An EW-type representation for constant extrinsic curvature surfaces in hyperbolic space.

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Background - the Enneper-Weierstrass representation:

Let (i, S) be an immersed surface in $\mathbb{R}^3$. That is, S is a surface and $i : S \to \mathbb{R}^3$ is an immersion.

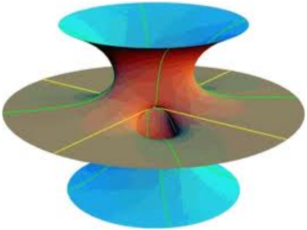
Let $N : S \to \mathbb{S}^2$ be the unit normal vector field over i. N is antiholomorphic

Let $x_3 : S \to \mathbb{R}^3$ be the third component. Let ω be the holomorphic 1-form such that $\mathrm{Re}(\omega) = dx_3$.

The pair (N, ω) uniquely determines i (up to translation).

This constitutes the **Enneper-Weierstrass** representation. Used, for example, by Enneper and Costa.

Example: the Costa surface:



Enneper-Weierstrass type representations:

The Granada School (selection):

Gálvez J. A., Martínez A., Milán F., *Math. Ann.*, (2000).

Aledo A. A., Espinar J. M., Gálvez J. A., *J. Geom. Phys.*, (2006).

Espinar J. M., Gálvez J. A., Mira P., *J. Eur. Math. Soc.*, (2011).

Labourie & co. (selection):

Labourie F., *GAFA*, (1997).

Labourie F., *Inv. Math.*, (2000).

Smith G., *Bull. Soc. Math. France*, (2006).

Hitchin & chums (Higgs bundle formalism).

The framework - part I:

Let (i, S) be an immersed surface in $\mathbb{H}^3$.

Let $N_i : S \to \mathrm{U}\mathbb{H}^3$ be the unit normal vector field over i.

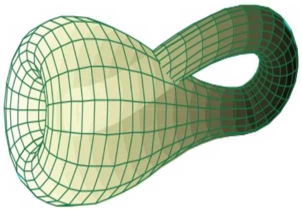
Let $A_i X := \nabla_X N_i$ be the shape operator of i.

Let $\mathrm{K}_i := \mathrm{Det}(A_i)$ be the extrinsic curvature of i.

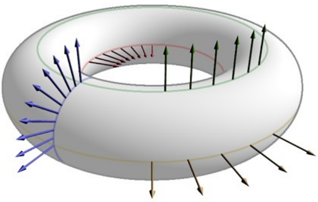
When $\mathrm{K}_i = k > 0$, we may suppose that $A_i > 0$ everywhere.

We say that i is **locally strictly convex** (LSC) when $A_i > 0$ everywhere.

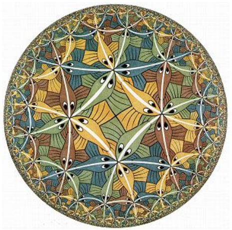
Example: a (non-oriented) immersion:



Example: the unit normal vector field:



Example: "Circle Limit" M.C. Escher:



Surfaces in hyperbolic space:

Let (i, S) be a proper immersed surface in $\mathbb{H}^3$ constant extrinsic curvature equal to k:

(1) When $k > 1$, (i, S) is a geodesic sphere (Hopf);

(2) when $k = 1$, (i, S) is either a horosphere or a cylinder of points equidistant to a complete geodesic (Volkov-Vladimirova, Sasaki);

(3) when $k = 0$, (i, S) is ruled; and

(4) no such surface exists for $k < 0$ (Weierstrass).

It is only for $k \in]0, 1[$ that really interesting things happen.

The hyperbolic Gauss map:

Let $\mathrm{U}\mathbb{H}^3$ be the unitary bundle over $\mathbb{H}^3$.

Let $\partial_\infty \mathbb{H}^3$ be the ideal boundary of $\mathbb{H}^3$.

We define the **Gauss map**:

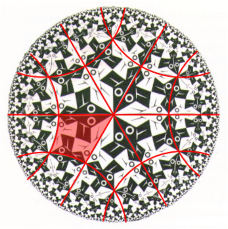
$$g : \mathrm{U}\mathbb{H}^3 \to \partial_\infty \mathbb{H}^3;\ V \mapsto \gamma_V(+\infty),$$

where $\gamma_V : \mathbb{R} \to \mathbb{H}^3$ is the unit geodesic such that:

$$\gamma_V'(0) = V.$$

$g(V)$ is the point in $\partial_\infty \mathbb{H}^3$ towards which V points.

Example: "Circle Limit" M.C. Escher (with geodesics):



The Weierstrass map:

We identify $\partial_\infty \mathbb{H}^3$ with $\hat{\mathbb{C}}$.

We define the **Weierstrass map**, $\varphi_i : S \to \hat{\mathbb{C}}$ by:

$$\varphi_i = g \circ N_i.$$

When i is LSC, φ_i is a local homeomorphism.

An inverse problem:

Let S be a surface. Fix $k \in]0,1[$. Let $\varphi : S \to \hat{\mathbb{C}}$ be a local homeomorphism.

Under what conditions on φ does there exist a complete smooth LSC immersion $i : S \to \mathbb{H}^3$ such that:

$$\mathrm{K}_i = k \qquad \& \qquad \phi_i = \phi?$$

Observe that N_i defines an immersion from S into $\mathrm{U}\mathbb{H}^3$.

We say that i is N-complete whenever the metric induced by N_i is complete.

Under what conditions on φ does there exist an N-complete smooth LSC immersion $i : S \to \mathbb{H}^3$ such that $\mathrm{K}_i = k$ and $\varphi_i = \varphi$?.

Solving the inverse problem:

Furnishing S with the conformal structure $\phi^*\mathbb{C}$, we may assume that S is a Riemann surface and that ϕ is a locally conformal mapping.

Theorem A, Smith (2004)

Let S be a Riemann surface. Let $\phi : S \to \mathbb{C}$ be a locally conformal map. If S is of hyperbolic type, then for all $k \in]0,1[$, there exists a unique N-complete LSC immmersion $i : S \to \mathbb{H}^3$ such that:

$$\mathrm{K}_i = k \qquad \& \qquad \phi_i = \phi.$$

Furthermore, i varies continuously with the data (S, ϕ).

Pointed ramified covers of the Riemann sphere:

A pointed ramified cover of $\hat{\mathbb{C}}$ is a triplet (Σ, P, ϕ) where:

(1) Σ is a compact Riemann surface;

(2) P is a finite subset of Σ;

(3) $\phi : \Sigma \to \hat{\mathbb{C}}$ is a non-constant holomorphic map; and

(4) The set of critical points of ϕ is contained in P.

Near $p \in P$, $\phi(z)$ is locally conjugate to $z \mapsto z^n$, for a unique n. We define:

$$\mathrm{Ord}(\phi; p) := n.$$

Moduli of pointed ramified covers:

If (Σ, P, ϕ) be a pointed ramified cover of $\hat{\mathbb{C}}$, then (Σ, P, ϕ) is uniquely determined by:

(1) the topological type of Σ;

(2) the cardinality of P;

(3) the unordered vector of ramification orders $(\mathrm{Ord}(\phi; p))_{p \in P}$;

(4) the unordered vector of images of the critical points $(\phi(p))_{p \in P}$; and

(5) discrete combinatorial data.

In particular, the space of ramified covers of $\hat{\mathbb{C}}$ is stratified by a countable family of finite-dimensional complex manifolds.

Main result:

Theorem B, Smith (2006 + ϵ)

Let (Σ, P, ϕ) be a ramified covering of $\hat{\mathbb{C}}$. Denote $S := \Sigma \setminus P$. For all $k \in]0,1[$ there exists a unique complete LSC immersion $i_k : \Sigma \setminus P \to \mathbb{H}^3$ such that:

$$\mathrm{K}_{i_k} = k, \qquad \phi_{i_k} = \phi.$$

Furthermore, i_k has finite area.

Conversely, let $i : S \to \mathbb{H}^3$ be a complete finite area LSC immersion such that $\mathrm{K}_i = k \in]0,1[$. Then the Riemann surface $(S, \phi_i^*\hat{\mathbb{C}})$ is conformally equivalent to a compact Riemann surface Σ with a finite set P of points removed and ϕ_i extends to a holomorphic map from Σ into $\hat{\mathbb{C}}$.

In other words...

We define:

$$\mathcal{I}_k := \left\{\begin{array}{l}(i,S) \text{ in } \mathbb{H}^3 \text{ s.t.}\\ i \text{ complete;}\\ i \text{ LSC;}\\ i \text{ finite area;}\\ \mathrm{K}_i = k.\end{array}\right\} \qquad \mathcal{H}_k := \left\{\begin{array}{l}(\Sigma,P,\phi) \text{ s.t.}\\ \Sigma \text{ a compact R.S.;}\\ P \subseteq \Sigma \text{ finite;}\\ \phi : \Sigma \to \hat{\mathbb{C}} \text{ holomorphic;}\\ \mathrm{Crit}(\phi) \subseteq P.\end{array}\right\}$$

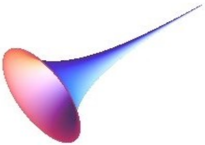
The Weierstrass map defines a bijection:

$$\mathcal{I}_k \to \mathcal{H}_k; \qquad i \mapsto \phi_i.$$

The geometry of the ends:

Let (Σ, P, ϕ) be a ramified covering of $\hat{\mathbb{C}}$. Let $S := \Sigma \setminus P$. Let $i : S \to \mathbb{H}^3$ as in Theorem B.

For $p \in P$, there exists a neighbourhood Ω of p in Σ such that the restriction of i to $\Omega \setminus p$ is a finite covering of a cylindrical cusp around a geodesic:



Behind the scenes:

$\mathrm{U}\mathbb{H}^3$ is a contact manifold.

The contact distribution carries a large family of almost-complex structures.

For a suitable choice of complex structure, a smooth LSC immersion $i : S \to \mathbb{H}^3$ has $\mathrm{K}_i = k$ if and only if N_i is pseudo-holomorphic.

The theory of smooth LSC immersed surfaces of constant extrinsic curvature is a special case of the theory of pseudo-holomorphic curves.

Labourie's approach - lifts and tubes:

Let (i, S) be a proper complete LSC immersed hypersurface in $\mathbb{H}^3$.

Define the **Gauss Lift**, $\hat{\imath} : S \to \mathrm{U}\mathbb{H}^3$ by:

$$\hat{\imath} := N.$$

$(\hat{\imath}, S)$ is an immersed surface in $\mathrm{U}\mathbb{H}^3$.

Let $\Gamma \subseteq \mathbb{H}^3$ be a complete geodesic.

Let $\mathrm{N}\Gamma \subseteq \mathrm{U}\mathbb{H}^3$ be the bundle of unit normal vectors over Γ.

Let $(\hat{\jmath}, S)$ be an immersed surface in $\mathrm{U}\mathbb{H}^3$.

We say that $(\hat{\jmath}, S)$ is a **tube** whenever $\hat{\jmath}$ is a covering map of $\mathrm{N}\Gamma$ for some Γ.

Labourie's compactness theorem:

Theorem C, Labourie (1997)

Let (S_n, i_n, x_n) be a sequence of proper complete LSC immersed hypersurfaces of constant extrinsic curvature equal to k. For all n, let $(S_n, \hat{\imath}_n, x_n)$ be the Gauss lift of (S_n, i_n, x_n). If there exists a compact subset $\subseteq \mathrm{U}\mathbb{H}^3$ such that $\hat{\imath}_n(x_n) \in K$ for all n, then there exists a complete immersed surface $(S_\infty, \hat{\imath}_\infty, x_\infty)$ towards which $(S_n, \hat{\imath}_n, x_n)$ subconverges.

Furthermore, either:

(1) $(S_\infty, \hat{\imath}_\infty)$ is a tube; or

(2) $i_\infty := \pi \circ \hat{\imath}_\infty$ is an immersion, where $\pi : \mathrm{U}\mathbb{H}^3 \to \mathbb{H}^3$ is the canonical projection.

Main steps of the proof:

Lemma D

Let (i, S) be a proper LSC immersed surface in $\mathbb{H}^3$ of constant extrinsic curvature equal to $k \in]0, 1[$. If (i, S) has finite area, then $\|A_i(x)\|$ tends to infinity as x diverges.

Lemma E

Let (i, S) be a proper LSC immersed surface in $\mathbb{H}^3$ of constant extrinsic curvature equal to $k \in]0, 1[$. Let $\mathcal{F}$ be the foliation of S obtained by integrating the principal directions of *least* principal curvature of i. For all $x \in S$, let L_x be the leaf of $\mathcal{F}$ passing through x. If (i, S) has finite area, then the geodesic curvature of $i(L_x)$ at x tends to 0 as x diverges.

Perspectives:

The energy of a pseudo-holomorphic is an important functional.

When $i : S \to \mathbb{H}^3$ is a smooth LSC immersion with $\mathrm{K}_i = k$, the energy of the associated pseudo-holomorphic curve is given by:

$$\mathcal{E}(i) = \int_S \mathrm{H}\mathrm{dVol},$$

where H is the mean curvature of the immersion i.

However $\mathcal{E}(i)$ is infinite!

Perspectives - ctd.:

Fix $P_0 \in \mathbb{H}^3$. For all $R > 0$, define:

$$S_R = S \cap B_R(P_0),$$

and:

$$\mathcal{E}_R(i) = \int_{S_R} \mathrm{HdVol}.$$

We expect $\mathcal{E}_R(i)$ to grow asymptotically by:

$$\mathcal{E}(i) = \mathcal{E}_1(i)R + \mathcal{E}_0(i) + \mathcal{E}_{-1}(i)R^{-1} + O(R^{-2}).$$

We call $\mathcal{E}_{-1}(i)$ the renormalised energy of i.

The renormalised energy perturbs known functionals (such as cross-ratio), and may have applications to the study of the Kähler geometry of the space of pointed ramified covers of $\hat{\mathbb{C}}$.

Obrigado!