Real submanifolds of codimension two of a complex space form

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The study of real hypersurfaces of Kählerian manifolds has been an important subject in geometry of submanifolds, especially when the ambient space is a complex space form.However, for arbitrary codimension, there are only a few recent results (see [1] for more details).

If a complex hypersurface M^n of a Kähler manifold $\overline{M}^{n+2}$ satisfies the condition,then M^n is a totally geodesic submanifold.

Let $\overline{M}$ be a non-Euclidean complex space form. If a submanifold M of real codimension two satisfies the condition, then one of the following holds.

(1) M is a totally geodesic complex hypersurface.

(2) M is a CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$.

Let M be a submanifold of real codimension two of a complex Euclidean space $\mathbf{C}^{\frac{n+2}{2}}$ with $\lambda = 0$ which satisfies the condition. If there exists a totally geodesic hypersurface M' of $\mathbf{C}^{\frac{n+2}{2}}$ such that $M \subset M'$, then M is one of the following:

(1) n-dimensional hyperplane $\mathbf{E}^n$,

(2) product manifold of an odd-dimensional sphere and a Euclidean space: $\mathbf{S}^{2p+1} \times \mathbf{E}^{n-2p-1}$.

Let M^n be a submanifold of real codimension two of a complex Euclidean space $\mathbf{C}^{\frac{n+2}{2}}$ with $\lambda = 0$ which satisfies the condition.If there exists a totally umbilical hypersurface M' of $\mathbf{C}^{\frac{n+2}{2}}$, i.e. $A' = cI$, $c \neq 0$, such that $M \subset M'$, then M is a product of two odd-dimensional spheres.

Let M be a connected submanifold of real codimension 2 of a complex Euclidean space $\overline{M} = \mathbf{C}^{\frac{n+2}{2}}$. If M satisfies the condition, then M is one of the following:

(1) n-dimensional sphere $\mathbf{S}^n$,

(2) n-dimensional Euclidean space $\mathbf{E}^n$,

(3) product manifold of an r-dimensional sphere and an $(n-r)$-dimensional Euclidean space $\mathbf{S}^r \times \mathbf{E}^{n-r}$, where r is an even number.

(4) CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$.

If for a real submanifold M of a complex manifold $(\overline{M}, J)$, the holomorphic tangent space $H_x(M) = JT_x(M) \cap T_x(M)$ has constant dimension with respect to $x \in M$, the submanifold M is called a CR submanifold and the constant complex dimension is called the CR dimension of M.In [1] we collected the elementary facts about complex manifolds and their submanifolds and introduced the reader to the study of CR submanifolds of complex manifolds, especially complex projective space.

Let M be a real submanifold of codimension 2 of a complex manifold $\overline{M}$ Then:

(1) M is a complex hypersurface if and only if $\lambda^2(x) = 1$ for any $x \in M$.

(2) M is a CR submanifold of CR dimension $\frac{n-2}{2}$ if $\lambda(x) = 0$ for any $x \in M$.

The converse in (2)?

We assume that M satisfies the condition

$$h(FX, Y) + h(X, FY) = 0,$$

$$\text{for all } X, Y \in T(M).$$

$$
\begin{aligned}
J\imath X &= \imath FX + u^1(X)\xi_1 + u^2(X)\xi_2,\\
J\xi_1 &= -\imath U_1 + \lambda\xi_2,\\
J\xi_2 &= -\imath U_2 - \lambda\xi_1,\\
F^2X &= -X + u^1(X)U_1 + u^2(X)U_2,\\
FU_1 &= -\lambda U_2, \qquad FU_2 = \lambda U_1,\\
\overline{\nabla}_X \imath Y &= \imath\nabla_X Y + h(X,Y)\\
&= \imath\nabla_X Y\\
&+ g(A_1X, Y)\xi_1 + g(A_2X, Y)\xi_2\\
\overline{\nabla}_X \xi_1 &= -\imath A_1 X + s(X)\xi_2,\\
\overline{\nabla}_X \xi_2 &= -\imath A_2 X - s(X)\xi_1,
\end{aligned}
$$

h is the second fundamental form and F is the structure tensor induced from the natural almost complex structure of a complex manifold

Let M^n be a submanifold of real codimension two of a complex projective space, which is not its totally geodesic complex hypersurface and let M satisfy the condition. If there exists a real hypersurface $M^C_{p,q}$ such that $M \subset M^C_{p,q}$, then M is congruent to $\pi(S^{2p+1} \times S^{2r+1} \times S^{2s+1})$, where $p + q + s = \frac{n+1}{2}$.

$$
\begin{array}{ccccc}
\pi^{-1}(M) & \xrightarrow{\imath_1'} & S^{2p+1} \times S^{2q+1} & \xrightarrow{\imath_2'} & S^{n+3}\\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi\\
M & \xrightarrow[\imath_1]{} & M^C_{p,q} & \xrightarrow[\imath_2]{} & P^{\frac{n+2}{2}}(\mathbf{C})
\end{array}
$$

References

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