Gap results for critical points of some geometrical variational problems

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The variational Problems

M^n a Riemannian manifold
$f : M^n \to N^{n+p}$ isometric immersion of a closed manifold M into N.

- H - the mean curvature vector of the immersion
- α - the second fundamental form of the immersion;

If $d\mu = d\mu_M$ denotes the Riemannian measure on M, we define the following functional

$$\Phi(M) = \int_M \|\alpha\|^2 \, d\mu \,, \tag{1}$$

and view it as functionals defined over the space of all isometric immersions of M into $\tilde{M}$.

Remark 1.

1. A totally geodesic submanifold of N is an minimizer for (1).
2. Φ is a measure of how far the submanifold is to be totally geodesic.

We also considerer the Wilmore functional given by

$$\Psi(M) = \int_M \|H\|^2 \, d\mu \,, \tag{2}$$

and also view it as functionals defined over the space of all isometric immersions of M into $\tilde{M}$.

Remark 2.

1. A minimal submanifold of N is an minimizer for (2).
2. Ψ is a measure of how far the submanifold is to be minimal.

In the case when the background manifold is the space form of curvature c, we introduce for consideration a third functional given by

$$\Theta_c(M) = \int_M (n(n-1)c + \|H\|^2) \, d\mu \,. \tag{3}$$

When $c = 1$, we denote Θ_c simply by Θ.

Critical points

- $f : (-a, a)M \to N$ be a one parameter family of deformations of M.
- $M_t = f(t, M)$ for $t \in (-a, a)$, and $M_0 = M$.
- Given a point $p \in M$, we let $\{x_1, \ldots, x_n, t\}$ be a coordinate system of $M \times (-a, a)$, valid in some neighborhood of $(p, 0)$, and such that $\{x_1, \ldots, x_n\}$ are normal coordinates of M at p.
- T the variational vector field of the deformation.

We would like to compute the t-derivative of $\Phi(M_t)$ at $t = 0$. The next result gives the Euler-Lagrange equation for (1). For convenience, we use the standard double index summation convention.

Theorem 3. *Let $f : (-a, a)M \to N$ be a deformation of an isometrically immersed submanifold $f : M \to f(M) \subset N$ into the Riemannian manifold $(N, \tilde{g})$. We set $M_t = f(t, M)$, have $M_0 = M$, and let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of M_t for all $t \in (-a, a)$. Then the infinitesimal variation of*

$$\Phi(M) = \int_M \|\alpha\|^2 \, d\mu \,,$$

is given by

$$\frac{d\Phi(t)}{dt} = \int_M \langle 2\nabla^{\tilde{g}}_{e_j}\nabla^{\tilde{g}}_{e_i}\alpha(e_i, e_j) + 2R^{\tilde{g}}(\alpha(e_i, e_j), e_j)e_i - \|\alpha\|^2 H, T\rangle d\mu$$
$$+ \int_M \langle 4\langle\alpha(e_i, e_j), \alpha(e_l, e_j)\rangle\alpha(e_l, e_i) - \nabla\|\alpha\|^2, T\rangle d\mu$$
$$-2\int_M (e_i\langle T, e_l\rangle + e_l\langle T, e_i\rangle)\langle\alpha(e_i, e_j), \alpha(e_l, e_j)\rangle d\mu.$$

M is a a critical point of the functional

$$\Psi(M) = \int_M \|H\|^2 \, d\mu \,,$$

if, and only if,

$$2\Delta h = 2cnh - 2h\|\nabla^{\nu}_{e_i}\nu_1\|^2 - h^3 + 2h \,\mathrm{trace}\, A^2_{\nu_1} \,,$$

and for all m in the range $2 \leq m \leq p$, we have that

$$0 = 4e_i(h)\langle\nabla^{\nu}_{e_i}\nu_1, \nu_m\rangle + 2he_i\langle\nabla^{\nu}_{e_i}\nu_1, \nu_m\rangle - 2h\langle\nabla^{\nu}_{e_i}\nu_1, \nabla^{\nu}_{e_i}\nu_m\rangle$$
$$+2h \,\mathrm{trace}\, A_{\nu_1}A_{\nu_m} \,.$$

And M is a critical point of the functional (3) if, and only, if, the last $p-1$ equations above hold, and the one before these is replaced by

$$2\Delta h = (3n - n^2)ch - 2h\|\nabla^{\nu}_{e_i}\nu_1\|^2 - h^3 + 2h \,\mathrm{trace}\, A^2_{\nu_1} \,.$$

In particular, a hypersurface M in S^{n+1}_c is a critical point for the functional

$$\Phi(M) = \int_M \|\alpha\|^2 \, d\mu \,,$$

if, and only if, its mean curvature function h satisfies the equation

$$2\Delta h = 2ch - h\|\alpha\|^2 + 2(k_1^3 + \cdots + k_n^3) \,,$$

while M is a critical point of the functional

$$\Psi(M) = \int_M \|H\|^2 \, d\mu \,,$$

if, and only if, its mean curvature function h satisfies the equation

$$2\Delta h = 2cnh + 2h\|\alpha\|^2 - h^3 \,.$$

Gap Theorems in spheres

Let us begin by recalling the gap theorem in a form suitable to our work (see [Theorem 5.3.2, Corollary 5.3.2](Simons) [Main Theorem](Chern-do Carmo-Kobayashi),[Corollary 2](Lawson).):

Theorem 4. *Suppose that $M^n \hookrightarrow \mathbb{S}^{n+p}$ is an isometric minimal immersion. Assume that the pointwise inequality $\|\alpha\|^2 \leq np/(2p-1)$ holds everywhere. Then*

1. *Either $\|\alpha\|^2 = 0$, or*
2. *$\|\alpha\|^2 = np/(2p-1)$ if, and only if, either $p = 1$ and M^n is the minimal Clifford torus $\mathbb{S}^m(\sqrt{m/n}) \times \mathbb{S}^{n-m}(\sqrt{(n-m)/n}) \subset \mathbb{S}^{n+1}$, $1 \leq m < n$, with $\|\alpha\|^2 = n$, or $n = p = 2$ and M is the real projective plane embedded into $\mathbb{S}^4$ by the Veronese map with $\|\alpha\|^2 = 4/3$.*

For an immersion $M \hookrightarrow \tilde{M}$, we let ν_H denote the normal vector in the direction of the mean curvature vector H, and denote by A_{ν_H} and ∇^{ν} the shape operator in the direction of ν_H and covariant derivative of the normal bundle, respectively. We consider immersions that satisfy the estimates

$$\begin{aligned} -\lambda\|H\|^2 - n &\leq \mathrm{trace}\, A^2_{\nu_H} - \|H\|^2 - \|\nabla^{\nu}\nu_H\|^2 \\ &\leq \|\alpha\|^2 - \|H\|^2 - \|\nabla^{\nu}\nu_H\|^2 \leq \frac{np}{2p-1} \end{aligned} \tag{4}$$

for some constant λ. Notice that $\|A_{\nu_H}\|^2 = \mathrm{trace}\, A^2_{\nu_H}$ is bounded above by $\|\alpha\|^2$, and so the second of the inequalities above is always true. Our first result is the following:

Theorem 5. *Suppose that (M^n, g) is a closed Riemannian manifold isometrically immersed into $\mathbb{S}^{n+p}$ as a critical point of the functional Φ above, and having constant mean curvature function $\|H\|$. Assume that the immersion is such that (4) holds for some constant $\lambda \in [0, 1/2)$. Then M is minimal, and so it is a critical point of the functional Θ also, $0 \leq \|\alpha\|^2 \leq np/(2p-1)$, and either*

1. *$\|\alpha\|^2 = 0$, in which case M lies in an equatorial sphere, or*
2. *$\|\alpha\|^2 = np/(2p-1)$, in which case either $p = 1$ and M^n is the Clifford torus $\mathbb{S}^m(\sqrt{m/n}) \times \mathbb{S}^{n-m}(\sqrt{(n-m)/n}) \subset \mathbb{S}^{n+1}$, $1 \leq m < n$, with $\|\alpha\|^2 = n$, or $n = p = 2$ and M^2 is the real projective plane embedded into $\mathbb{S}^4$ by the Veronese map with $\|\alpha\|^2 = 4/3$ and scalar curvature $2/3$, all cases of metrics with nonnegative Ricci tensor.*

We now consider immersions that satisfy the estimates

$$-\lambda\|H\|^2 - 1 \leq \|\alpha\|^2 - \|H\|^2 - \|\nabla^{\nu}\nu_H\|^2 \leq \frac{np}{2p-1} \tag{5}$$

for some constant λ. Here, $\{\nu_j\}$ is an orthonormal frame of the normal bundle.

Our second result distinguishes further the critical points of Ψ obtained in Theorem 5.

Theorem 6. *Suppose that (M^n, g) is a closed Riemannian manifold isometrically immersed into $\mathbb{S}^{n+p}$ as a critical point of the functional Ψ above, and having constant mean curvature function. Assume that the immersion is such that (5) holds for some constant $\lambda \in [0, 1)$. Then M is minimal, and so a critical point of Ψ and Θ also, $0 \leq \|\alpha\|^2 \leq np/(2p-1)$, and either*

1. *$\|\alpha\|^2 = 0$, in which case M lies inside an equatorial sphere, or*
2. *$\|\alpha\|^2 = np/(2p-1)$, in which case either $n = p = 2$ and M is a minimal real projective plane with an Einstein metric embedded into $\mathbb{S}^4$, or $p = 1$, $n = 2m$ and M is the Clifford torus $\mathbb{S}^m(\sqrt{1/2}) \times \mathbb{S}^m(\sqrt{1/2}) \subset \mathbb{S}^{n+1}$ with its Einstein product metric.*

The following observation follows easily, but it emphasizes the fact that among these submanifolds, we have some that are critical points of the total scalar curvature functional among metrics in M realized by isometric immersions into the sphere $\mathbb{S}^{n+p}$.

Corollary 7. *Let (M, g) be a closed Riemannian manifold that is canonically placed in $\mathbb{S}^{n+p}$ with constant mean curvature function and satisfying (5) for some $\lambda \in [0, 1)$. Then M is a minimal critical point of the total scalar curvature functional under deformations of the isometric immersion, and either $\|\alpha\|^2 = 0$ or $\|\alpha\|^2 = np/(2p-1)$. In the latter case, (M, g) is Einstein and the two possible surface cases in codimension $p = 1$ and $p = 2$ correspond to Einstein manifolds that are associated to different critical values of the total scalar curvature.*

It is slightly easier to prove our first gap result by replacing the role that the functional Φ plays by that of the functional Θ, and derive the same conclusion. The point is not the use of critical points of Θ versus those of Φ. Rather, since the curvature of the sphere is positive, if the second fundamental form is pointwise small in relation to the mean curvature vector, the constant mean curvature function condition forces the critical points of these functionals to be the same, and minimal.

Gap theorems in space of negative curvature

The natural gap theorem for the functionals Ψ or Φ themselves that we can derive is somewhat dual to that proven for minimal immersions into spheres, and occur on quotients of space forms of negative curvature instead.

We recall that a closed hyperbolic manifold is of the form $\mathbb{H}^m/\Gamma$ for Γ a torsion-free discrete group of isometries of $\mathbb{H}^m$. We have the following:

Theorem 8. *Let M be a critical point of (2) on a hyperbolic compact manifold $\mathbb{H}^{n+p}/\Gamma$. If the pointwise inequality $0 \leq \|\alpha\|^2 - \frac{1}{2}\|H\|^2 - \|\nabla^{\nu}\nu_H\|^2 \leq n$ holds on M, then either $\|H\|^2 = 0$ and M is minimal, or $\|\alpha\|^2 = \frac{1}{2}\|H\|^2 + n = \|A_{\nu_H}\|^2$ and M is a nonminimal submanifold whose mean curvature vector is a covariantly constant section of its normal bundle.*

For the functional (1), we have the following gap result in this framework:

Theorem 9. *Let M be a critical point of (1) on a hyperbolic compact manifold $\mathbb{H}^{n+p}/\Gamma$. If the pointwise inequality*

$$\|\alpha\|^2\left(\left(3 - \frac{n}{2}\right)\|\alpha\|^2 - \|H\|^2\right) \leq (n\|\alpha\|^2 + 2\|H\|^2)$$

holds, then either $\|H\|^2 = 0$ and M is a minimal submanifold, or $n \leq 5$, the equality above holds, $\|\alpha\|^2 = \|A_{\nu_H}\|^2$, and M is a submanifold whose mean curvature vector is a covariantly constant section of its normal bundle.

For the proof we need the following Lemma:

Lemma 10. *Let (M, g) be a Riemannian manifold isometrically immersed into $(\tilde{M}, \tilde{g})$, and consider the degree 1-homogeneous function*

$$\frac{\mathrm{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right)}{\|\alpha\|^2}.$$

At a critical point we have that

$$\mathrm{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right) = \frac{\|H\|(\|\alpha\|^2 + 2\|A_{\nu_1}\|^2)}{n + 2\|H\|^2/\|\alpha\|^2}\,,$$

the maximum occurs when $\|\alpha\|^2 = \|A_{\nu_1}\|^2$, and so

$$\mathrm{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right) \leq \frac{3\|H\|\|\alpha\|^2}{2}.$$

Proof of the Lemma. We have that

$$\mathrm{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right) = \sum_{k=1}^{p}\sum_{i,l,s=1}^{n} h^1_{is}h^k_{sl}h^k_{li}\,,$$

and so the function of the h^k_{ij}s under consideration is defined by

$$\frac{\mathrm{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right)}{\|\alpha\|^2} = \frac{\sum_{k=1}^{p}\sum_{i,l,s=1}^{n} h^1_{is}h^k_{sl}h^k_{li}}{\sum_k\sum_{ij}(h^k_{ij})^2}$$

outside the origin, and extended by continuity everywhere. Its critical points subject to the constraints, are the solutions of the system of equations, where $\lambda_1, \ldots, \lambda_p$ are the Lagrange multipliers:

$$\sum_{k=1}^{p}\sum_{l=1}^{n} h^k_{ul}h^k_{lv}\delta_{1r} + \sum_{i=1}^{n} h^1_{ui}h^r_{iv} + \sum_{s=1}^{n} h^1_{vs}h^r_{su}$$

$$= (\sum_k\sum_{ij}(h^k_{ij})^2)^2\delta_{uv}\sum_{j=1}^{p}\lambda_j\delta_{jr}\,,$$

If we multiply by h^r_{uv} and add in u, v and r, we obtain the relation

$$\|\alpha\|^2\mathrm{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right) = \lambda_1 h\|\alpha\|^4\,,$$

while if we set $u = v$, $r = 1$ and add in u, we obtain that

$$\|\alpha\|^2(\|\alpha\|^2 + 2\|A^2_{\nu_1}\|^2) - 2h\,\mathrm{trace}\left(A_{\nu_1}\sum_j A^2_{\nu_j}\right) = \lambda_1 n\|\alpha\|^4\,.$$

A simple algebraic manipulation yields the stated equality at critical points, and the statement about the maximum is then clear. As we have assumed that $n \geq 2$, the inequality follows. □

Proof of the Theorem. We use once again the critical point equation given by Theorem 3, and obtain that

$$0 \leq 2\int h\Delta h \, d\mu_g = \int h^2(-2 - 2\|\nabla^{\nu}_{e_i}\nu_1\|^2$$
$$-\|\alpha\|^2 + 2\frac{1}{h}\mathrm{trace}\, A_{\nu_1}\sum_j A^2_{\nu_j})d\mu_g\,.$$

By Lemma 10, we have that

$$-2 - \|\alpha\|^2 + 2\frac{1}{h}\mathrm{trace}\, A_{\nu_1}\sum_j A^2_{\nu_j} \leq -2 - \|\alpha\|^2 + 2\frac{3\|\alpha\|^2}{n + 2\|H\|^2/\|\alpha\|^2}\,,$$

and the stated inequality is equivalent to the right side of this expression being nonpositive. Thus, either $h = 0$ or the right hand side of the expression above vanishes and the equality holds, and $\nabla^{\nu}\nu_1 = 0$. In the latter case, $h^r_{ij} = 0$ for all $r \geq 2$ and $H = h\nu_1$ is a covariantly constant section of $\nu(M)$. □