1)

$A = \{x \in \mathbb{R} \text{ t.c. } x \leq 4\} = (-\infty, 4]$

$B = \{x \in \mathbb{R} \text{ t.c. } 3x^2 > 0\} = \mathbb{R} \setminus \{0\}$

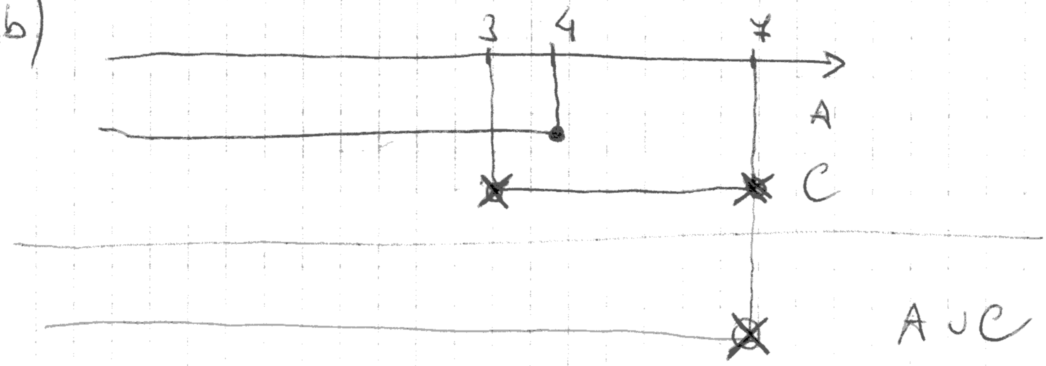
$C = (3, 7)$

a) $A \subseteq B$? no, perché $0 \in A$ ma $0 \notin B$

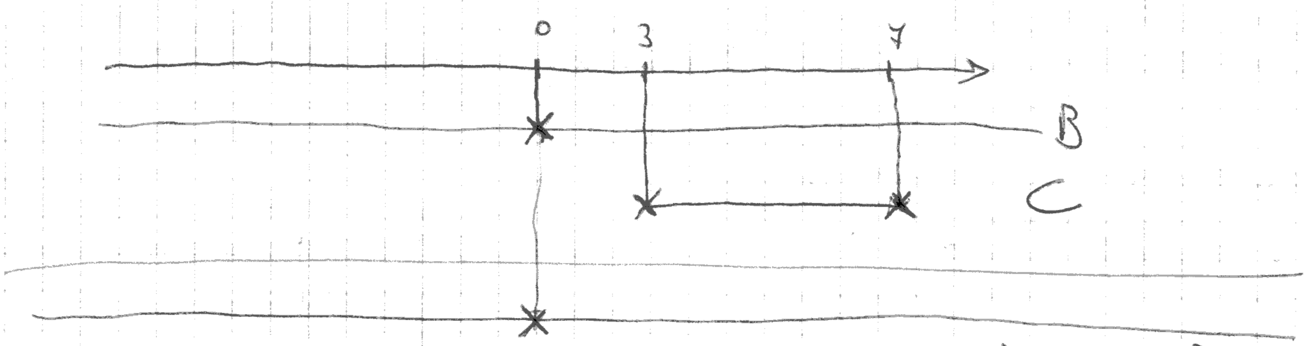
$C \subseteq B$? sì perché tutti i numeri da 3 a 7 stanno in $\mathbb{R} \setminus \{0\}$

~~ALT. Sì perché l'unico numero numero che non sta in B è 0, ed esso non sta neanche in (3,7)~~

b)

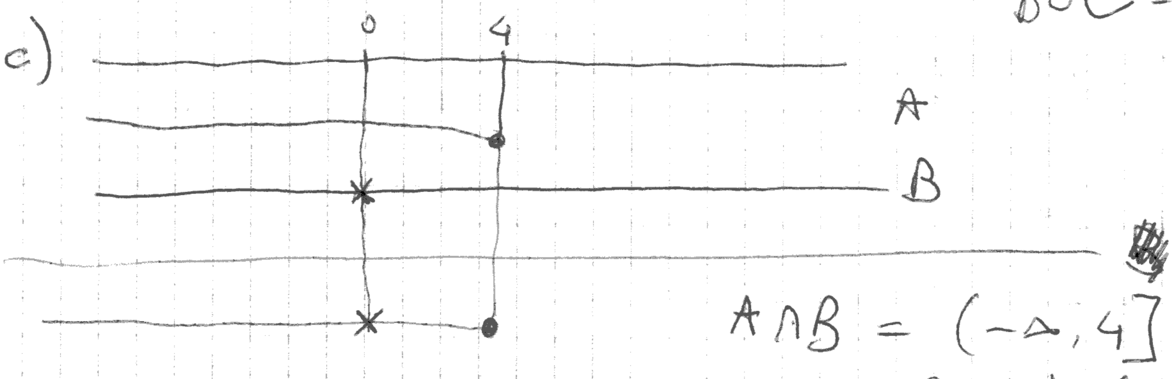


$A \cup C = (-\infty, 7)$

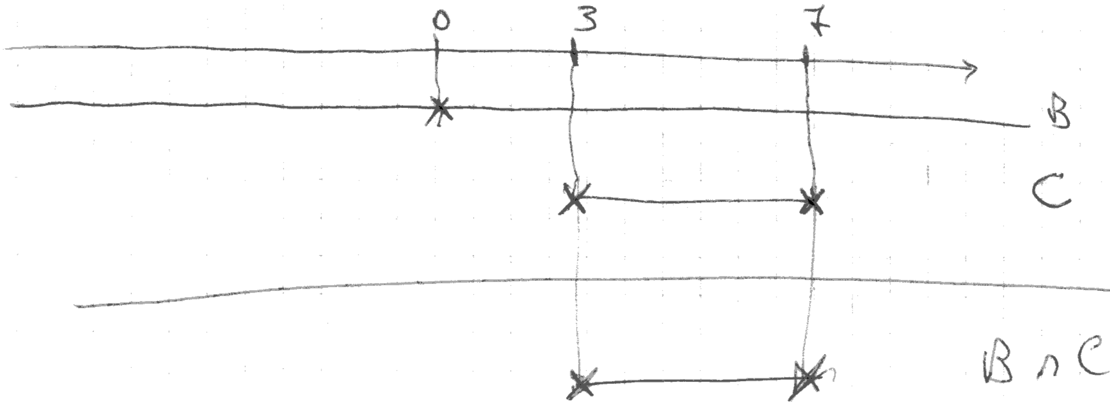


$B \cup C = \mathbb{R} \setminus 0 = B$
(coerente con $C \subseteq B$)

c)



$A \cap B = (-\infty, 4] \setminus \{0\} =$
$= (-\infty, 0) \cup (0, 4]$



$B \cap C = (3,7) = C$
(coerente con $C \subseteq B$)

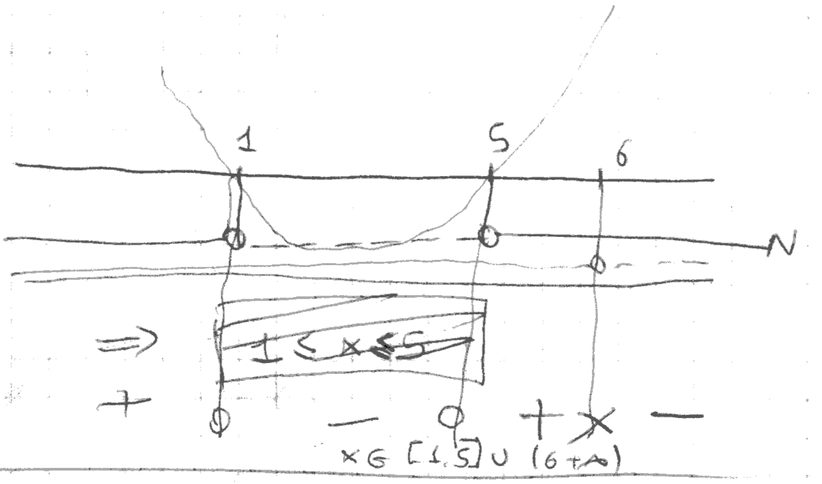
2) $\dfrac{5}{6-x} \leq x$ $\quad$ $CE = \{x \neq 6\}$

$\dfrac{5}{6-x} \leq \dfrac{6x - x^2}{6-x}$

$\dfrac{x^2 - 6x + 5}{6-x} \leq 0$

$\dfrac{(x-5)(x-1)}{6-x}$

(somma e prodotto, formula o verifica diretta con $\pm 1, \pm 5$)

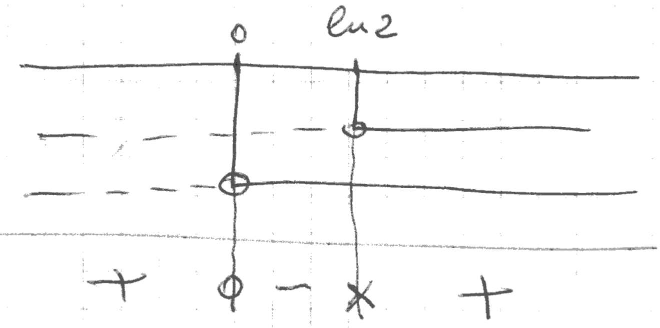


3) $f(x) = \dfrac{e^x - 1}{e^x - 2}$

a) $e^x - 2 \neq 0 \iff e^x \neq 2 \iff \boxed{x \neq \ln 2}$ $\quad$ $CE = \mathbb{R} \setminus \{\ln 2\}$

b) $e^x - 2 > 0 \iff x > \ln 2$

$e^x - 1 > 0 \iff x > 0$



c) $\lim_{x \to -\infty} f(x) = \dfrac{-1}{-2} = \dfrac{1}{2}$

$\lim_{x \to \ln 2^-} f(x) = -\infty$

$\lim_{x \to \ln 2^+} f(x) = +\infty$

$\lim_{x \to +\infty} f(x) \overset{H}{=} \lim_{x \to +\infty} \dfrac{e^x}{e^x} = 1$

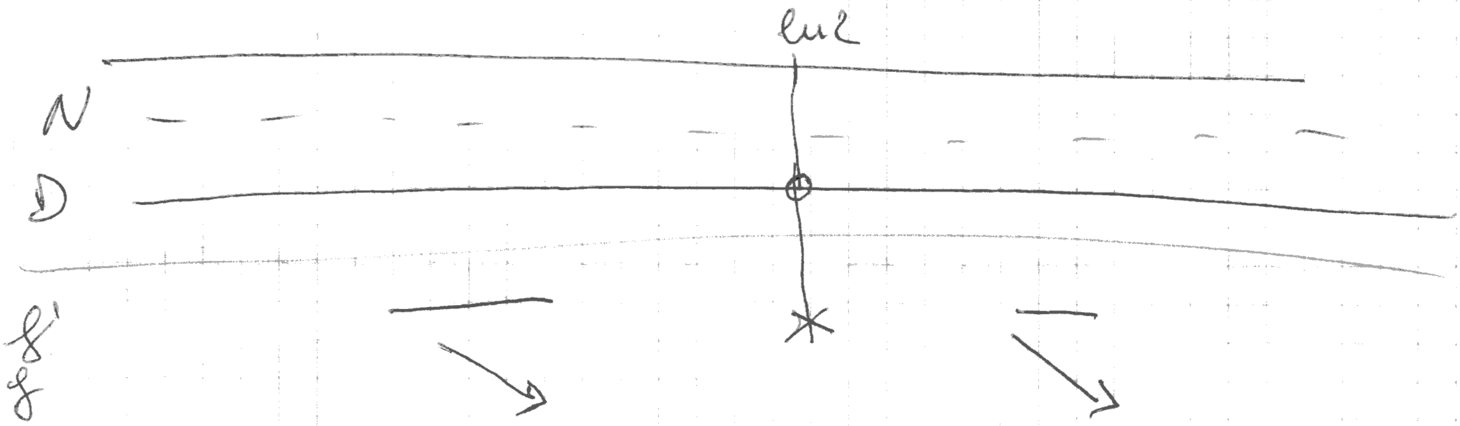
d) $$f'(x) = \frac{e^x(e^x-2)-(e^x-1)e^x}{(e^x-2)^2} =$$

$$= \frac{\cancel{e^{2x}} - 2e^x + e^x}{(e^x-2)^2} = \frac{-e^x}{(e^x-2)^2}$$

segno $f'(x)$:

$-e^x < 0$ sempre. $\cancel{e^x > 0}$

$(e^x-2)^2 \geq 0$ sempre. $(e^x-2)^2 = 0 \iff x = \ln 2$



e) no MIN, no MAX, perchè $f' \neq 0$ sempre e f è ~~continua~~ derivabile in tutto il suo insieme di definizione.

f) $$f''(x) = -\left[\frac{e^x(e^x-2)^2 - 2e^{2x}(e^x-2)}{(e^x-2)^4}\right] =$$

$$= \frac{\cancel{2}e^{3x} - \cancel{4e^{2x}} - \cancel{e^{3x}} + \cancel{4e^{2x}} - 4e^x}{(e^x-2)^4} =$$

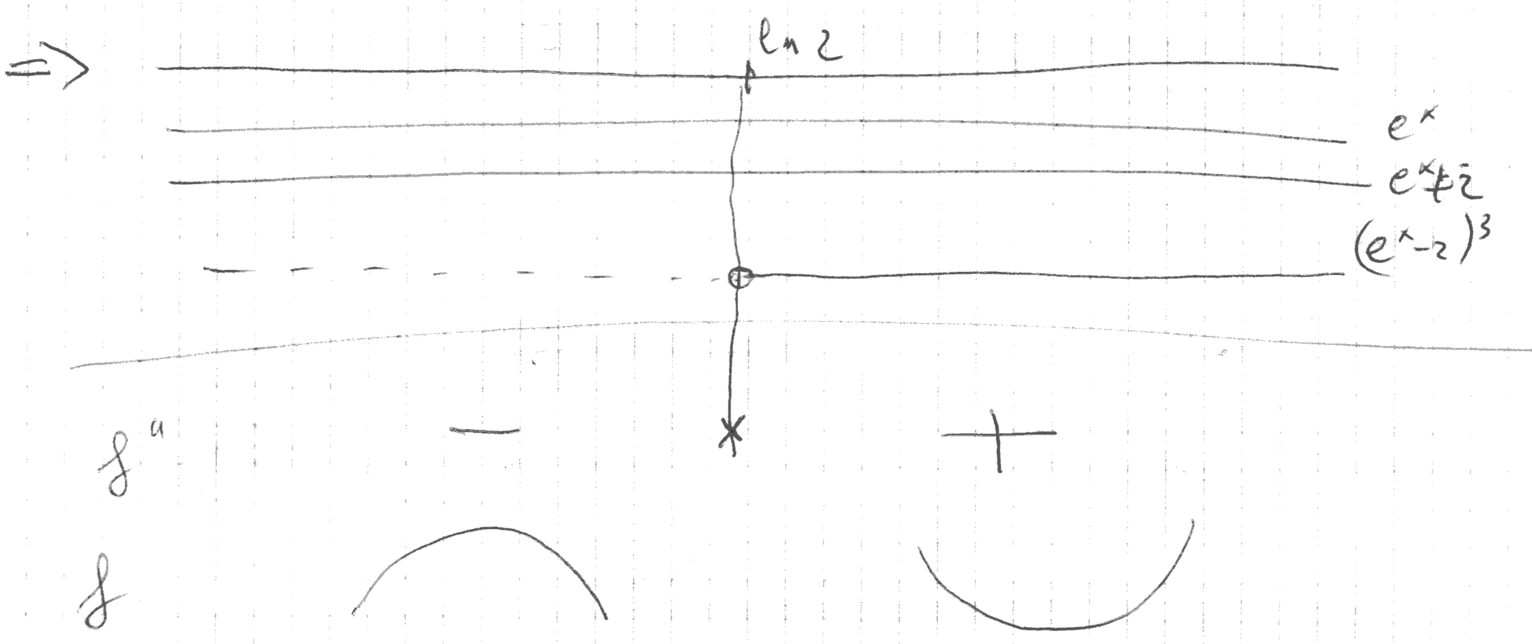
$$= \frac{e^{3x}-4e^x}{(e^x-2)^4} = \frac{e^x(e^{2x}-4)}{(e^x-2)^4} = \frac{e^x(e^x-2)(e^x+2)}{(e^x-2)^4} =$$

$$= \frac{e^x(e^x+2)}{(e^x-2)^3}$$

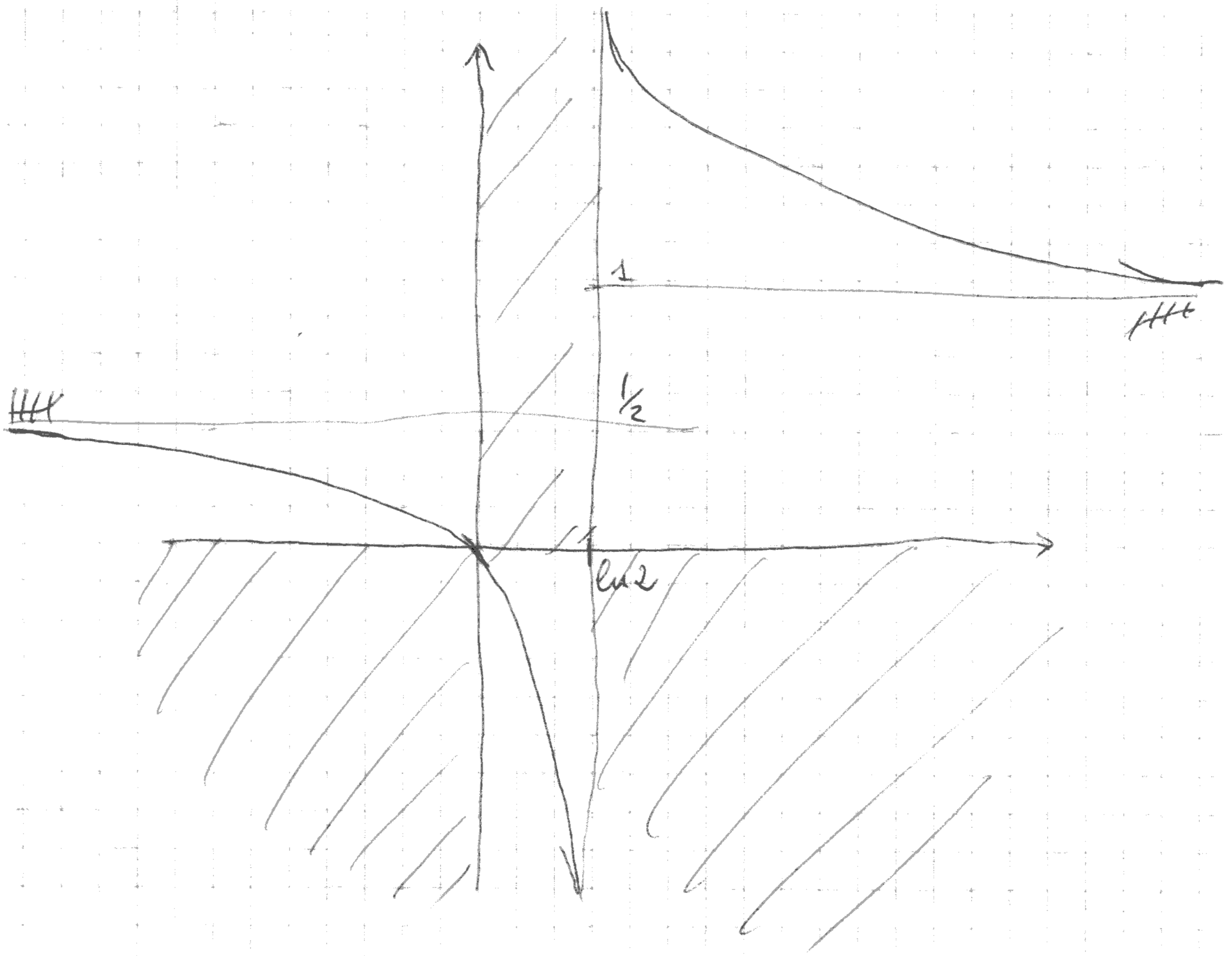
$e^x > 0$ sempre

$e^x + 2 > 0$ sempre

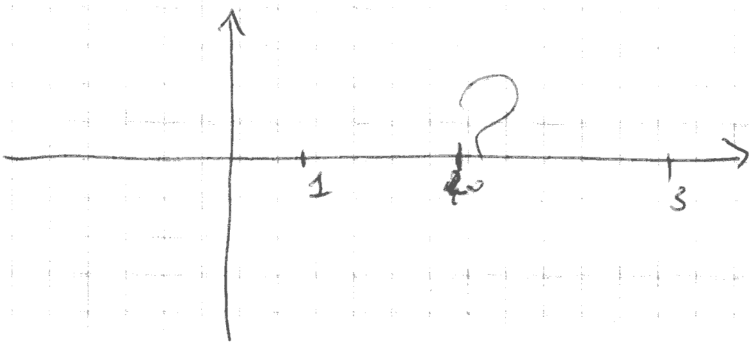
$(e^x - 2)^3 > 0 \iff e^x - 2 > 0 \iff e^x > \ln 2$



In particolare, non ci sono pti di flesso ($\ln 2 \notin CE$)

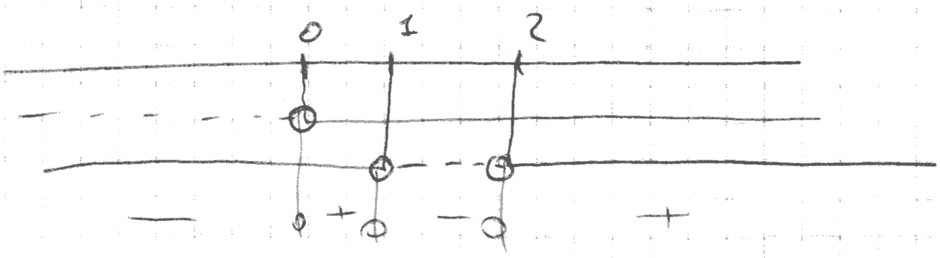

1
1/2
ln 2

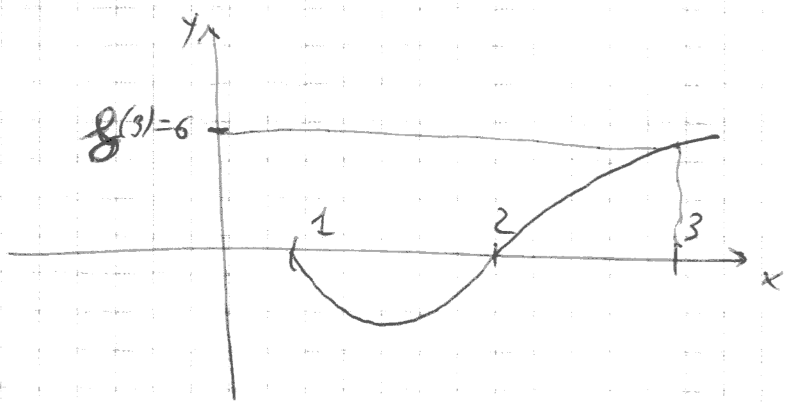
4) $f(x) = x^3 - 3x^2 + 2x$ in $(1, 3)$



segno $f(x)$

$$x^3 - 3x^2 + 2x = x(x^2 - 3x + 2)$$
$$= x(x-1)(x-2)$$





$$\Rightarrow \text{Area} = \int_1^2 -f(x) + \int_2^3 f(x)$$

$$F(x) = \int f(x) = \frac{x^4}{4} - x^3 + x^2$$

$$\Rightarrow \text{Area} = \left[-\frac{x^4}{4} + x^3 - x^2\right]_1^2 + \left[\frac{x^4}{4} - x^3 + x^2\right]_2^3 =$$

$$= \left(-\frac{16}{4} + 8 - 4\right) - \left(-\frac{1}{4} \cancel{+1} \cancel{-1}\right) + \left(\frac{81}{4} - 27 + 9\right) - \left(\frac{16}{4} - 8 + 4\right)$$

$$= \cancel{-4} \cancel{+8} \cancel{-4} + \frac{1}{4} + \frac{81}{4} - 27 + 9 - 0 =$$

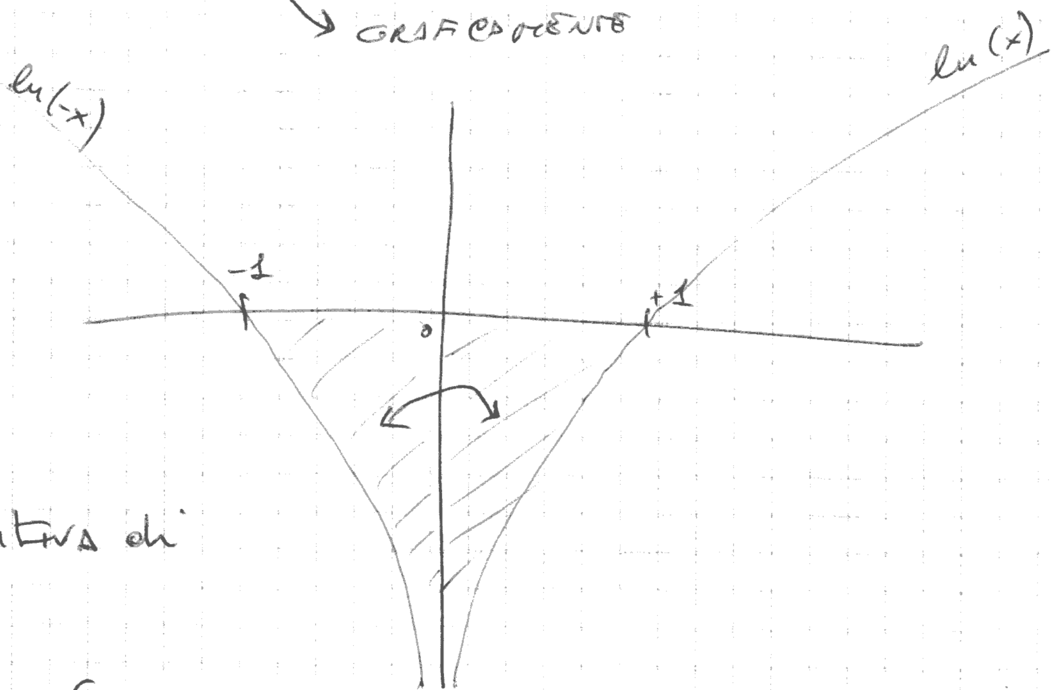
$$= \frac{82}{4} - 18 = \frac{41}{2} - 18 = 20.5 - 18 = \boxed{2.5}$$

5) $\int_{-1}^{0} \ln(-x)\,dx$. Poiché $0 \notin CE(\ln(-x))$, è un INTEGRALE IMPROPRIO.

~~Calcoliamo la primitiva~~:

~~$\int \ln(-x)\,dx =$~~ Con la sostituzione $t=-x$, abbiamo
$dt=-dx$

$$\underbrace{\int_{-1}^{0} \ln(-x)\,dx = \int_{1}^{0} \lg(t)\cdot(-dt) = -\int_{1}^{0} \ln t\,dt = \int_{0}^{1} \ln t\,dt}_{\text{GRAFICAMENTE}}$$



⇒ Serve la primitiva di $\ln t$.

$$\int \ln t\,dt = t\ln t - \int t\cdot\frac{1}{t}\,dt =$$

$$= t\ln t - t + c = \underbrace{t(\ln t - 1)}_{\text{DA SAPERE A MEMORIA PER IL COMPITO.}} + c$$

VERIFICA: $(t\ln t - t)' = \ln t + t\cdot\frac{1}{t} - 1 = \ln t$ ok ✓

$$\Rightarrow \int_{-1}^{0} \ln(-x)\,dx = \int_{0}^{1} \ln t\,dt = \lim_{\varepsilon\to 0^+}\Big[t\ln t - t\Big]_{\varepsilon}^{1} = \cancel{-1-(\varepsilon\ln\varepsilon-\varepsilon)}$$

$$= -1 - \lim_{\varepsilon\to 0^+} (\underbrace{\varepsilon\ln\varepsilon}_{0\cdot(-\infty)=F.I.} - \varepsilon)^{\to 0}$$

$\Rightarrow$ Serve $\lim_{x \to 0^+} x \ln x$.

$$x \ln x = \frac{\ln x}{1/x} \quad \begin{matrix} \ln x \to -\infty \\ 1/x \to +\infty \end{matrix} \Rightarrow$$ POSSO APPLICARE L'HOPITAL

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} \overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-1/x^2} =$$

$$= \lim_{x \to 0^+} -\frac{x^{\cancel{2}}}{\cancel{x}} = 0$$

$$\Rightarrow \lim_{\varepsilon \to 0^+} \varepsilon \ln \varepsilon - \varepsilon = 0 - 0 = 0$$

$$\Rightarrow \int_{-1}^{0} \ln(-x)\,dx = -1 - 0 = \boxed{-1}$$ IL SEGNO è COERENTE CON IL GRAFICO.

6) $$\sum_{n=1}^{+\infty} \frac{5^{n+1}}{2^{3n}} = \sum_{1}^{+\infty} \frac{5^{n+1}}{8^n} = 5 \sum_{1}^{+\infty} \left(\frac{5}{8}\right)^n = 5\left[\left(\frac{1}{1-5/8}\right) - 1\right]$$

$$= 5\left(\frac{8}{3} - \frac{3}{3}\right) = \frac{25}{3}$$

$$\sum_{n=2}^{+\infty} \frac{3}{n} = 3 \sum_{n=2}^{+\infty} \frac{1}{n} \to +\infty$$ perché è la serie armonica.

7) a) $\det A = ^{(\text{SARRUS})} -8 + 1 + 2 = -5 \neq 0 \Rightarrow A$ è invertibile

b) $$A \cdot B = \begin{pmatrix} 0 \cdot 2 + 2 - 1 \\ 2 - 1 + 2 \\ 4 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

c) $rg(A) = 3$, perché $\det A \neq 0$
$rg(B) = 1$, perché ha almeno un elemento $\neq 0$.