Einstein Metrics,

Harmonic Forms, 

Symplectic Four-Manifolds

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**Definition.** A Riemannian metric $h$ is said to be **Einstein** if it has **constant Ricci curvature**.
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“...the greatest blunder of my life!”

— A. Einstein, to G. Gamow
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$$s = r^j_\ j = \mathcal{R}^{ij}{}_{ij}.$$  

$$\frac{\text{vol}_g(B_{\varepsilon}(p))}{c_n \varepsilon^n} = 1 - s \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$
Geometrization Problem:
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Perhaps reasonable in other dimensions?
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When $n \geq 4$, situation is more encouraging...
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\[ \mathcal{E}(M) = \{ \text{Einstein } h \} / (\text{Diffeos } \times \mathbb{R}^+) \]
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Berger,
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One key question:
Four Dimensions is Exceptional

When $n = 4$, Einstein metrics are genuinely non-trivial: not typically spaces of constant curvature.

There are beautiful and subtle global obstructions to the existence of Einstein metrics on 4-manifolds.

But might allow for geometrization of 4-manifolds by decomposition into Einstein and collapsed pieces.

One key question:

Does enough rigidity really hold in dimension four to make this a genuine geometrization?
Symplectic 4-manifolds:
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On such manifolds, Seiberg-Witten theory mimics Kähler geometry when treating non-Kähler metrics.
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Some Suggestive Questions. If $(M^4, \omega)$ is a symplectic 4-manifold, when does $M^4$ admit an Einstein metric $h$ (unrelated to $\omega$)?
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Some Suggestive Questions. If $(M^4, \omega)$ is a symplectic 4-manifold, when does $M^4$ admit an Einstein metric $h$ (unrelated to $\omega$)? What if we also require $\lambda \geq 0$?
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Theorem (L ’09). Suppose that $M$ is a smooth compact oriented 4-manifold which admits a symplectic structure $\omega$. Then $M$ also admits an Einstein metric $h$ with $\lambda \geq 0$ if and only if $M$ is diffeomorphic to

$$M \cong \begin{cases} 
\mathbb{CP}^2 \# k \mathbb{CP}^2, & 0 \leq k \leq 8, \\
S^2 \times S^2, & K^3, \\
K^3/\mathbb{Z}_2, & T^4, \\
T^4/\mathbb{Z}_2, & T^4/\mathbb{Z}_3, \\
T^4/\mathbb{Z}_4, & T^4/\mathbb{Z}_6, \\
T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), & T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \\
or & T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). 
\end{cases}$$
Theorem (L ’09). Suppose that $M$ is a smooth compact oriented 4-manifold which admits a symplectic structure $\omega$. Then $M$ also admits an Einstein metric $h$ with $\lambda \geq 0$ if and only if

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K3, & \\
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T^4, & \\
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Conventions:

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![Diagram of connected sum](image_url)
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T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\
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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.
\[ \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \]
\[ S^2 \times S^2, \]
\[ K3, \]
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Definitive list . . .

$\mathbb{CP}^2 \#^k \overline{\mathbb{CP}^2}, \quad 0 \leq k \leq 8,$

$S^2 \times S^2,$

$K3,$

$K3/\mathbb{Z}_2,$

$T^4,$

$T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6,$

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But we understand some cases better than others!

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Every Einstein metric is Ricci-flat Kähler.
But we understand some cases better than others!

\[
\begin{align*}
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S^2 \times S^2, & \\
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Moduli space \( \mathcal{E}(M) = \{\text{Einstein } h\}/(\text{Diffeos} \times \mathbb{R}^+) \)
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Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) completely understood.
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Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) connected!
\[ \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, \quad 0 \leq k \leq 8, \]
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Moduli space \( \mathcal{E}(M) \) connected!
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\[ \mathbb{CP}_2 \# k \overline{\mathbb{CP}_2}, \quad 0 \leq k \leq 8, \]
\[ S^2 \times S^2, \]
\[ K3, \]
\[ K3/\mathbb{Z}_2, \]
\[ T^4, \]
\[ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \]
\[ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \]

Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) connected!
Above the line:

Know an Einstein metric on each manifold.

\[
\begin{align*}
\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}, & \quad 0 \leq k \leq 8, \\
S^2 \times S^2, & \\
K3, & \\
K3/\mathbb{Z}_2, & \\
T^4, & \\
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T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), & \text{or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4).
\end{align*}
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Every Einstein metric is Ricci-flat Kähler.

Moduli space \( \mathcal{E}(M) \) connected!
Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$.

\[
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T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\
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\end{align*}
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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!
Above the line:

Moduli space $\mathcal{E}(M) \neq \emptyset$. But is it connected?

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S^2 \times S^2, \\
K3, \\
K3/\mathbb{Z}_2, \\
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T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\
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Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathcal{E}(M)$ connected!
Objective of this lecture:
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Describe modest recent progress on this issue.
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For $M$ a Del Pezzo surface, will define explicit open $U \subset \{\text{Riemannian metrics on } M\}$ such that

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- These form a connected family, mod diffeos;
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Describe modest recent progress on this issue.

For $M$ a Del Pezzo surface, will define explicit open

$$\mathcal{U} \subset \{\text{Riemannian metrics on } M\}$$

such that

- Every known Einstein metric belongs to $\mathcal{U}$;
- These form a connected family, mod diffeos; and
- No other Einstein metrics belong to $\mathcal{U}$!
Formulation will depend on...
Special character of dimension 4:
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The Lie group $SO(4)$ is not simple
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$$so(4) \cong so(3) \oplus so(3).$$
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$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$
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$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

$\Lambda^+$ self-dual 2-forms.

$\Lambda^-$ anti-self-dual 2-forms.
Riemann curvature of $g$

$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$
Riemann curvature of $g$

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splits into 4 irreducible pieces:
Riemann curvature of $g$

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<table>
<thead>
<tr>
<th>$\Lambda^+$</th>
<th>$\Lambda^+\ast$</th>
<th>$\Lambda^\ast$</th>
<th>$\Lambda^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_+ + \frac{s}{12}$</td>
<td>$\hat{\varphi}$</td>
<td>$\hat{\varphi}$</td>
<td>$W_ - + \frac{s}{12}$</td>
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where

- $s = \text{scalar curvature}$
- $\mathring{\mathcal{R}} = \text{trace-free Ricci curvature}$
- $W_+ = \text{self-dual Weyl curvature}$
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$s = \text{scalar curvature}$

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Smooth compact $M^4$ has invariants $b_{\pm}(M)$, defined in terms of intersection pairing.
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Diagonalize:

\[
\begin{bmatrix}
+1 \\
\vdots \\
+1 \\
-1 \\
\vdots \\
-1
\end{bmatrix}
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\vdots \\
\left\{ \begin{array}{c}
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\end{array} \right\}
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b_{+}(M) \\
b_{-}(M) \\
\end{array} \right\}
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-1 \\
\vdots \\
-1
\end{bmatrix}.
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Hodge theory:

\[ H^2(M, \mathbb{R}) = \{ \varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \ d\star\varphi = 0 \}. \]
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self-dual & anti-self-dual harmonic forms.
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self-dual & anti-self-dual harmonic forms. Then

\[ b^\pm(M) = \dim \mathcal{H}^\pm_g. \]
\[ \mathcal{H}_g^+ \subset H^2(M, \mathbb{R}) \]
\{ a \mid a \cdot a = 0 \} \subset H^2(M, \mathbb{R})
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Kähler if the 2-form

\[ \omega = h(J\cdot, \cdot) \]

is closed:

\[ d\omega = 0. \]

But we do not assume this!
More Technical Question. When does a compact complex surface $(M^4, J)$ admit an Einstein metric $h$ which is Hermitian, in the sense that

$$h(\cdot, \cdot) = h(J\cdot, J\cdot)?$$
Theorem. A compact complex surface \((M^4, J)\) admits an Einstein metric \(h\) which is Hermitian with respect to \(J\) \(\iff\) \(c_1(M^4, J)\) “has a sign.”
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More precisely, \(\exists\) such \(h\) with Einstein constant \(\lambda\) \iff there is a Kähler form \(\omega\) such that

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Only two metrics arise in non-Kähler case!
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Key Point.
Theorem. A compact complex surface $(M^4, J)$ admits an Einstein metric $h$ which is Hermitian with respect to $J$ $\iff$ $c_1(M^4, J)$ “has a sign.”

More precisely, $\exists$ such $h$ with Einstein constant $\lambda$ $\iff$ there is a Kähler form $\omega$ such that

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Key Point. Metrics are actually conformally Kähler.
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\[
h = s^{-2}g
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Strictly 4-dimensional phenomenon!
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Key Point. Metrics are actually conformally Kähler.

Strictly 4-dimensional phenomenon!

“Riemannian Goldberg-Sachs Theorem.”
So, which complex surfaces admit
So, which complex surfaces admit Einstein Hermitian metrics with $\lambda > 0$?
Del Pezzo surfaces:
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$$(M^4, J)$$ for which $c_1$ is a Kähler class $[\omega]$. 
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$(M^4, J)$ for which $c_1$ is a Kähler class $[\omega]$.

Shorthand: “$c_1 > 0$.”
Del Pezzo surfaces:

$$(M^4, J)$$ for which $c_1$ is a Kähler class $[\omega]$. Shorthand: “$c_1 > 0.$”

Blow-up of $\mathbb{C}P_2$ at $k$ distinct points, in general position,
Del Pezzo surfaces:

\((M^4, J)\) for which \(c_1\) is a Kähler class \([\omega]\).

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Blow-up of \(\mathbb{CP}^2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position,
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Blow-up of $\mathbb{CP}^2$ at $k$ distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$. 
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Shorthand: “\(c_1 > 0\).”

Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{CP}_1 \times \mathbb{CP}_1\).
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$$M \cong N \# \mathbb{CP}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$. 

![Diagram showing blow-up of $N$ by $\mathbb{CP}_1$.]
Blowing up:

If $N$ is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

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![Diagram of blow-up process]
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![Diagram of blowing up with $M$ and $N$]
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\[ 
\begin{array}{c}
  \text{Diagram:}\n  \\
  M \quad \downarrow \quad \text{N}\n\end{array}
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No 3 on a line,
Del Pezzo surfaces:

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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{CP}_1 \times \mathbb{CP}_1\).

No 3 on a line, no 6 on conic,
Del Pezzo surfaces:

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Blow-up of \(\mathbb{CP}_2\) at \(k\) distinct points, \(0 \leq k \leq 8\), in general position, or \(\mathbb{CP}_1 \times \mathbb{CP}_1\).

No 3 on a line, no 6 on conic, no 8 on nodal cubic.
Del Pezzo surfaces:

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For each topological type:
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For each topological type:

Moduli space of such \((M^4, J)\) is connected.
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$(M^4, J)$ for which $c_1$ is a Kähler class $[\omega]$.

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For each topological type:

Moduli space of such $(M^4, J)$ is connected.

Just a point if $b_2(M) \leq 5$. 
Every del Pezzo surface has $b_+ = 1$. 
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\forall h, \exists! \text{ self-dual harmonic 2-form } \omega:
Every del Pezzo surface has $b_+ = 1$. \(\iff\)

\[ \forall h, \exists! \text{ self-dual harmonic 2-form } \omega: \]

\[ d\omega = 0, \quad \star \omega = \omega. \]
Every del Pezzo surface has $b_+ = 1$. ⇐⇒

Up to scale, $\forall \ h, \ \exists \ !$ self-dual harmonic 2-form $\omega$:

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Up to scale, $\forall \ h, \ \exists !$ self-dual harmonic 2-form $\omega$:

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Such a form defines a symplectic structure,
Every del Pezzo surface has $b_+ = 1$. 

Up to scale, $\forall \ h, \ \exists!$ self-dual harmonic 2-form $\omega$:

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Such a form defines a symplectic structure, except at points where $\omega = 0$. 
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$$d\omega = 0, \quad \star \omega = \omega.$$ 

Such a form defines a symplectic structure, except at points where $\omega = 0$.

**Definition.** Let $M$ be smooth 4-manifold with $b_+(M) = 1$. 

Every del Pezzo surface has $b_+ = 1$.  \iff

Up to scale, $\forall \ h, \ \exists!$ self-dual harmonic 2-form $\omega$:

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Up to scale, $\forall h$, $\exists!$ self-dual harmonic 2-form $\omega$:

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**Definition.** Let \( M \) be smooth 4-manifold with \( b_+(M) = 1 \). We will say that a Riemannian metric \( h \) on \( M \) is of symplectic type if associated SD harmonic \( \omega \).
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- $W_+ \neq 0$ everywhere; and
- $W_+$ and $\omega$ are everywhere roughly aligned.
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\[
\text{Kähler} \implies W_+ = \begin{pmatrix}
-\frac{s}{12} & -\frac{s}{12} \\
-\frac{s}{12} & \frac{s}{6}
\end{pmatrix}
\]
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- open condition;
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Theorem A.

Let \((M, h)\) be a smooth compact 4-dimensional Einstein manifold. If \(h\) is of positive symplectic type, then it's a conformally Kähler, Einstein metric on a Del Pezzo surface \((M, J)\).
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Conversely, all these are of positive symplectic type.
For $M^4$ a Del Pezzo surface,
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considered as smooth compact oriented 4-manifold,
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Corollary. $\mathcal{E}_\omega^+(M)$ is connected.
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**Corollary.** $\mathcal{E}_\omega^+(M)$ is connected. Moreover, if $b_2(M) \leq 5$, 
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**Corollary.** $\mathcal{E}_\omega^+(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathcal{E}_\omega^+(M) = \{\text{point}\}$. 
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**Corollary.** $\mathcal{E}_\omega^+(M)$ is exactly one connected component of $\mathcal{E}(M)$. 
Method of Proof.
Key Observation:
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Our strategy:
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Our strategy:

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as proxy for Einstein equation.
Now suppose that $\omega \neq 0$ everywhere.
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Rescale $h$ to obtain $g$ with $|\omega| \equiv \sqrt{2}$:

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for some $g$-preserving almost-complex structure $J$. 
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$$0 = \nabla^* \nabla \omega - 2W^+(\omega, \cdot) + \frac{s}{3} \omega$$
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$$0 = \nabla^* \nabla (fW^+) + \frac{s}{2} fW^+ - 6 fW^+ \circ W^+ + 2 f |W^+|^2 I$$
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for $fW^+ \in \text{End}(\Lambda^+)$. 
Now take inner product of Weitzenböck formula

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with \(2\omega \otimes \omega\)
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with \(2\omega \otimes \omega\) and integrate by parts,
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with \(2\omega \otimes \omega\) and integrate by parts, using identity

\[
\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle = [W^+(\omega, \omega)]^2 + 4|W^+(\omega)|^2 - s W^+(\omega, \omega).
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This yields

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0 = \int_M \left( -s W^+ (\omega, \omega) + 8|W^+|^2 - 4|W^+ (\omega)\perp|^2 \right) f \, d\mu,
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where \(W^+(\omega)^\perp\) = projection of \(W^+(\omega, \cdot)\) to \(\omega^\perp\).
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This yields

\[
0 \geq \int_M \left( -sW^+(\omega, \omega) + 3 [W^+(\omega, \omega)]^2 \right) f \, d\mu
\]
Now take inner product of Weitzenböck formula

\[
0 = \nabla^* \nabla (fW^+) + \frac{s}{2} fW^+ - 6 f W^+ \circ W^+ + 2 f |W^+|^2 I
\]

with \(2\omega \otimes \omega\) and integrate by parts, using identity

\[
\langle W^+, \nabla^* \nabla (\omega \otimes \omega) \rangle = [W^+ (\omega, \omega)]^2 + 4|W^+ (\omega)|^2 - sW^+ (\omega, \omega).
\]

This yields

\[
0 \geq 3 \int_M W^+ (\omega, \omega) \left( W^+ (\omega, \omega) - \frac{s}{3} \right) f \ d\mu
\]
Now take inner product of Weitzenböck formula

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0 = \nabla^* \nabla (f W^+) + \frac{s}{2} f W^+ - 6 f W^+ \circ W^+ + 2 f |W^+|^2 I
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\]

This yields

\[
0 \geq 3 \int_M W^+(\omega, \omega) \left( \frac{1}{2} |\nabla \omega|^2 \right) f \, d\mu
\]
Proposition.
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Corollary.
Proposition. If compact almost-Kähler \((M^4, g, \omega)\) satisfies \(\delta(f W^+) = 0\) for some \(f > 0\), then

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Corollary. Let \((M^4, g, \omega)\) be a compact almost-Kähler manifold
**Proposition.** If compact almost-Kähler $(M^4, g, \omega)$ satisfies $\delta(fW^+) = 0$ for some $f > 0$, then

$$0 \geq \int_M W^+(\omega, \omega)|\nabla \omega|^2 f \ d\mu$$

**Corollary.** Let $(M^4, g, \omega)$ be a compact almost-Kähler manifold with $W^+(\omega, \omega) > 0$. 
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\[
\text{Kähler} \quad \implies \quad W_+ = \begin{pmatrix}
-\frac{s}{12} & \frac{s}{12} \\
\frac{s}{12} & s
\end{pmatrix}
\]
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Corollary. Let \((M^4, g, \omega)\) be a compact almost-Kähler manifold with \(W^+(\omega, \omega) > 0\). If

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**Proposition.** If compact almost-Kähler $(M^4, g, \omega)$ satisfies $\delta(fW^+) = 0$ for some $f > 0$, then

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**Corollary.** Let $(M^4, g, \omega)$ be a compact almost-Kähler manifold with $W^+(\omega, \omega) > 0$. If

$$\delta(fW^+) = 0$$

for some $f > 0$, then $g$ is a Kähler metric with scalar curvature $s > 0$. Moreover, $f = c/s$ for some constant $c > 0$.

Conversely, any Kähler $(M^4, g, \omega)$ with $s > 0$ satisfies $W^+(\omega, \omega) > 0$, and $f = c/s$ solves

$$\nabla(fW^+) = 0 \implies \delta(fW^+) = 0.$$
Theorem.
Theorem. Let $(M, h)$
Theorem. Let $(M, h)$ be a compact oriented
Theorem. Let $(M, h)$ be a compact oriented Riemannian 4-manifold with $\delta W + = 0$. If $W^+ (\omega, \omega) > 0$ for some self-dual harmonic 2-form $\omega$, then $h = s^{-2} g$ for a unique Kähler metric $g$ of scalar curvature $s > 0$.

Conversely, for any Kähler metric $g$ of positive scalar curvature, the conformally related metric $h = s^{-2} g$ satisfies $\delta W + = 0$ and $W^+ (\omega, \omega) > 0$. 295
Theorem. Let \((M, h)\) be a compact oriented Riemannian 4-manifold with \(\delta W^+ = 0\). For some self-dual harmonic 2-form \(\omega\), then \(h = s^{-2}g\) for a unique Kähler metric \(g\) of scalar curvature \(s > 0\).

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**Remark.** If such metrics exist, $b_+(M) = 1$. 
**Theorem.** Let \((M, h)\) be a compact oriented Riemannian 4-manifold with \(\delta W^+ = 0\). If
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Theorem A follows by restricting to Einstein case.
Theorem A. Let $(M, h)$ be a smooth compact 4-dimensional Einstein manifold. If $h$ is of positive symplectic type, then it’s a conformally Kähler, Einstein metric on a Del Pezzo surface $(M, J)$. 
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Corollary. $\mathcal{E}^+_{\omega}(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathcal{E}^+_{\omega}(M) = \{\text{point}\}$.
Theorem A. Let $(M, h)$ be a smooth compact 4-dimensional Einstein manifold. If $h$ is of positive symplectic type, then it’s a conformally Kähler, Einstein metric on a Del Pezzo surface $(M, J)$.

Corollary. $\mathcal{E}_\omega^+(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathcal{E}_\omega^+(M) = \{\text{point}\}$.

Corollary. $\mathcal{E}_\omega^+(M)$ is exactly one connected component of $\mathcal{E}(M)$.
Application to Almost-Kähler Geometry:
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\[
s \geq 0,
\]
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Theorem. Let \((M, g, \omega)\) be a compact almost-Kähler 4-manifold with
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**Theorem.** Let \((M, g, \omega)\) be a compact almost-Kähler 4-manifold with

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Then \((M, g, \omega)\) is a constant-scalar-curvature Kähler ("cscK") manifold.
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In particular, gives a new proof of the following:
Theorem. Let \((M, g, \omega)\) be a compact almost-Kähler 4-manifold with
\[ s \geq 0, \quad \delta W^+ = 0. \]
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In particular, gives a new proof of the following:

**Corollary (Sekigawa).** Every compact almost-Kähler Einstein 4-manifold with non-negative Einstein constant is Kähler-Einstein.
Theorem. Let \((M, g, \omega)\) be a compact almost-Kähler 4-manifold with 
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(Special case of "Goldberg conjecture.")
Theorem. Let \((M, g, \omega)\) be a compact almost-Kähler 4-manifold with 
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In particular, gives a new proof of the following:

Corollary (Sekigawa). Every compact almost-Kähler Einstein 4-manifold with non-negative Einstein constant is Kähler-Einstein.

Helped motivate the discovery of Theorem A...
Tanti auguri, Stefano!
Tanti auguri, Stefano!

E grazie agli organizzatori
Tanti auguri, Stefano!

E grazie agli organizzatori

per un convegno così bello!