The topology of limits of embedded minimal disks.
(joint work with Jacob Bernstein)

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- Notations;
- Background;
- Main result;
- Possible questions;
- Another proof.
Let $\mathbf{M}$ be an oriented surface in $\mathbb{R}^3$, let $\xi$ be the unit vector field normal to $\mathbf{M}$:

$$
\mathbf{A} = -d\xi : T_p\mathbf{M} \rightarrow T_{\xi(p)}\mathbb{S}^2 \simeq T_p\mathbf{M}
$$

is the **shape operator** of $\mathbf{M}$. 
Definition

- The eigenvalues $k_1, k_2$ of $A_p$ are the principal curvatures of $M$ at $p$.
- $H = \frac{1}{2} \text{tr}(A) = \frac{k_1 + k_2}{2}$ is the mean curvature.
- $|A| = \sqrt{k_1^2 + k_2^2}$ is the norm of the second fundamental form.

Minimal Surface: critical points for the area functional.

$H = 0$
Surface given as a graph of a function

\[ \frac{|Hess(u)|^2}{(1+|\nabla u|^2)^2} \leq |A|^2 \leq 2 \frac{|Hess(u)|^2}{1+|\nabla u|^2} \]

Minimal Graph

\[ 0 = \text{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \quad \text{Quasi-linear elliptic PDE} \]
Motivational Question:
What classes of smooth minimal surfaces have good (pre-)compactness properties?
Suppose

- $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots$ are open subsets of $\mathbb{R}^3$; $\Omega = \bigcup_i \Omega_i$.
  (take $\Omega_i = \Omega$)
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1. $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots$ are open subsets of $\mathbb{R}^3$; $\Omega = \bigcup_i \Omega_i$. (take $\Omega_i = \Omega$)

2. $D_i \subset \Omega_i$ is a sequence of properly (relatively closed, $\overline{D_i} = D_i$) embedded minimal surfaces.

Well-known compactness result:

If for each compact subset of $\Omega$, there exist constants $C_1(K)$, $C_2(K) < 1$ so that

$$\sup_{K \subseteq D_i} |\text{Area}(\partial D_i \setminus K)| < C_1(K),$$
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then, up to passing to a subsequence, $D_i$ converges, with finite multiplicity, a minimal surface $D$ properly embedded in $\Omega$.
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- $D_i \subset \Omega_i$ is a sequence of properly (relatively closed, $\overline{D_i} = D_i$) **embedded** minimal surfaces.

**Well-known compactness result:**
If for each $K$, compact subset of $\Omega$, there exist constants $C_1(K), C_2(K) < \infty$ so that

$$\sup_{K \cap D_i} |A| \leq C_1(K), \quad \text{Area}(D_i \cap K) < C_2(K)$$

then, up to passing to a subsequence, $D_i$ converges, with finite multiplicity, a minimal surface $D$ properly embedded in $\Omega$. 
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- In general, a neighborhood of a point $p \in M$ is always a graph over $T_pM$. However, the size of such neighborhood depends on $p$. 

Let $u$ be such graph then $kuk_{C^2} \leq 10^{-C}$ if $u$ is a minimal graph then $\div u p + |ru|_2 = 0 = ku_{C^2}$, $\epsilon$ is uniformly bounded independently of $p$. 


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- If $\sup_M |A| = \sup_M |d\xi| \leq C$ then the size of such neighborhood only depends on $C$ and NOT on $p$:

$$d_{S^2}(\xi(p), \xi(q)) \leq \int_{\gamma_{p,q}} |\nabla \xi| \leq \text{length}(\gamma_{p,q}) \sup_{\gamma_{p,q}} |A| \leq RC,$$

if $q \in B_R(p)$. Take $RC < \frac{\pi}{10}$. 

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if $q \in B_R(p)$. Take $RC < \frac{\pi}{10}$.

- Let $u$ be such graph then
  - $\|u\|_{C^2} \leq 10C$
  - if $u$ is a minimal graph then
    $$\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0 \implies \|u\|_{C^2,\alpha} \text{ is uniformly bounded independently of } p.$$
Proof of the well-known compactness result:

- \( \sup_{D_i} |A| \leq C \) uniformly \( \implies \) nearby a point be we have a sequence of graphs \( u_i \) with \( \|u_i\|_{C^{2,\alpha}} \) uniformly bounded.
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- Area bound $\implies$ there are finitely many of such graphs nearby $p$ (properness).
Proof of the well-known compactness result:

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- Arzela-Ascoli \implies \text{subsequence converging } C^2 \text{ to a graph that is minimal.}

- Area bound \implies \text{there are finitely many of such graphs nearby } p \text{ (properness).}

- Embeddedness is preserved by the maximum principle.
Natural question:
What happens if we remove such bounds?
Minimal Lamination

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- $\mathcal{L}$ is relatively closed in $\Omega$;
- $\mathcal{L} = \bigcup_{\alpha} L_{\alpha}$ where $L_{\alpha}$ are connected pair-wise disjoint embedded minimal surfaces in $\Omega$ – called leaves of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset $U_p$ of $\Omega$, a closed subset $K_p$ of $(-1, 1)$ and a Lipschitz diffeomorphism, "straightening map,"

$$
\psi_p : (U_p, p) \to (B_1, 0)
$$

so

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\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}.
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$$\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}.$$

If $\mathcal{L} = \Omega$, then this is a \textit{minimal foliation} of $\Omega$. 
Well-known compactness result:

If for each $K$, compact subset of $\Omega$, there is a constant $C(K) < \infty$ so that

$$\sup_{K \cap D_i} |A| \leq C(K),$$

then, up to passing to a subsequence, the $D_i$ converge to $\mathcal{L}$, a smooth minimal lamination of $\Omega$. 
In light of the previous result, we say that the curvatures of the $D_i$ blow-up at $p \in \Omega$ if there is a sequence of points $p_i \in D_i$ such that

$$p_i \to p \quad \text{and} \quad |A|(p_i) \to \infty.$$ 

Blow-up points or singular points.
and by passing to a subsequence we may assume that there is a relatively closed subset \( S \subset \Omega \) such that

- the curvatures of the \( D_i \) blow-up at each \( p \in S \);
- \( D_i \setminus S \) converges on \( \Omega \setminus S \) to a minimal lamination \( \mathcal{L} \) of \( \Omega \setminus S \).
Natural question:
What sets $S$ and laminations $\mathcal{L}$ can arise in this way?
Rescalings of a triply periodic minimal surfaces in $\mathbb{R}^3$. $S = \mathbb{R}^3$, $L = \emptyset$. 

$\frac{1}{i}$ (triply periodic minimal surface)
Rescalings of a triply periodic minimal surfaces in $\mathbb{R}^3$. $S = \mathbb{R}^3$, $\mathcal{L} = \emptyset$.

Let us focus on sequence of surfaces with finite topology.
Rescalings of a catenoid. $S = \tilde{0}$, $\mathcal{L}$ has a single leaf $\{z = 0\} \setminus \tilde{0}$. **NB:** The leaf extends smoothly to a surface in $\mathbb{R}^3$. 

\[ \frac{1}{i} \text{(Catenoid)} \]
Rescalings of a helicoid.
$S = z\text{-axis},\ \mathcal{L}$ is a foliation of $\mathbb{R}^3$ minus the z-axis by horizontal planes.
**NB:** The leaves extend smoothly to surfaces in $\mathbb{R}^3$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$. 
Rescalings of a catenoid.

$S = 0$, $\mathcal{L}$ has a single leaf $\{z = 0\} \setminus \vec{0}$.

**NB:** The leaf extends smoothly to a surface in $\mathbb{R}^3$. 
Theorem (Anderson, White (1985))

If the total curvatures of the surfaces $D_i$, i.e. \( \int_{D_i} |A|^2 \), are uniformly bounded, then

- $S$ is finite;
- $L$ extends smoothly across $S$.

If the $D_i$ are disks then $S = \emptyset$. 
Theorem (Anderson, White (1985))

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What if the $D_i$ have unbounded total curvatures?

Let us assume that the $D_i$ are (properly embedded) DISKS.
Key example

Rescalings of a helicoid $S = z - axis$, $\mathcal{L}$ is a foliation of $\mathbb{R}^3$ minus the z-axis by horizontal planes. **NB:** The leaves extend smoothly to surfaces in $\mathbb{R}^3$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbb{R}^3$. 
Theorem (Colding-Minicozzi, 2004)

Suppose each $D_i$ is a properly embedded disk and $\Omega = \mathbb{R}^3$. If $S \neq \emptyset$ then

- $\mathcal{L}$ is a foliation of $\mathbb{R}^3 \setminus S$ by parallel planes;
- $S$ is a line perpendicular to those planes. (Meeks)
The situation is very different when $\Omega \subseteq \mathbb{R}^3$. What can be said about the set $S$ and about the leaves of $L$? (reminder $D$'s are embedded DISKS)
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- a point (Colding-Minicozzi);
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The set $S$ can be:

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- a line segment (Dean);
- an arbitrary finite subset of a line segment (Kahn);
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The set $S$ can be:
- a point (Colding-Minicozzi);
- a line segment (Dean);
- an arbitrary finite subset of a line segment (Kahn);
- any closed subset of the z-axis (Hoffman-White, later Kleene).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to $\Omega = \mathbb{H}^3$. 
an arbitrary \( C^{1,1} \) curve (Meeks-Weber).

Sequence of minimal annuli in a solid torus of revolution whose singular set is the central circle of the solid torus.
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- an arbitrary $C^{1,1}$ curve (Meeks-Weber).
In the local case, Colding-Minicozzi (essentially) show

**Theorem (Colding-Minicozzi 2004)**

*The structure of $S$*

- *$S$ is contained in a properly embedded Lipshitz curve $S'$ of $\Omega$.)*
- *For any $p \in S$ there exists a leaf $L$ such that $p \in \overline{L}$ and $\overline{L}$ is a properly embedded minimal surface.*
- *If $\overline{L}$ is a properly embedded minimal surface, and $\overline{L} \cap S \neq \emptyset$, then $\overline{L}$ meets $S$ transversely.*
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Meeks showed that if $S = S'$ (i.e., $S$ has no “gaps”), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)

White showed that $S$ is contained in a $C^1$ curve.
What can be said about the leaves of $\mathcal{L}$?

In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if $\mathcal{L}$ is a leaf of an example then it can be an open disk $\mathcal{D} \cap S$; a proper disk in $\mathcal{D}$; proper annulus in $\mathcal{D} \cap S$ with $\overline{\mathcal{L}} \setminus S = 6$; and $\overline{\mathcal{L}}$ is a proper disk in $\mathcal{D}$.

Question (Ho-man-White)

Can a surface of genus $> 0$ occur? A planar domain with more than two ends?

Answer (Bernstein-T.)

No, under natural geometric condition on $\mathcal{D}$ it cannot.
What can be said about the the leaves of $\mathcal{L}$? In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if $L$ is a leaf of an example then it can be
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- a non-proper disk in $\Omega \setminus \mathcal{S}$;
- a proper disk or annulus (Hoffman-White) in $\Omega$;
- a proper annulus in $\Omega \setminus \mathcal{S}$ with $\bar{L} \cap \mathcal{S} \neq \emptyset$ and $\bar{L}$ is a proper disk in $\Omega$. 

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No, under natural geometric condition on $\Omega$ it cannot.
Let $\Omega$ be the interior of an oriented compact three-manifold with boundary so that:

- $\partial \Omega$ is strictly mean convex;
- There are no closed minimal surfaces in $\Omega$. 

Theorem

Let $L$ be a leaf of $\mathcal{L}$ then:

- $L$ is either a disk or an annulus.

Second bullet: Colding-Minicozzi = the set of singular points meeting $L$ is a discrete set in $L$. 

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**Theorem**

Let $L$ be a leaf of $\mathcal{L}$ then:

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- If $\bar{L}$ is a properly embedded minimal surface, then $L$ is either a puncture disk, or a disk or an annulus disjoint from $S$. 

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Second bullet: Colding-Minicozzi $\Rightarrow$ the set of singular points meeting $\overline{L}$ is a discrete set in $\overline{L}$. 
Example

Example of a torus being the limit of (non-minimal) disks.
Idea of proof

- The disks $D_i$ in the sequence act as an “effective” universal cover of $L$. 

- Specifically, one can “lift” closed curves in $L$ to curves in the $D_i$.

- The geometry of the minimal, mean convex set — restricts the topology of the $L$, essentially forcing it to have an abelian fundamental group.

- An uncomplicated geometric feature: the conditions on $\mathcal{I}$ ensure, by a result of White, that minimal surfaces in $\mathcal{I}$ satisfy an isoperimetric inequality.
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Idea of proof

- The disks $D_i$ in the sequence act as an “effective” universal cover of $L$.
- Specifically, one can “lift” closed curves in $L$ to curves in the $D_i$.
- The geometry of the $D_i$ – minimally embedded and in a mean convex set – restricts the topology of the $L$, essentially forcing it to have abelian fundamental group.
- A more complicated geometric feature we use: the conditions on $\Omega$ ensure – by a result of White – that minimal surfaces in $\Omega$ satisfy an isoperimetric inequality.
Definition

If \( \gamma : S^1 \to L \) is a piece-wise \( C^1 \) closed curve, then \( \gamma \) has the \textit{closed-lift property} if there exists a sequence of closed “lifts” \( \gamma_i : S^1 \to D_i \) converging to \( \gamma \). Otherwise, \( \gamma \) has the \textit{open-lift property}. 

\( \text{If } S^1 \text{ is embedded so are its lifts.} \)
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If \( \gamma \) is embedded so are its lifts.
Separating Lemma

If $\gamma: S^1 \to L$ is a closed embedded curve in $L$ with the closed lift property, then $\gamma$ is separating.
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Proof

- Let $\gamma_i: S^1 \to D_i$ be a sequence of embedded closed lifts converging to $\gamma$;
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Proof

- Let \( \gamma_i: S^1 \to D_i \) be a sequence of embedded closed lifts converging to \( \gamma \);
- Each \( \gamma_i \) is the boundary of a close minimal disk \( \Delta_i \subset D_i \);
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Proof

- Let \( \gamma_i : S^1 \to D_i \) be a sequence of embedded closed lifts converging to \( \gamma \);
- Each \( \gamma_i \) is the boundary of a close minimal disk \( \Delta_i \subset D_i \);
- \( \text{Area}(\Delta_i) < C_1 \text{Length}(\gamma_i) < C_2 \text{Length}(\gamma) \);
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- Let \( \gamma_i : S^1 \to D_i \) be a sequence of embedded closed lifts converging to \( \gamma \);
- Each \( \gamma_i \) is the boundary of a close minimal disk \( \Delta_i \subset D_i \);
- \( \text{Area}(\Delta_i) < C_1 \text{Length}(\gamma_i) < C_2 \text{Length}(\gamma) \);
- \( \Delta_i \to \Delta \) in \( C^\infty_{loc}(\Omega) \) and \( \Delta \subset L \setminus \gamma \) is open and close;
Separating Lemma

If $\gamma: S^1 \rightarrow L$ is a closed embedded curve in $L$ with the closed lift property, then $\gamma$ is separating.

Proof

- Let $\gamma_i: S^1 \rightarrow D_i$ be a sequence of embedded closed lifts converging to $\gamma$;
- Each $\gamma_i$ is the boundary of a close minimal disk $\Delta_i \subset D_i$;
- $\text{Area}(\Delta_i) < C_1 \text{Length}(\gamma_i) < C_2 \text{Length}(\gamma)$;
- $\Delta_i \rightarrow \Delta$ in $C^\infty_{loc}(\Omega)$ and $\Delta \subset L \setminus \gamma$ is open and close;
- If $\gamma$ does not separate $L$ then $\Delta = L \setminus \gamma$;
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Proof

- Let $\gamma_i: S^1 \to D_i$ be a sequence of embedded closed lifts converging to $\gamma$;
- Each $\gamma_i$ is the boundary of a close minimal disk $\Delta_i \subset D_i$;
- $\text{Area}(\Delta_i) < C_1 \text{Length}(\gamma_i) < C_2 \text{Length}(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{\text{loc}}(\Omega)$ and $\Delta \subset L \setminus \gamma$ is open and close;
- If $\gamma$ does not separate $L$ then $\Delta = L \setminus \gamma$;
- Contradiction because $\partial \Omega$ strictly mean convex implies $\Delta_i$ cannot get close to $\partial \Omega$. 


Commutator Lemma

Let $L$ be two-sided and let

$$\alpha : [0, 1] \to L \text{ and } \beta : [0, 1] \to L$$

be closed piece-wise $C^1$ Jordan curves. If $\alpha$ and $\beta$ have the open lift property and $\alpha \cap \beta = p_0$ where $p_0 = \alpha(0) = \beta(0)$, then

$$\nu := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$$

has the closed lift property.
Proposition

If \( L \) is two-sided then \( L \) has genus zero.
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Otherwise,
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Proof

Otherwise,

- Let $\alpha$ and $\beta$ be two non-separating curves in $L$ meeting at one point.
- Each curve has the open-lift property by the separating lemma.
Proposition

If \( L \) is two-sided then \( L \) has genus zero.

Proof

Otherwise,

- Let \( \alpha \) and \( \beta \) be two non-separating curves in \( L \) meeting at one point.
- Each curve has the open-lift property by the separating lemma.
- \( \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1} \) has the closed lift property but it is non-separating.

Contradiction.
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Proposition [No pants]
If $L$ is two-sided then $L$ is either a disk or an annulus.
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Proof

We have already shown that $L$ must be a planar domain. Assume $L$ is not an annulus.
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Proof
We have already shown that $L$ must be a planar domain. Assume $L$ is not an annulus. Then,
- There exist embedded closed curves $\alpha$ and $\beta$ separating $L$ in 3 connected components, $L_1, L_2$ and $L_3$. 
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We have already shown that $L$ must be a planar domain. Assume $L$ is not an annulus. Then,

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- Let $L_3$ be the component such that $\alpha \cup \beta \subset \partial L_3$ and $L_3$ is not an annulus.
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- Let $L_3$ be the component such that $\alpha \cup \beta \subset \partial L_3$ and $L_3$ is not an annulus.
- $\alpha$ and $\beta$ must have the open lift property.
- Let $\sigma$ be an embedded arc in $L_3$ with endpoints in $\alpha$ and $\beta$ and consider the two curves $\sigma \circ \alpha \circ \sigma^{-1}$ and $\beta$. 
Proposition [No pants]

If \( L \) is two-sided then \( L \) is either a disk or an annulus.

Proof

We have already shown that \( L \) must be a planar domain. Assume \( L \) is not an annulus. Then,

- There exist embedded closed curves \( \alpha \) and \( \beta \) separating \( L \) in 3 connected components, \( L_1, L_2 \) and \( L_3 \).
- Let \( L_3 \) be the component such that \( \alpha \cup \beta \subset \partial L_3 \) and \( L_3 \) is not an annulus.
- \( \alpha \) and \( \beta \) must have the open lift property.
- Let \( \sigma \) be an embedded arc in \( L_3 \) with endpoints in \( \alpha \) and \( \beta \) and consider the two curves \( \sigma \circ \alpha \circ \sigma^{-1} \) and \( \beta \).
- By the Commutator Lemma \( \sigma \circ \alpha \circ \sigma^{-1} \circ \beta \circ (\sigma \circ \alpha \circ \sigma^{-1})^{-1} \circ \beta^{-1} \) has the closed lift property.
Proof

- A sequence of embedded minimal disks $\Delta_i$ must converge to an open and close subset of $L \setminus (\beta \circ \sigma \circ \alpha)$. 
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In particular, the limit must contain either $L_1$, or $L_2$ or $L_3$. 
Proof

- A sequence of embedded minimal disks $\Delta_i$ must converge to an open and closed subset of $L \setminus (\beta \circ \sigma \circ \alpha)$.
- In particular, the limit must contain either $L_1$, or $L_2$ or $L_3$.
- Contradiction because $\partial \Omega$ strictly mean convex implies $\Delta_i$ cannot get close to $\partial \Omega$. 
Proposition

A leaf \( L \) is two-sided.
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Proof

- If $L$ is one-sided, then there is a closed non-separating curve along which $L$ does not have well defined normal;
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- If $L$ is one-sided, then there is a closed non-separating curve along which $L$ does not have well defined normal;
- Non-separating $\iff$ lift of this curve is open;
Proposition
A leaf $L$ is two-sided.

Proof
- If $L$ is one-sided, then there is a closed non-separating curve along which $L$ does not have well defined normal;
- Non-separating $\implies$ lift of this curve is open;
- Following lift around in a $D_i$ violates either properness or embeddedness.
Understanding Geometric Conditions

Question

- *Is our theorem sharp?*
- *To what extent can the assumptions on \( \Omega \) be relaxed?*
Understanding Geometric Conditions

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- *Is our theorem sharp?*
- *To what extent can the assumptions on $\Omega$ be relaxed?*

Let $D$ be an embedded but NOT properly embedded minimal disk in $\Omega$ with the property that the closure, $\bar{D}$, of $D$ in $\Omega$ is a proper minimal lamination of $\Omega$. (In fact more general.)
Topology of Minimal Disk Closures

The leaves of $\bar{D}$ behave almost identically to those of the limit leaves of a sequence of minimal disks.

**Theorem**

Let $\Omega$ be the interior of an oriented compact three-manifold with boundary so that:
- $\partial \Omega$ is strictly mean convex;
- There are no closed minimal surfaces in $\Omega$.

Then each leaf $L$ of $\bar{D}$ is either a disk, an annulus or a Möbius band.
The preceding theorem is sharp in the following sense: There is an embedded minimal disk $D$ that contains a Möbius band in its closure. Note: the lamination $\bar{D}$ cannot occur as the lamination that is the limit of a sequence of minimal disks. There is an embedded minimal disk whose closure contains a minimal torus in $\mathcal{T}$. 
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Sharpness

The preceding theorem is sharp in the following sense:

- There is an embedded minimal disk $D$ that contains a Möbius band in its closure. Note: the lamination $\bar{D}$ cannot occur as the lamination that is the limit of a sequence of minimal disks.

- There is an embedded minimal disk $D$ whose closure contains a minimal torus in $\Omega$. 
Some further questions:

- To what extent are both theorems true even for regions which contain closed minimal surfaces?
- To what extent is the theorem for a sequence of minimal disks sharp? For instance, it is hard to picture a minimal torus arises in this context.
Commutator Lemma

Let $L$ be two-sided and let

$$\alpha : [0, 1] \rightarrow L \text{ and } \beta : [0, 1] \rightarrow L$$

be closed piece-wise $C^1$ Jordan curves. If $\alpha$ and $\beta$ have the open lift property and $\alpha \cap \beta = p_0$ where $p_0 = \alpha(0) = \beta(0)$, then

$$\nu := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$$

has the closed lift property.
Proof

Let $\alpha_i^+ (\beta_i^+)$ be a lift of $\alpha (\beta)$ and let $\alpha_i^- (\beta_i^-)$ be a lift of $\alpha^{-1} (\beta^{-1})$. 
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Let $\alpha_i^+ (\beta_i^+)$ be a lift of $\alpha (\beta)$ and let $\alpha_i^- (\beta_i^-)$ be a lift of $\alpha^{-1} (\beta^{-1})$.

- Using embeddedness, the graphs converging to a small neighborhood of $p_0$ can be ordered by “height.”
Proof

Let $\alpha_i^+ (\beta_i^+)$ be a lift of $\alpha (\beta)$ and let $\alpha_i^- (\beta_i^-)$ be a lift of $\alpha^{-1} (\beta^{-1})$.

- Using embeddedness, the graphs converging to a small neighborhood of $p_0$ can be order by “height.”
- If $\alpha_i^+$ moves “upward” $m_i$ sheets, $\alpha_i^-$ moves “downward” $m_i$ sheets.
Proof

Let $\alpha_i^+ (\beta_i^+)$ be a lift of $\alpha (\beta)$ and let $\alpha_i^- (\beta_i^-)$ be a lift of $\alpha^{-1} (\beta^{-1})$.

- Using embeddedness, the graphs converging to a small neighborhood of $p_0$ can be order by “height.”
- If $\alpha_i^+$ moves “upward” $m_i$ sheets, $\alpha_i^-$ moves “downward” $m_i$ sheets.
- If $\beta_i^+$ moves “upward” $n_i$ sheets, $\beta_i^-$ moves “downward” $n_i$ sheets.