Quaternionic Kähler metrics from G_2 geometry

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New trends in Differential Geometry, Villasimius 2014

Main result

Theorem

Let X^5 be a 5-manifold with a 2-dimensional generic distribution $D \subset TX$ (real analytic). To such (X, D) is canonically associated a 8-dimensional quaternionic Kähler metric g.

- generic : $[D, D]$ is 3-dimensional, $[D, [D, D]] = TX$
- there is a disc bundle

$$\begin{array}{ccc} \Delta & \longrightarrow & F \\ & & \downarrow p \\ & & X \end{array}$$

such that g is defined on a $F \times (0, \epsilon)$.

Fiber Δ = space of conformal metrics on D.

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Behavior of g

The asymptotics of g are fixed by (X, D):

$$g \sim \frac{ds^2}{s^2} + \frac{\gamma_3}{s^3} + \frac{\gamma_2}{s^2} + \frac{\gamma_1}{s} + \gamma_0$$

The orders of growth correspond to the filtration

$$T_{-3}F = TF \supset T_{-2}F = p^*[D, D] \supset T_{-1}F = p^*D \supset T_0F = \ker p_* = T\Delta,$$

and γ_i is defined on $T_{-i}F$ with $\ker \gamma_i = T_{-i+1}F$:

- $\ker \gamma_3 = p^*[D, D]$
- γ_2 is defined on $p^*[D, D]$ and $\ker \gamma_2 = p^*D$
- γ_1 is defined only on p^*D and $\ker \gamma_1 = \ker p_* = T\Delta$:
  $\gamma_1 =$ tautological metric on p^*D
- γ_0 is defined along the fibres of p:
  $\gamma_0 =$ hyperbolic metric on fibre Δ

and γ_2 and γ_3 are defined algebraically.

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Former similar constructions of qK metrics

- LeBrun (82): $(X^3, \text{conformal metric}) \rightsquigarrow Q^4$
  $G = SO(4,1)$, $G/H = \mathbb{R}H^4$
- LeBrun (89): $(X^{3+k}, \text{conformal } (3,k) \text{ metric}) \rightsquigarrow Q^{4(k+1)}$
  $G = SO(4, k+1)$
- B. (2000): $(X^{4k-1}, \text{quaternionic contact structure}) \rightsquigarrow Q^{4k}$
  $G = Sp(k,1)$, $G/H = \mathbb{H}H^k$
- B. (2007): $(X^3, \text{CR structure}) \rightsquigarrow Q^4$
  $G = SU(1,2)$, $G/H = \mathbb{C}H^2$

Each time: there is a model $G/P \rightsquigarrow G/H$, with $P \subset G$ parabolic subgroup of a real group, and G/H a quaternionic Kähler symmetric space of noncompact type.
New case: $G = G_2^r$, with $G_2^r/P = \{\text{isotropic lines in } \mathbb{R}^{3,4}\}$, symmetric space $G_2^r/SO(4)$.

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Nonlinear Poisson transform

Program with R. *Mazzeo*.
Let G/H be a symmetric space of noncompact type.

Poisson transform

There is a 1:1 correspondence between

1. bounded harmonic functions on G/H
2. 'functions' on maximal Furstenberg boundary $G/P_{\min}$.

Nonlinear Poisson transform

There should be a 1:1 correspondence between

1. complete 'asympt. symmetric' Einstein deformations of G/H
2. certain deformations of the parabolic geometry of $G/P_{\min}$.

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Nonlinear Poisson transform is established for:

- (Graham-Lee 1991) $G = SO(k,1)$
- (B. 2000) all other rank 1 cases
- (B.-Mazzeo 2011) reducible rank 2 cases

Question 1

All the previous constructions of quaternionic Kähler metrics are local versions of this nonlinear Poisson transform: one recovers the other parabolic subgroups by lifting to the minimal parabolic via

$$G/P_{\mathrm{min}} \longrightarrow G/P.$$

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Find a unified construction of a local quaternionic Kähler metric starting from a parabolic geometry modeled on a boundary G/P of a quaternionic Kähler symmetric space G/H of noncompact type.

(Remind there is one such symmetric space for each simple complex Lie group).

More modest question: unify the existing constructions.

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Geodesics

Some of these constructions rely on the use of a space of geodesics (LeBrun's constructions, G_2).

Proposition

The only examples which admit a space of geodesics are:

- conformal geometries (LeBrun's examples)
- the G_2/P example
- $SO(3,4)/P_3$ with P_3 fixing a totally isotropic 3-plane.

Beginning of the construction: the geodesic flow

Start from (X^5, D^2) wih D generic, and consider the circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & F^6 = \mathbb{P}(D) \\ & & \downarrow p \\ & & X^5 \end{array}$$

Proposition

F carries a canonical 1-dimensional distribution.

This can be proved using the Cartan connection of the geometry. In the model, $F^6 = G_2/P_{\min}$ and there are two circle fibrations:

$$\begin{array}{ccc} & G_2/P_{\min} & \\ \swarrow p & & \searrow q \\ G_2/P = G_2/P_1 & & G_2/P_2 \end{array}$$

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Zoll distributions

Nonlinear Poisson transform predicts:

geodesics are closed $\Longrightarrow$ there exists a complete Einstein metric on $G_2/SO(4)$ filling in the parabolic geometry at infinity

Definition

A generic distribution $D^2 \subset TX^5$ is **Zoll** if all the geodesics are closed.

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2. if yes, when can the quaternionic Kähler metric be extended to a complete quaternionic Kähler metric ? (ie when is the Einstein metric quaternionic Kähler ?)

   $\longrightarrow$ might get 'positive frequencies'

Same question in the other cases with geodesics: in particular for $G = SO(4,2)$, the question becomes:

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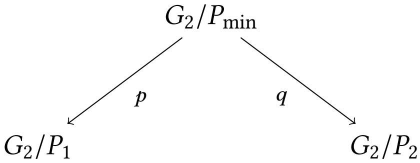
Sketch of proof of the Theorem



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Complexify the whole situation:

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Complexify the whole situation:

$$\begin{array}{ccc} & & \mathbb{C}P^1 \\ & \swarrow & \\ & F^{\mathbb{C}} & \\ \swarrow p & & q \searrow \\ X^{\mathbb{C}} & & N \end{array}$$

Rational curves

Must construct a 8-dimensional family of rational curves extending the 5-dimensional family

$$C_x = q(p^{-1}(x)), \qquad \text{normal bundle: } \mathscr{O} \oplus \mathscr{O} \oplus \mathscr{O} \oplus \mathscr{O}(1).$$

One must consider C_x as non reduced (doubled in the $\mathscr{O}(1)$ direction).

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Parabolic dilations

One uses the parabolic geometry of (X, D): the parabolic dilations

$$h_t(x_1, x_2, x_3, x_4, x_5) = (tx_1, tx_2, t^2x_3, t^3x_4, t^3x_5)$$

on a small open set U have the property that

$$h_t^* D \underset{t\to 0}{\longrightarrow} \text{model } G_2/P.$$

It follows that

$$h_t^* N(U) = N(h_t^* U) \underset{t\to 0}{\longrightarrow} \text{model } N(G_2/P).$$

In particular the rational curves converge to that of the model and therefore have the same normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$.

The behavior of the metric is found by inverse twistor transform.

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