Conification of Kähler and hyper-Kähler manifolds and supergravity c-map

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Villasimius, September 20, 2014
Joint work with V. Cortes, M. Dyckmanns and Th.Mohaupt
CMPh-2013, ArXiv 1305.3549v1

Dedicated to Stefano Marchiafava with love and admiration

Abstract

Given a Kähler manifold M endowed with a Hamiltonian Killing vector field Z, we construct a conical Kähler manifold $\hat{M}$ such that M is the Kähler quotient of $\hat{M}$.
Similarly, given a hyper-Kähler manifold $(M^{4n}, g, J_1, J_2, J_3)$ endowed with a Killing vector field Z, Hamiltonian with respect to the Kähler form of J_1 and satisfying $\mathcal{L}_Z J_2 = -2J_3$.
We construct a hyper-Kähler cone $\hat{M}^{4n+4}$ such that $M^{4n} =$ is a hyper-Kähler quotient of $\hat{M}$.

Application to c-map

Rigid c-map associates with a special Kähler manifold M^{2n} the hyper-Kähler manifold $N^{4n} = T^*M$.
Supergravity c-map associates with a projective special Kähler manifold $\bar{M}^{2n-2} = M^{2n}/\mathbb{C}^*$ a quaternionic Kähler manifold $\bar{N}^{4n}$ with the Ferrara-Sabharwal metric.
We show that it can be describe using conification $\hat{N}^{4n}$ of the hyper-Kähler manifold N^{4n} via diagram:

$$\begin{array}{ccc} M^{2n} & \xrightarrow{c} N^{4n} = T^*M \xrightarrow{con} & \hat{N}^{4n+4} \\ \downarrow \mathbb{C}^* & & \downarrow (\mathbb{H}^*/\pm 1) \\ \bar{M}^{2n-2} & \xrightarrow{\bar{c}} & \bar{N}^{4n} \end{array}$$

Outline

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Conification of a Kähler manifold

Data:
Let $(M, g, J, \omega = g \circ J, Z)$ be a pseudo-Kähler manifold endowed with a time-like or space-like Hamiltonian Killing vector field Z with Hamiltonian $-f$ s.t. $df = -\omega(Z, \cdot)$. We will assume that f and $f_1 := f - \frac{1}{2}g(Z, Z)$ are nowhere vanishing.
Let $\pi : P \to M$ be an S^1-principal bundle with a principal connection η with the curvature $d\eta = \pi^*(\omega - \frac{1}{2}d(g \circ Z))$ and pseudo-Euclidean metric

$$g_P := \frac{2}{f_1}\eta^2 + \pi^* g.$$

Conification of a Kähler manifold

On the manifold $\hat{\pi} : \hat{M} := P \times \mathbb{R}$ we define tensor fields

$$\begin{aligned} \xi &:= \partial_t \in \mathfrak{X}(\hat{M}), && (1) \\ \hat{g} &:= e^{2t}(g_P + 2fdt^2 + 2(df)\cdot dt) \in \Gamma(S^2 T^* \hat{M}), && (2) \\ \theta &:= e^{2t}(\eta + \frac{1}{2}(g \circ Z)) \in \Omega^1(\hat{M}), && (3) \\ \hat{\omega} &:= d\theta \in \Omega^2(\hat{M}), && (4) \end{aligned}$$

Conical manifold and Theorem

A conical pseudo-Riemannian manifold (M, g, ξ) is a pseudo-Riemannian manifold (M, g) endowed with a time-like or space-like vector field ξ such that $D\xi = \mathrm{Id}$.

Theorem Given (M, g, J, Z) as above, then $(\hat{M} = P \times \mathbb{R}, \hat{g}, \hat{J} := \hat{g}^{-1}\hat{\omega}, \xi)$ is a conical pseudo-Kähler manifold. The induced CR-structure on the hypersurface $P \subset \hat{M}$ coincides with the horizontal distribution T^hP for the connection η and $\pi : P \to M$ is holomorphic. The projection $\hat{\pi} : \hat{M} \to M$ is not holomorphic. The metric $\hat{g}$ has signature $(2k+2, 2\ell)$ if $f_1 > 0$ and $(2k, 2\ell+2)$ if $f_1 < 0$, where $(2k, 2\ell)$ is the signature of the metric g.

The Swann bundle revisited

We derive explicit formulas relating the metric of a quaternionic Kähler manifold to the pseudo-hyper-Kähler metric of its Swann bundle. This will be used to obtain an explicit formula for the quaternionic Kähler metric in the HK/QK correspondence from the conical pseudo-hyper-Kähler metric

3-Sasaki bundle S of a quaternionic Kähler manifold

Let (M, g, Q) be a (possibly indefinite) quaternionic Kähler manifold of nonzero scalar curvature and $\pi : S \to M$ the 3-Sasaki bundle , i.e. principal $\mathrm{SO}(3)$- bundle 3 of frames (J_1, J_2, J_3) in Q s.t. $J_3 = J_1 J_2$ and $J_\alpha^2 = -\mathrm{Id}$, $\alpha = 1, 2, 3$ and $A \in \mathrm{SO}(3)$ acts by

$$s = (J_1, J_2, J_3) \mapsto \tau(A, s) := (J_1, J_2, J_3)A^{-1}.$$

Let $(e_\alpha) \in \mathfrak{so}(3)$ be the standard basis and Z_α the fundamental vector fields s.t.

$$[e_\alpha, e_\beta] = 2e_\gamma, \quad [Z_\alpha, Z_\beta] = -2Z_\gamma. \tag{5}$$

The Levi-Civita connection ∇ of (M, g) induces a principal connection

$$\theta = \sum \theta_\alpha e_\alpha : TS \to \mathfrak{so}(3)$$

on S with curvature $\Omega := d\theta - \frac{1}{2}[\theta \wedge \theta]$, where $\frac{1}{2}[\theta \wedge \theta](X, Y) := [\theta(X), \theta(Y)], \quad X, Y \in T_sS, \quad s \in S.$

3-Sasaki bundle S of a quaternionic Kähler manifold

A local section $\sigma = (J_1, J_2, J_3) \in \Gamma(\pi)$, defines a vector-valued 1-form on M

$$\bar{\theta} = \sigma^* \theta = \sum \bar{\theta}_\alpha e_\alpha$$

s.t. $\nabla J_\alpha = 2(\bar{\theta}_\beta \otimes J_\gamma - \bar{\theta}_\gamma \otimes J_\beta)$.
The curvature $R^Q \in \Gamma(\wedge^2 T^* M \otimes Q)$ of θ is

$$R^Q = \sum \bar{\Omega}_\alpha J_\alpha, \quad \bar{\Omega}_\alpha = -d\bar{\theta}_\alpha + 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma.$$

We have $\bar{\Omega}_\alpha = -\frac{\nu}{2}\omega_\alpha$, where $\omega_\alpha = gJ_\alpha$ and
$\nu := \frac{scal}{4n(n+2)}$ $(\dim M = 4n)$ is the reduced scalar curvature.

Swann cone $\hat{M}$

We endow the manifold S with the pseudo-Riemannian metric

$$g_S = \sum \theta_\alpha^2 + \frac{\nu}{4}\pi^* g.$$

Consider the Swann cone $(\hat{M} = \mathbb{R}^+ \times S, \hat{g} = dr^2 + r^2 g_S)$ with the Euler field $\xi = Z_0 = r\partial_r$ and three exact forms

$$\omega_\alpha := d\hat{\theta}_\alpha,\ \hat{\theta}_\alpha := \frac{r^2}{2}\theta_\alpha.$$

Proposition The Swann cone $\hat{M}$ is a (pseudo)hyper-Kähler manifold with the Kähler forms $\hat{\omega}_\alpha$. The signature is $(4+4k, 4l)$ if $\nu > 0$ and $(4+4l, 4k)$ if $\nu < 0$, where $(4k, 4l)$ is the signature of the quaternionic Kähler metric g on M.

The moment map of an infinitesimal automorphism

Let $\hat{M}$ be the Swann cone of M.
Let X be a tri-holomorphic space-like or time-like Killing vector field on $\hat{M}$, which commutes with the Euler vector field $\xi = r\partial_r = Z_0$.

Proposition The vector field X is tri-Hamiltonian with moment map $-\mu$, where

$$\mu : \hat{M} \to \mathbb{R}^3, \quad x \mapsto (\mu_1(x), \mu_2(x), \mu_3(x)),$$

$\mu_\alpha := \hat{\theta}_\alpha(X),\ d\mu_\alpha = -\iota_X \hat{\omega}_\alpha.$

Data on the level set P of the moment map

Now we recover the quaternionic Kähler metric on M from the geometric data on the level set, see A.Haydys

$$P = \{\mu_1 = 1,\ \mu_2 = \mu_3 = 0\} \subset \hat{M}.$$

of the moment map μ.
The group $\mathbb{R}^+ \times \mathrm{SO}(3)$ generated by ξ, Z_1, Z_2, Z_3 acts as conformal linear group $\mathrm{CO}(3)$ on $\mathbb{R}^3 = \mathrm{span}(\mu_\alpha)$. the three-dimensional vector space spanned by the functions μ_α, in particular,

$$\mathcal{L}_{Z_0}\mu_\alpha = 2\mu_\alpha, \quad \mathcal{L}_{Z_\alpha}\mu_\beta = -2\mu_\gamma,$$

Data on P

This implies

$$\hat{M} = \bigcup_{a \in \mathbb{R}^+ \times \mathrm{SO}(3)} aP.$$

and $P \subset \hat{M}$ is a smooth submanifold of codimension 3. On P we have the following data:

$$\begin{aligned}
g_P &:= g_{\hat{M}}|_P = \hat{g}|_P \in \Gamma(S^2 T^* P) \\
\theta_\alpha^P &:= \hat{\theta}_\alpha|_P \in \Omega^1(P)\ (\alpha = 1, 2, 3) \\
f &:= \left.\frac{r^2}{2}\right|_P \in C^\infty(P) \\
\theta_0^P &:= -\frac{1}{2} df \in \Omega^1(P) \\
X_P &:= X|_P \in \mathfrak{X}(P) \\
Z_1^P &:= Z_1|_P \in \mathfrak{X}(P).
\end{aligned}$$

Relation between P and M

The quaternionic Kähler metric g on M is related as follows to the geometric data on the level set $P \subset \hat{M}$ of the moment map:

$$\nu\pi^* g|_P = \frac{2}{f}\left(g_P - \frac{2}{f}\sum_{a=0}^{3}({\theta_a}^P)^2\right). \tag{6}$$

The above symmetric form is Z_1^P-invariant and has 1-dimensional kernel $\mathbb{R}Z_1^P$.

Theorem Let M' be a hypersurface in P transversal to the vector field Z_1^P. Then

$$g' := \frac{1}{2|f|}\tilde{g}_P|_{M'}$$

is a possibly indefinite quaternionic Kähler metric on M'.

Generalization:Conification of hyper-Kähler manifolds

Data: $(M, g, J_\alpha, \omega_\alpha := gJ_\alpha \alpha = 1, 2, 3)$ a pseudo-hyper-Kähler manifold;
Z Killing vector field, g(Z,Z)$\neq 0$,ω_1-hamiltonian s.t. $df = -\omega_1 Z$ where f, $f_1 := f - \frac{g(Z,Z)}{2}$ are non-vanishing functions;
Consider the conification $\hat{M}_1 := P \times \mathbb{R}$ of the pseudo-Kähler manifold (M, g, J_1, ω_1) endowed with the ω_1-Hamiltonian Killing vector field Z. (Here $\pi : P \to M$ is the circle bundle with the connection η and the pseudo-Riemannian metric g_P.)
Recall that $\hat{M}_1 := P \times \mathbb{R}$ is endowed with the structure $(\hat{g}, \hat{J}_1, \xi)$ of a conical pseudo-Kähler manifold.

Conification of hyper-Kähler manifolds

We construct a conical hyper-Kähler manifold $(\hat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ such that $\hat{M}_1 \subset \hat{M}$ with the conical pseudo-Kähler structure induced by $(\hat{g}, \hat{J}_1, \xi)$.
We define the vector field

$$Z_1 := Z + f_1 X_P$$

and the one-forms

$$\begin{aligned} \theta_1^P &:= \eta + \frac{1}{2} gZ \\ \theta_2^P &:= \frac{1}{2}\omega_3 Z \\ \theta_3^P &:= -\frac{1}{2}\omega_2 Z \end{aligned} \tag{7}$$

on P.

Conification of hyper-Kähler manifolds

We consider $\theta_\alpha := f^{-1}\theta^P_\alpha$ as the components of a one-form $\theta := \sum_\alpha \theta_\alpha i_\alpha$ with values in $\mathrm{im}\mathbb{H}$ and extend θ to a one-form $\tilde{\theta}$ on $\tilde{M} := \mathbb{H}^* \times P \supset \{1\} \times P \cong P$ by

$$\tilde{\theta}_\alpha(q,p) := \varphi_\alpha(q) + (\mathrm{Ad}_q\theta(p))_\alpha, \quad (q,p) \in \tilde{M},$$

where $\varphi = \varphi_0 + \sum_\alpha \varphi_\alpha i_\alpha$ is the right-invariant Maurer-Cartan form of $\mathbb{H}^*$ and $\mathrm{Ad}_q x = qxq^{-1} = x_0 + \sum_\alpha (\mathrm{Ad}_q x)_\alpha i_\alpha$ for all $x = x_0 + \sum x_\alpha i_\alpha \in \mathbb{H}$. Notice that

$$\varphi_a(e_b) = \partial_{ab},$$

where $(e_0, \ldots, e_3)$, is the right-invariant frame of $\mathbb{H}^*$ which coincides with the standard basis of $\mathbb{H} = \mathrm{Lie}(\mathbb{H}^*)$ at $q = 1$.

We define

$$\tilde{\omega}_\alpha := d(\rho^2 \tilde{\theta}^P_\alpha),$$

where $\tilde{\theta}^P_\alpha := f\tilde{\theta}_\alpha$ and $\rho := |q|$. Let us denote by e^L_1 the left-invariant vector field on $\mathbb{H}^*$ which coincides with e_1 at $q = 1$ and by $\hat{M}$ the space of integral curves of the vector field $V_1 := e^L_1 - Z_1$. We will assume that the quotient map $\tilde{\pi} : \tilde{M} \to \hat{M}$ is a submersion onto a Hausdorff manifold. (Locally this is always the case, since the vector field has no zeroes.)

Theorem

Let (M, g, J_1, J_2, J_3, Z) be a pseudo-hyper-Kähler manifold endowed with a Killing vector field Z satisfying the above assumptions and $\mathcal{L}_Z J_2 = -2J_3$. Then there exists a pseudo-hyper-Kähler structure $(\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3)$ on $\hat{M}$ with exact Kähler forms $\hat{\omega}_\alpha$ determined by

$$\tilde{\pi}^* \hat{\omega}_\alpha = \tilde{\omega}_\alpha.$$

The vector field $r\partial_r$ on $\tilde{M}$ projects to a vector field ξ on $\hat{M}$ such that

$$(\hat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$$

is a conical hyper-Kähler manifold.
The signature of the metric $\hat{g}$ is $(4k, 4\ell + 4)$ if $f_1 < 0$ and $(4k + 4, 4\ell)$ if $f_1 > 0$, where $(4k, 4\ell)$ is the signature of the metric g.

HK/QK correspondence

The correspondence between hyper-Kähler manifold (M, g, J_α) with the data (Z, f) and quaternionic Kähler manifold $\bar{M} = \hat{M}/(\mathbb{H}^*/(\pm 1))$ associated with the hyper-Káhler cone $\hat{M}$ is called HK/QK correspondence.
If Z_1^P generates a free and proper action of a one-dimensional Lie group A ($\cong S^1$ or $\mathbb{R}$) and if M' is a global section for the A-action, then $M' \simeq P/A$, which inherits the quaternionic Kähler metric g'.

Application : Description of supergravity c-map in terms of rigid c-map and conification

We apply the above construction to the hyper-Kähler manifold $N = c(M) = T^*M$ which the image of a conical special Kähler manifold M under rigid c-map. We will show that the supergravity c-map which associates with a projective special Kähler manifold $\bar{M} = M/\mathbb{C}^*$ a quaternionic Kähler manifold can be obtained as a composition of the rigid c-map $M \to N = T^*M$, conification $N \to \hat{N}$ and Swann projection $\hat{N}/(\mathbb{H}^*/(\pm 1))$.

Conical and projective affine special Kähler manifold

A *conical affine special Kähler manifold* (M, J, g_M, ∇, ξ) is a pseudo-Kähler manifold (M, J, g_M) endowed with a flat torsionfree connection ∇ and a vector field ξ such that

i) $\nabla\omega_M = 0$, where $\omega_M := g_M(J\cdot, \cdot)$ is the Kähler form,
ii) $(\nabla_X J)Y = (\nabla_Y J)X$ for all $X, Y \in \Gamma(TM)$,
iii) $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi-Civita connection,
iv) g_M is positive definite on $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ and negative definite on $\mathcal{D}^\perp$.

Projective special Kähler manifold

Then ξ and $J\xi$ are commuting holomorphic vector fields that are homothetic and Killing respectively. We assume that the holomorphic Killing vector field $J\xi$ induces a free S^1-action and that the holomorphic homothety ξ induces a free $\mathbb{R}^+$-action on M. Then (M, g_M) is a metric cone over (S, g_S), where

$$S := \{p \in M | g_M(\xi(p), \xi(p)) = 1\}, \; g_S := g_M|_S$$

and $-g_S$ induces a Riemannian metric $g_{\bar{M}}$ on $\bar{M} := S/S^1_{J\xi}$. $(\bar{M}, -g_{\bar{M}})$ is obtained from (M, J, g) via a Kähler reduction with respect to $J\xi$ and, hence, $g_{\bar{M}}$ is a Kähler metric. The corresponding Kähler form $\omega_{\bar{M}}$ is obtained from ω_M by symplectic reduction. This determines the complex structure $J_{\bar{M}}$.

More precisely, S is a (Lorentzian) Sasakian manifold in term of the radial coordinate $r := \sqrt{g(\xi, \xi)}$, we have

$$g_M = dr^2 + r^2 \pi^* g_S, \quad g_S = g_M|_S = \tilde{\eta} \otimes \tilde{\eta}|_S - \bar{\pi}^* g_{\bar{M}}, \qquad (8)$$

where

$$\tilde{\eta} := \frac{1}{r^2} g_M(J\xi, \cdot) = d^c \log r = i(\overline{\partial} - \partial) \log r \qquad (9)$$

is the contact one-form form when restricted to S and $\pi : M \to S = M/\mathbb{R}^{>0}_{\xi}$, $\bar{\pi} : S \to \bar{M} = S/S^1_{J\xi}$ are the canonical projection maps. We will drop π^* and $\bar{\pi}^*$ and identify, e.g., $g_{\bar{M}}$ with a $(0,2)$ tensor field on M that has the distribution $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ as its kernel.

Special holomorphic coordinates and the prepotential

Locally, there exist so-called **conical special holomorphic coordinates** $z = (z^I) = (z^0, \ldots, z^n) : U \overset{\sim}{\to} \tilde{U} \subset \mathbb{C}^{n+1}$ such that the geometric data in U is encoded in a holomorphic homogenepous of degree 2 function $F : \tilde{U} \to \mathbb{C}$ (prepotential). Namely, we have locally

$$g_M = \sum_{I,J} N_{IJ} dz^I d\bar{z}^J, \quad N_{IJ}(z, \bar{z}) := 2\mathrm{Im}\, F_{IJ}(z) := 2\mathrm{Im}\, \frac{\partial^2 F(z)}{\partial z^I \partial z^J}$$

$$(I, J = 0, \ldots, n),\ \xi|_U = \sum z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I}.$$

The Kähler potential for g_M is given by
$r^2|_U = g_M(\xi, \xi) = \sum z^I N_{IJ} \bar{z}^J$.
The $\mathbb{C}^*$-invariant functions $X^\mu := \frac{z^\mu}{z^0}$, $\mu = 1, \ldots, n$, define a local holomorphic coordinate system on $\bar{M}$. The Kähler potential for $g_{\bar{M}}$ is $\mathcal{K} := -\log \sum_{I,J=0}^n X^I N_{IJ}(X) \bar{X}^J$, where $X := (X^0, \ldots, X^n)$ with $X^0 := 1$. }

The rigid c-map

Now, we introduce the **rigid c-map**, which assigns to each affine special (pseudo-)Kähler manifold (M, J, g_M, ∇) and in particular to any conical affine special Kähler manifold (M, J, g_M, ∇, ξ) of real dimension $2n+2$ a (pseudo-)hyper-Kähler manifold $(N = T^*M, g_N,\ J_1, J_2, J_3)$ of dimension $4n+4$ given, in terms of the decomposition
$TpN = T_p^v N \oplus T_p^h N \simeq T_x M \oplus T_x^* M$, by

$$g_N = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix},\ J_1 = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix},\ J_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},\ J_3 = J_1 J_2.$$

Special real coordinates

Locally we may assume that $(M \subset \mathbb{C}^{n+1}, J = J_{can}, g_M, \nabla, \xi)$ is a conical (i.e. $\mathbb{C}^*$-invariant) affine special Kähler manifold that is globally described by a homogeneous holomorphic function $F = F(z^0, \cdots, z^n) : \mathbb{C}^{n+1} \supset M \to \mathbb{C}$ of degree 2.
The real coordinates

$$(q^a)_{a=1,\ldots,2n+2} := (x^I, y_J) := (\mathrm{Re}\, z^I,\, \mathrm{Re}\, F_J(z) := \mathrm{Re}\, \frac{\partial F(z)}{\partial z^J})$$

on M are ∇-affine and fulfil $\omega_M = -2 \sum dx^I \wedge dy_I$, where $\omega_M = g(J\cdot, \cdot)$ is the Kähler form on M. On the cotangent bundle $\pi_N : N := T^*M \to M$ the real functions $(p_a) := (\bar{\zeta}_I, \zeta^J)$ together with q^a), form a system of canonical coordinates.

HyperKähler metric of $N = T^*M$ in canonical coordinates

Proposition

In the above coordinates (z^I, p_a), the hyper-Kähler structure on $N = T^*M$ obtained from the rigid c-map is given by

$$g_N = \sum dz^I N_{IJ} d\bar{z}^J + \sum A_I N^{IJ} \bar{A}_J, \tag{10}$$

$$\omega_1 = \frac{i}{2} \sum N_{IJ} dz^I \wedge d\bar{z}^J + \frac{i}{2} \sum N^{IJ} A_I \wedge \bar{A}_J, \tag{11}$$

$$\omega_2 = -\frac{i}{2} \sum (d\bar{z}^I \wedge \bar{A}_I - dz^I \wedge A_I), \tag{12}$$

$$\omega_3 = \frac{1}{2} \sum (dz^I \wedge A_I + d\bar{z}^I \wedge \bar{A}_I), \tag{13}$$

where $A_I := d\bar{\zeta}_I + \sum_J F_{IJ}(z) d\zeta^J$ $(I = 0, \ldots, n)$ are complex-valued one-forms on N and $\omega_\alpha = g_N(J_\alpha \cdot, \cdot)$.

The supergravity c-map and Ferrara-Subharwal metric

Let $(\bar{M}, g_{\bar{M}})$ be a $2n$ -dimensional projective special Kähler manifold defined by a holomorphic prepotential F. The **supergravity c-map** associates with $(\bar{M}, g_{\bar{M}})$ a quaternionic Kähler manifold $(\bar{N} = \bar{M} \times \mathbb{R}^+ \times \mathbb{R}^{2n+3}, g_{\bar{N}})$ of dimension $4n+4$ with the following Ferrara-Subharwal metric

$$
\begin{aligned}
g_{\bar{N}} &= g_{\bar{M}} + g_G , \\
g_G &= \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2}(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I))^2 \\
&\quad + \frac{1}{2\rho} \sum \mathcal{I}_{IJ}(m) d\zeta^I d\zeta^J \\
&\quad + \frac{1}{2\rho} \sum \mathcal{I}^{IJ}(m)(d\tilde{\zeta}_I \mathcal{R}_{IK}(m) d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL}(m) d\zeta^L).
\end{aligned}
$$

Here $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)$, $I = 0, 1, \ldots, n$, are standard coordinates on $\mathbb{R}^+ \times \mathbb{R}^{2n+3}$.

Ferrara-Subharwal metric

The real-valued matrices $\mathcal{I}(m) := (\mathcal{I}_{IJ}(m))$ and $\mathcal{R}(m) := (\mathcal{R}_{IJ}(m))$ depend only on $m \in \bar{M}$ and $\mathcal{I}(m)$ and is defined in terms of the prepotential F:

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i\frac{\sum_K N_{IK} z^K \sum_L N_{JL} z^L}{\sum_{IJ} N_{IJ} z^I z^J}, \quad N_{IJ} := 2\mathrm{Im} F_{IJ}. \tag{14}$$

HK/QK correspondence for the c-map

Let $(M \subset \mathbb{C}^{n+1}, J = J_{can}, g_M, \nabla, \xi)$ be a conical affine special Kähler manifold that is globally described by a homogeneous holomorphic function F of degree 2 in standard holomorphic coordinates $z = (z^I) = (z^0, \ldots, z^n)$. Then we can apply the HK/QK correspondence to the hyper-Kähler manifold $(N = T^*M, g_N, J_1, J_2, J_3)$ of signature $(4, 4n)$ obtained from the rigid c-map. The vector field $Z := 2(J\xi)^h = 2J_1\xi^h$ on N fulfils the assumptions of the HK/QK correspondence, i.e. it is a space-like ω_1-Hamiltonian Killing vector field with $\mathcal{L}_Z J_2 = -2J_3$. Here, $X^h \in \Gamma(TN)$ is defined for any vector field $X \in \Gamma(TM)$ by $X^h(\pi_N^* q^a) = \pi_N^* X(q^a)$ and $X^h(p_a) = 0$ for all $a = 1, \ldots, 2n+2$. (X^h is the horizontal lift with respect to the flat connection ∇).

Applying the HK/QK correspondence to (N, g_N, J_1, J_2, J_3) endowed with the ω_1-Hamiltonian Killing vector field Z gives (up to a constant conventional factor) the one-parameter family g^c_{FS} of quaternionic pseudo-Kähler metrics, which includes the Ferrara-Sabharwal metric g_{FS}. The metric g^c_{FS} is positive definite and of negative scalar curvature on the domain $\{\rho > -2c\} \subset \bar{N}$ (which coincides with $\bar{N}$ if $c \geq 0$). If $c < 0$ the metric g^c_{FS} is of signature $(4n, 4)$ on the domain $\{-c < \rho < -2c\} \subset \bar{N}$. Furthermore, if $c > 0$ the metric g^c_{FS} is of signature $(4, 4n)$ on the domain $\bar{M} \times \{-c < \rho < 0\} \times \mathbb{R}^{2n+3} \subset \bar{M} \times \mathbb{R}^{<0} \times \mathbb{R}^{2n+3}$.