Overdetermined elliptic problems in the plane

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$\Omega \subset \mathbb{R}^2$ smooth C^2 domain, $\qquad f \in C^1(\mathbb{R})$.

$$\begin{cases} \Delta u + f(u) = 0, & 0 < u < \text{ const. in } \Omega \\ u = 0 \text{ and } \frac{\partial u}{\partial n} = \alpha & \text{on } \partial\Omega, \quad \alpha > 0. \quad (\alpha = 0) \end{cases}$$

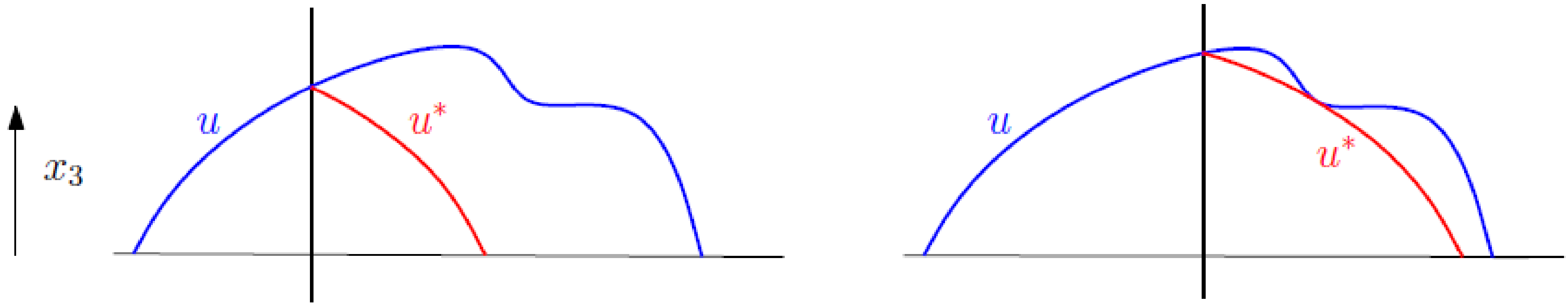
Theorem (Serrin)

If Ω is bounded, then it is a ball and u is radial.

Theorem (Alexandrov)

If $S \subset \mathbb{R}^3$ is a closed surface with constant mean curvature, then S is a round sphere.

Proof: Moving plane technique



If 0 is a critical point for u and u^* then use either

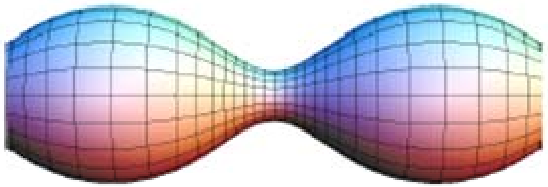
- Interior maximum principle, or
- Boundary maximum principle (Hopf) or
- Corner Lemma (Serrin): If 0 lies in $\partial\Omega \cap \Pi$ (Π is the vertical symmetry plane) and $u(0) = u^*(0)$, $(\nabla u)(0) = (\nabla u^*)(0)$ and $(\nabla^2 u)(0) = (\nabla^2 u^*)(0)$, then $u = u^*$ around 0.

Conjecture (Berestycki, Caffarelli, Nirenberg)

If $\partial\Omega$ is connected, then Ω is either a round disk, the exterior of a disk or a halfplane. Moreover, u has the symmetries of Ω.

Theorem (Sicbaldi) Schlenk)

BCN conjecture fails in high dimension.



Theorem (Del Pino, Pacard, Wei)

There are counterexamples: $n > 8$, epigraphs and

$f(0) = f(1)$, $f'(1) < 0$, $f(s) > 0$ for $0 < s < 1$

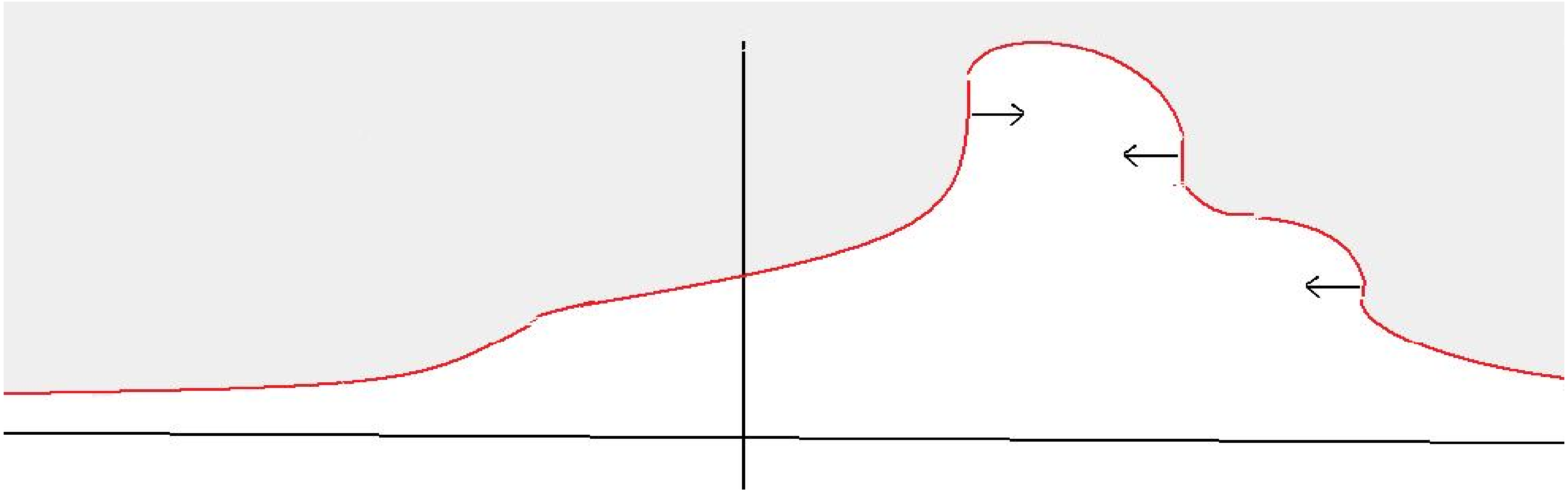
Lemma

The inwards pointing normal halflines lie in Ω.

Proposition (??)

Either $\overline{\Omega}$ contains a straight line tangent to $\Gamma = \partial\Omega$, or Ω is an epigraph:

$$\Omega = \{(x, y) \mid y > \varphi(x)\}, \qquad \varphi \colon \mathbb{R} \to \mathbb{R}.$$



Lemma

If Ω *is an epigraph and* $\limsup_{x\to+\infty}\varphi(x) = \limsup_{x\to-\infty}\varphi(x) = +\infty$, *then the moving plane technique applies to every horizontal line.*

Conjecture 1 (Berestycki, Caffarelli, Nirenberg)

If $\partial\Omega$ *is connected, then* Ω *is either a round disk, the exterior of a disk or a halfplane. Moreover,* u *has the symmetries of* Ω.

Theorem (R–, Sicbaldi)

If the boundary of Ω *is a proper arc, then BCN conjecture holds,*
$\Omega = \{(x,y) \mid y > 0\}$ *and* $u(x,y) = u(y)$.

Lemma

If Ω *is an epigraph and* $\limsup_{x\to-\infty}\varphi(x)=+\infty$, $\limsup_{x\to+\infty}\varphi(x)=0$
then the moving plane technique applies to every horizontal line.

Lemma

If Ω *is an epigraph and* $\limsup_{x\to+\infty}\varphi(x)$, $\limsup_{x\to-\infty}\varphi(x)\in\mathbf{R}$
then the moving plane technique applies to every horizontal line.

Proposition

Either $\overline{\Omega}$ *contains a straight line tangent to* $\Gamma=\partial\Omega$, *or*
the moving plane technique applies to every horizontal line.

Liouville property (BCN Ambrosio-Cabré)

If $\Delta u + f(u) = 0$ in $\mathbb{R}^2$, $|\nabla u| \leq C$, $u_y > 0$, then u is parallel: $u(x,y) = u(y)$.

u_x, u_y eigenfuntions: $\Delta v + f'(u)v = 0$

$v = u_y$, $w = u_x/u_y$. Then $\mathrm{div}(v^2 \nabla w) = 0$.

$\xi \in C_0^2(\mathbb{R}^2)$.

$$\int \xi^2 v^2 |\nabla w|^2 = -2 \int \xi v^2 w < \nabla \xi, \nabla w >$$

$$\leq \left(\int \xi^2 v^2 |\nabla w|^2 \right)^{1/2} \left(\int v^2 w^2 |\nabla \xi|^2 \right)^{1/2}$$

$\Rightarrow \quad |\nabla w| = 0 \quad \Rightarrow \quad u_x = a u_y$, and u is parallel.

Proposition

If u is an overdetermined solution in Ω with bounded gradient and $u_y > 0$, then Ω is a halfplane and u is parallel.

Proof.

$$\langle \nabla w, n \rangle = 0.$$

Farina-Valdinoci using stability argument

Proposition

Every overdetermined solution has bounded geometry.

Proposition

If u is an overdetermined solution in Ω, then either u is parallel or $\overline{\Omega}$ contains a straight line tangent to $\partial\Omega$.

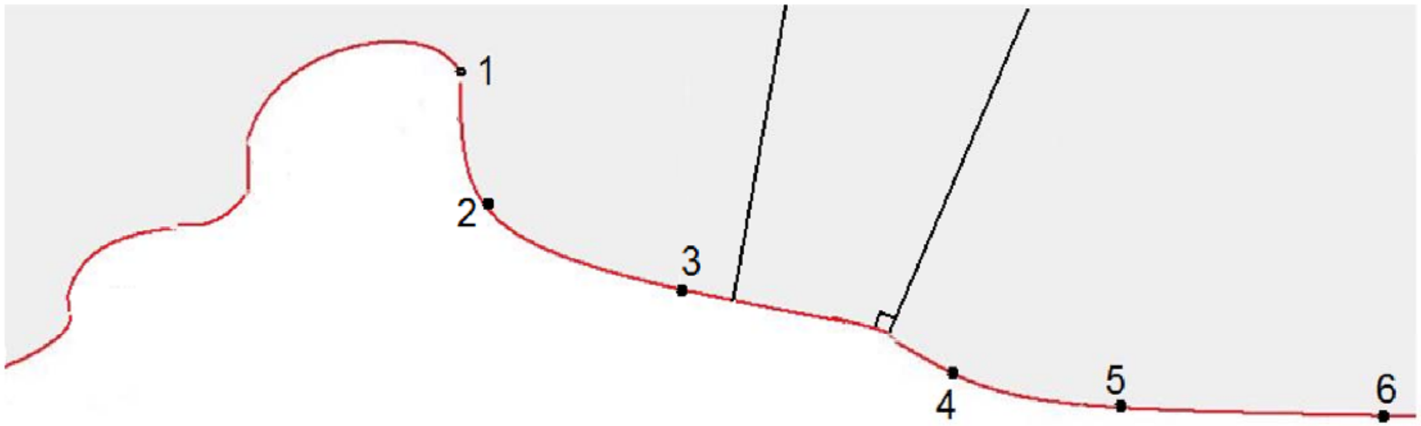
limit direction

$$p_n \in \partial\Omega \qquad p_n \to +\infty \qquad \vec{v} = \lim_n \frac{p_n}{|p_n|}$$

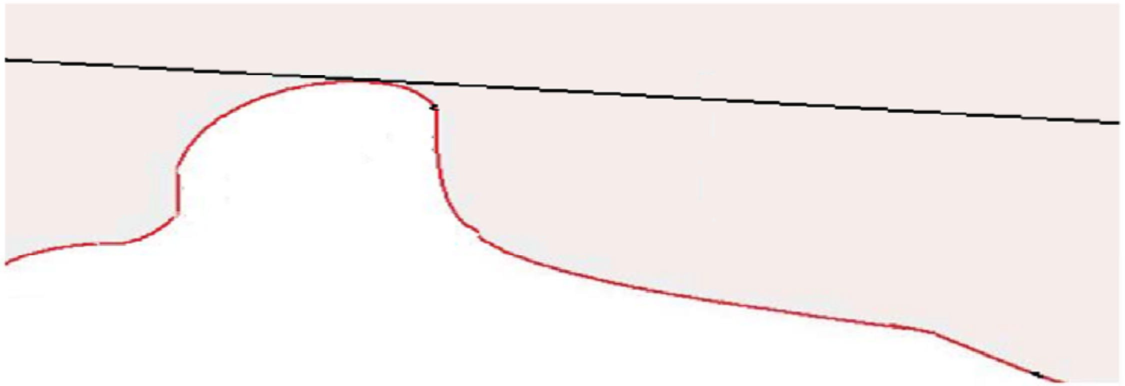
$$\lim_m \frac{p_m - p_n}{|p_m - p_n|} = \vec{v} \qquad \lim_n \frac{0 - p_n}{|0 - p_n|} = -\vec{v}$$

$$\Omega_n = \Omega - p_n \qquad u_n(p) = u(p + p_n)$$

$$u_\infty = \lim u_n : \Omega_\infty \longrightarrow \mathbf{R} \qquad \Delta u_\infty + f(u_\infty) = 0$$



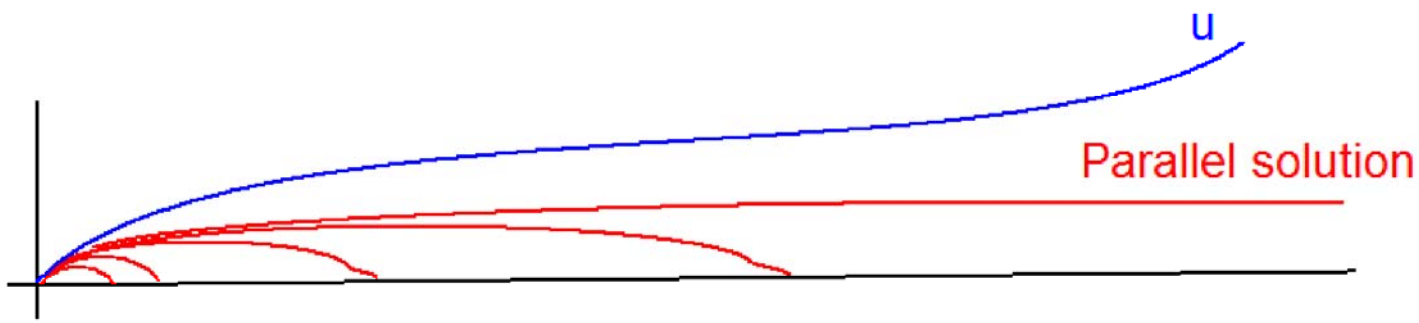
Proposition. If u is an overdetermined solution in Ω and $\overline{\Omega}$ contains a straight line tangent to $\partial\Omega$ then a translation limit of (Ω, u) is parallel.



Lemma. There is at most a bounded nonnegative solution $u(t)$, $0 \leq t < \infty$, of the ODE

$$u'' + f(u) = 0, \quad u(0) = 0, \quad u'(0) \geq 0$$

Proposition If $\overline{\Omega}$ contains a straight line tangent to its boundary then $\Omega = \{(x,y)/y > 0\}$ and $u(x,y) = u(y)$.

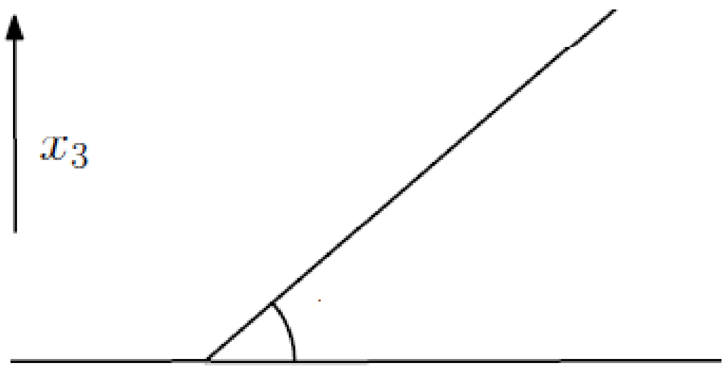


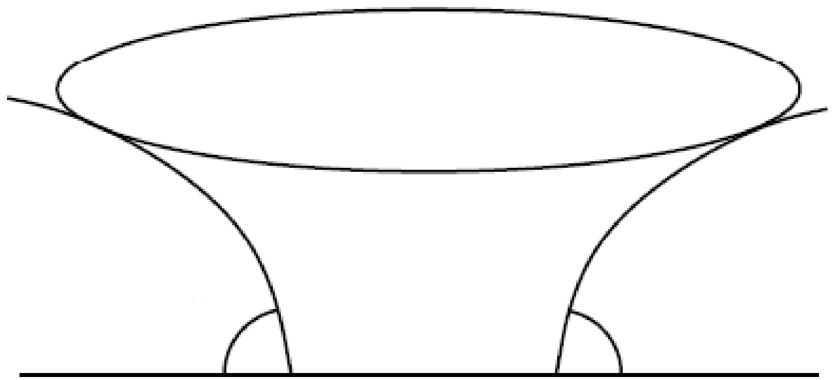
Theorem 7 (Traizet)

If u is an overdetermined positive harmonic function

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 \quad \text{and} & \frac{\partial u}{\partial n} = \alpha \text{ on } \partial\Omega, \end{cases}$$

and $\partial\Omega$ is connected, then either $u(x, y) = \alpha y$ is a linear function, or $u(r) = a \log r$ is logarithmic.





The case of lattices

Theorem Let Σ CMC, symmetry $P1$. If Σ is stable, then

1. either Σ consists of parallel planes, or
2. Σ is an array of round spheres, or
3. an array of circular cylinders, or
4. doubly periodic with genus$(\Sigma/P1) = 2$, or
5. triply periodic with genus$(\Sigma/P1) = 3$