Higher codimension CR structures, Levi–Kähler reduction and toric geometry

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New trends in differential geometry

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Joint work with

- Vestislav Apostolov (UQAM)
- Paul Gauduchon (Ecole Polytechnique)
- Eveline Legendre (Toulouse)

What this talk is about

Defn. A *CR structure* on a manifold N is a distribution $\mathcal{D} \subseteq TN$ equipped with a complex structure $J\colon \mathcal{D} \to \mathcal{D}$ satisfying an integrability condition. Define $L_{\mathcal{D}}\colon \wedge^2\mathcal{D} \to TN/\mathcal{D}$ by

$$L_{\mathcal{D}}(X, Y) = -[X, Y] \mod \mathcal{D}.$$

This (or the associated hermitian tensor) is called the *Levi form*.

Example. N a real submanifold of $\mathbb{C}^d$ with standard complex structure I: set $\mathcal{D} = TN \cap ITN$ and $J = I|_{\mathcal{D}}$.

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Main idea. For G acting freely on N by "transversal" CR automorphisms of $(\mathcal{D}, J)$, G-invariant positive definite components of $L_{\mathcal{D}}$ descend to Kähler metrics on N/G.

Note: N is then a principal G-bundle over M; transversality means that $\mathcal{D}$ is a connection on $N \to M$.

Motivating examples: CR spheres

Any odd dimensional sphere

$$\mathbb{S}^{2m+1} = \{z \in \mathbb{C}^{m+1} : |z_0|^2 + |z_1|^2 + \cdots + |z_{m+1}|^2 = 1\}$$

is a CR submanifold of $\mathbb{C}^{m+1}$, with positive definite Levi form.

Quotient of $\mathbb{S}^{2m+1}$ by weighted $\mathbb{S}^1$ action on $\mathbb{C}^{m+1}$ with weights $\mathbf{a} = (a_0, a_1, \ldots a_m)$ is the weighted projective space $\mathbb{C}P^m_{\mathbf{a}}$ (which is $\mathbb{C}P^m$ when $a_i = a_j$ for all i, j).

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Generator $\xi_{\mathbf{a}}$ of action is transverse to $\mathcal{D}$ and so trivializes $T\mathbb{S}^{2m+1}/\mathcal{D}$. The Levi form defines a hermitian metric on $\mathcal{D}$, which descends to a Kähler metric $g_{\mathbf{a}}$ on $\mathbb{C}P^m_{\mathbf{a}}$.

Following Tanaka, Chern–Moser, Webster, Bryant, David–Gauduchon et al. we discover that:

Theorem. The Kähler metric $g_{\mathbf{a}}$ on $\mathbb{C}P^m_{\mathbf{a}}$ is Bochner-flat (i.e., has vanishing Bochner tensor).

Motivating observations

- The theorem holds because $\mathbb{S}^{2m+1}$ is "CR-flat" (thus it has vanishing Chern–Moser tensor).
- The Kähler quotient of $\mathbb{C}^{m+1}$ by a weighted $\mathbb{S}^1$ action is Bochner-flat only in the standard case of equal weights.
- Construction not limited to codimension one: ℓ-fold product of weighted $\mathbb{S}^1$ actions on CR-spheres yields an ℓ-fold product of weighted projective spaces.
- The (skew, or imaginary part of the) Levi form depends only on $\mathcal{D}$: the complex structure J on $\mathcal{D}$ is largely a passenger.

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Plan.

1. Contact geometry in arbitrary codimension
2. CR structures and Levi–Kähler reduction
3. Application to toric Kähler geometry

1. Levi nondegenerate distributions and symplectization

On a manifold N with distribution $\mathcal{D}$, have an exact sequence

$$0 \to \mathcal{D} \to TN \to TN/\mathcal{D} \to 0.$$

Let $L_{\mathcal{D}} \colon \wedge^2 \mathcal{D} \to TN/\mathcal{D}$ be its Levi form.

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Let $\mathcal{D}^0 \xrightarrow{i} T^*N$ be the inclusion and τ the tautological 1-form on $T^*N \xrightarrow{p} N$ (with $\tau_\alpha = \alpha \circ p_* \colon T_\alpha T^*N \to \mathbb{R}$ for $\alpha \in T^*N$).

Proposition. The pullback $\Omega^{\mathcal{D}} = i^* \mathrm{d}\tau = \mathrm{d} i^* \tau$ of the standard symplectic form $\Omega = \mathrm{d}\tau$ to $\mathcal{D}^0$ is symplectic on the open subset

$$U_{\mathcal{D}} = \{\alpha \in \mathcal{D}^0 \cong (TM/\mathcal{D})^* \mid \alpha \circ L_{\mathcal{D}} \text{ is nondegenerate}\}$$

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Defn. Say $(N, \mathcal{D})$ is *Levi-nondegenerate* or *contact* of rank m and codimension $\ell = \mathrm{rank}(\mathcal{D}^0)$ if $\forall\, z \in N$, $U_{\mathcal{D}} \cap p^{-1}(z) \neq \varnothing$.

Thus $\mathrm{rank}(\mathcal{D}) = 2m$ is even, and $U_{\mathcal{D}} \xrightarrow{p} N$ has (local) sections called *contact forms*.

1. Local contact actions

$(N, \mathcal{D})$ contact of rank m and codimension ℓ.

$\mathfrak{con}(N, \mathcal{D}) \subseteq \Gamma(TN)$: Lie algebra of infinitesimal contactomorphisms of $(N, \mathcal{D})$, i.e., vector fields X with

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Any $X \in \Gamma(TN)$: lift $\tilde{X}$ to vector field on T^*N with hamiltonian $\tau(\tilde{X})$, i.e., $\alpha \mapsto \tau_\alpha(\tilde{X}) = \alpha(X)$.

If $X \in \mathfrak{con}(N, \mathcal{D})$, then $\tilde{X}$ is tangent to $\mathcal{D}^0 \subseteq T^*N$.

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Defn. A (*local, effective*) *contact action* of a Lie algebra $\mathfrak{g}$ on $(N, \mathcal{D})$ is a Lie algebra monomorphism $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{con}(N, \mathcal{D})$. For $v \in \mathfrak{g}$, write K_v for the vector field $\mathbf{K}(v)$.

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Lemma. $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{con}(N, \mathcal{D})$ a local contact action of $\mathfrak{g}$ on $(N, \mathcal{D})$; define $\mu_{\mathfrak{g}}\colon \mathcal{D}^0 \to \mathfrak{g}^*$ by $\langle \mu_{\mathfrak{g}}(\alpha), v\rangle = \alpha(K_v)$ for $\alpha \in \mathcal{D}^0$ and $v \in \mathfrak{g}$. Then the lift of $\mathbf{K}$ to T^*N preserves $\mathcal{D}^0$, and is hamiltonian on $U_{\mathcal{D}}$ with momentum map $\mu_{\mathfrak{g}}$.

1. Transversal actions

$(N, \mathcal{D})$ contact of codimension ℓ.

A local contact action $\mathbf{K} \colon \mathfrak{g} \to \mathfrak{con}(N, \mathcal{D})$ of an ℓ-dimensional Lie algebra $\mathfrak{g}$ is *transversal* iff pointwise image $\mathcal{K}^{\mathfrak{g}}$ of $\mathbf{K}$ is a rank ℓ distribution transverse to $\mathcal{D}$ called the *Reeb distribution*:

Condition 1. At every point of N, $\mathcal{D} + \mathcal{K}^{\mathfrak{g}} = TN$.

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Define $\eta\colon TN \to \mathfrak{g}$ (uniquely) by

$$\ker(\eta) = \mathcal{D} \quad \text{and} \quad \eta(K_v) = v, \quad \forall\, v \in \mathfrak{g}.$$

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Then $(p, \mu_{\mathfrak{g}}) \colon \mathcal{D}^0 \to N \times \mathfrak{g}^*$ is a bundle isomorphism. Also:

- For any $v \in \mathfrak{g}$, $\mathcal{L}_{K_v}\eta + [v, \eta]_{\mathfrak{g}} = 0$;
- $d\eta + \frac{1}{2}[\eta \wedge \eta]_{\mathfrak{g}} = \eta \circ L_{\mathcal{D}}$, where $L_{\mathcal{D}}$ is extended by zero from $\mathcal{D}$ to $TN = \mathcal{D} \oplus \mathcal{K}^{\mathfrak{g}}$.

Example. η could be a connection 1-form on a principal G-bundle.

1. Contact torus actions

$(N, \mathcal{D})$ be contact manifold of rank m and codimension ℓ.

Let $\mathbb{T}_N = \mathfrak{t}_N / 2\pi\Lambda$ be a (real) torus with Lie algebra $\mathfrak{t}_N = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, where Λ is the lattice of circle subgroups of $\mathbb{T}_N$.

Defn. A *contact torus action* of $\mathbb{T}_N$ on M is a local contact action $\mathbf{K} \colon \mathfrak{t}_N \to \mathfrak{con}(N, \mathcal{D})$ which integrates to an effective action of $\mathbb{T}_N$. It is *toric* if $\dim \mathbb{T}_N = d := m + \ell$.

Say $(N, \mathcal{D}, \mathbf{K})$ has *tube type* iff $\mathfrak{t}_N$ has an ℓ-dimensional subalgebra $\mathfrak{g}$ acting transversally on N via $\mathbf{K}$, i.e., $\mathcal{K}^{\mathfrak{g}} = \mathrm{span}\{K_{v,z} \mid v \in \mathfrak{g}\}$ satisfies Condition 1.

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Example. $\mathfrak{g}$ the Lie algebra of a closed subgroup G of $\mathbb{T}_N$:
Condition 1 $\Rightarrow$ action of G is locally free on N
$\Rightarrow M := N/G$ is a compact orbifold.
Action of $\mathbb{T}_N$ induces action of quotient torus $\mathbb{T} := \mathbb{T}_N / G$ on M.

Condition 1 also ensures: $\mathcal{D} \cong$ pullback of TM to N, and hence G-invariant data on $\mathcal{D}$ descend to M.

1. Levi quotients and symplectic quotients

$(N, \mathcal{D}, \mathbf{K})$ tube type: $\mathfrak{g} \subseteq \mathfrak{t}_N$ transversal, Reeb distribution $\mathcal{K}^{\mathfrak{g}}$, connection 1-form $\eta\colon TN \to \mathfrak{g}$.
For any $\lambda \in \mathfrak{g}^* \setminus 0$, define $\eta^\lambda\colon N \to \mathcal{D}^0$ by $\eta^\lambda_z(X) = \langle \eta_z(X), \lambda \rangle$, and let $L_{\mathcal{D},\lambda} = \eta^\lambda \circ L_{\mathcal{D}} = \langle \mathrm{d}\eta|_{\mathcal{D}}, \lambda \rangle$.

Defn. Say $(\mathcal{D}, L_{\mathcal{D},\lambda})$ is the *Levi structure* induced by $(\mathfrak{g}, \lambda)$; it is *nondegenerate* over the open subset $N_\lambda \subseteq N$ where η^λ is a contact form (i.e., $U_{\mathcal{D}}$-valued). If $\mathfrak{g}$ is the Lie algebra of a closed subgroup G of $\mathbb{T}_N$, refer to $M = N/G$ as *Levi quotient* of N by G.

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Action of $\mathbb{T}_N$ on N lifts to hamiltonian action on $U_{\mathcal{D}}$.
Momentum map $\mu_N := \mu_{\mathfrak{t}_N} \colon U_{\mathcal{D}} \to \mathfrak{t}_N^*$ with $\langle \mu_N(\alpha), v \rangle = \alpha(K_v)$.
Given $\iota \colon \mathfrak{g} \hookrightarrow \mathfrak{t}_N$, have $\mu_{\mathfrak{g}} = \iota^\top \mu_N \colon U_{\mathcal{D}} \to \mathfrak{g}^*$ and
$N_\lambda = \{z \in N \mid U_{\mathcal{D},z} \cap {\mu_{\mathfrak{g}}}^{-1}(\lambda) \neq \varnothing\}$.

Proposition. $L_{\mathcal{D},\lambda}$ descends to a symplectic form on $M := N/G$ if and only if $N_\lambda = N$. In this case M is the symplectic quotient ${\mu_{\mathfrak{g}}}^{-1}(\lambda)/G$ of $U_{\mathcal{D}}$ by the lifted G action.

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Remark. $\{\lambda \in \mathfrak{g}^* \setminus 0 \mid N_\lambda = N\}$ is an open cone $\mathcal{C} \subseteq \mathfrak{g}^*$.

1. Levi quotient formalism and horizontal momentum map

Fix an epimorphism $\mathfrak{h} \to \mathfrak{t}$ with kernel $\mathbb{R}$ between abelian Lie algebras of dimensions $m+1$ and m. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\iota} & \mathfrak{t}_N & \xrightarrow{\mathbf{u}} & \mathfrak{t} & \longrightarrow & 0 \\ & & \lambda\big\downarrow & & \mathbf{L}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \mathbb{R} & \xrightarrow{\varepsilon} & \mathfrak{h} & \xrightarrow{\mathrm{d}} & \mathfrak{t} & \longrightarrow & 0 \end{array}$$

associates pairs $(\mathfrak{g}, \lambda)$ to epimorphisms $\mathbf{L} \colon \mathfrak{t}_N \to \mathfrak{h}$ ($\mathfrak{g}$ is the kernel of $\mathbf{u} := \mathrm{d} \circ \mathbf{L}$, and λ is induced by $\mathbf{L}|_{\mathfrak{g}}$). Let $\mathcal{A} = (\varepsilon^{\top})^{-1}(1)$.

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Note that $\langle \mu_N(\eta_z^\lambda), v \rangle = \eta_z^\lambda(K_v)$ equals $\langle v, \lambda \rangle$ for $v \in \mathfrak{g}$. This vanishes for $v \in \ker \lambda \subseteq \mathfrak{g}$, hence induces $\mu^\lambda\colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ with

$$\langle \mu^\lambda(z), \mathbf{L}(v) \rangle = \eta_z^\lambda(K_v) \tag{1}$$

for all $z \in N$ and $v \in \mathfrak{t}_N$.

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for all $z \in N$ and $v \in \mathfrak{t}_N$.

μ^λ is the *horizontal* (*natural*) *momentum map* of $(\mathcal{D}, L_{\mathcal{D},\lambda}, \mathfrak{g})$.

Aside: Natural momentum maps in toric geometry

Setting: a hamiltonian action of a Lie group $\mathbb{T}$ on a symplectic orbifold (M, ω), with Lie algebra $\mathfrak{t} \hookrightarrow C^\infty(M, TM)$.

Let $\mathfrak{h} \subseteq C^\infty(M, \mathbb{R})$ be the subspace of hamiltonian generators f, i.e., with $\mathrm{grad}_\omega f \in \mathfrak{t}$. This defines an exact sequence

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where $\mathbb{R}$ is the subspace of constant functions. Hence (dually)

$$0 \to \mathfrak{t}^* \to \mathfrak{h}^* \xrightarrow{\varepsilon^\top} \mathbb{R} \to 0$$

and we have a canonical map $\mu \colon M \to \mathfrak{h}^*$ given by $\langle \mu(x), f \rangle = f(x)$ for $x \in M$ and $f \in \mathfrak{h}$. It takes values in the affine subspace $\mathcal{A} := (\varepsilon^\top)^{-1}(1)$ of $\mathfrak{h}^*$.

This **natural momentum map** μ determines a momentum map in the usual sense after choosing a splitting $\mathfrak{t} \to \mathfrak{h}$ (a basepoint in $\mathcal{A}$).

1. Convexity and connectedness

Theorem. Suppose $(N, \mathcal{D})$ is a (compact, connected) toric contact manifold under $\mathbb{T}_N = \mathfrak{t}_N/2\pi\Lambda$. Given a transversal subalgebra $\iota\colon \mathfrak{g} \to \mathfrak{t}_N$ and $\lambda \in \mathfrak{g}^*$ with $N_\lambda = N$, let $\mu^\lambda\colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ be the induced horizontal momentum map. Then the image of μ^λ is a compact convex simple polytope Δ in $\mathcal{A}$, the convex hull of the points $\mu^\lambda(z)$ where $\mathcal{K}_z^{\mathfrak{g}} = \mathcal{K}_z^{\mathfrak{t}_N}$.

Furthermore, μ^λ is a submersion over the interior of any face of Δ, and the fibres of μ^λ are $\mathbb{T}_N$-orbits.

1. Convexity and connectedness

Theorem. Suppose $(N, \mathcal{D})$ is a (compact, connected) toric contact manifold under $\mathbb{T}_N = \mathfrak{t}_N / 2\pi\Lambda$. Given a transversal subalgebra $\iota\colon \mathfrak{g} \to \mathfrak{t}_N$ and $\lambda \in \mathfrak{g}^*$ with $N_\lambda = N$, let $\mu^\lambda\colon N \to \mathcal{A} \subseteq \mathfrak{h}^*$ be the induced horizontal momentum map. Then the image of μ^λ is a compact convex simple polytope Δ in $\mathcal{A}$, the convex hull of the points $\mu^\lambda(z)$ where $\mathcal{K}_z^{\mathfrak{g}} = \mathcal{K}_z^{\mathfrak{t}_N}$.

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Proof follows Atiyah. The essential ingredient is that for any $v \in \mathfrak{t}_N$, $f := \eta^\lambda(K_v)$ is a Morse–Bott function on N whose critical submanifolds all have even index.

Hence for any vectors $v_1, \ldots, v_k \in \mathfrak{t}_N$, the map $f\colon N \to \mathbb{R}^k$ with $f_i = \eta^\lambda(K_{v_i})$ satisfies

(*A*) all fibres $f^{-1}(p)$ are empty or connected;

(*B*) the image $f(N)$ is convex.

2. CR structures

Defn. A *rank* m*, codimension* ℓ *CR structure* on a $(2m+\ell)$-manifold N is a rank $2m$ distribution $\mathcal{D} \subseteq TN$ equipped with an almost complex structure $J\colon \mathcal{D} \to \mathcal{D}$, which satisfies the integrability conditions

$$
\begin{aligned}
&[X, Y] - [JX, JY] \in \Gamma(\mathcal{D}),\\
&[X, JY] + [JX, Y] = J([X, Y] - [JX, JY]), \quad \forall X, Y \in \Gamma(\mathcal{D}).
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$(N, \mathcal{D}, J)$ is called a *CR manifold* (of codimension ℓ) and is said to be *Levi nondegenerate* if $\mathcal{D}$ is.

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Levi form $L_{\mathcal{D}}$ is J-invariant or "type (1,1)" on $\mathcal{D}$. It follows that $h_{\mathcal{D}}(X, Y) := L_{\mathcal{D}}(X, JY)$ is a section of $S^2\mathcal{D}^* \otimes TN/\mathcal{D}$. Say $(N, \mathcal{D}, J)$ is *Levi definite* if there is a contact form α such that $\alpha \circ h_{\mathcal{D}} \in S^2\mathcal{D}^*$ is positive definite.

Set $U^+_{\mathcal{D}} := \{\alpha \in \mathcal{D}^0 \mid \alpha \circ h_{\mathcal{D}} \text{ is positive definite}\} \subseteq U_{\mathcal{D}}$.

2. CR torus actions and Levi–Kähler reduction

$(N, \mathcal{D}, J)$ a CR manifold: the Lie algebra $\mathfrak{cr}(N, \mathcal{D}, J)$ of CR vector fields consists of those $X \in \mathfrak{con}(N, \mathcal{D})$ such that $\mathcal{L}_X J = 0$. A local action $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{con}(N, \mathcal{D})$ is called a local CR action iff it takes values in $\mathfrak{cr}(N, \mathcal{D}, J)$.

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A *CR torus action* of $\mathbb{T}_N = \mathfrak{t}_N / 2\pi\Lambda$ on a CR manifold $(N, \mathcal{D}, J)$ is contact torus action of $\mathbb{T}_N$ on $(N, \mathcal{D})$ induced by a local CR action $\mathbf{K}\colon \mathfrak{t}_N \to \mathfrak{cr}(N, \mathcal{D}, J)$. Say $(N, \mathcal{D}, J, \mathbf{K})$ is *tube type* or is *toric* if the underlying contact torus action is.

Suppose $\mathbf{K}\colon \mathfrak{g} \to \mathfrak{cr}(N, \mathcal{D}, J)$ is a transversal CR action with connection 1-form $\eta\colon TN \to \mathfrak{g}$. For $\lambda \in \mathfrak{g}^*$, set $h_{\mathcal{D},\lambda} := L_{\mathcal{D},\lambda}(\cdot, J\cdot)$.

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Defn. Say $(N, \mathcal{D}, J, \mathfrak{g})$ is *Levi–Kähler* at momentum level $\lambda \in \mathfrak{g}^* \setminus \{0\}$ iff $h_{\mathcal{D},\lambda}$ is positive definite on $\mathcal{D}$, i.e., $\langle \eta, \lambda \rangle$ is a section of $U^+_{\mathcal{D}}$. If also $\mathfrak{g}$ is the Lie algebra of a Lie group G acting on N such that M/G is a smooth manifold (or orbifold), then the Kähler metric on M induced by $(h_{\mathcal{D},\lambda}, J, L_{\mathcal{D},\lambda})$ is called the *Levi–Kähler quotient* of $(N, \mathcal{D}, J)$ by $(\mathfrak{g}, \lambda)$.

2. Flat space

$\mathcal{S}$ a d element set (e.g., $\mathcal{S} = \{1, 2, \ldots d\}$).
Let $\mathbb{Z}_\mathcal{S} \cong \mathbb{Z}^d$ be the free abelian group generated by $\mathcal{S}$. Let $\mathfrak{t}_\mathcal{S} = \mathbb{Z}_\mathcal{S} \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^d$ and $\mathbb{C}_\mathcal{S} = \mathbb{Z}_\mathcal{S} \otimes_\mathbb{Z} \mathbb{C} \cong \mathbb{C}^d$ be corresponding free vector spaces over $\mathbb{R}$ and $\mathbb{C}$.

Denote the generators of $\mathbb{Z}_\mathcal{S} \subseteq \mathfrak{t}_\mathcal{S} \subseteq \mathbb{C}_\mathcal{S}$ by $e_s : s \in \mathcal{S}$, and by $z_s \colon \mathbb{C}_\mathcal{S} \to \mathbb{C}$, the standard (linear) complex coordinates on $\mathbb{C}_\mathcal{S}$.

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Then $\mathbb{T}_{\mathcal{S}} = \mathfrak{t}_{\mathcal{S}}/2\pi\mathbb{Z}_{\mathcal{S}} \cong (\mathbb{S}^1)^d$ acts diagonally on $\mathbb{C}_{\mathcal{S}}$, via $[\sum_s t_s e_s] \cdot (\sum_s z_s e_s) = \sum_s \exp(it_s) z_s e_s$; action is hamiltonian (wrt. standard symplectic form $\omega_{\mathcal{S}}$) with momentum map $\boldsymbol{\sigma} \colon \mathbb{C}_{\mathcal{S}} \to \mathfrak{t}_{\mathcal{S}}^*$:

$$\langle \boldsymbol{\sigma}(z), e_s \rangle = \sigma_s(z) = \tfrac{1}{2}|z_s|^2.$$

The flat Kähler metric in action-angle coordinates on $\mathbb{C}_{\mathcal{S}}$ is then

$$g_{\mathcal{S}} = \sum_{s \in \mathcal{S}} \Big(\frac{d\sigma_s{}^2}{2\sigma_s} + 2\sigma_s d\boldsymbol{\vartheta}_s{}^2 \Big), \quad \omega_{\mathcal{S}} = \sum_{s \in \mathcal{S}} d\sigma_s \wedge d\boldsymbol{\vartheta}_s,$$

where $\boldsymbol{\vartheta} \colon \mathbb{C}_{\mathcal{S}}^{\times} \to \mathbb{T}_{\mathcal{S}}$ are angle coordinates with $J d\sigma_s = 2\sigma_s d\boldsymbol{\vartheta}_s$.

2. Toric CR submanifolds of flat space

Defn. A *toric CR submanifold* $(N, \mathcal{D}, J)$ of $\mathbb{C}_{\mathcal{S}}$ is a compact connected CR submanifold which is invariant under the $\mathbb{T}_{\mathcal{S}}$ action.

A (toric, codimension ℓ) *Levi–Kähler reduction* M of $\mathbb{C}_{\mathcal{S}}$ is a Levi–Kähler quotient of $(N, \mathcal{D}, J)$ by $(\mathfrak{g}, \lambda)$, where $(N, \mathcal{D}, J)$ is a toric CR submanifold of $\mathbb{C}_{\mathcal{S}}$ of codimension ℓ, and $\mathfrak{g} \subseteq \mathfrak{t}_{\mathcal{S}}$ is the Lie algebra of an ℓ-dimensional subgroup G of $\mathbb{T}_{\mathcal{S}}$.

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Specify the choice of $(\mathfrak{g}, \lambda)$ via an epimorphism $\mathbf{L}\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{h}$, or equivalently, an indexed family $L_s : s \in \mathcal{S}$ of vectors in $\mathfrak{h}$ which span (where $L_s = \mathbf{L}(e_s)$). Thus $\mathfrak{g}$ is the kernel of $\mathbf{u} = \mathrm{d} \circ \mathbf{L}\colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{t}$, sending $\sum_s t_s e_s$ to $\sum_s t_s u_s$ for an indexed family $u_s : s \in \mathcal{S}$ of vectors in $\mathfrak{t}$ which span.

Data $(N, \mathcal{D}, J)$ and $(\mathfrak{g}, \lambda)$ are then linked by Condition 1, which may now be viewed as a constraint on $(N, \mathcal{D}, J)$ given $(\mathfrak{g}, \lambda)$ or vice versa.

3. Toric geometry: Rational Delzant theory

A *toric symplectic orbifold* is a symplectic $2m$-orbifold (M, ω) with a hamiltonian action of an m-torus $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$, where $\mathfrak{t}$ is the Lie algebra of $\mathbb{T}$ and Λ is the lattice of circle subgroups.

Fact (Delzant, Lerman–Tolman). The image of the natural momentum map $\mu\colon M \to \mathcal{A} \subseteq \mathfrak{h}^*$ is a compact convex polytope

$$\Delta := \{\xi \in \mathcal{A} \mid \forall s \in \mathcal{S},\ \ L_s(\xi) \geq 0\} \subseteq \mathcal{A}$$

for **affine normals** $L_s \in \mathfrak{h}$ $(s \in \mathcal{S})$, defining affine functions on $\mathcal{A} \subseteq \mathfrak{h}^*$.

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for **affine normals** $L_s \in \mathfrak{h}$ $(s \in \mathcal{S})$, defining affine functions on $\mathcal{A} \subseteq \mathfrak{h}^*$. The L_s are determined uniquely up to scale by Δ, and the orbifold structure of M determines these scales such that:

- $\forall\, s \in \mathcal{S}$ the **inward normals** $u_s := \mathrm{d}L_s$ are in $\Lambda \subseteq \mathfrak{t}$;
- $\forall \xi \in \Delta$, $\{u_s \in \mathfrak{t} : L_s(\xi) = 0\} \subseteq \mathfrak{t}$ is linearly independent.

$(\Delta, \mathbf{L})$ is called a **rational Delzant polytope**; it determines $(M, \omega, \mathbb{T})$ up to equiv. symplectomorphism and orbifold covering.

3. Toric geometry: the Delzant construction

Given a rational Delzant polytope $(\Delta, \mathbf{L})$, we construct a symplectic toric orbifold as a symplectic quotient of the flat space $\mathbb{C}_{\mathcal{S}}$ generated by the set $\mathcal{S}$ parameterizing the facets of Δ. For this we use $\mathbf{L}$, viewed as a linear map $\mathfrak{t}_{\mathcal{S}} \to \mathfrak{h}$, to define a pair $(\mathfrak{g}, \lambda)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\iota_{\mathfrak{g}}} & \mathfrak{t}_{\mathcal{S}} & \xrightarrow{\mathbf{u}} & \mathfrak{t} & \longrightarrow & 0 \\ & & \lambda\downarrow & & \mathbf{L}\downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{R} & \xrightarrow{\varepsilon} & \mathfrak{h} & \xrightarrow{\mathrm{d}} & \mathfrak{t} & \longrightarrow & 0 \end{array}$$

with $\mathfrak{g} \subseteq \mathfrak{t}_{\mathcal{S}}$ and $\lambda \in \mathfrak{g}^*$. The rationality conditions on Δ ensure $\mathfrak{g}$ is the Lie algebra of a subtorus G of $\mathbb{T}_{\mathcal{S}}$, and M is given as a symplectic quotient $(\iota_{\mathfrak{g}}^{\top} \circ \boldsymbol{\sigma})^{-1}(0)/G$.

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Since $\langle \boldsymbol{\sigma}(z), e_s \rangle = \frac{1}{2}|z_s|^2$, $N := (\iota_{\mathfrak{g}}^{\top} \circ \boldsymbol{\sigma})^{-1}(0)$ is an intersection of hermitian quadrics, hence a toric CR submanifold of $\mathbb{C}_{\mathcal{S}}$.

3. Example: spheres and projective spaces

$\mathcal{S} = \{0, 1, \dots m\}$, $\mathfrak{h} = \mathfrak{t}_{\mathcal{S}}$ with $\mathfrak{t} = \mathfrak{h}/\ell$, where ℓ is the span of $\sum_{s \in \mathcal{S}} e_s$. Thus $\mathcal{A} \subseteq \mathfrak{h}^*$ is the affine subspace whose coordinates sum to one, and $L_s = e_s$ so Δ is the standard simplex in $\mathcal{A}$. Thus $C_{\mathcal{S}} \cong \mathbb{C}^{m+1}$, G diagonal subgroup of $\mathbb{T}_{\mathcal{S}}$, $N = \mathbb{S}^{2m+1}$ and $M = \mathbb{C}P^m$.

Can vary example by taking $L_s = w_s e_s$ for some $w_s \in \mathbb{Q}$. The diagonal G is replaced by a weighted action, and M is a weighted projective space.

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Can also take products. For this, fix $\ell \in \mathbb{Z}^+$ and $m_1, \dots m_\ell \in \mathbb{Z}^+$, and let $\mathcal{I} = \{1, \dots \ell\}$, $I_i = \{0, \dots m_i\}$ and $\mathcal{S} = \{(i, r) \mid i \in \mathcal{I}$ and $r \in I_i\}$. Let $m = \sum_{i=1}^{\ell} m_i$ and $d = m + \ell$.
Thus $\mathbb{C}_{\mathcal{S}} \cong \mathbb{C}^{m_1+1} \times \mathbb{C}^{m_2+1} \times \cdots \times \mathbb{C}^{m_\ell+1} \cong \mathbb{C}^d$ and $\mathfrak{t}_{\mathcal{S}}$ has a subspace $\mathfrak{g}_o = \{x \in \mathfrak{t}_{\mathcal{S}} \mid x_{iq} = x_{ir}$ for all $i \in \mathcal{I}$, $q, r \in I_i\}$.

Let x_i be the common value of the x_{ir} and thus identify $\mathfrak{g}_o$ with $\mathbb{R}^\ell$. On $\mathfrak{g}_o$ we have a natural linear form λ_o sending $(x_1, x_2, \dots x_\ell)$ to $x_1 + x_2 + \cdots + x_\ell \in \mathbb{R}$, and we let $\mathbf{L}^o \colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{h} = \mathfrak{t}_{\mathcal{S}}/\ker \lambda_o$ and $\mathbf{u}^o \colon \mathfrak{t}_{\mathcal{S}} \to \mathfrak{t} = \mathfrak{t}_{\mathcal{S}}/\mathfrak{g}_o$ be the quotient maps.

3. Products of spheres

Theorem. Let $N = \mathbb{S}^{2m_1+1} \times \cdots \times \mathbb{S}^{2m_\ell+1} \subseteq \mathbb{C}_{\mathcal{S}}$ be a product of standard CR spheres and let $(\Delta, \mathbf{L})$ be a rational Delzant polytope of the same combinatorial type as the product of simplices such that the kernel $\mathfrak{g}$ of $\mathbf{u} = \mathrm{d} \circ \mathbf{L}$ satisfies Condition 1.
Then N is Levi–Kähler with respect to $\mathfrak{g}$ at the momentum level λ determined by $\mathbf{L}$, and the quotient is a compact toric Kähler orbifold with whose Kähler metric has symplectic potential

$$\begin{aligned} G &= \frac{1}{2} \sum_{i=1}^{\ell} \Big(\sum_{r=0}^{m_i} L_{ir} \log L_{ir} - \Big(\sum_{r=0}^{m_i} L_{ir} \Big) \log \Big(\sum_{r=0}^{m_i} L_{ir} \Big) \Big) \\ &= \frac{1}{2} \sum_{i=1}^{\ell} \sum_{r=0}^{m_i} L_{ir} \log \Big(\frac{L_{ir}}{\sum_{s=0}^{m_i} L_{is}} \Big). \end{aligned}$$

Equivalently, the reduced metric on the image of μ^λ is given by

$$g_{\mathrm{red}} = \frac{1}{2} \sum_{i=1}^{\ell} \Big(\sum_{r=0}^{m_i} \frac{\mathrm{d}L_{ir}{}^2}{L_{ir}} - \frac{\big(\sum_{r=0}^{m_i} \mathrm{d}L_{ir}\big)^2}{\sum_{r=0}^{m_i} L_{ir}} \Big).$$

3. Products of spheres continued

For $\ell = 1$, we reobtain (from the preceding theorem) R. Bryant's description of Bochner-flat Kähler metrics on weighted projective spaces. We have also studied the case that $N = \mathbb{S}^3 \times \mathbb{S}^3$.

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For $\ell = 1$, we reobtain (from the preceding theorem) R. Bryant's description of Bochner-flat Kähler metrics on weighted projective spaces. We have also studied the case that $N = \mathbb{S}^3 \times \mathbb{S}^3$.

Theorem. Let (M, g, ω, J) a compact simply-connected Kähler 4-orbifold. Then the following conditions are equivalent.

1. (M, g, ω, J) is a Levi–Kähler quotient of $\mathbb{S}^3 \times \mathbb{S}^3$.
2. (M, g, ω, J) is toric with respect to a 2-torus $\mathbb{T}$, $g = g_+$ is compatible with a second complex structures J_- which commutes with $J_+ = J$ but induces the opposite orientation on M, and g_+ is conformal to a metric g_- which is Kähler with respect to J_-, such that $(\mathbb{T}, g_\pm, J_\pm, \omega_\pm)$ is *ambitoric* in the sense of Apostolov–C–Gauduchon. Furthermore, the scalar curvature of g_- is a Killing potential with respect to (g_+, ω_+) for a vector field induced by the action of $\mathbb{T}$.

That's all folks

Thank you!