘nilpotent’ type in the language of [4]. Observe that the system (2) automatically satisfies $d^2 = 0$ and therefore defines a Lie algebra irrespective of the values of A, B, C, D, E . The resulting isomorphism classes are listed in §3, where we distinguish those compatible with (3).

We shall divide the proof of Theorem 1.2 into two parts. The second part is devoted to a mainly computational derivation of the structure equations (2), and is relegated to the next section. However, it is instructive to begin by assuming (2) and deducing (3) from it. This we do immediately, and in passing we shall see that the choice of metric is irrelevant.

Let Ω be the fundamental 2-form of some J -Hermitian metric, and set

$$\Omega = \sum_{i,j=1}^3 x_{i\bar{j}} \alpha^{i\bar{j}}, \quad (4)$$

where $x_{i\bar{j}} \in \mathbb{C}$ are constant coefficients with $\bar{x}_{\bar{j}i} = -x_{i\bar{j}}$. The positive definiteness of g implies that the restriction of Ω to any complex line is non-zero. Equivalently,

$$\Omega(V, J\bar{V}) = g(V, \bar{V}) > 0 \quad (5)$$

for any vector $V \neq 0$ in the complexified tangent space.

Given that $Jd\alpha^3$ differs from $d\alpha^3$ by changing the sign of E , an easy calculation gives

$$Jd\alpha^3 \wedge d\alpha^{\bar{3}} = (|A|^2 + B\bar{C} + C\bar{B} + |D|^2 + |E|^2)\alpha^{1\bar{1}2\bar{2}}. \quad (6)$$

The vanishing of this 4-form is precisely (3), which can now be deduced from (2) via

Lemma 1.3 Given (2), (M, J, g) is SKT for any invariant Hermitian metric g if and only if $Jd\alpha^3 \wedge d\alpha^{\bar{3}} = 0$.

Proof: If (V_i) is a basis of $(1,0)$ vectors dual to (α^i) then (5) implies that $x_{3\bar{3}} = \Omega(V_3, J\bar{V}_3) > 0$. On the other hand, all the terms in (4), with the exception of $x_{3\bar{3}}\alpha^{3\bar{3}}$, are eliminated by two differentiations. Thus, the SKT condition is satisfied for any compatible metric iff $\square\alpha^{3\bar{3}} = 0$.

Let $\Psi = Jd\alpha^3 \wedge d\alpha^{\bar{3}}$, and observe that

$$\begin{aligned} 2\square\alpha^{3\bar{3}} &= dJd(i\alpha^{3\bar{3}}) = idJ(d\alpha^3 \wedge \alpha^{\bar{3}} - \alpha^3 \wedge d\alpha^{\bar{3}}) \\ &= d(Jd\alpha^3 \wedge \alpha^{\bar{3}} + \alpha^3 \wedge Jd\alpha^{\bar{3}}) \\ &= Jd\alpha^3 \wedge d\alpha^{\bar{3}} + d\alpha^3 \wedge Jd\alpha^{\bar{3}} \\ &= \Psi + J\Psi \end{aligned}$$

since $J^2 = 1$ on 2-forms. But Ψ is a form of type $(2,2)$ relative to J (or any other almost complex structure on the real 4-dimensional space underlying $\langle \alpha^1, \alpha^2 \rangle$), so $J\Psi = \Psi$. \square

The Lee form of a Hermitian manifold (M, J, g) real dimension $2n$ is the 1-form

$$\theta = J * d * \Omega = -Jd^*\Omega$$

where d^* is the formal adjoint of d with respect to g . The formula $*\Omega = \Omega^{n-1}/(n-1)!$ implies that $d(\Omega^{n-1}) = \theta \wedge \Omega^{n-1}$. Equivalently,

$$\xi = d\Omega - \frac{1}{n-1}\theta \wedge \Omega$$

satisfies $\xi \wedge \Omega^{n-2} = 0$, and is therefore a *primitive* form. Under a conformal change $\tilde{g} = e^{2f}g$, the Lee form transforms as

$$\tilde{\theta} = \theta + 2(n-1)df.$$

Almost Hermitian manifolds with $\theta = 0$ have in the past been called *semi-Kähler* or *cosymplectic*, though a Hermitian structure is also called *balanced* if $\theta = 0$. In this case, we are therefore talking about Hermitian manifolds of Gray-Hervella class \mathcal{W}_3 [11].

The Hermitian structure is *conformally balanced* if θ is exact, for in that case f can be chosen so that $\tilde{\theta} = 0$. However, in the invariant setting, θ is exact if and only if $\theta = 0$. Since $\langle d^*\Omega, \sigma \rangle = \langle \Omega, d\sigma \rangle$ for all 2-forms σ , the vanishing of θ is equivalent to Ω being orthogonal to the image of d in $\bigwedge^2 \mathfrak{g}^*$, a fact exploited in the study [1].

In real dimension 4, the SKT condition is equivalent to $d^*\theta = 0$. On the other hand,

Proposition 1.4 A Hermitian manifold (M, J, g) of real dimension $2n \geq 6$ can only be SKT if $\theta \neq 0$.

Proof: Suppose that $\theta = 0$. Since $d^*\Omega = *d*\Omega$ and $*\Omega$ is proportional to Ω^{n-1} , we may conclude that the $(n-1)$ form $\Phi = \Omega^{n-2} \wedge \partial\Omega$ vanishes. This implies that $\partial\Omega$ is a *primitive* $(2,1)$ form, and

$$0 = \bar{\partial}\Phi = (n-2)\Omega^{n-3} \wedge \bar{\partial}\Omega \wedge \partial\Omega + \Omega^{n-2} \wedge \bar{\partial}\partial\Omega.$$

Primitivity implies that $\Omega^{n-3} \wedge \bar{\partial}\Omega \wedge \partial\Omega$ is proportional to $\|\partial\Omega\|^2$, so $\bar{\partial}\partial\Omega = 0$ now implies that $\partial\Omega = 0$ and M is Kähler. But this is excluded in Definition 1.1. \square

It is amusing to view this result in the light of Theorem 1.2. Suppose that $n = 3$ and that (2) holds. Referring to (4), we may set

$$x_{1\bar{2}} = z, \quad x_{2\bar{1}} = -\bar{z}, \quad x_{1\bar{1}} = ix, \quad x_{2\bar{2}} = iy$$

with $x, y > 0$ and $xy > |z|^2$ to reflect positivity. The condition that Ω be orthogonal to $d\alpha^3$ becomes

$$Az + i(Bx - Cy) + D\bar{z} = 0. \quad (7)$$

Proposition 1.4 implies that this is incompatible with the inequality

$$|A|^2 + |D|^2 + 2\operatorname{Re}(\bar{B}C) \leq 0 \quad (8)$$

from (3) unless all the coefficients vanish.

The incompatibility between (7) and (8) is clear if Ω assumes the standard form

$$\Omega_0 = \frac{1}{2}i(\omega^{1\bar{1}} + \omega^{2\bar{2}} + \omega^{3\bar{3}}). \quad (9)$$

For then $z = 0$, $x = y$; thus $B = C$ and $\operatorname{Re}(B\bar{C}) = |B|^2$. The general case is far less obvious, but follows by setting $B = y = 1$ (which is no real restriction) and applying

Remark 1.5 Let $A, C, D, z \in \mathbb{C}$ be such that $x = C + i(Az + D\bar{z})$ is real. Then

$$|A|^2 + |D|^2 + 2\operatorname{Re}C \leq 0 \quad \Rightarrow \quad x \leq |z|^2.$$

To verify this, set $F = i(Az + D\bar{z})$ so that

$$x = \operatorname{Re}C + \operatorname{Re}F \leq \operatorname{Re}C + |F| \leq -\frac{1}{2}(|A|^2 + |D|^2) + (|A| + |D|)|z|.$$

If $x > |z|^2$ then

$$2|z|^2 - 2(|A| + |D|)|z| + |A|^2 + |D|^2 < 0$$

which (as a quadratic in $|z|$ with non-positive discriminant) is impossible.

2 Reducing the coefficients

This section is devoted to completing the proof of Theorem 1.2 by arriving at (2). Our starting point is [24, Theorem 1.3], which we mildly re-state as

Theorem 2.1 *A maximally complex subspace Λ of \mathfrak{g}_c^* is the $(1, 0)$ -space of a complex structure if and only if Λ has a basis (α^i) for which $d\alpha^1 = 0$ and $d\alpha^i \in \mathcal{I}(\{\alpha^1, \dots, \alpha^{i-1}\})$ for $i \geq 2$.*

Recall that \mathcal{I} stands for ‘ideal’. In dimension 6 this result provides the generic structure equations

$$\begin{aligned} d\alpha^1 &= 0, \\ d\alpha^2 &= a_1\alpha^{12} + a_2\alpha^{13} + a_3\alpha^{1\bar{1}} + a_4\alpha^{1\bar{2}} + a_5\alpha^{1\bar{3}}, \\ d\alpha^3 &= b_1\alpha^{12} + b_2\alpha^{13} + b_3\alpha^{1\bar{1}} + b_4\alpha^{1\bar{2}} + b_5\alpha^{1\bar{3}} + c_1\alpha^{23} + c_2\alpha^{2\bar{1}} + c_3\alpha^{2\bar{2}} + c_4\alpha^{2\bar{3}} \end{aligned} \quad (10)$$

where $a_i, b_j, c_k \in \mathbb{C}$. We therefore need to establish the vanishing of nine coefficients, namely a_* and b_2, b_5, c_1, c_4 . (The new names $C, D, -A, -B$ for b_3, b_4, c_2, c_3 are carefully chosen to simplify formulae in subsequent sections.) We follow a type of decision tree in order to eliminate the

coefficients one by one, but suppress the detailed calculations. The latter can be carried out by hand, though the procedure itself was refined by computer (Maple and Mathematica versions are available from the authors).

The system defined by (10) does not in general define a Lie algebra as d^2 (it is clearer to type this as dd) may not vanish. A valid solution must therefore satisfy both $dd\alpha^i = 0$ and the SKT condition $\square\Omega = 0$. On a computer, the operator $\square = -\frac{1}{2}d \circ (-iJ) \circ d$ can be executed by treating $-iJ$ as the substitution $\alpha^{\bar{j}} \mapsto -\alpha^{\bar{j}}$. If $dd = 0$ then $\square = \partial\bar{\partial}$, but in general $\square\Omega$ may have a non-zero (3,1) component. To avoid this embarrassment, we shall restrict attention to

$$\alpha^{i\bar{j}}\square\Omega = \alpha^i \wedge \alpha^{\bar{j}} \wedge \square\Omega \quad (11)$$

which we identify with its coefficient relative to the standard 6-form $\alpha^{123\bar{1}\bar{2}\bar{3}}$. (Actually we shall only ever take $i = j$ in (11).)

A first calculation reveals that $\alpha^{1\bar{1}}\square\Omega = x_{3\bar{3}}|c_1|^2$. The positive definiteness of Ω implies that $c_1 = 0$. Independently, the $\alpha^{23\bar{2}}$ component of $dd\alpha^3$ equals $|c_4|^2$, so $c_4 = 0$. To simplify matters, we now consider two cases: $a_5 \neq 0$ and $a_5 = 0$ (the choice of subscript 5 is partly a matter of taste).

Case 1. Suppose $a_5 \neq 0$. Subtracting a multiple of α^2 from α^3 we may suppose that $b_5 = 0$. Using $dd\alpha^3$ (we mean of course ‘the vanishing of $dd\alpha^3$ ’) gives $c_3 = 0$. Using $dd\alpha^2$ now gives $a_4 = 0$ and $b_1 = b_2 = 0$. To sum up, $a_4, b_1, b_2, b_5, c_1, c_3, c_4$ are all zero. If p denotes the $\alpha^{13\bar{1}}$ component of $dd\alpha^3$ then $p = a_2c_2 + b_4\bar{a}_5$ and

$$0 = \alpha^{2\bar{2}}\square\Omega = (|a_2|^2 + |a_5|^2)x_{2\bar{2}} - (\text{Re } p)x_{3\bar{3}}.$$

This contradicts $a_5 \neq 0$. \curvearrowright

Case 2. Suppose $a_5 = 0$. We divide into two subcases: $c_3 \neq 0$ and $c_3 = 0$.

Case 2.1. Suppose $c_3 \neq 0$. Inspecting only $dd\alpha^3$ already gives $a_1 = a_2 = 0$ and $b_5 = 0$. Using $dd\alpha^2$ then gives $a_4 = 0$. Returning to $dd\omega^3$ gives $b_2 = a_3 = 0$. To sum up, a_*, b_2, b_5, c_1, c_4 are all zero so we recover (2). The proof of Lemma 1.3 amounts to nothing more than an application of the equation $\alpha^{3\bar{3}}\square\Omega = 0$, which yields (3). In this way we arrive at the conclusion of Theorem 1.2 that represents the generic solution to the SKT hypothesis. \curvearrowright

Case 2.2. Suppose $c_3 = 0$. Using $dd\alpha^2$ shows that $a_4 = 0$ (even if $a_1 = a_2 = 0$). We now divide into two subsubcases: $c_2 \neq 0$ and $c_2 = 0$.

Case 2.2.1. Suppose $c_2 \neq 0$. Using $dd\alpha^2$ and $dd\alpha^3$ respectively gives $a_2 = 0$ and $b_5 = 0$. Then $\alpha^{2\bar{2}}\square\Omega = x_{3\bar{3}}|b_2|^2$, whence $b_2 = 0$. Using $dd\alpha^3$ gives $a_1 = 0$. It follows that

$$0 = \alpha^{3\bar{3}}\square\Omega = x_{3\bar{3}}(|b_1|^2 + |b_4|^2 + |c_2|^2)$$

and the solution reduces to

$$\begin{cases} d\alpha^1 &= 0 \\ d\alpha^2 &= a_3\omega^{1\bar{1}} \\ d\alpha^3 &= b_3\alpha^{1\bar{1}}. \end{cases} \quad (12)$$

By subtracting a multiple of α^3 from α^2 (or swapping the two if $b_3 = 0$), this solution can be subsumed into that of Theorem 1.2. \curvearrowright

Case 2.2.2. Suppose $c_2 = 0$ so that $c_* = 0$. The vanishing of $dd\alpha^3$ implies that $b_5 = 0$ and consequently that either $a_1 = a_2 = 0$ or $b_4 = 0$. In the former case, the vanishing of $\alpha^{2\bar{2}}\square\Omega$

and $\alpha^{3\bar{3}}\square\Omega$ gives $b_2 = 0$ and $b_1 = b_4 = 0$ respectively, and we obtain (12). The final situation to deal with is therefore that a_4, a_5, b_4, b_5, c_* all vanish. This implies that

$$\begin{aligned} 0 = \alpha^{2\bar{2}}\square\Omega &= x_{2\bar{2}}|a_2|^2 + x_{2\bar{3}}a_2\bar{b}_2 + x_{3\bar{2}}\bar{a}_2b_2 + x_{3\bar{3}}|b_2|^2 \\ &= \Omega(a_2V_2 + b_2V_3, \bar{a}_2\bar{V}_2 + \bar{b}_2\bar{V}_3), \end{aligned} \quad (13)$$

in the notation of (5). Unless $a_2 = b_2 = 0$, the restriction of Ω to the complex line spanned by $a_2V_2 + b_2V_3$ is zero, which is impossible. Exactly the same argument applied to $\alpha^{3\bar{3}}\square\Omega$ gives $a_1 = b_1 = 0$ and we are left with only a_3, b_3 non-zero, whence (12). \smile

Remark 2.2 An invariant complex structure J always induces a complex Lie algebra structure on the i -eigenspace $\mathfrak{g}^{1,0}$ of \mathfrak{g}_c . In the 6-dimensional nilpotent case, $\mathfrak{g}^{1,0}$ is either abelian or isomorphic to the complex Heisenberg algebra. In the former case, J is itself called *abelian*, and this is equivalent to asserting that d maps the subspace $\Lambda^{1,0}$ of \mathfrak{g}_c^* into $\Lambda^{1,1}$. The complex structure given by Theorem 1.2 is therefore abelian if and only if $E = 0$. So SKT does not imply that the complex structure is abelian. This is in contrast with the result that if a 2-step nilpotent Lie group admits an invariant HKT structure then the hypercomplex structure must be abelian [6].

3 Real Lie algebras

Theorem 1.2 can be used to classify explicitly the real 6-dimensional nilpotent Lie algebras admitting an SKT structure. Before explaining this, we shall introduce a formalism that will eventually help to understand and manipulate (2). Namely, we shall identify forms of type (1,1) with 2×2 matrices by setting

$$\mathbf{Y}_\alpha = A\alpha^{\bar{1}2} + B\alpha^{\bar{2}2} + C\alpha^{1\bar{1}} + D\alpha^{1\bar{2}} \quad (14)$$

where

$$\mathbf{Y} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (15)$$

so that

$$d\alpha^3 = \mathbf{Y}_\alpha + E\alpha^{12}. \quad (16)$$

The exact positioning of the coefficients may seem strange, but follows a logic that is revealed in (23) below. The subscript α indicates the basis relative to which the construction is made.

Operations on 2-forms now translate into matrix operations in a natural way. For example,

$$\mathbf{X}_\alpha \wedge \mathbf{Y}_\alpha = \text{tr}(\mathbf{X}\mathbf{Y}^\#)\alpha^{1\bar{1}2\bar{2}}$$

where

$$\mathbf{Y}^\# = \text{adj } \mathbf{Y} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = (\det \mathbf{Y})\mathbf{Y}^{-1}$$

is the transpose of the matrix of cofactors. Moreover,

$$\overline{\mathbf{Y}_\alpha} = \overline{\mathbf{Y}^\#}_\alpha$$

and so

$$\mathbf{Y}_\alpha \wedge \overline{\mathbf{Y}_\alpha} = \text{tr}(\mathbf{Y}\overline{\mathbf{Y}})\alpha^{1\bar{1}2\bar{2}}.$$

Example 3.1 The complex coefficients A, B, C, D are meant to be thought of as those in (2). In an illustration, we compute

$$\begin{aligned} d\alpha^3 \wedge d\alpha^{\bar{3}} &= (\operatorname{tr}(\mathbf{Y}\bar{\mathbf{Y}}) - |E|^2)\alpha^{1\bar{1}2\bar{2}} \\ Jd\alpha^3 \wedge d\alpha^{\bar{3}} &= (\operatorname{tr}(\mathbf{Y}\bar{\mathbf{Y}}) + |E|^2)\alpha^{1\bar{1}2\bar{2}} \\ d\alpha^3 \wedge d\alpha^3 &= \operatorname{tr}(\mathbf{Y}\mathbf{Y}^\#)\alpha^{1\bar{1}2\bar{2}} = 2(\det \mathbf{Y})\alpha^{1\bar{1}2\bar{2}}. \end{aligned} \quad (17)$$

The middle equation is a more succinct version of the SKT formula (6).

The following result relies on the classification of [24], whose notation we freely adopt.

Theorem 3.2 *A 6-dimensional nilmanifold $M = \Gamma \backslash G$ admits an invariant SKT structure if and only if the Lie algebra \mathfrak{g} is isomorphic to one of*

$$\begin{aligned} &(0, 0, 0, 0, 13 + 42, 14 + 23) \\ &(0, 0, 0, 0, 12, 14 + 23) \\ &(0, 0, 0, 0, 12, 34) \\ &(0, 0, 0, 0, 0, 12). \end{aligned}$$

In particular, \mathfrak{g} is 2-step and has first Betti number $b_1(\mathfrak{g}) = b_1(M)$ at least 4.

Proof: First note that the possibility that \mathfrak{g} is abelian is precluded by Definition 1.1. The vanishing of the real and imaginary components of $d\alpha^1, d\alpha^2$ in (2) implies immediately that $b_1(\mathfrak{g}) \geq 4$. The fact that $d\alpha^3 \in \wedge^2(\ker d)$ means (in the notation of [24]) that $(\mathfrak{g}^2)^\circ = V_2$ equals \mathfrak{g} , which is therefore 2-step.

Using the methods of [24], any 2-step nilpotent Lie algebra with $b_1 \geq 4$ is isomorphic to one of

$$\begin{aligned} \text{(i)} \quad &(0, 0, 0, 0, 12, 13) & \text{(ii)} \quad &(0, 0, 0, 0, 13 + 42, 14 + 23), \\ \text{(iii)} \quad &(0, 0, 0, 0, 12, 14 + 23) & \text{(iv)} \quad &(0, 0, 0, 0, 12, 34) \\ \text{(v)} \quad &(0, 0, 0, 0, 0, 12) & \text{(vi)} \quad &(0, 0, 0, 0, 0, 12 + 34) \end{aligned} \quad (18)$$

For example, in case (i) there is a real basis (e^i) of 1-forms for which $de^i = 0$ for $1 \leq i \leq 4$, $de^5 = e^{12}$ and $de^6 = e^{13}$. We need to eliminate (i) and (vi), and prove existence in the other cases.

Given (2), write $d\alpha^3 = \sigma^1 + i\sigma^2$ in real and imaginary components, and consider the real 2×2 matrix $\mathbf{B} = (b^{ij})$ associated to the bilinear form

$$\sigma^i \wedge \sigma^j = b^{ij}\alpha^{1\bar{1}2\bar{2}}.$$

Under the SKT assumption, equations (17) give

$$-\mathbf{B} = \begin{pmatrix} |E|^2 + \operatorname{Re} U & \operatorname{Im} U \\ \operatorname{Im} U & |E|^2 - \operatorname{Re} U \end{pmatrix} \quad (19)$$

where $U = -\det \mathbf{Y} = BC - AD$. Using de^5, de^6 in place of σ^1, σ^2 is merely a change of real basis and must yield a matrix congruent to \mathbf{B} . It follows that, in the above examples,

- \mathbf{B} is the zero matrix for (i) and (v)
- \mathbf{B} has rank 1 for (iii) and (vi)
- $\det \mathbf{B} \neq 0$ for (ii) and (iv).

In case (vi), we may rescale α^3 so that $d\alpha^3$ is real. This implies that $d\alpha^3$ is a (1,1) form and $E = 0$. Since \mathbf{B} has rank 1, the matrix (19) has zero determinant so $U = 0$. This

means that $\mathbf{B} = 0$, which is a contradiction. In case (i) we already know that $\mathbf{B} = 0$ so that $E = 0 = U$. Thus, $d\alpha^3 = \mathbf{Y}_\alpha$ is a (1,1) form with rank $\mathbf{Y} \leq 1$. But the image $\langle e^{12}, e^{13} \rangle$ of d in $\wedge^2 \mathfrak{g}^*$ is divisible by the real 1-form e^1 and must therefore be generated by $e^1 \wedge J e^1$. But this contradicts the fact that $d(\mathfrak{g}^*)$ is actually 2-dimensional.

The remaining cases are distinguished by the rank and signature of \mathbf{B} , and it is easy to check that the coefficients in (3) can be chosen to realize the four possibilities. \square

Example 3.3 The irrelevance of the choice of Hermitian metric is special to the nilpotent situation. The third Lie algebra listed in Theorem 3.2 corresponds to the product $H_3 \times H_3$ where H_3 is the real Heisenberg group. A simple example in which the SKT condition is metric dependent is provided by $H_3 \times S^3$, where S^3 is identified with $SU(2)$. We may choose a real basis of 1-forms such that

$$de^1 = 0, \quad de^2 = 0, \quad de^3 = e^{12}, \quad de^4 = e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45}.$$

Setting $\omega^1 = e^1 + ie^2$, $\omega^2 = e^3 + ie^4$, $\omega^3 = e^5 + ie^6$ gives

$$\begin{cases} d\omega^1 = 0 \\ d\omega^2 = \frac{1}{2}(i\omega^{1\bar{1}} - \omega^{3\bar{3}}) \\ d\omega^3 = \frac{1}{2}(\omega^{2\bar{3}} + \omega^{3\bar{2}}) \end{cases}$$

It follows that $d(\omega^{3\bar{3}}) = 0$ and $\partial\bar{\partial}\omega^{2\bar{2}} = d\omega^2 \wedge \overline{d\omega^2} = 0$, so (9) satisfies $\square\Omega_0 = 0$. On the other hand, $\square\omega^{1\bar{3}} = -\frac{1}{4}i\omega^{12\bar{2}\bar{3}}$ and the general 2-form (4) determines an SKT metric if and only if $x_{1\bar{3}} = -\overline{x_{3\bar{1}}} = 0$.

4 Invariant forms on Iwasawa

We first summarize the relevant facts concerning left-invariant complex structures on the Iwasawa manifold. We have attempted to give a self-contained account, though important background for §§4,5 can be found in [17, 24]. The reader is implicitly referred to these papers for further explanation of a number of points.

Let

$$G = \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} : z^i \in \mathbb{C} \right\}$$

denote the complex Heisenberg group and Γ the discrete subgroup for which z^i are Gaussian integers. We define M to be the set $\Gamma \backslash G = \{\Gamma g : g \in G\}$ of right cosets. It is a homogeneous space relative to the action of G by right translation that persists on the quotient, though we shall be interested in the projections of tensors that are invariant by left translation on G .

The complex 1-forms $\omega^1 = dz^1$, $\omega^2 = dz^3$, $\omega^3 = -dz^3 + z^1 dz^2$ satisfy $d\omega^3 = \omega^{1\bar{2}}$ and span the the (1,0) space Λ_0 of the bi-invariant complex structure J_0 on M . It is known that, in addition to J_0 , any left-invariant complex structure on G leaves invariant the real 4-dimensional subspace

$$\mathbb{D} = \langle \operatorname{Re} \omega^1, \operatorname{Re} \omega^2, \operatorname{Im} \omega^1, \operatorname{Im} \omega^2 \rangle \quad (20)$$

that arises from a principal T^2 -fibration $M \rightarrow T^4$. It therefore makes sense to consider the space $\Lambda = \langle \alpha^1, \alpha^2, \alpha^3 \rangle$ generated by the modified 1-forms

$$\begin{cases} \alpha^1 = \omega^1 + a\omega^{\bar{1}} + b\omega^{\bar{2}} \\ \alpha^2 = \omega^2 + c\omega^{\bar{1}} + d\omega^{\bar{2}} \\ \alpha^3 = \omega^3 + x\omega^{\bar{1}} + y\omega^{\bar{2}} + u\omega^{\bar{3}} \end{cases} \quad (21)$$

with $a, b, c, d, x, y, u \in \mathbb{C}$. If Λ is maximally complex it defines an invariant almost complex structure on M that we denote by $J_{\mathbf{X},x,y}$.

The effect of $J_{\mathbf{X},x,y}$ on a real basis (e^i) can be deduced by setting

$$\omega^1 = e^1 + ie^2, \quad \omega^2 = e^3 + ie^4, \quad \omega^4 = e^5 + ie^6,$$

though little is to be gained from this. The integrability condition for $J_{\mathbf{X},x,y}$ is readily expressed in terms of (21) as $d\alpha^3 \wedge \alpha^{123} = 0$ (equivalently $d\alpha^3 \wedge \alpha^{12} = 0$). This reduces to the equation

$$u = bc - ad = -\det \mathbf{X} \quad (22)$$

where

$$\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Using (14), we may consider the simple 2-form

$$\begin{aligned} \alpha^{12} &= \omega^{12} + a\omega^{\bar{1}\bar{2}} + b\omega^{\bar{2}\bar{2}} + c\omega^{1\bar{1}} + d\omega^{1\bar{2}} - u\omega^{\bar{1}\bar{2}} \\ &= \omega^{12} + c\omega^{1\bar{1}} + d\omega^{1\bar{2}} - a\omega^{2\bar{1}} - b\omega^{2\bar{2}} - u\omega^{\bar{1}\bar{2}} \\ &= \mathbf{X}_\omega + \omega^{12} - u\omega^{\bar{1}\bar{2}}. \end{aligned} \quad (23)$$

We write the characteristic polynomial of $\mathbf{X}\bar{\mathbf{X}}$ as $c(x) = x^2 - \gamma x + \delta$, so that

$$\gamma = \text{tr}(\mathbf{X}\bar{\mathbf{X}}), \quad \delta = \det(\mathbf{X}\bar{\mathbf{X}}) = |u|^2 \quad (24)$$

(notation of [17]).

The formulae

$$\begin{aligned} \alpha^{1\bar{1}2\bar{2}} &= -\alpha^{12} \wedge \alpha^{\bar{1}\bar{2}} = (1 - \gamma + \delta)\omega^{1\bar{1}2\bar{2}} = c(1)\omega^{1\bar{1}2\bar{2}} \\ \alpha^{1\bar{1}2\bar{2}3\bar{3}} &= \alpha^{1\bar{1}2\bar{2}} \wedge \alpha^{3\bar{3}} = c(1)(1 - \delta)\omega^{1\bar{1}2\bar{2}3\bar{3}} \end{aligned} \quad (25)$$

express volume changes associated to a switch of basis from ω to α . As a consequence, $\Lambda \cap \bar{\Lambda} = \{0\}$ if and only if

$$\delta \neq 1 \quad \text{and} \quad c(1) \neq 0 \quad (26)$$

and these are the conditions that ensure that $J_{\mathbf{X},x,y}$ is well defined. From now on, we assume that (22) and (26) hold. For simplicity, we also suppose $x = y = 0$, and denote $J_{\mathbf{X},0,0}$ simply by $J_{\mathbf{X}}$. It will become obvious that reducing to this case causes no loss of generality, the key point being that $d\alpha^3$ (and so (29) below) does not involve x, y . An underlying reason for the irrelevance of x, y is provided in §5.

One may invert (23) so as to express ω^{12} in terms of the α 's. Up to the overall factor $c(1)$, the corresponding formula is given by reversing the signs of a, b, c, d :

Lemma 4.1 $c(1)\omega^{12} = -\mathbf{X}_\alpha + \alpha^{12} - u\alpha^{\bar{1}\bar{2}}$.

Proof: Consider the bases $(\omega^1, \omega^2, \omega^{\bar{1}}, \omega^{\bar{2}}), (\alpha^1, \alpha^2, \alpha^{\bar{1}}, \alpha^{\bar{2}})$. The second is related to the first by the block matrix

$$\mathbf{M} = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{X} \\ \hline \overline{\mathbf{X}} & \mathbf{I} \end{array} \right)$$

in a row-by-row fashion. Set $\mathbf{Z} = (\mathbf{I} - \mathbf{X}\overline{\mathbf{X}})^{-1}$ so that $c(1) = \det(\mathbf{Z}^{-1})$. By first observing that

$$\mathbf{X}\overline{\mathbf{Z}} = \mathbf{Z}(\mathbf{Z}^{-1}\mathbf{X})\overline{\mathbf{Z}} = \mathbf{Z}(\mathbf{I} - \mathbf{X}\overline{\mathbf{X}})\mathbf{X}\overline{\mathbf{Z}} = \mathbf{Z}\mathbf{X}(\mathbf{I} - \overline{\mathbf{X}}\mathbf{X})\overline{\mathbf{Z}} = \mathbf{Z}\mathbf{X},$$

it is easy to verify that

$$\mathbf{M}^{-1} = \left(\begin{array}{c|c} \mathbf{Z} & -\mathbf{Z}\mathbf{X} \\ \hline -\overline{\mathbf{X}}\mathbf{Z} & \overline{\mathbf{Z}} \end{array} \right).$$

The coefficients $1, c, d, -a, -b, -u$ featuring in (23) are the 2×2 minors of the 2×4 matrix $\mathbf{M} = (\mathbf{I} | \mathbf{X})$, corresponding to the Mathematica command `Minors[M, 2]`. The coefficients of ω^{12} are therefore given by

$$\text{Minors}[(\mathbf{Z} | -\mathbf{Z}\mathbf{X}), 2] = (\det \mathbf{Z}) \text{Minors}[(\mathbf{I} | -\mathbf{X}), 2],$$

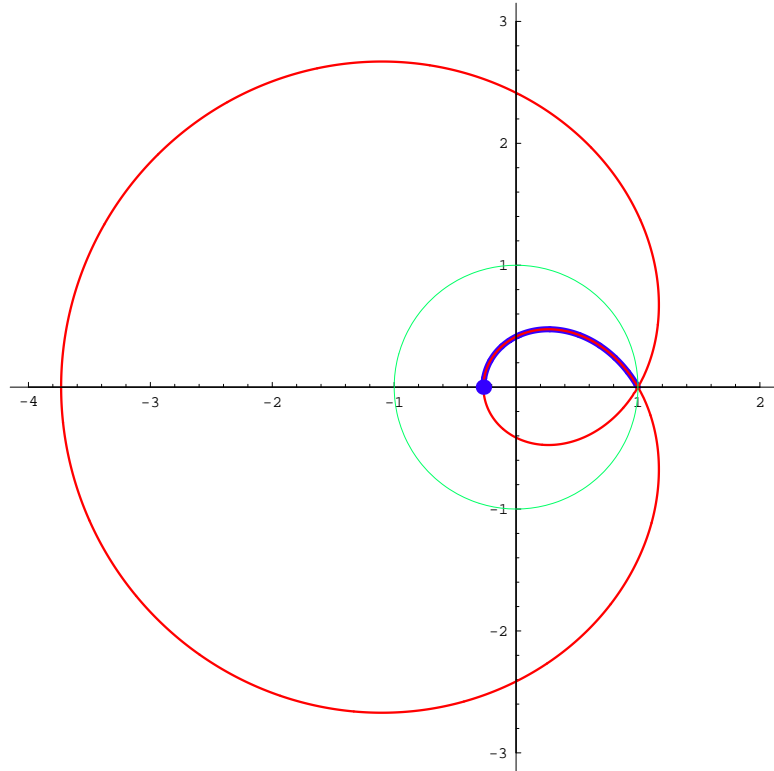
using a well-known property of minors. ◊

Proposition 4.2 *The complex structure J_X is SKT if and only if the eigenvalues of $\mathbf{X}\overline{\mathbf{X}}$ satisfy the equation*

$$(1 + |z|^2) |1 + z|^2 = 8|z|^2 \tag{27}$$

illustrated below.

It follows easily that all points of the curve are realized except for $z = 1$ that is excluded by (26) (see Theorem 5.2 below).



Proof: We first prove that $J_{\mathbf{X}}$ is SKT if and only if

$$1 - 6\delta + \delta^2 + \gamma + \gamma\delta = 0. \quad (28)$$

Using Lemma 4.1,

$$\begin{aligned} c(1)d\alpha^3 &= c(1)\omega^{12} + uc(1)\omega^{\overline{12}} \\ &= -\mathbf{X}_\alpha + \alpha^{12} - u\alpha^{\overline{12}} + u(-\overline{\mathbf{X}}_\alpha^\# + \alpha^{\overline{12}} - \overline{u}\alpha^{12}) \\ &= -(\mathbf{X}_\alpha + u\overline{\mathbf{X}}_\alpha^\#) + (1 - \delta)\alpha^{12} \\ &= (-\mathbf{X} + \delta\overline{\mathbf{X}}^{-1})_\alpha + (1 - \delta)\alpha^{12}. \end{aligned} \quad (29)$$

Comparing this with (16) yields

$$c(1)\mathbf{Y} = -\mathbf{X} + \delta\overline{\mathbf{X}}^{-1}, \quad c(1)E = 1 - \delta. \quad (30)$$

Thus

$$c(1)^2 \operatorname{tr}(\mathbf{Y}\overline{\mathbf{Y}}) = \operatorname{tr}(\mathbf{X}\overline{\mathbf{X}} - 2\delta\mathbf{I} + \delta^2(\mathbf{X}\overline{\mathbf{X}})^{-1}) = \gamma - 4\delta + \delta^2(\gamma/\delta)$$

and (28) follows from Lemma 1.3 and (17).

Now let λ, μ denote the eigenvalues of $\mathbf{X}\overline{\mathbf{X}}$, so that $\gamma = \lambda + \mu$ and $\delta = \lambda\mu$. Thus, $\lambda\mu \geq 0$ so λ and μ cannot be real with opposite signs. It is an elementary but non-trivial fact that if λ, μ are real and non-positive then they are equal [14, Problem 6, §4.6]. Under the SKT assumption,

$$\gamma = \frac{-1 + 6\delta - \delta^2}{1 + \delta}$$

is non-negative if and only if $3 - 2\sqrt{2} \leq \delta \leq 3 + 2\sqrt{2}$. The eigenvalues λ, μ can only be real if

$$0 \leq (\lambda - \mu)^2 = \gamma^2 - 4\delta = \left(\frac{1 - \delta}{1 + \delta}\right)^2 (\delta^2 - 14\delta + 1)$$

which implies that $\delta \leq 7 - \sqrt{48}$ or $\delta \geq 7 + \sqrt{48}$. The various inequalities are incompatible, and there are no solutions with $\lambda, \mu > 0$ and it follows that $\lambda = \overline{\mu}$. Thus,

$$1 - \gamma + \delta = 1 + \lambda + \overline{\lambda} + |\lambda|^2 = |1 + \lambda|^2 > 0$$

and (28) translates into (27). ◊

Remark 4.3 It is easy to check that equation (27) is unchanged by substituting $1/z$ in place of z . We shall see below that this corresponds to reversing the sign of J . Another curve with a similar shape invariant by $z \mapsto 1/z$ and $z \mapsto \overline{z}$ is given in polar coordinates by $r = e^{\sin \theta}$. (This can be generalized by replacing $\sin \theta$ by an odd Fourier series.) If we shift the origin to the point $z = 1$ of self-intersection, (27) becomes

$$r = -3 \cos \theta \pm \sqrt{2 + \cos^2 \theta}.$$

A true limaçon has the somewhat simpler equation $r = 2 \cos \theta + 1$, but the corresponding translate is not invariant under inversion.

5 Moduli space interpretation

Let \mathcal{C} denote the set of all invariant complex structures on the Iwasawa manifold M . This is a subset of the set of all almost complex structures on \mathfrak{g} (equivalently, maximally complex subspaces of \mathfrak{g}_c), that can in turn be identified with the homogeneous space $GL(6, \mathbb{R})/GL(3, \mathbb{C})$. It is known that \mathcal{C} has four connected components, and these can be described as follows.

Changing the sign of J corresponds to an overall reversal of orientation, and corresponds to the transformation

$$(\alpha^1, \alpha^2, \alpha^3) \mapsto (\alpha^{\bar{1}}, \alpha^{\bar{2}}, \alpha^{\bar{3}}). \quad (31)$$

This identifies the components of \mathcal{C} in pairs. Let \hat{J} denote the restriction of J to the real subspace (20) underlying $\langle \omega^1, \omega^2 \rangle$. The remaining two components of \mathcal{C} are distinguished by the orientation of \hat{J} or equivalently, by (25), the sign of $c(1)$. For example

$$\mathcal{C}_+ = \{J_{\mathbf{X}, x, y} : x, y \in \mathbb{C}, |u| < 1, c(1) > 0\} \quad (32)$$

is the connected component that contains J_0 .

To fully understand the SKT constraint on complex structures on M , we need to describe various group actions on \mathcal{C} and their effect on the matrix \mathbf{X} .

(i) Involution. Referring to (21), can say that the almost complex structure \hat{J} is represented by the matrix $(\mathbf{I} | \mathbf{X})$. This is replaced by

$$(\bar{\mathbf{X}} | \mathbf{I}) \equiv (\mathbf{I} | \bar{\mathbf{X}}^{-1}) \quad (33)$$

under (31) when J, \hat{J} are replaced by $-J, -\hat{J}$. The equivalence relation \equiv indicates premultiplication on both halves by an invertible matrix, and reflects a re-adjustment

$$(\alpha^{\bar{1}}, \alpha^{\bar{2}}) \mapsto (\tilde{\alpha}^1, \tilde{\alpha}^2)$$

into the row echelon form (21) in which $\tilde{\alpha}^i$ has leading term ω^i . Notice that $\det(\bar{\mathbf{X}}^{-1}) = -1/\bar{u}$, consistent with setting $\tilde{\alpha}^3 = \alpha^{\bar{3}}/\bar{u} = \omega^3 + \omega^{\bar{3}}/\bar{u}$. It now follows that

$$-(J_{\mathbf{X}}) = J_{\bar{\mathbf{X}}^{-1}}.$$

If the eigenvalues of \mathbf{X} are $\lambda, \bar{\lambda}$ then those of $\bar{\mathbf{X}}^{-1}$ are $1/\bar{\lambda}, 1/\lambda$, and this justifies Remark 4.3 and helps to explain why the solution curve has a limaçon shape. The inner part $|z| < 1$ corresponds to solutions in the same component as J_0 and the outer part $|z| > 1$ to the component of $-J_0$. One is an inversion of the other. The thicker part with $|z| < 1$ and $\text{Im } z \geq 0$ fits (after rotation by 45°) into the semicircular region of the diagram in [17], and represents the SKT solutions most faithfully. (The rest of the complex plane above does not fit into the diagram that is a schematic pasting of the real and complex plane best contemplated in 3 dimensions.)

The integrability condition for a left-invariant almost complex structure on an arbitrary Lie algebra \mathfrak{g} is invariant under the action of the automorphism group

$$\mathcal{G} = \{f \in \text{End } \mathfrak{g} : [f(v), f(w)] = f[v, w].\}$$

In the Iwasawa case, \mathcal{G} can be identified with the semidirect product $GL(2, \mathbb{C}) \ltimes \mathbb{C}^2$ consisting of complex 3×3 matrices

$$\left(\begin{array}{c|c} \mathbf{P} & \mathbf{q} \\ \hline 0 & \det \mathbf{P} \end{array} \right), \quad \mathbf{P} \in GL(2, \mathbb{C}), \quad \mathbf{q} \in \mathbb{C}^2 \quad (34)$$

(see [24]). The action on \mathcal{C} of \mathbf{P} and \mathbf{q} can be considered separately.

(ii) Right translation. The normal subgroup \mathbb{C}^2 of \mathcal{G} can be identified with the group $\text{Ad } G$ of inner automorphisms of \mathfrak{g} . Since $J \in \mathcal{C}$ is by definition left-invariant, $\text{Ad}(g)J$ equals the right translate of J by $g^{-1} \in G$. Whilst J_0 is fixed by this action (G being a *complex* Lie group), all the other orbits have positive dimension. Moreover, if $u \neq 0$ then $J_{\mathbf{X},x,y}$ lies in the same orbit as $J_{\mathbf{X}}$ for any $x, y \in \mathbb{C}$. This is explained in [17], but to avoid duplication we next insert an infinitesimal version of this fact.

Remark 5.1 The action of the diffeomorphism group on a complex structure J is detected by the image of

$$\bar{\partial}: \Gamma(T \otimes \Lambda^{0,0}) \rightarrow \Gamma(T \otimes \Lambda^{0,1})$$

where T denotes the holomorphic tangent bundle of (M, J) . Restricting to invariant tensors, a p -form f with values in T can be regarded as a linear mapping $\Lambda^{1,0} \rightarrow \Lambda^{p,0}$ and $\bar{\partial}f$ is calculated by means of the formula

$$(\bar{\partial}f)(\alpha^i) = \bar{\partial}(f(\alpha^i)) - f(\bar{\partial}\alpha^i) = -f(\bar{\partial}\alpha^i).$$

Since $\bar{\partial}\alpha^3 = \mathbf{Y}_\alpha$, the dimension of $\text{Im } \bar{\partial}$ coincides with the rank of \mathbf{Y} which equals 2 if and only if $u \neq 0$ (see (16) and (30)). This discussion is relevant to the jumping of Hodge numbers discussed in [19].

In the SKT context, it is no restriction to impose the condition $\det \mathbf{X} \neq 0$ since $z = 0$ is not a solution of (27). In the light of the above remarks, one might call the structures $J_{\mathbf{X},x,y}$ for which $\det \mathbf{X} \neq 0$ the *stable* points of \mathcal{C} . Since G acts smoothly on M by right translation, $J_{\mathbf{X}}$ and $J_{\mathbf{X},x,y}$ then determine the same point in the moduli space of complex structures modulo diffeomorphism, and for calculations we may assume that $x = y = 0$ so that $J = J_{\mathbf{X}}$ is completely determined by \hat{J} and the matrix \mathbf{X} .

(iii) Outer automorphisms. The quotient $\mathcal{G}/\text{Ad}(G)$ can be identified with $GL(2, \mathbb{C})$, an element of which acts by a change of basis

$$\begin{cases} \omega^1 \mapsto p_1^1 \omega^1 + p_2^1 \omega^2 \\ \omega^2 \mapsto p_1^2 \omega^1 + p_2^2 \omega^2 \end{cases}$$

By analogy to (33), the matrix $(\mathbf{I} | \mathbf{X})$ representing \hat{J} is transformed into

$$(\mathbf{P} | \mathbf{X}\bar{\mathbf{P}}) \equiv (\mathbf{I} | \mathbf{P}^{-1}\mathbf{X}\bar{\mathbf{P}})$$

and therefore a left action on \mathcal{C} is defined by

$$\mathbf{P}^{-1} \cdot J = J_{\mathbf{P}^{-1}\mathbf{X}\bar{\mathbf{P}}}.$$

The presence of $\det \mathbf{P}$ in (34) ensures that the extension from \mathbb{D} to \mathfrak{g} is well defined.

The remaining action by $GL(2, \mathbb{C})$ is less geometrical and the resulting quotient \mathcal{C}/\mathcal{G} has singularities, an example of which is given after Example 5.3. Now

$$\mathbf{X} \mapsto \mathbf{P}^{-1}\mathbf{X}\bar{\mathbf{P}}, \quad \mathbf{P} \in GL(n, \mathbb{C}) \quad (35)$$

is an action that gives rise to the theory of *consimilarity* for $n \times n$ matrices \mathbf{X} . If $[\mathbf{X}]$ denotes the consimilarity class of \mathbf{X} (i.e. an orbit for the above action) and $[\mathbf{Y}]$ the similarity class of \mathbf{Y} , there is a well defined mapping

$$\phi: [\mathbf{X}] \mapsto [\mathbf{X}\bar{\mathbf{X}}]. \quad (36)$$

This is not a bijection as $\mathbf{X}\bar{\mathbf{X}}$ can be zero without \mathbf{X} being zero. However, the general theory of consimilarity developed in [14, §4.6] implies that ϕ restricts to a bijection between classes subject to an equal rank condition. In the simple case of $n = 2$, we may condense the discussion of this section into

Theorem 5.2 *An invariant complex structure J on the Iwasawa manifold is SKT if and only if it equals $J_{\mathbf{X},x,y}$ where*

$$\mathbf{X} = \mathbf{P}^{-1} \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \bar{\mathbf{P}}$$

with $\mathbf{P} \in GL(2, \mathbb{C})$, z a solution of (27) different from 1 and $x, y \in \mathbb{C}$. Moreover, such structures lie in the connected components of J_0 and $-J_0$ in \mathcal{C} .

Proof: If \mathbf{X} has the form given then $\mathbf{X}\bar{\mathbf{X}}$ is similar to a diagonal matrix with entries z, \bar{z} . Since $z \neq 1$, $J_{\mathbf{X},x,y}$ is well defined. Bearing in mind (from (ii)) that x, y are irrelevant, Proposition 4.2 implies that $J_{\mathbf{X},x,y}$ is SKT.

Conversely, suppose that $J \in \mathcal{C}$ is SKT. The complex structures $J_{\mathbf{X},x,y}$ represented by (21) constitute a dense set of \mathcal{C} , and the only missing points are those arising when one or more of the coefficients become infinite. The SKT condition involves only $d\alpha^3$, so the only case potentially not covered by (21) is that in which α^3 belongs to the span of ω^3 and \mathbb{D} . But then $Jd\alpha^3 \wedge d\alpha^3 \neq 0$ and J cannot be SKT by Lemma 1.3. The statement about connected components now follows from the fact that the solutions in Proposition 4.2 all satisfy $c(1) > 0$ (see (32)).

We may therefore suppose that $J = J_{\mathbf{X},0,0} = J_{\mathbf{X}}$, and that the eigenvalues z, \bar{z} of $\mathbf{X}\bar{\mathbf{X}}$ satisfy (27). Assume firstly that z is not real. Choose $\mathbf{P} \in GL(2, \mathbb{C})$ such that $\mathbf{Y} = \mathbf{P}\mathbf{X}\bar{\mathbf{P}}^{-1}$ satisfies

$$\mathbf{Y}\bar{\mathbf{Y}} = \mathbf{P}(\mathbf{X}\bar{\mathbf{X}})\bar{\mathbf{P}}^{-1} = \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

In the notation (15), $A\bar{B} + B\bar{D} = 0 = A\bar{C} + C\bar{D}$, which implies

$$A(\bar{C} - \bar{B}C) = 0 = D(\bar{C} - \bar{B}C),$$

and $A = D = 0$. We can now premultiply \mathbf{P} by a diagonal matrix so as to convert B to z and C to 1, as required.

Now assume that $z \in \mathbb{R}$. The equation $\mathbf{Y}\bar{\mathbf{Y}} = \begin{pmatrix} z & 0 \\ 1 & z \end{pmatrix}$ would imply that $B = 0$ and $z > 0$, which contradicts (27). Thus, $\mathbf{X}\bar{\mathbf{X}}$ is again diagonalizable and we need to solve

$$\mathbf{Y}\bar{\mathbf{Y}} = z\mathbf{I}, \tag{37}$$

that implies that $|A| = |D|$. But we can find $\mathbf{Q} \in GL(2, \mathbb{C})$ such that $\tilde{\mathbf{Y}} = \mathbf{Q}\mathbf{Y}\bar{\mathbf{Q}}^{-1}$ has its last entry \tilde{D} zero. Since $\tilde{\mathbf{Y}}$ satisfies (37) in place of \mathbf{Y} , we also have $\tilde{A} = 0$, and we can modify \mathbf{Q} so that $\tilde{C} = 1$ and $\tilde{\mathbf{Y}} = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$. \circ

Example 5.3 The point $z = 1$ on the curve is not admissible, because in this case

$$\alpha^1 = \omega^1 + \omega^2 = \bar{\alpha}^2,$$

and any corresponding tensor J is degenerate. The solutions

$$z = -2 \pm \sqrt{3} \in \mathbb{R}$$

are admissible, but since $-2 - \sqrt{3} = 1/(-2 + \sqrt{3})$ the J 's coincide up to an overall complex conjugation, in accordance with the discussion in (i). These are the simplest solutions to the SKT equation. The purely imaginary solutions are $z = \pm i\sqrt{3 \pm 2\sqrt{2}}$, and we presume that there are no rational solutions to (27).

Let H_z denote the stabilizer of $\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$ in $GL(2, \mathbb{C})$ for the action (35). It is easy to verify that if $z \in \mathbb{R}$ and $z \neq 0$ then

$$H_z = \left\{ \begin{pmatrix} a & -\bar{c}x \\ c & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 \neq 0 \right\} \cong GL(1, \mathbb{H}) \quad (38)$$

If $z \in \mathbb{C} \setminus \mathbb{R}$ then H_z is isomorphic to the common subgroup \mathbb{C}^* obtained by setting $c = 0$ in (38). It follows that the SKT structures $z = -2 \pm \sqrt{3}$ (one of which is a blob in the Figure) represent singular points in \mathcal{C}/\mathcal{G} .

6 Balanced structures and reduced holonomy

Let M be a nilmanifold of dimension $2n$ with an invariant complex structure. The basis arising from Theorem 2.1 furnishes us with a closed form

$$\eta = \alpha^{12 \cdots n} \quad (39)$$

of type $(n, 0)$. Being invariant, η also has constant norm, so it is natural to ask whether it is parallel with respect to a suitable connection.

Throughout this final section, ∇ denotes any Hermitian connection, that is one satisfying $\nabla g = 0$ and $\nabla J = 0$. We use D to denote the Levi-Civita connection, so that $Dg = 0$ but in general $DJ \neq 0$. The holonomy of ∇ is contained in $SU(n)$ if and only if $\nabla \eta = 0$.

Proposition 6.1 *Suppose that (J, g) is an invariant Hermitian structure on a nilmanifold M of dimension $2n$. Then the holonomy of ∇ reduces to $SU(n)$ iff g is balanced.*

Proof: Since ∇ preserves the complex structure, we may write

$$\nabla \eta = i\beta \otimes \eta.$$

The connection 1-form $i\beta$ belongs to the Lie algebra $\mathfrak{u}(1)$ and β is *real* (this can be checked directly by applying ∇ to $\|\eta\|^2 = \langle \eta, \bar{\eta} \rangle$).

The tensor DJ can be identified with both $D\Omega$ and $d\Omega$, and gets converted into the torsion of ∇ in passing from D to ∇ . It follows that

$$\nabla \eta = D\eta + d\Omega \cdot \eta$$

where \cdot stands for a suitable linear mapping. If we skew-symmetrize both sides, the Levi-Civita connection D gets converted to d , and extracting $(n, 1)$ components gives

$$\beta^{0,1} \wedge \eta = d\eta + (d\Omega \cdot \eta)^{n,1} = (\bar{\partial}\Omega \cdot \eta)^{n,1}. \quad (40)$$

The last expression determines a $U(n)$ equivariant mapping

$$\Lambda^{1,2} \otimes \Lambda^{n,0} \rightarrow \Lambda^{n,1} \cong \Lambda^{n-1,0} \quad (41)$$

and the $(2,1)$ -component $\partial\Omega$ is omitted because it can only define the zero map $\Lambda^{n,0} \rightarrow \Lambda^{n,1}$. But (41) is equivalent to the $SU(n)$ -equivariant contraction $\Lambda^{1,2} \rightarrow \Lambda^{0,1} \cong \Lambda^{n-1,0}$ that extracts the ‘trace’ of $\bar{\partial}\Omega$. It follows that η is parallel if and only if $\theta = 0$. \square

Combined with Proposition 1.4, this is consistent with a result of [22], namely that a compact $2n$ -dimensional manifold endowed with a conformally balanced Hermitian structure for which $\square\Omega = 0$ and ∇ has holonomy in $SU(n)$ is in fact Kähler and therefore Calabi-Yau.

Returning to six dimensions, it is an easy matter to list Lie algebras admitting a balanced KT structure for which the holonomy of ∇ therefore reduces. We restrict attention to those considered in §3. Case (v) is realized by taking $C = 1$ and A, B, D, E zero which is incompatible with (7). But this is the only one excluded:

Corollary 6.2 *Each of the Lie algebras*

$$\begin{aligned} &(0, 0, 0, 0, 13 + 42, 14 + 23) \\ &(0, 0, 0, 0, 12, 14 + 23) \\ &(0, 0, 0, 0, 12, 34) \\ &(0, 0, 0, 0, 0, 12, 13) \\ &(0, 0, 0, 0, 0, 12 + 34) \end{aligned}$$

admits a Hermitian structure for which ∇ has $SU(3)$ holonomy.

Proof: It suffices to consider the standard metric corresponding to (9). The ‘balanced’ condition $B = C$ is then independent of A, D, E , and this gives us the flexibility to realize the various cases as follows:

If $B = C = 0$ then

$A = D = 0$ and $E = 1$ gives case (ii)

$A = D = 1$ and $E = 0$ gives (vi)

$A = E = 1$ and $D = 0$ gives (i)

$A = D = 1$ and $E = 2$ gives (iii)

If $B = C = i$ then

$A = 1, D = -1$ and $E = 2$ gives (iv). ◻

By analogy to the Kähler case, the reduction of the holonomy of ∇ to $SU(n)$ is equivalent to the vanishing of the Ricci form

$$\rho(X, Y) = \frac{1}{2} \sum_{i=1}^n g(R_{XY}e_i, Je_i), \quad X, Y \in \mathfrak{g}$$

where R is the curvature tensor of ∇ . The holonomy reduction of Proposition 6.1 has the advantage of dealing with a positive definite metric, although the connection is not torsion-free. However, the situation is just as intriguing for a pseudo-Riemannian metric and the Levi-Civita connection.

Lemma 6.3 *Suppose that (J, h) is an invariant pseudo-Kähler structure on a nilmanifold M of real dimension $2n$. Then the Ricci tensor of h vanishes.*

Proof: The hypothesis means that M admits both an invariant complex structure J and a closed 2-form Ω for which h (defined by (1) with h in place of g) is a pseudo-Riemannian metric. In the presence of a compatible complex structure, the equation $d\Omega = 0$ is sufficient to imply that $D\Omega = 0$ and thus $DJ = 0$, just as in the familiar (positive-definite) Kähler case. We can therefore apply (40) with $d\Omega = 0$ to deduce that $D\eta = 0$, and the result follows. ◻

As an example,

Proposition 6.4 *The Lie algebra $(0, 0, 0, 0, 13 + 42, 14 + 23)$ associated to the Iwasawa manifold admits a pseudo-Kähler metric h for which D has holonomy in $SU(2, 1)$ and h is Ricci-flat but not flat.*

Proof: We take

$$\Omega = e^{16} + e^{25} + e^{34}, \quad Je^1 = e^2, \quad Je^3 = -e^4, \quad Je^5 = -e^6.$$

It is necessary to check that

$$d\Omega = -e^1 \wedge de^6 - e^2 \wedge de^5 = -e^1 \wedge (e^{14} + e^{23}) - e^2 \wedge (e^{13} + e^{42}) = 0$$

and that

$$J \cdot \Omega = (Je^1) \wedge (Je^6) + (Je^2) \wedge (Je^5) + (Je^3) \wedge (Je^4) = \Omega.$$

Setting $\alpha^1 = e^1 + ie^2$, $\alpha^2 = e^3 - ie^4$, $\alpha^3 = e^5 - ie^6$ gives $d\alpha^3 = \alpha^{1\bar{2}}$, so J is integrable and abelian. Observe that \hat{J} has negative orientation, so its connected component in \mathcal{C} does not contain $\pm J_0$. The resulting pseudo-metric (1) assumes the matrix form

$$(h_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and has signature $(4, 2)$.

A computation reveals that

$$R_{1212} = \sum_{m=1}^6 h_{1m} R^m_{212} = R^5_{212} = -2,$$

and so the full Riemann curvature is non-zero. ◻

A contrasting situation is provided by the Lie algebra $(0, 0, 0, 0, 0, 12 + 34)$. This carries no invariant symplectic form [24], and does not therefore possess an invariant pseudo-Kähler metric. We suspect that the remaining four algebras in (18) do admit pseudo-Kähler metrics.

A completely different class of Ricci-flat structures on nilmanifolds, indeed ones with signature $(2n, 2n)$ that are not invariant, are discussed in [7].

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