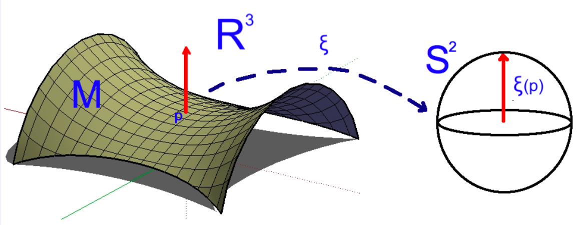
The topology of limits of embedded minimal disks.
(joint work with Jacob Bernstein)

Giuseppe Tinaglia
King's College London

- Notations;
- Background;
- Main result;
- Possible questions;
- Another proof.



Let **M** be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to **M**:

$$\mathbf{A} = -d\xi\colon\ T_p\mathbf{M} \to T_{\xi(p)}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

is the **shape operator** of **M**.

Definition

- The eigenvalues k_1, k_2 of $\mathbf{A}_p$ are the **principal curvatures** of $\mathbf{M}$ at p.
- $\mathbf{H} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) = \frac{k_1+k_2}{2}$ is the **mean curvature**.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the second fundamental form**.

Minimal Surface: critical points for the area functional.

$$\mathbf{H} = 0$$

Surface given as a graph of a function

- $\frac{|Hess(u)|^2}{(1+|\nabla u|^2)^2} \leq |\mathbf{A}|^2 \leq 2\frac{|Hess(u)|^2}{1+|\nabla u|^2}$

Minimal Graph

- $0 = \text{div}\frac{\nabla \mathbf{u}}{\sqrt{1+|\nabla \mathbf{u}|^2}}$ Quasi-linear elliptic PDE

Motivational Question:

What classes of smooth minimal surfaces have good (pre-)compactness properties?

Suppose

- $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots$ are open subsets of $\mathbf{R}^3$; $\Omega = \bigcup_i \Omega_i$. (take $\Omega_i = \Omega$)

Suppose

- $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots$ are open subsets of $\mathbf{R^3}$; $\Omega = \bigcup_i \Omega_i$. (take $\Omega_i = \Omega$)
- $D_i \subset \Omega_i$ is a sequence of properly (relatively closed, $\overline{D}_i = D_i$) **embedded** minimal surfaces.

Suppose

- $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \dots$ are open subsets of $\mathbf{R}^3$; $\Omega = \bigcup_i \Omega_i$. (take $\Omega_i = \Omega$)
- $D_i \subset \Omega_i$ is a sequence of properly (relatively closed, $\overline{D}_i = D_i$) **embedded** minimal surfaces.

Well-known compactness result:

If for each K, compact subset of Ω, there exist constants $C_1(K), C_2(K) < \infty$ so that

$$\sup_{K \cap D_i} |\mathbf{A}| \leq C_1(K), \text{Area}(D_i \cap K) < C_2(K)$$

then, up to passing to a subsequence, D_i converges, with finite multiplicity, a minimal surface D properly embedded in Ω.

What does a uniform bound on $|\mathbf{A}|$ imply?

What does a uniform bound on $|\mathbf{A}|$ imply?

- In general, a neighborhood of a point $p \in \mathbf{M}$ is always a graph over $T_p\mathbf{M}$. However, the size of such neighborhood depends on p.

What does a uniform bound on $|\mathbf{A}|$ imply?

- In general, a neighborhood of a point $p \in \mathbf{M}$ is always a graph over $T_p\mathbf{M}$. However, the size of such neighborhood depends on p.
- If $\sup_{\mathbf{M}} |\mathbf{A}| = \sup_{\mathbf{M}} |d\xi| \leq \mathbf{C}$ then the size of such neighborhood only depends on $\mathbf{C}$ and NOT on p:

$$d_{\mathbf{S}^2}(\xi(p), \xi(q)) \leq \int_{\gamma_{p,q}} |\nabla \xi| \leq length(\gamma_{p,q}) \sup_{\gamma_{p,q}} |\mathbf{A}| \leq R\mathbf{C},$$

if $q \in \mathcal{B}_R(p)$. Take $R\mathbf{C} < \frac{\pi}{10}$.

What does a uniform bound on $|\mathbf{A}|$ imply?

- In general, a neighborhood of a point $p \in \mathbf{M}$ is always a graph over $T_p\mathbf{M}$. However, the size of such neighborhood depends on p.
- If $\sup_{\mathbf{M}} |\mathbf{A}| = \sup_{\mathbf{M}} |d\xi| \leq \mathbf{C}$ then the size of such neighborhood only depends on $\mathbf{C}$ and NOT on p:

$$d_{\mathbf{S}^2}(\xi(p), \xi(q)) \leq \int_{\gamma_{p,q}} |\nabla \xi| \leq length(\gamma_{p,q}) \sup_{\gamma_{p,q}} |\mathbf{A}| \leq R\mathbf{C},$$

  if $q \in \mathcal{B}_R(p)$. Take $R\mathbf{C} < \frac{\pi}{10}$.
- Let $\mathbf{u}$ be such graph then
  - $\|\mathbf{u}\|_{C^2} \leq 10\mathbf{C}$

What does a uniform bound on $|\mathbf{A}|$ imply?

- In general, a neighborhood of a point $p \in \mathbf{M}$ is always a graph over $T_p\mathbf{M}$. However, the size of such neighborhood depends on p.
- If $\sup_{\mathbf{M}} |\mathbf{A}| = \sup_{\mathbf{M}} |d\xi| \leq \mathbf{C}$ then the size of such neighborhood only depends on $\mathbf{C}$ and NOT on p:

$$d_{\mathbf{S}^2}(\xi(p), \xi(q)) \leq \int_{\gamma_{p,q}} |\nabla \xi| \leq \mathit{length}(\gamma_{p,q}) \sup_{\gamma_{p,q}} |\mathbf{A}| \leq R\mathbf{C},$$

  if $q \in \mathcal{B}_R(p)$. Take $R\mathbf{C} < \frac{\pi}{10}$.
- Let $\mathbf{u}$ be such graph then
  - $\|\mathbf{u}\|_{C^2} \leq 10\mathbf{C}$
  - if $\mathbf{u}$ is a minimal graph then $\operatorname{div} \frac{\nabla \mathbf{u}}{\sqrt{1+|\nabla \mathbf{u}|^2}} = 0 \implies$ $\|\mathbf{u}\|_{C^{2,\alpha}}$ is uniformly bounded independently of p.

Proof of the well-known compactness result:

- $\sup_{D_i} |\mathbf{A}| \leq \mathbf{C}$ uniformly $\implies$ nearby a point be we have a sequence of graphs $\mathbf{u}_i$ with $\|\mathbf{u}_i\|_{C^{2,\alpha}}$ uniformly bounded.

Proof of the well-known compactness result:

- $\sup_{D_i} |\mathbf{A}| \leq \mathbf{C}$ uniformly $\implies$ nearby a point be we have a sequence of graphs $\mathbf{u}_i$ with $\|\mathbf{u}_i\|_{C^{2,\alpha}}$ uniformly bounded.
- Arzela-Ascoli $\implies$ subsequence converging C^2 to a graph that is minimal.

Proof of the well-known compactness result:

- $\sup_{D_i} |\mathbf{A}| \leq \mathbf{C}$ uniformly $\implies$ nearby a point be we have a sequence of graphs $\mathbf{u}_i$ with $\|\mathbf{u}_i\|_{C^{2,\alpha}}$ uniformly bounded.
- Arzela-Ascoli $\implies$ subsequence converging C^2 to a graph that is minimal.
- Area bound $\implies$ there are finitely many of such graphs nearby p (properness).

Proof of the well-known compactness result:

- $\sup_{D_i} |\mathbf{A}| \leq \mathbf{C}$ uniformly $\implies$ nearby a point be we have a sequence of graphs $\mathbf{u}_i$ with $\|\mathbf{u}_i\|_{C^{2,\alpha}}$ uniformly bounded.
- Arzela-Ascoli $\implies$ subsequence converging C^2 to a graph that is minimal.
- Area bound $\implies$ there are finitely many of such graphs nearby p (properness).
- Embeddedness is preserved by the maximum principle.

Natural question:

What happens if we remove such bounds?

Minimal Lamination

A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω (open set) if

- $\mathcal{L}$ is relatively closed in Ω;

Minimal Lamination

A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω (open set) if

- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;

Minimal Lamination

A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω (open set) if

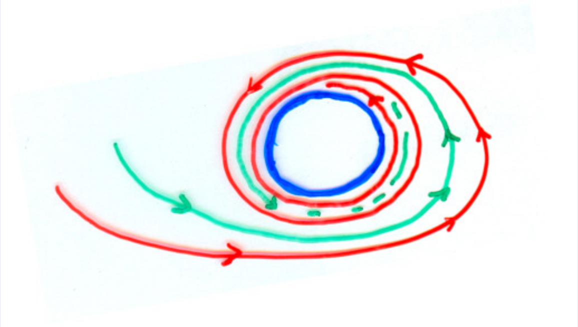
- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset U_p of Ω, a closed subset K_p of $(-1, 1)$ and a Lipschitz diffeomorphism, "straightening map,"
  $\psi_p : (U_p, p) \to (B_1, 0)$ so
  $\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}$.

Minimal Lamination

A subset $\mathcal{L}$ is a *proper minimal lamination* of Ω (open set) if

- $\mathcal{L}$ is relatively closed in Ω;
- $\mathcal{L} = \bigcup_\alpha L_\alpha$ where L_α are connected pair-wise disjoint embedded minimal surfaces in Ω – called *leaves* of $\mathcal{L}$;
- For each $p \in \mathcal{L}$ there is an open subset U_p of Ω, a closed subset K_p of $(-1, 1)$ and a Lipschitz diffeomorphism, "straightening map,"
  $\psi_p : (U_p, p) \to (B_1, 0)$ so
  $\psi_p(\mathcal{L} \cap U_p) = B_1 \cap \{x_3 = t\}_{t \in K_p}$.

If $\mathcal{L} = \Omega$, then this is a *minimal foliation* of Ω.



Well-known compactness result:

If for each K, compact subset of Ω, there is a constant $C(K) < \infty$ so that

$$\sup_{K \cap D_i} |\mathbf{A}| \leq C(K),$$

then, up to passing to a subsequence, the D_i converge to $\mathcal{L}$, a smooth minimal lamination of Ω.

In light of the previous result,
we say that the curvatures of the D_i blow-up at $p \in \Omega$ if there is a sequence of points $p_i \in D_i$ such that

$$p_i \to p \quad \text{and} \quad |\mathbf{A}|(p_i) \to \infty.$$

Blow-up points or **singular points**.

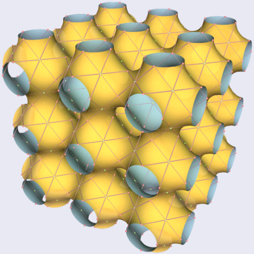
and by passing to a subsequence we may assume that there is a relatively closed subset $\mathcal{S} \subset \Omega$ such that

- the curvatures of the D_i blow-up at each $p \in \mathcal{S}$;
- $D_i \setminus \mathcal{S}$ converges on $\Omega \setminus \mathcal{S}$ to a minimal lamination $\mathcal{L}$ of $\Omega \setminus \mathcal{S}$.

Natural question:

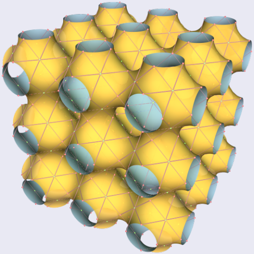
What sets $\mathcal{S}$ and laminations $\mathcal{L}$ can arise in this way?

$\frac{1}{i}$(triply periodic minimal surface)



Rescalings of a triply periodic minimal surfaces in $\mathbf{R^3}$.
$\mathcal{S} = \mathbf{R^3}$, $\mathcal{L} = \emptyset$.

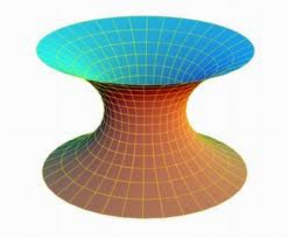
$\frac{1}{i}$(triply periodic minimal surface)



Rescalings of a triply periodic minimal surfaces in $\mathbf{R^3}$.
$\mathcal{S} = \mathbf{R^3}$, $\mathcal{L} = \emptyset$.

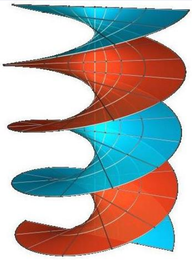
Let us focus on sequence of surfaces with finite topology.

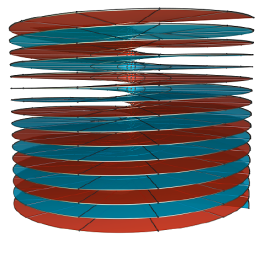
$\frac{1}{i}$(Catenoid)



Rescalings of a catenoid.
$\mathcal{S} = \vec{0}$, $\mathcal{L}$ has a single leaf $\{z = 0\} \setminus \vec{0}$.
NB: The leaf extends smoothly to a surface in $\mathbf{R}^3$.

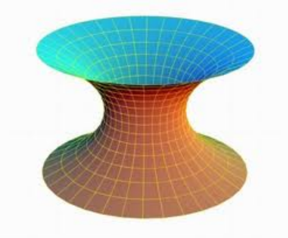
$\frac{1}{i}$(Helicoid)





Rescalings of a helicoid.
$\mathcal{S} = z - axis$, $\mathcal{L}$ is a foliation of $\mathbf{R}^3$ minus the z-axis by horizontal planes.
NB: The leaves extend smoothly to surfaces in $\mathbf{R}^3$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbf{R}^3$.

$\frac{1}{i}$(Catenoid)



Rescalings of a catenoid.
$\mathcal{S} = \vec{0}$, $\mathcal{L}$ has a single leaf $\{z = 0\} \setminus \vec{0}$.
NB: The leaf extends smoothly to a surface in $\mathbf{R}^3$.

Theorem (Anderson, White (1985))

If the total curvatures of the surfaces D_i, i.e. $\int_{D_i} |\mathbf{A}|^2$, are uniformly bounded, then

- $\mathcal{S}$ *is finite;*
- $\mathcal{L}$ *extends smoothly across* $\mathcal{S}$.

If the D_i *are disks then* $\mathcal{S} = \emptyset$.

Theorem (Anderson, White (1985))

If the total curvatures of the surfaces D_i, i.e. $\int_{D_i} |\mathbf{A}|^2$, are uniformly bounded, then

- *$\mathcal{S}$ is finite;*
- *$\mathcal{L}$ extends smoothly across $\mathcal{S}$.*

If the D_i are disks then $\mathcal{S} = \emptyset$.

What if the D_i have unbounded total curvatures?

Theorem (Anderson, White (1985))

If the total curvatures of the surfaces D_i, i.e. $\int_{D_i} |\mathbf{A}|^2$, are uniformly bounded, then

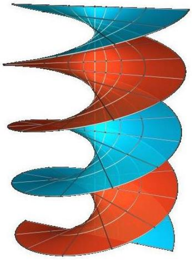
- *$\mathcal{S}$ is finite;*
- *$\mathcal{L}$ extends smoothly across $\mathcal{S}$.*

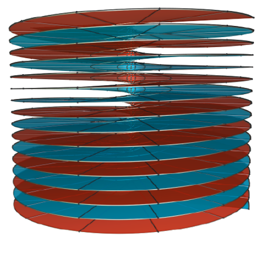
If the D_i are disks then $\mathcal{S} = \emptyset$.

What if the D_i have unbounded total curvatures?

Let us assume that the D_i are (properly embedded) DISKS.

Key example





Rescalings of a helicoid
$\mathcal{S} = z - axis$, $\mathcal{L}$ is a foliation of $\mathbf{R}^3$ minus the z-axis by horizontal planes.
NB: The leaves extend smoothly to surfaces in $\mathbf{R}^3$. Likewise, the lamination $\mathcal{L}$ extends to a proper foliation of $\mathbf{R}^3$.

Colding-Minicozzi Theory

Theorem (Colding-Minicozzi, 2004)

Suppose each D_i is a properly embedded disk and $\Omega = \mathbf{R}^3$. If $\mathcal{S} \neq \emptyset$ then

- *$\mathcal{L}$ is a foliation of $\mathbf{R}^3 \setminus \mathcal{S}$ by parallel planes;*
- *$\mathcal{S}$ is a line perpendicular to those planes. (Meeks)*

The situation is very different when $\Omega \subsetneq \mathbf{R^3}$.
What can be said about the set $\mathcal{S}$ and about the leaves of $\mathcal{L}$?
(reminder D_is are **embedded DISKS**)

The situation is very different when $\Omega \subsetneq \mathbf{R}^3$.
What can be said about the set $\mathcal{S}$ and about the leaves of $\mathcal{L}$?
(reminder D_is are **embedded DISKS**)

The set $\mathcal{S}$ can be:

- a point (Colding-Minicozzi);

The situation is very different when $\Omega \subsetneq \mathbf{R^3}$.
What can be said about the set $\mathcal{S}$ and about the leaves of $\mathcal{L}$?
(reminder D_is are **embedded DISKS**)

The set $\mathcal{S}$ can be:

- a point (Colding-Minicozzi);
- a line segment (Dean);

The situation is very different when $\Omega \subsetneq \mathbf{R}^3$.
What can be said about the set $\mathcal{S}$ and about the leaves of $\mathcal{L}$?
(reminder D_is are **embedded DISKS**)

The set $\mathcal{S}$ can be:

- a point (Colding-Minicozzi);
- a line segment (Dean);
- an arbitrary finite subset of a line segment (Kahn);

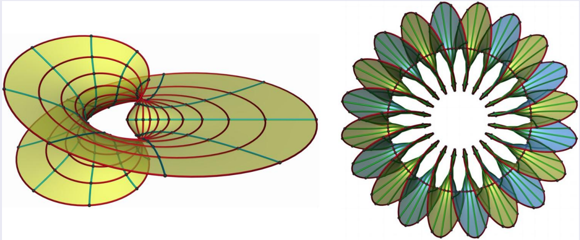
The situation is very different when $\Omega \subsetneq \mathbf{R^3}$.
What can be said about the set $\mathcal{S}$ and about the leaves of $\mathcal{L}$?
(reminder D_is are **embedded DISKS**)

The set $\mathcal{S}$ can be:

- a point (Colding-Minicozzi);
- a line segment (Dean);
- an arbitrary finite subset of a line segment (Kahn);
- any closed subset of the z-axis (Hoffman-White, later Kleene).

In contrast to the other constructions, Hoffman-White use variational methods which carry over to $\Omega = \mathbb{H}^3$.

- an arbitrary $C^{1,1}$ curve (Meeks-Weber).



Sequence of minimal annuli in a solid torus of revolution whose singular set is the central circle of the solid torus.

The set $\mathcal{S}$ can be:

- a point (Colding-Minicozzi);
- a line segment (Dean);
- an arbitrary finite subset of a line segment (Kahn);
- any closed subset of the z-axis (Hoffman-White, later Kleene);
- an arbitrary $C^{1,1}$ curve (Meeks-Weber).

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi 2004)

The structure of $\mathcal{S}$

- $\mathcal{S}$ *is contained in a properly embedded Lipshitz curve* $\mathcal{S}'$ *of* Ω.
- *For any* $p \in \mathcal{S}$ *there exists a leaf* L *such that* $p \in \overline{L}$ *and* $\overline{L}$ *is a properly embedded minimal surface.*
- *If* $\overline{L}$ *is a properly embedded minimal surface, and* $\overline{L} \cap \mathcal{S} \neq \emptyset$, *then* $\overline{L}$ *meets* $\mathcal{S}$ *transversely.*

In the local case, Colding-Minicozzi (essentially) show

Theorem (Colding-Minicozzi 2004)

The structure of $\mathcal{S}$

- $\mathcal{S}$ *is contained in a properly embedded Lipshitz curve* $\mathcal{S}'$ *of* Ω.
- *For any* $p \in \mathcal{S}$ *there exists a leaf* L *such that* $p \in \overline{L}$ *and* $\overline{L}$ *is a properly embedded minimal surface.*
- *If* $\overline{L}$ *is a properly embedded minimal surface, and* $\overline{L} \cap \mathcal{S} \neq \emptyset$, *then* $\overline{L}$ *meets* $\mathcal{S}$ *transversely.*

- Meeks showed that if $\mathcal{S} = \mathcal{S}'$ (i.e., $\mathcal{S}$ has no "gaps"), then it is a $C^{1,1}$ curve (tangent to curve is orthogonal to leaves)
- White showed that $\mathcal{S}$ is contained in a C^1 curve.

What can be said about the the leaves of $\mathcal{L}$?

What can be said about the the leaves of $\mathcal{L}$?
In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \setminus \mathcal{S}$;

What can be said about the the leaves of $\mathcal{L}$?
In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \setminus \mathcal{S}$;
- a proper disk or annulus (Hoffman-White) in Ω;

What can be said about the the leaves of $\mathcal{L}$?
In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \setminus \mathcal{S}$;
- a proper disk or annulus (Hoffman-White) in Ω;
- a proper annulus in $\Omega \setminus \mathcal{S}$ with $\bar{L} \cap \mathcal{S} \neq \emptyset$ and $\bar{L}$ is a proper disk in Ω.

What can be said about the the leaves of $\mathcal{L}$?
In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \setminus \mathcal{S}$;
- a proper disk or annulus (Hoffman-White) in Ω;
- a proper annulus in $\Omega \setminus \mathcal{S}$ with $\bar{L} \cap \mathcal{S} \neq \emptyset$ and $\bar{L}$ is a proper disk in Ω.

Question (Hoffman-White)

Can a surface of genus>0 occur? A planar domain with more than two ends?

What can be said about the the leaves of $\mathcal{L}$?
In all known examples, the leaves of $\mathcal{L}$ are either disks or annuli. Indeed, if L is a leaf of an example then it can be

- a non-proper disk in $\Omega \setminus \mathcal{S}$;
- a proper disk or annulus (Hoffman-White) in Ω;
- a proper annulus in $\Omega \setminus \mathcal{S}$ with $\bar{L} \cap \mathcal{S} \neq \emptyset$ and $\bar{L}$ is a proper disk in Ω.

Question (Hoffman-White)

Can a surface of genus>0 occur? A planar domain with more than two ends?

Answer (Bernstein-T.)

No, under natural geometric condition on Ω it cannot.

Let Ω be the interior of an oriented compact three-manifold with boundary so that:

- $\partial\Omega$ is strictly mean convex;
- There are no closed minimal surfaces in Ω.

Let Ω be the interior of an oriented compact three-manifold with boundary so that:

- $\partial\Omega$ is strictly mean convex;
- There are no closed minimal surfaces in Ω.

Theorem

Let L be a leaf of $\mathcal{L}$ then:

- L is either a disk or an annulus.
- If $\overline{L}$ is a properly embedded minimal surface, then L is either a puncture disk, or a disk or an annulus disjoint from $\mathcal{S}$.

Let Ω be the interior of an oriented compact three-manifold with boundary so that:

- $\partial\Omega$ is strictly mean convex;
- There are no closed minimal surfaces in Ω.

Theorem

Let L be a leaf of $\mathcal{L}$ then:

- L is either a disk or an annulus.
- If $\overline{L}$ is a properly embedded minimal surface, then L is either a puncture disk, or a disk or an annulus disjoint from $\mathcal{S}$.

Second bullet: Colding-Minicozzi $\implies$ the set of singular points meeting $\overline{L}$ is a discrete set in $\overline{L}$.

Example

Example of a torus being the limit of (non-minimal) disks.

Idea of proof

- The disks D_i in the sequence act as an "effective" universal cover of L.

Idea of proof

- The disks D_i in the sequence act as an "effective" universal cover of L.
- Specifically, one can "lift" closed curves in L to curves in the D_i.

Idea of proof

- The disks D_i in the sequence act as an "effective" universal cover of L.
- Specifically, one can "lift" closed curves in L to curves in the D_i.
- The geometry of the D_i – minimally embedded and in a mean convex set – restricts the topology of the L, essentially forcing it to have abelian fundamental group.

Idea of proof

- The disks D_i in the sequence act as an "effective" universal cover of L.
- Specifically, one can "lift" closed curves in L to curves in the D_i.
- The geometry of the D_i – minimally embedded and in a mean convex set – restricts the topology of the L, essentially forcing it to have abelian fundamental group.
- A more complicated geometric feature we use: the conditions on Ω ensure – by a result of White – that minimal surfaces in Ω satisfy an isoperimetric inequality.

Definition

If $\gamma\colon S^1 \to L$ is a piece-wise C^1 closed curve, then γ has the *closed-lift property* if there exists a sequence of closed "lifts" $\gamma_i\colon S^1 \to D_i$ converging to γ. Otherwise, γ has the *open-lift property*.

Definition

If $\gamma\colon S^1 \to L$ is a piece-wise C^1 closed curve, then γ has the *closed-lift property* if there exists a sequence of closed "lifts" $\gamma_i\colon S^1 \to D_i$ converging to γ. Otherwise, γ has the *open-lift property*.

If γ is embedded so are its lifts.

Separating Lemma

If $\gamma \colon S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Separating Lemma

If $\gamma \colon S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof

- Let $\gamma_i \colon S^1 \to D_i$ be a sequence of embedded closed lifts converging to γ;

Separating Lemma

If $\gamma \colon S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof

- Let $\gamma_i \colon S^1 \to D_i$ be a sequence of embedded closed lifts converging to γ;
- Each γ_i is the boundary of a close minimal disk $\Delta_i \subset D_i$;

Separating Lemma

If $\gamma \colon S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof

- Let $\gamma_i \colon S^1 \to D_i$ be a sequence of embedded closed lifts converging to γ;
- Each γ_i is the boundary of a close minimal disk $\Delta_i \subset D_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;

Separating Lemma

If $\gamma\colon S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof

- Let $\gamma_i\colon S^1 \to D_i$ be a sequence of embedded closed lifts converging to γ;
- Each γ_i is the boundary of a close minimal disk $\Delta_i \subset D_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega)$ and $\Delta \subset L \setminus \gamma$ is open and close;

Separating Lemma

If $\gamma\colon S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof

- Let $\gamma_i\colon S^1 \to D_i$ be a sequence of embedded closed lifts converging to γ;
- Each γ_i is the boundary of a close minimal disk $\Delta_i \subset D_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega)$ and $\Delta \subset L \setminus \gamma$ is open and close;
- If γ does not separate L then $\Delta = L \setminus \gamma$;

Separating Lemma

If $\gamma\colon S^1 \to L$ is a closed embedded curve in L with the closed lift property, then γ is separating.

Proof

- Let $\gamma_i\colon S^1 \to D_i$ be a sequence of embedded closed lifts converging to γ;
- Each γ_i is the boundary of a close minimal disk $\Delta_i \subset D_i$;
- $Area(\Delta_i) < C_1 Length(\gamma_i) < C_2 Length(\gamma)$;
- $\Delta_i \to \Delta$ in $C^\infty_{loc}(\Omega)$ and $\Delta \subset L \setminus \gamma$ is open and close;
- If γ does not separate L then $\Delta = L \setminus \gamma$;
- Contradiction because $\partial\Omega$ strictly mean convex implies Δ_i cannot get close to $\partial\Omega$.

Commutator Lemma

Let L be two-sided and let

$$\alpha : [0,1] \to L \text{ and } \beta : [0,1] \to L$$

be closed piece-wise C^1 Jordan curves. If α and β have the open lift property and $\alpha \cap \beta = p_0$ where $p_0 = \alpha(0) = \beta(0)$, then

$$\nu := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$$

has the closed lift property.

Proposition

If L is two-sided then L has genus zero.

Proposition

If L is two-sided then L has genus zero.

Proof

Otherwise,

- Let α and β be two non-separating curves in L meeting at one point.

Proposition

If L is two-sided then L has genus zero.

Proof

Otherwise,

- Let α and β be two non-separating curves in L meeting at one point.
- Each curve has the open-lift property by the separating lemma.

Proposition

If L is two-sided then L has genus zero.

Proof

Otherwise,

- Let α and β be two non-separating curves in L meeting at one point.
- Each curve has the open-lift property by the separating lemma.
- $\alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$ has the closed lift property but it is non-separating.

Proposition

If L is two-sided then L has genus zero.

Proof

Otherwise,

- Let α and β be two non-separating curves in L meeting at one point.
- Each curve has the open-lift property by the separating lemma.
- $\alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$ has the closed lift property but it is non-separating.
- Contradiction.

Proposition [No pants]

If L is two-sided then L is either a disk or an annulus.

Proposition [No pants]

If L is two-sided then L is either a disk or an annulus.

Proof

We have already shown that L must be a planar domain.
Assume L is not an annulus.

Proposition [No pants]

If L is two-sided then L is either a disk or an annulus.

Proof

We have already shown that L must be a planar domain. Assume L is not an annulus. Then,

- There exist embedded closed curves α and β separating L in 3 connected components, L_1,L_2 and L_3.

Proposition [No pants]

If L is two-sided then L is either a disk or an annulus.

Proof

We have already shown that L must be a planar domain. Assume L is not an annulus. Then,

- There exist embedded closed curves α and β separating L in 3 connected components, L_1,L_2 and L_3.
- Let L_3 be the component such that $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus.

Proposition [No pants]

If L is two-sided then L is either a disk or an annulus.

Proof

We have already shown that L must be a planar domain. Assume L is not an annulus. Then,

- There exist embedded closed curves α and β separating L in 3 connected components, L_1,L_2 and L_3.
- Let L_3 be the component such that $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus.
- α and β must have the open lift property.

Proposition [No pants]

If L is two-sided then L is either a disk or an annulus.

Proof

We have already shown that L must be a planar domain. Assume L is not an annulus. Then,

- There exist embedded closed curves α and β separating L in 3 connected components, L_1,L_2 and L_3.
- Let L_3 be the component such that $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus.
- α and β must have the open lift property.
- Let σ be an embedded arc in L_3 with endpoints in α and β and consider the two curves $\sigma \circ \alpha \circ \sigma^{-1}$ and β.

Proposition [No pants]

If L is two-sided then L is either a disk or an annulus.

Proof

We have already shown that L must be a planar domain. Assume L is not an annulus. Then,

- There exist embedded closed curves α and β separating L in 3 connected components, L_1,L_2 and L_3.
- Let L_3 be the component such that $\alpha \cup \beta \subset \partial L_3$ and L_3 is not an annulus.
- α and β must have the open lift property.
- Let σ be an embedded arc in L_3 with endpoints in α and β and consider the two curves $\sigma \circ \alpha \circ \sigma^{-1}$ and β.
- By the Commutator Lemma $\sigma \circ \alpha \circ \sigma^{-1} \circ \beta \circ (\sigma \circ \alpha \circ \sigma^{-1})^{-1} \circ \beta^{-1}$ has the closed lift property.

Proof

- A sequence of embedded minimal disks Δ_i must converge to an open and close subset of $L \setminus (\beta \circ \sigma \circ \alpha)$.

Proof

- A sequence of embedded minimal disks Δ_i must converge to an open and close subset of $L \setminus (\beta \circ \sigma \circ \alpha)$.
- In particular, the limit must contain either L_1, or L_2 or L_3.

Proof

- A sequence of embedded minimal disks Δ_i must converge to an open and close subset of $L \setminus (\beta \circ \sigma \circ \alpha)$.
- In particular, the limit must contain either L_1, or L_2 or L_3.
- Contradiction because $\partial\Omega$ strictly mean convex implies Δ_i cannot get close to $\partial\Omega$.

Proposition

A leaf L is two-sided.

Proposition

A leaf L is two-sided.

Proof

- If L is one-sided, then there is a closed non-separating curve along which L does not have well defined normal;

Proposition

A leaf L is two-sided.

Proof

- If L is one-sided, then there is a closed non-separating curve along which L does not have well defined normal;
- Non-separating $\implies$ lift of this curve is open;

Proposition

A leaf L is two-sided.

Proof

- If L is one-sided, then there is a closed non-separating curve along which L does not have well defined normal;
- Non-separating $\implies$ lift of this curve is open;
- Following lift around in a D_i violates either properness or embeddedness.

Understanding Geometric Conditions

Question

- *Is our theorem sharp?*
- *To what extent can the assumptions on Ω be relaxed?*

Understanding Geometric Conditions

Question

- *Is our theorem sharp?*
- *To what extent can the assumptions on Ω be relaxed?*

Let D be an embedded but NOT properly embedded minimal disk in Ω with the property that the closure, $\bar{D}$, of D in Ω is a proper minimal lamination of Ω. (In fact more general.)

Topology of Minimal Disk Closures

The leaves of $\bar{D}$ behave almost identically to those of the limit leaves of a sequence of minimal disks.

Theorem

Let Ω be the interior of an oriented compact three-manifold with boundary so that:

- *$\partial\Omega$ is strictly mean convex;*
- *There are no closed minimal surfaces in Ω.*

Then each leaf L of $\bar{D}$ is either a disk, an annulus or a Möbius band.

Sharpness

The preceding theorem is sharp in the following sense:

Sharpness

The preceding theorem is sharp in the following sense:

- There is an embedded minimal disk D that contains a Möbius band in its closure. Note: the lamination $\bar{D}$ cannot occur as the lamination that is the limit of a sequence of minimal disks.

Sharpness

The preceding theorem is sharp in the following sense:

- There is an embedded minimal disk D that contains a Möbius band in its closure. Note: the lamination $\bar{D}$ cannot occur as the lamination that is the limit of a sequence of minimal disks.
- There is an embedded minimal disk D whose closure contains a minimal torus in Ω.

Further Questions

Some further questions:

- To what extent are both theorems true even for regions which contain closed minimal surfaces?
- To what extent is the theorem for a sequence of minimal disks sharp? For instance, it is hard to picture a minimal torus arises in this context.

Commutator Lemma

Let L be two-sided and let

$$\alpha : [0,1] \to L \text{ and } \beta : [0,1] \to L$$

be closed piece-wise C^1 Jordan curves. If α and β have the open lift property and $\alpha \cap \beta = p_0$ where $p_0 = \alpha(0) = \beta(0)$, then

$$\nu := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$$

has the closed lift property.

Proof

Let α_i^+ (β_i^+) be a lift of α (β) and let α_i^- (β_i^-) be a lift of α^{-1} (β^{-1}).

Proof

Let α_i^+ (β_i^+) be a lift of α (β) and let α_i^- (β_i^-) be a lift of α^{-1} (β^{-1}).

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."

Proof

Let α_i^+ (β_i^+) be a lift of α (β) and let α_i^- (β_i^-) be a lift of α^{-1} (β^{-1}).

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."
- If α_i^+ moves "upward" m_i sheets, α_i^- moves "downward" m_i sheets.

Proof

Let α_i^+ (β_i^+) be a lift of α (β) and let α_i^- (β_i^-) be a lift of α^{-1} (β^{-1}).

- Using embeddedness, the graphs converging to a small neighborhood of p_0 can be order by "height."
- If α_i^+ moves "upward" m_i sheets, α_i^- moves "downward" m_i sheets.
- If β_i^+ moves "upward" n_i sheets, β_i^- moves "downward" n_i sheets.