Nonsmooth differential geometry

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- First order differential structure of metric measure spaces
  - L^2-normed L^∞-modules
  - Cotangent module
  - Tangent module
  - Behavior under transformations

- RCD spaces
  - Definition
  - Some geometric properties
  - Second order calculus

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The setting

$(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ is a complete and separable metric space equipped with a non-negative and locally finite Borel measure

L^2-normed $L^\infty(\mathfrak{m})$-modules

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- a multiplication with $L^\infty(\mathfrak{m})$ functions, i.e. a bilinear map $L^\infty(\mathfrak{m}) \times M \to M$ satisfying

$$f(gv) = (fg)v,$$
$$\mathbf{1}v = v,$$

for every $f, g \in L^\infty(\mathfrak{m})$ and $v \in M$.

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  for every $f, g \in L^\infty(\mathfrak{m})$ and $v \in M$.

- a pointwise L^2-norm, i.e. a map $|\cdot| : M \to L^2(\mathfrak{m})$ satisfying

$$|v| \geq 0, \qquad \mathfrak{m} - a.e.,$$
$$|fv| = |f|\,|v|, \qquad \mathfrak{m} - a.e.,$$
$$\|v\|_M = \sqrt{\int |v|^2 \, \mathrm{d}\mathfrak{m}}$$

The basic example

The space of L^2 vector fields on a Riemannian manifold.

More generally: the space of L^2 sections of a normed vector bundle.

The idea

In the smooth setting, one can fully describe a vector bundle by looking either to its fibers, or to its sections.

In the non-smooth one, we take the latter viewpoint and thus declare L^2-normed $L^\infty(\mathfrak{m})$-modules to ‘be’ vector bundles on our metric measure space

Basic features of modules: locality

For $v, w \in M$ and a Borel set $E \subset \mathcal{X}$ we say that

$$v = w, \qquad \mathfrak{m} - a.e. \text{ on } E$$

provided

$$\chi_E(v - w) = 0.$$

or equivalently

$$|v - w| = 0, \qquad \mathfrak{m} - a.e. \text{ on } E.$$

The set $\{v = w\} \subset \mathcal{X}$ is then defined as $\{|v - w| = 0\}$.

Basic features of modules: duality

The dual M^* of M is the space of linear continuous maps $L : M \to L^1(\mathfrak{m})$ which are local, i.e. such that

$$L(fv) = f\,L(v), \qquad \forall v \in M,\ f \in L^\infty(\mathfrak{m}).$$

M^* is also an L^2-normed L^∞-module, the pointwise norm being given by

$$|L|_* := \operatorname*{ess\text{-}sup}_{v\,:\,|v|\leq 1\ \mathfrak{m}-a.e.} L(v)$$

A special case: Hilbert modules

M is an Hilbert module provided, when seen as Banach space, is an Hilbert space.

This happens if and only if

$$|v + w|^2 + |v - w|^2 = 2|v|^2 + 2|w|^2, \qquad \mathfrak{m} - a.e.,$$

for every $v, w \in M$.
Thus by polarization we have a pointwise scalar product

$$M \ni v, w \qquad \mapsto \qquad \langle v, w \rangle \in L^1(\mathfrak{m}),$$

symmetric and L^∞-linear.

Example: the dual of $L^2(\mathfrak{m})$

The dual of $L^2(\mathfrak{m})$ as Hilbert space is $L^2(\mathfrak{m})$, i.e. for $L : L^2(\mathfrak{m}) \to \mathbb{R}$ linear and continuous there is a unique $g \in L^2(\mathfrak{m})$ such that

$$L(f) = \int fg \, \mathrm{d}\mathfrak{m} \qquad \forall f \in L^2(\mathfrak{m})$$

and $\|L\|_{L^2(\mathfrak{m})'} = \|g\|_{L^2(\mathfrak{m})}$. And viceversa.

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The dual of $L^2(\mathfrak{m})$ as Hilbert module is $L^2(\mathfrak{m})$, i.e. for $T : L^2(\mathfrak{m}) \to L^1(\mathfrak{m})$ linear, continuous and local, there is a unique $g \in L^2(\mathfrak{m})$ such that

$$T(f) = fg \quad \mathfrak{m} - a.e. \qquad \forall f \in L^2(\mathfrak{m})$$

and $|T|_* = |g|$ $\mathfrak{m}$-a.e.. And viceversa.

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Cornerstones of the approach

Forget about:	Focus on:
Fibers	Sections
Geometry	Analysis
Charts	Intrinsic calculus
Lipschitz functions	Sobolev functions

Variational definition of $|Df|$ on $\mathbb{R}^d$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth.

Then $|Df|$ is the minimum continuous function G for which

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t$$

holds for any smooth curve γ

Test plans

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],\mathcal{X}))$. We say that $\boldsymbol{\pi}$ is a test plan provided:

- for some $C > 0$ it holds
$$(\mathrm{e}_t)_*\boldsymbol{\pi} \leq C\mathfrak{m}, \qquad \forall t \in [0,1].$$
- it holds
$$\iint_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi} < \infty$$

The Sobolev class $S^2(\mathcal{X})$

We say that $f : X \to \mathbb{R}$ belongs to $S^2(\mathcal{X})$ provided there exists $G \in L^2(\mathfrak{m})$, $G \geq 0$ such that

$$\int \big|f(\gamma_1) - f(\gamma_0)\big| \, \mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma)$$

for any test plan $\boldsymbol{\pi}$.

Any such G is called weak upper gradient of f.

The minimal G in the $\mathfrak{m}$-a.e. sense will be denoted by $|Df|$

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We also put $W^{1,2}(\mathcal{X}) := L^2 \cap S^2(\mathcal{X})$ endowed with the norm

$$\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + \||Df|\|^2_{L^2}.$$

$W^{1,2}(\mathcal{X})$ is always a Banach space.

Calculus rules for $|Df|$

Lower semicontinuity:

$$\left.\begin{array}{rl} (f_n) & \subset S^2(\mathcal{X}) \\ f_n & \to f \qquad \mathfrak{m}-a.e. \\ |Df_n| & \to G \ \text{in } L^2(\mathfrak{m}) \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rl} f & \in S^2(\mathcal{X}) \\ |Df| & \leq G \end{array}\right.$$

Subadditivity: $|D(\alpha f + \beta g)| \leq |\alpha||Df| + |\beta||Dg| \qquad \mathfrak{m}-a.e.$

Locality: $|Df| = |Dg| \qquad \mathfrak{m}-a.e.\ on\ \{f = g\}$

Chain rule: $|D(\varphi \circ f)| = |\varphi'| \circ f\,|Df|,$ for $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

Leibniz rule: $|D(fg)| \leq |f||Dg| + |g||Df|,$ for $f, g \in S^2 \cap L^\infty(\mathcal{X})$

The ‘Pre-cotangent module’

Consider the set

$$\mathrm{Pcm} := \Big\{ (A_i, f_i)_{i\in\mathbb{N}} \ : (A_i) \text{ is a Borel partition of } \mathcal{X}$$
$$f_i \in S^2(\mathcal{X}) \text{ for every } i \in \mathbb{N}$$
$$\sum_i \int_{A_i} |Df_i|^2 \, \mathrm{d}\mathfrak{m} < \infty \Big\}$$

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Define an equivalence relation $\sim$ on Pcm by declaring

$$(A_i, f_i)_{i\in\mathbb{N}} \sim (B_j, g_j)_{j\in\mathbb{N}}$$

provided for any $i, j \in \mathbb{N}$ we have

$$|D(f_i - g_j)| = 0 \quad \mathfrak{m} - a.e. \text{ on } \{A_i \cap B_j\}$$

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Denote by $[A_i, f_i]$ the equivalence class of $(A_i, f_i)_{i\in\mathbb{N}}$

Operations on $\mathrm{Pcm}/\sim$

Sum

$$[A_i, f_i] + [B_j, g_j] := [A_i \cap B_j, f_i + g_j]$$

Multiplication by a simple function For $h = \sum_j \alpha_j \chi_{E_j}$ we put

$$h \cdot [A_i, f_i] := [A_i \cap E_j, \alpha_j f_i]$$

Pointwise norm

$$|[A_i, f_i]| := |Df_i|, \qquad \mathfrak{m} - a.e.\ on\ A_i$$

Norm

$$\|[A_i, f_i]\| := \sqrt{\int_{\mathcal{X}} |[A_i, f_i]|^2 \, \mathrm{d}\mathfrak{m}} = \sqrt{\sum_i \int_{A_i} |Df_i|^2 \, \mathrm{d}\mathfrak{m}}$$

The cotangent module $L^2(T^*\mathcal{X})$

We define $L^2(T^*\mathcal{X})$ to be the completion of $(\mathrm{Pcm}/\sim, \|\cdot\|)$. Its elements are called 1-forms.

All the operations can be extended by continuity endowing $L^2(T^*\mathcal{X})$ with the structure of L^2-normed L^∞-module.

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Note: in the smooth case, the construction is canonically identifiable with the space of L^2 sections of the cotangent bundle.

Calculus rules for $\mathrm{d}f$

Closure:

$$\left.\begin{array}{rl} (f_n) & \subset S^2(\mathcal{X}) \\ f_n & \to f \qquad \mathfrak{m}-a.e. \\ \mathrm{d}f_n & \to \omega \ \text{ in } L^2(T^*\mathcal{X}) \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{rl} f & \in S^2(\mathcal{X}) \\ \mathrm{d}f & = \omega \end{array}\right.$$

Linearity: $\mathrm{d}(\alpha f + \beta g) = \alpha \mathrm{d}f + \beta \mathrm{d}g$

Locality: $\mathrm{d}f = \mathrm{d}g \qquad \mathfrak{m} - a.e.\ on\ \{f = g\}$

Chain rule: $\mathrm{d}(\varphi \circ f) = \varphi' \circ f\, \mathrm{d}f,$ for $\varphi : \mathbb{R} \to \mathbb{R}$ Lipschitz

Leibniz rule: $\mathrm{d}(fg) = f\, \mathrm{d}g + g\, \mathrm{d}f,$ for $f, g \in S^2 \cap L^\infty(\mathcal{X})$

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The tangent module

Definition The tangent module $L^2(T\mathcal{X})$ is the dual of the cotangent one. Its elements are called vector fields.

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- Vector fields are in 1-1 correspondence with L^2 derivations, i.e. maps $L : S^2(\mathcal{X}) \to L^1(\mathfrak{m})$ satisfying the Leibniz rule and such that
$$|L(f)| \leq l|Df| \qquad \mathfrak{m} - a.e.,$$
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$$|L(f)| \leq l|Df| \qquad \mathfrak{m} - a.e.,$$
for some $l \in L^2(\mathfrak{m})$
- For any vector field $X \in L^2(T\mathcal{X})$ we have its pointwise norm $|X| \in L^2(\mathcal{X})$. It can be seen that such norm induces, in an appropriate weak sense, the original distance d on $\mathcal{X}$.

Infinitesimally Hilbertian spaces

We say that $(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ is 'infinitesimally Hilbertian' if $L^2(T^*\mathcal{X})$, and thus also $L^2(T\mathcal{X})$, is an Hilbert module.

On these spaces the pointwise scalar product of vector fields

$$L^2(T\mathcal{X}) \ni X, Y \quad \mapsto \quad \langle X, Y \rangle \in L^1(\mathfrak{m})$$

should be thought of as the metric tensor on $(\mathcal{X}, \mathsf{d}, \mathfrak{m})$.

Gradients

On an infinitesimally Hilbertian space, for $f \in S^2(\mathcal{X})$ there is a unique $X \in L^2(T\mathcal{X})$ such that

$$|\mathrm{d}f|^2 = \mathrm{d}f(X) = |X|^2 \qquad \mathfrak{m} - a.e.$$

Such X is called gradient and denoted by ∇f.

The map

$$S^2(\mathcal{X}) \ni f \quad \mapsto \quad \nabla f \in L^2(T\mathcal{X})$$

is linear.

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Maps of bounded deformation

A map $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ is of bounded deformation provided

$$\begin{aligned} &\mathrm{Lip}(\varphi) < \infty \\ &\varphi_* \mathfrak{m}_1 \leq C \mathfrak{m}_2, \qquad \text{for some } C > 0. \end{aligned}$$

Transformation of test plans and Sobolev functions

Let $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ be of bounded deformation.

Then φ induces by left composition a map $\varphi : C([0,1], \mathcal{X}_1) \to C([0,1], \mathcal{X}_2)$ which alters speed of at most a factor $\mathrm{Lip}(\varphi)$.

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By direct verification we see that if $\boldsymbol{\pi}$ is a test plan on $\mathcal{X}_1$, then $\varphi_* \boldsymbol{\pi}$ is a test plan on $\mathcal{X}_2$.

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By direct verification we see that if $\boldsymbol{\pi}$ is a test plan on $\mathcal{X}_1$, then $\varphi_* \boldsymbol{\pi}$ is a test plan on $\mathcal{X}_2$.

By duality, if $f \in S^2(\mathcal{X}_2)$ we have $f \circ \varphi$ is in $S^2(\mathcal{X}_1)$ with

$$|D(f \circ \varphi)| \leq \mathrm{Lip}(\varphi) |Df| \circ \varphi$$

Pullback of 1-forms

Theorem (G. '14) Let $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$ be of bounded deformation. Then there exists a unique linear continuous map $\varphi^* : L^2(T^*\mathcal{X}_2) \to L^2(T^*\mathcal{X}_1)$ such that

$$\varphi^* \mathrm{d}f = \mathrm{d}(f \circ \varphi)$$
$$\varphi^*(g\omega) = g \circ \varphi \, \varphi^* \omega$$

Such map satisfies

$$|\varphi^* \omega| \le \mathrm{Lip}(\varphi) |\omega| \circ \varphi, \qquad \mathfrak{m}_1 - a.e..$$

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$$\varphi^*(g\omega) = g \circ \varphi \, \varphi^* \omega$$

Such map satisfies

$$|\varphi^*\omega| \leq \mathrm{Lip}(\varphi)|\omega| \circ \varphi, \qquad \mathfrak{m}_1 - a.e..$$

If φ is invertible with inverse of bounded deformation, then the transpose of φ^* is the differential $\mathrm{d}\varphi : L^2(T\mathcal{X}_1) \to L^2(T\mathcal{X}_2)$.
A similar statement holds for non-invertible maps involving the notion of pullback module.

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Laplacian and heat flow

Let $(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ be infinitesimally Hilbertian.
$D(\Delta) \subset W^{1,2}(\mathcal{X})$ is the space of f's for which there is $h \in L^2(\mathfrak{m})$ such that

$$\int \langle \nabla f, \nabla g \rangle \, \mathrm{d}\mathfrak{m} = -\int hg \, \mathrm{d}\mathfrak{m}, \qquad \forall g \in W^{1,2}(\mathcal{X}).$$

We call h the Laplacian of f and denote it by Δf.
$D(\Delta)$ is a vector space and $f \mapsto \Delta f$ linear.

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We call h the Laplacian of f and denote it by Δf.
$D(\Delta)$ is a vector space and $f \mapsto \Delta f$ linear.

There exists a unique 1-parameter semigroup of linear operators $\mathsf{h}_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ such that for every $f \in L^2(\mathfrak{m})$ the curve $t \mapsto \mathsf{h}_t f \in L^2(\mathfrak{m})$ is absolutely continuous and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}_t f = \Delta \mathsf{h}_t f$$

We call the h_t's the heat flow.

RCD(K,∞) spaces

Definition (Ambrosio, G., Savaré '11) Let $K \in \mathbb{R}$. Then $(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ is an RCD(K,∞) space provided:

i) it is infinitesimally Hilbertian

ii) $\mathfrak{m}(B_r(x)) \leq e^{Cr^2}$ for some $x \in \mathcal{X}$ and $C > 0$

iii) Every $f \in W^{1,2}(\mathcal{X})$ with $|Df| \leq 1$ $\mathfrak{m}$-a.e. admits a 1-Lipschitz representative

iv) For every $f \in W^{1,2}(\mathcal{X})$ and $t \geq 0$ we have

$$|D(\mathsf{h}_t f)|^2 \leq e^{-2Kt}\mathsf{h}_t(|Df|^2)$$

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Theorem (Ambrosio, G., Savaré '11 - based on Lott-Villani '05, Sturm '05, G. '09)
pmGH limits of RCD(K, ∞) spaces are still RCD(K, ∞) spaces.

Bochner inequality

On $\mathrm{RCD}(K,\infty)$ spaces the Bochner inequality

$$\Delta \frac{|\nabla f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2$$

holds in the weak sense, i.e.:

$$\frac{1}{2}\int \Delta g |\nabla f|^2 \, \mathrm{d}\mathfrak{m} \geq \int g\big(\langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2\big) \, \mathrm{d}\mathfrak{m}$$

for $f, g \in D(\Delta)$ with $\Delta f \in W^{1,2}(\mathcal{X})$, $g \geq 0$ and $g, \Delta g \in L^\infty(\mathcal{X})$.

Requiring the weak version of

$$\Delta \frac{|\nabla f|^2}{2} \geq \frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2$$

leads to the notion of $\mathrm{RCD}(K,N)$ space (Ambrosio-G.-Savaré '12, G. '12, Erbar-Kuwada-Sturm '13)

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Geometric results known for RCD(K, N) spaces

Abresch-Gromoll inequality (G.-Mosconi '12)

Splitting theorem (G. '13)

Maximal diameter theorem (Ketterer '13)

Rectifiability results (Mondino-Naber '14)

About rectifiability

Theorem ($\sim$Mondino-Naber '14) Let $(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K, N)$ space and $\varepsilon > 0$.
Then there is a Borel partition (A_n) of $\mathcal{X}$ and maps $\varphi_n : A_n \to \mathbb{R}^{d_n}$ with $d_n \leq N$ such that

$$\mathrm{Lip}(\varphi_n),\ \mathrm{Lip}(\varphi_n^{-1}) \leq 1 + \varepsilon$$

and putting $\mu_n := (\varphi_n)_*(\mathfrak{m}|_{A_n})$ we have $\mu_n = \rho_n \mathcal{L}^{d_n}$ with

$$\operatorname*{ess\text{-}sup}_{\varphi_n(A_n)} \rho_n - \operatorname*{ess\text{-}inf}_{\varphi_n(A_n)} \rho_n \leq \varepsilon$$

About rectifiability

Theorem ($\sim$Mondino-Naber '14) Let $(\mathcal{X}, \mathsf{d}, \mathfrak{m})$ be a RCD(K, N) space and $\varepsilon > 0$.
Then there is a Borel partition (A_n) of $\mathcal{X}$ and maps $\varphi_n : A_n \to \mathbb{R}^{d_n}$ with $d_n \leq N$ such that

$$\mathrm{Lip}(\varphi_n),\ \mathrm{Lip}(\varphi_n^{-1}) \leq 1 + \varepsilon$$

and putting $\mu_n := (\varphi_n)_*(\mathfrak{m}|_{A_n})$ we have $\mu_n = \rho_n \mathcal{L}^{d_n}$ with

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In particular, recalling the properties of the pullback of 1-forms we get:

Corollary The tangent module $L^2(T\mathcal{X})$ is canonically isomorphic to the space of Borel and L^2 maps assigning to $\mathfrak{m}$-a.e. $x \in \mathcal{X}$ an element of the pmGH-limit of rescaled spaces.

Content

- First order differential structure of metric measure spaces
  - L^2-normed L^∞-modules
  - Cotangent module
  - Tangent module
  - Behavior under transformations
- RCD spaces
  - Definition
  - Some geometric properties
  - Second order calculus

3 simple formulas

$$\mathrm{H}f(\nabla g, \nabla g) = \langle \nabla \langle \nabla f, \nabla g \rangle, \nabla g \rangle - \tfrac{1}{2} \left\langle \nabla f, \nabla |\nabla g|^2 \right\rangle$$

$$\langle \nabla_{\nabla f} X, \nabla g \rangle = \langle \nabla \langle X, \nabla g \rangle \nabla f \rangle - \mathrm{H}g(X, \nabla f)$$

$$\mathrm{d}\omega(X, Y) = X\big(\omega(Y)\big) - X\big(\omega(Y)\big) - \omega([X, Y])$$

Why these can be used on RCD(K, ∞) spaces

Starting from

$$\Delta \frac{|\nabla f|^2}{2} \geq \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2$$

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Why these can be used on $\mathrm{RCD}(K,\infty)$ spaces

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so that in particular

$$\int |\mathrm{H}f|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \leq \int |\Delta f|^2 - K|\nabla f|^2\,\mathrm{d}\mathfrak{m}$$

(Bakry '85, Savaré '12, Sturm '14, G. '14)

Where this brings

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Definition of the Ricci curvature via the formula

$$\mathrm{Ric}(X,X) := \Delta\frac{|X|^2}{2} - |\nabla X|^2_{\mathsf{HS}} + \langle X, \Delta_{\mathsf{H}} X\rangle$$

which is a measure-valued operator satisfying

$$\mathrm{Ric}(X,X) \geq K|X|^2\mathfrak{m}$$

Thank you