Constructions of H_r-hypersurfaces, barriers and Alexandrov Theorem in $\mathbb{H}^n \times \mathbb{R}$

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Preliminaries

M^n and $\bar{M}^{n+1}$ be oriented Riemannian manifolds

$X : M^n \to \bar{M}^{n+1}$ be an isometric immersion

$p \in M$, $A : T_pM \to T_pM$ be the the shape operator

$k_1, ..., k_n$ principal curvatures

$$H_1(p) = \frac{1}{2}(k_1 + k_2 + \ldots + k_n)$$

$$H_2(p) = \frac{1}{\binom{n}{2}}(k_1k_2 + k_1k_3 + \ldots + k_nk_{n-1})$$

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$$H_r(p) = \frac{1}{\binom{n}{r}} \sum_{i_1<\ldots<i_r} k_{i_1}...k_{i_r} = \frac{1}{\binom{n}{r}} S_r(p)$$

r-mean curvature (Higher order mean curvatures)

H_r-hypersurfaces

In Euclidean spaces:

H-hyperurfaces are critical points of the area functional $A = \int_M dM$ for volume preserving variations .

H_r-hypersurfaces are critical points of the r-area functional $A_r = \int_M S_r\, dM$ for volume preserving variations

For space forms, L.Barbosa and G. Colares stablished the variational problem whose critical points are H_r-hypersurfaces and addressed the stability properties.

What about other ambient spaces?

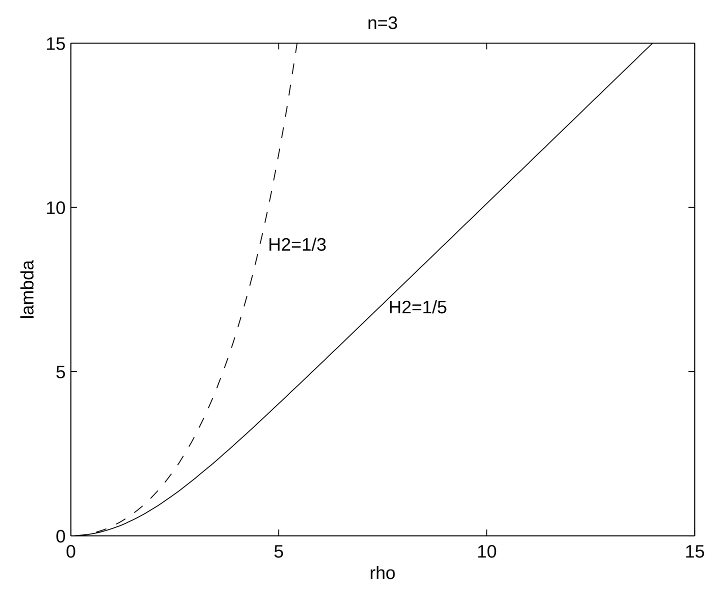
Equations and constructions of examples

Considering vertical graphs in $M \times \mathbb{R}$, where $M \subset \mathbb{R}^n$ with metric conformal to the euclidean one, we obtained explicit equations for H_r including one in the divergence form.

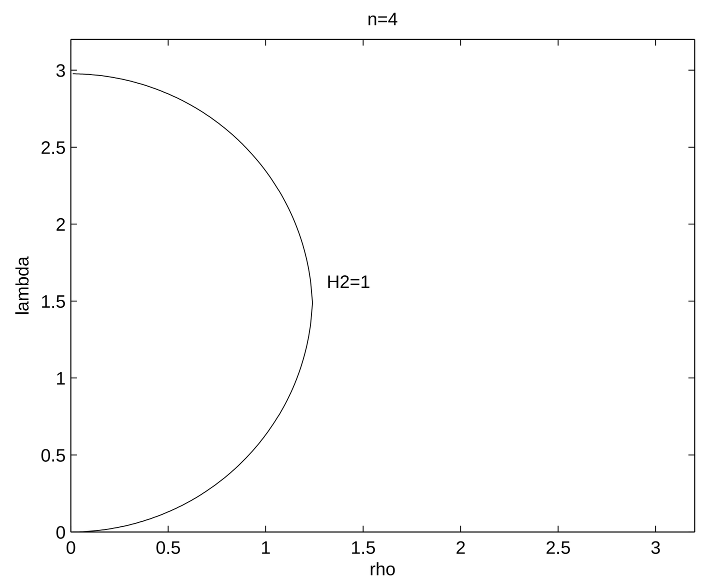
We have constructed some examples of H_r-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ either invariant by rotations or by parabolic screw motions. In particular, we have obtained a classification result for hypersurfaces with $H_r = 0$ and invariant by parabolic translations.

Rotational H_r-hypersurfaces

For each $0 < H_r \leq \frac{(n-r)}{n}$, there exists, up to translations or reflections, a unique entire rotational H_r-graph in $\mathbb{H}^n \times \mathbb{R}$ and it is strictly convex.



For each $H_r > \frac{(n-r)}{n}$, there exists, up to translations or reflections, a unique embedded compact rotational H_r-hypersurface and it is strictly convex.



Moreover, in $\mathbb{H}^n \times \mathbb{R}$, an embedded compact rotational H_r-hypersurface must have $H_r > \frac{(n-r)}{n}$ and an entire rotational H_r-graph must have $0 < H_r \leq \frac{(n-r)}{n}$.

Relations with other results

Last Theorem shows that the value $\frac{(n-r)}{n}$ is a critical value for the theory for rotational H_r-hypersurfaces.

For H in $\mathbb{H}^n \times \mathbb{R}$, the critical value is $\frac{(n-1)}{n}$. (R. Sa Earp, E. Toubiana and P. Berard)

$r = n = 2$, extrinsic curvature: J.M. Espinar, J.A. Galvez and H. Rosenberg proved: A complete immersion in $\mathbb{H}^2 \times \mathbb{R}$ with $H_2 = cte > 0$ is a rotational sphere.

Question: Is a complete immersion in $\mathbb{H}^n \times \mathbb{R}$ with $H_n = cte > 0$ the rotational "n-sphere"?

Barriers

By using the rotational examples as barriers and a suitable version of the Maximum Principle (J. Hounie - M.L. Leite and F. Fontenele - S. Silva) we obtain some interesting geometric results, as for example a Convex Hull Lemma, a Height estimate and

Alexandrov Theorem

Theorem Let M be a compact (without boundary) connected embedded hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with constant r-mean curvature function $H_r > 0$. Then M is, up to translations, the sphere-like rotational hypersurface we have obtained. In particular, we have $H_r > \frac{(n-r)}{n}$.