f-Kenmotsu manifolds with the Schouten-van Kampen connection

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Abstract

The object of the present paper is to the study 3-dimensional f-Kenmotsu manifolds with the Schouten-van Kampen connection.

Introduction

The Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [2, 5, 14]. A. F. Solov'ev has investigated hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection [15, 16, 17, 18]. Then Z. Olszak has studied the Schouten-van Kampen connection to adapted to an almost contact metric structure [10]. He has characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection and he has finded certain curvature properties of this connection on these manifolds.

On the other hand, let M be an almost contact manifold, i.e. M is connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ,ξ,η,g) [1]. As usually, denote by Φ the fundamental 2-form of M, $\Phi(X,Y)=g(X,\phi Y)$, $X,Y\in\chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on M.

For further use, we recall the following definitions [13], [3], [1]. The manifold M and its structure (ϕ,ξ,η,g) is said to be:

i) normal if the almost complex structure defined on the product manifold $M\times\mathbb{R}$ is integrable (equivalently $[\phi,\phi]+2d\eta\otimes\xi=0$),

ii) almost cosymplectic if $d\eta=0$ and $d\Phi=0$,

iii) cosymplectic if it is normal and almost cosymplectic (equivalently, $\nabla\phi=0$, ∇ being covariant differentiation with respect to the Levi-Civita connection).

The manifold M is called locally conformal, cosymplectic respectively almost cosymplectic if M has an open covering $\{U_t\}$ endowed with differentiable functions $\sigma_t:U_i\to\mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t,\xi_t,\eta_t,g_t)$ defined by

$$\phi_t=\phi,\quad \xi_t=e^{\sigma_t}\xi,\quad \eta_t=e^{-\sigma_t}\eta,\quad g_t=e^{-2\sigma_t}g$$

is cosymplectic (respectively almost cosymplectic).

Olszak and Rosca [11] studied normal locally conformal almost cosymplectic manifold. They gave a geometric interpretation of f-Kenmotsu manifolds and studied some curvature properties. Amongothers they proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.

By an f-Kenmotsu manifold we mean almost contact metric manifold which is normal and locally conformal almost cosymplectic.

In the present paper we have studied some curvature properties of a 3-dimensional f-Kenmotsu manifold with the Schouten-van Kampen connection. The paper has been organized as follows: After introduction we have given the Schouten-van Kampen connection and f-Kenmotsu manifolds. Then we have adapted the Schouten-van Kampen connection on a 3-dimensional f-Kenmotsu manifold. In section 5, we have studied projectively flat a 3-dimensional f-Kenmotsu manifold with the Schouten-van Kampen connection. In section 6, we have considered a conharmonically flat 3-dimensional f-Kenmotsu manifold with the Schouten-van Kampen connection. Finally we have given an example of a 3-dimensional f-Kenmotsu manifold with the Schouten-van Kampen connection which is verified Theorem 1 and Theorem 2.

The Schouten-van Kampen connection

Let M a connected pseudo-Riemannian manifold of an arbitrary signature $(p,n-p)$, $0\le p\le n$, $n=\dim M\ge 2$. By g will be denoted the pseudo-Riemannian metric on M, and by ∇ the Levi-Civita connection coming from the metric g. Assume that H and V are two complementary, orthogonal distributions on M such that $dimH=n-1$, $dimV=1$, and the distribution V is non-null. Thus $TM=H\oplus V$, $H\cap V=\{0\}$ and $H\perp V$. Assume that ξ is a unit vector field and η is a linear form such that $\eta(\xi)=1$, $g(\xi,\xi)=\varepsilon=\pm1$ and

$$H=\ker\eta,\qquad V=\operatorname{span}\{\xi\}. \tag{1}$$

We can always choose such ξ and η at least locally (in a certain neighborhood of an arbitrary chosen point of M). We also have $\eta(X)=\varepsilon g(X,\xi)$. Moreover, it holds that $\nabla_X\xi\in H$.

For any $X\in TM$, by X^h and X^v we denote the projections of X onto H and V, respectively. Thus, we have $X=X^h+X^v$ with

$$X^h=X-\eta(X)\xi,\qquad X^v=\eta(X)\xi. \tag{2}$$

The Schouten-van Kampen connection $\tilde\nabla$ associated to the Levi-Civita connection ∇ and adapted to the pair of the distributions (H,V) is defined by [2]

$$\tilde\nabla_XY=(\nabla_XY^h)^h+(\nabla_XY^v)^v, \tag{3}$$

and the corresponding second fundamental form B is defined by $B=\nabla-\tilde\nabla$. Note that condition (3) implies the parallelism of the distributions H and V with respect to the Schouten-van Kampen connection $\tilde\nabla$.

From (2), one can compute

$$\begin{aligned}(\nabla_XY^h)^h&=\nabla_XY-\eta(\nabla_XY)\xi-\eta(Y)\nabla_X\xi,\\ (\nabla_XY^v)^v&=(\nabla_X\eta)(Y)\xi+\eta(\nabla_XY)\xi,\end{aligned}$$

which enables us to express the Schouten-van Kampen connection with help of the Levi-Civita connection in the following way [15]

$$\tilde\nabla_XY=\nabla_XY-\eta(Y)\nabla_X\xi+(\nabla_X\eta)(Y)\xi. \tag{4}$$

Thus, the second fundamental form B and the torsion $\tilde T$ of $\tilde\nabla$ are [15, 16]

$$B(X,Y)=\eta(Y)\nabla_X\xi-(\nabla_X\eta)(Y)\xi,$$

$$\tilde T((X,Y)=\eta(X)\nabla_Y\xi-\eta(Y)\nabla_X\xi+2d\eta(X,Y)\xi.$$

With the help of the Schouten-van Kampen connection (4), many properties of some geometric objects connected with the distributions H, V can be characterized [15, 16, 17]. Probably, the most spectacular is the following statement: g, ξ and η are parallel with respect to $\tilde\nabla$, that is, $\tilde\nabla\xi=0$, $\tilde\nabla g=0$, $\tilde\nabla\eta=0$.

f-Kenmotsu manifolds

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ,ξ,η,g) satisfying

$$\begin{aligned}\phi^2&=-I+\eta\otimes\xi,\quad \eta(\xi)=1,\\ \phi\xi&=0,\quad \eta\circ\phi=0,\quad \eta(X)=g(X,\xi),\\ g(\phi X,\phi Y)&=g(X,Y)-\eta(X)\eta(Y),\end{aligned} \tag{5}$$

for any vector fields $X,Y\in\chi(M)$, where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field. We say that (M,ϕ,ξ,η,g) is a f-Kenmotsu manifold if the Levi-Civita connection of g satisfy [9]:

$$(\nabla_X\phi)(Y)=f\{g(\phi X,Y)\xi-\eta(Y)\phi X\}, \tag{6}$$

where $f\in C^\infty(M)$ such that $df\wedge\eta=0$. If $f=\alpha=\text{constant}\neq0$ then the manifold is α-Kenmotsu manifold [6]. 1-Kenmotsu manifold is Kenmotsu manifold [7]. If $f=0$, then the manifold is cosymplectic [6]. An f-Kenmotsu manifold is said to be regular if $f^2+f'\neq0$, where $f'=\xi(f)$.

For an f-Kenmotsu manifold from (5) it follows that

$$\nabla_X\xi=f\{X-\eta(X)\xi\}. \tag{7}$$

Then using (7), we have

$$(\nabla_X\eta)(Y)=f\{g(X,Y)-\eta(X)\eta(Y)\}. \tag{8}$$

The condition $df\wedge\eta=0$ holds if $\dim M\ge5$. This does not hold in general if $\dim M=3$ [11].

As is well known a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned}R(X,Y)Z&=g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y\\ &\quad-\frac{\tau}{2}\{g(Y,Z)X-g(X,Z)Y\}.\end{aligned} \tag{9}$$

In a 3-dimensional f-Kenmotsu manifold M, we have [11]

$$\begin{aligned}R(X,Y)Z&=(\frac{\tau}{2}+2f^2+2f')\{g(Y,Z)X-g(X,Z)Y\}\\ &\quad-(\frac{\tau}{2}+3f^2+3f')\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\\ &\quad+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\},\end{aligned} \tag{10}$$

$$S(X,Y)=(\frac{\tau}{2}+f^2+f')g(X,Y)-(\frac{\tau}{2}+3f^2+3f')\eta(X)\eta(Y), \tag{11}$$

$$QX=(\frac{\tau}{2}+f^2+f')X-(\frac{\tau}{2}+3f^2+3f')\eta(X)\xi, \tag{12}$$

where R denotes the curvature tensor, S is the Ricci tensor, Q is the Ricci operator and τ is the scalar curvature of M.

From (10), we obtain

$$R(X,Y)\xi=-(f^2+f')\{\eta(Y)X-\eta(X)Y\}, \tag{13}$$

and (11) yields

$$S(X,\xi)=-2(f^2+f')\eta(X). \tag{14}$$

3-dimensional f-Kenmotsu manifolds with the Schouten-van Kampen connection

Let M be a 3-dimensional f-Kenmotsu manifold with the Schouten-van Kampen connection. Then using (7) and (8) in (4), we get

$$\tilde\nabla_XY=\nabla_XY+f(g(X,Y)\xi-\eta(Y)X). \tag{15}$$

Let R and $\tilde R$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\tilde\nabla$,

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]},\qquad \tilde R(X,Y)=[\tilde\nabla_X,\tilde\nabla_Y]-\tilde\nabla_{[X,Y]}.$$

Using (15), by direct calculations, we obtain the following formula connecting R and $\tilde R$ on a 3-dimensional f-Kenmotsu manifold M

$$\begin{aligned}\tilde R(X,Y)Z&=R(X,Y)Z\\ &\quad+f^2\{g(Y,Z)X-g(X,Z)Y\}\\ &\quad+f'\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}.\end{aligned} \tag{16}$$

We will also consider the Riemann curvature $(0,4)$-tensors $\tilde R, R$, the Ricci tensors $\tilde S, S$, the Ricci operators $\tilde Q, Q$ and the scalar curvatures $\tilde\tau,\tau$ of the connections $\tilde\nabla$ and ∇ are defined by

$$\begin{aligned}\tilde R(X,Y,Z,W)&=R(X,Y,Z,W)\\ &\quad+f^2\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}\\ &\quad+f'\{g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)\\ &\quad+g(X,W)\eta(Y)\eta(Z)-g(Y,W)\eta(X)\eta(Z)\},\end{aligned} \tag{17}$$

$$\begin{aligned}\tilde S(Y,Z)&=S(Y,Z)\\ &\quad+(2f^2+f')g(Y,Z)+f'\eta(Y)\eta(Z)),\end{aligned} \tag{18}$$

$$\tilde QX=QX+(2f^2+f')X+f'\eta(X)\xi, \tag{19}$$

$$\tilde\tau=\tau+6f^2+4f', \tag{20}$$

respectively, where $\tilde R(X,Y,Z,W)=g(\tilde R(X,Y)Z,W)$ and $R(X,Y,Z,W)=g(R(X,Y)Z,W)$.

Projectively flat 3-dimensional f-Kenmotsu manifolds with the Schouten-van Kampen connection

In this section, we study projectively flat 3-dimensional f-Kenmotsu manifolds with respect to the Schouten-van Kampen connection. In a 3-dimensional f-Kenmotsu manifold the projective curvature tensor with respect to the Schouten-van Kampen connection is given by

$$\tilde P(X,Y)Z=\tilde R(X,Y)Z-\frac12\{\tilde S(Y,Z)X-\tilde S(X,Z)Y\}. \tag{21}$$

If $\tilde P=0$, then the manifold M is called projectively flat manifold with respect to the Schouten-van Kampen connection.

Let M be a projectively flat manifold with respect to the Schouten-van Kampen connection. From (21), we have

$$\tilde R(X,Y)Z=\frac12\{\tilde S(Y,Z)X-\tilde S(X,Z)Y\}. \tag{22}$$

Using (17) and (18) in (22), we get

$$\begin{aligned}&g(R(X,Y)Z,W)+f^2\{g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}\\ &+f'\{g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)\\ &+g(X,W)\eta(Y)\eta(Z)-g(Y,W)\eta(X)\eta(Z)\}\\ =&\ \frac12\{S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\\ &+[2f^2+f'][g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ &+f'[\eta(Y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W)]\}.\end{aligned} \tag{23}$$

Now putting $W=\xi$ in (23), we obtain

$$\begin{aligned}&(f^2+f')\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}\\ &+(f^2+f')\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\\ =&\ \frac12\{S(Y,Z)\eta(X)-S(X,Z)\eta(Y)+(2f^2+f')[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\},\end{aligned}$$

which gives that

$$\begin{aligned}&S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\\ &+(2f^2+f')[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]=0.\end{aligned} \tag{24}$$

Again putting $X=\xi$ in (24), we get

$$S(Y,Z)=-(2f^2+f')g(Y,Z)-f'\eta(Y)\eta(Z). \tag{25}$$

Thus M is an η-Einstein manifold with respect to the Levi-Civita connection.

Also using (25) in (18), we obtain

$$\tilde S(Y,Z)=0. \tag{26}$$

Hence the manifold M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection.

Conversely, let M be a Ricci-flat manifold with respect to the Schouten-van Kampen connection. Then from (26), we have

$$\tilde\tau=0,$$

which implies that

$$\tau=-6f^2-4f'. \tag{27}$$

Using (26) in (21), we obtain

$$\begin{aligned}\tilde P(X,Y)Z&=\tilde R(X,Y)Z\\ &=R(X,Y)Z+f^2\{g(Y,Z)X-g(X,Z)Y\}\\ &\quad+f'\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}.\end{aligned} \tag{28}$$

Now using (10) in (28), we get

$$\begin{aligned}\tilde P(X,Y)Z&=(\frac{\tau}{2}+3f^2+2f')\{g(Y,Z)X-g(X,Z)Y-g(Y,Z)\eta(X)\xi\\ &\quad+g(X,Z)\eta(Y)\xi-\eta(Y)\eta(Z)X+\eta(X)\eta(Z)Y\}.\end{aligned} \tag{29}$$

Again using (27) in (29), we have

$$\tilde P(X,Y)Z=0.$$

Thus we have the following:

Theorem 1. *Let M be a 3-dimensional f-Kenmotsu manifold with respect to the Schouten-van Kampen connection. Then M is a projectively flat manifold with respect to the Schouten-van Kampen connection if and only if M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection.*

Conharmonically flat 3-dimensional f-Kenmotsu manifolds with the Schouten-van Kampen connection

In this section, we study conharmonically flat 3-dimensional f-Kenmotsu manifolds with respect to the Schouten-van Kampen connection. In a 3-dimensional f-Kenmotsu manifold the conharmonic curvature tensor with respect to the Schouten-van Kampen connection is given by

$$\tilde K(X,Y)Z=\tilde R(X,Y)Z-\{\tilde S(Y,Z)X-\tilde S(X,Z)Y+g(Y,Z)\tilde QX-g(X,Z)\tilde QY\}. \tag{30}$$

If $\tilde K=0$, then the manifold M is called conharmonically flat manifold with respect to the Schouten-van Kampen connection.

Let M be a conharmonically flat manifold with respect to the Schouten-van Kampen connection. From (30), we have

$$\tilde R(X,Y)Z=\tilde S(Y,Z)X-\tilde S(X,Z)Y+g(Y,Z)\tilde QX-g(X,Z)\tilde QY. \tag{31}$$

Using (17), (18) and (19) in (31), we get

$$\begin{aligned}&R(X,Y)Z+f^2\{g(Y,Z)X-g(X,Z)Y\}\\ &+f'\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ =&\ S(Y,Z)X-S(X,Z)Y\\ &+(4f^2+2f'+\frac{\tau}{2}+f^2+f')\{g(Y,Z)X-g(X,Z)Y\}\\ &+f'\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ &+(f'-\frac{\tau}{2}-3f^2-3f')\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\}.\end{aligned} \tag{32}$$

Now putting $X=\xi$ in (32), we obtain

$$\begin{aligned}&R(\xi,Y)Z+(f^2+f')\{g(Y,Z)\xi-\eta(Z)Y\}\\ =&\ S(Y,Z)\xi-S(\xi,Z)Y\\ &+(4f^2+2f'+\frac{\tau}{2}+f^2+f')\{g(Y,Z)\xi-\eta(Z)Y\}\\ &+f'\{\eta(Y)\eta(Z)\xi-\eta(Z)Y\}\\ &+(f'-\frac{\tau}{2}-3f^2-3f')\{g(Y,Z)\xi-\eta(Z)\eta(Y)\xi\}.\end{aligned} \tag{33}$$

Using (10) and (14) in (33), we get

$$\begin{aligned}0&=S(Y,Z)\xi-S(\xi,Z)Y+(4f^2+2f'+\frac{\tau}{2}+f^2+f')\{g(Y,Z)\xi-\eta(Z)Y\}\\ &\quad+f'\{\eta(Y)\eta(Z)\xi-\eta(Z)Y\}\\ &\quad+(f'-\frac{\tau}{2}-3f^2-3f')\{g(Y,Z)\xi-\eta(Z)\eta(Y)\xi\}.\end{aligned} \tag{34}$$

Taking the inner product with ξ in (34), we have

$$\begin{aligned}0&=S(Y,Z)+2(f^2+f')\eta(Y)\eta(Z)\\ &\quad+(2f^2+f')\{g(Y,Z)-\eta(Y)\eta(Z)\},\end{aligned}$$

which gives that

$$S(Y,Z)=-(2f^2+f')g(Y,Z)-f'\eta(Y)\eta(Z). \tag{35}$$

Thus M is an η-Einstein manifold with respect to the Levi-Civita connection.

Using (35) in (18), we obtain

$$\tilde S(Y,Z)=0. \tag{36}$$

Hence the manifold M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection.

Conversely, let M be a Ricci-flat manifold with respect to the Schouten-van Kampen connection. Then from (26), again we have (27).

Now using (36) in (30), we obtain

$$\begin{aligned}\tilde K(X,Y)Z&=\tilde R(X,Y)Z\\ &=R(X,Y)Z+f^2\{g(Y,Z)X-g(X,Z)Y\}\\ &\quad+f'\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ &=(\frac{\tau}{2}+3f^2+2f')\{g(Y,Z)X-g(X,Z)Y-g(Y,Z)\eta(X)\xi\\ &\quad+g(X,Z)\eta(Y)\xi-\eta(Y)\eta(Z)X+\eta(X)\eta(Z)Y\}.\end{aligned} \tag{37}$$

Again using (27) in (37), we have

$$\tilde K(X,Y)Z=0.$$

Thus we have the following:

Theorem 2. *Let M be a 3-dimensional f-Kenmotsu manifold with respect to the Schouten-van Kampen connection. Then M is a conharmonically flat manifold with respect to the Schouten-van Kampen connection if and only if M is a Ricci-flat manifold with respect to the Schouten-van Kampen connection.*

An example of a 3-dimensional f-Kenmotsu manifold with the Schouten-van Kampen connection

We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3,\ z\neq0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1=z^2\frac{\partial}{\partial x},\quad e_2=z^2\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Lat g be the Riemannian metric defined by

$$\begin{aligned}g(e_1,e_3)&=g(e_2,e_3)=g(e_1,e_2)=0,\\ g(e_1,e_1)&=g(e_2,e_2)=g(e_3,e_3)=1.\end{aligned}$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1)=-e_2$, $\phi(e_2)=e_1$, $\phi(e_3)=0$. Then using linearity of ϕ and g we have

$$\eta(e_3)=1,\quad \phi^2Z=-Z+\eta(Z)e_3,$$

$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any $Z,W\in\chi(M)$. Now, by direct computations we obtain

$$[e_1,e_2]=0,\quad [e_2,e_3]=-\frac2z e_2,\quad [e_1,e_3]=-\frac2z e_1.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$\begin{aligned}2g(\nabla_XY,Z)&=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\ &\quad-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).\end{aligned} \tag{38}$$

Using (38), we have

$$2g(\nabla_{e_1}e_3,e_1)=2g(-\frac2z e_1,e_1),$$

$$2g(\nabla_{e_1}e_3,e_2)=0\quad\text{and}\quad 2g(\nabla_{e_1}e_3,e_3)=0.$$

Hence $\nabla_{e_1}e_3=-\frac2z e_1$. Similarly, $\nabla_{e_2}e_3=-\frac2z e_2$ and $\nabla_{e_3}e_3=0$. (38) further yields

$$\begin{aligned}\nabla_{e_1}e_2&=0, & \nabla_{e_1}e_1&=\frac2z e_3,\\ \nabla_{e_2}e_2&=\frac2z e_3, & \nabla_{e_2}e_1&=0,\\ \nabla_{e_3}e_2&=0, & \nabla_{e_3}e_1&=0.\end{aligned} \tag{39}$$

From (39), we see that the manifold satisfies $\nabla_X\xi=f\{X-\eta(X)\xi\}$ for $\xi=e_3$, where $f=-\frac2z$. Hence we conclude that M is an f-Kenmotsu manifold. Also $f^2+f'\neq0$. Hence M is a regular f-Kenmotsu manifold [20].

It is known that

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z. \tag{40}$$

With the help of the above formula and using (40), it can be easily verified that

$$\begin{aligned}R(e_1,e_2)e_3&=0, & R(e_2,e_3)e_3&=-\frac{6}{z^2}e_2,\\ R(e_1,e_3)e_3&=-\frac{6}{z^2}e_1, & R(e_1,e_2)e_2&=-\frac{4}{z^2}e_1,\\ R(e_3,e_2)e_2&=-\frac{6}{z^2}e_3, & R(e_1,e_3)e_2&=0,\\ R(e_1,e_2)e_1&=\frac{4}{z^2}e_2, & R(e_2,e_3)e_1&=0,\\ R(e_1,e_3)e_1&=\frac{6}{z^2}e_3.\end{aligned} \tag{41}$$

The Schouten-van Kampen connection on M is given by

$$\begin{aligned}\tilde\nabla_{e_1}e_3&=(-\frac2z-f)e_1, & \tilde\nabla_{e_2}e_3&=(-\frac2z-f)e_2,\\ \tilde\nabla_{e_3}e_3&=-f(e_3-\xi), & \tilde\nabla_{e_1}e_2&=0,\\ \tilde\nabla_{e_2}e_2&=\frac2z(e_3-\xi), & \tilde\nabla_{e_3}e_2&=0,\\ \tilde\nabla_{e_1}e_1&=\frac2z(e_3-\xi), & \tilde\nabla_{e_2}e_1&=0,\\ \tilde\nabla_{e_3}e_1&=0.\end{aligned} \tag{42}$$

From (42), we can see that $\tilde\nabla_{e_i}e_j=0$ $(1\le i,j\le3)$ for $\xi=e_3$ and $f=-\frac2z$. Hence M is a 3-dimensional f-Kenmotsu manifold with the Schouten-van Kampen connection. Also using (41), it can be seen that $\tilde R=0$. Thus the manifold M is a flat manifold with respect to the Schouten-van Kampen connection. Since a flat manifold is a Ricci-flat manifold with respect to the Schouten-van Kampen connection, the manifold M is both a projectively flat and a conharmonically flat 3-dimensional f-Kenmotsu manifold with respect to the Schouten-van Kampen connection. So, from Theorem 1 and Theorem 2, M is an η-Einstein manifold with respect to the Levi-Civita connection.

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