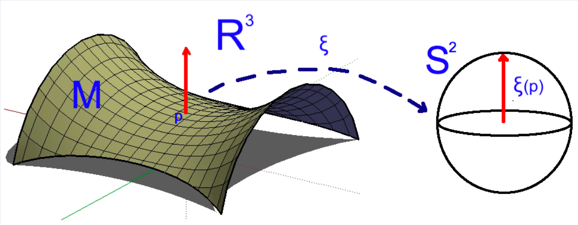
The geometry of constant mean curvature surfaces embedded in $\mathbb{R}^3$.

(joint work with Meeks)

Giuseppe Tinaglia
King's College London

Outline:

- Introduction to the theory of constant mean curvature (CMC) surfaces.
- Historical perspective
- Main results.
- Future directions.



Let **M** be an oriented surface in $\mathbf{R}^3$, let ξ be the unit vector field normal to **M**:

$$\mathbf{A} = -d\xi\colon\ T_p\mathbf{M} \to T_{\xi(p)}\mathbf{S}^2 \simeq T_p\mathbf{M}$$

is the **shape operator** of **M** (**second fundamental form**).

Introduction to the theory of CMC surfaces

Definition

- The eigenvalues k_1, k_2 of $\mathbf{A}_p$ are the **principal curvatures** of $\mathbf{M}$ at p.
- $\mathbf{H} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) = \frac{k_1+k_2}{2}$ is the **mean curvature**.
- $|\mathbf{A}| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the second fundamental form.**

Gauss equation

$$4\mathbf{H}^2 = |\mathbf{A}|^2 + 2\mathbf{K_G} \quad (\mathbf{K_G} = \text{Gaussian curvature})$$

Introduction to the theory of CMC surfaces

First Variation Formula

$$\mathbf{M}_t = \{p + t\phi(p)\xi(p) \mid p \in \mathbf{M}\}, \quad \phi \in C_0^\infty(\mathbf{M})$$

$$\frac{d}{dt}\text{Area}(\mathbf{M}_t)\Big|_{t=0} = -2\int_{\mathbf{M}} \mathbf{H}\phi$$

Definition

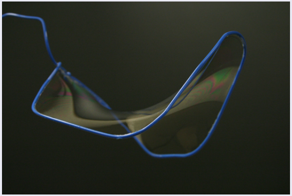
M is a **minimal surface** $\iff$ **M** is a critical point for the area functional $\iff$ $\mathbf{H} \equiv 0$.

Definition

M is a **CMC surface** $\iff$ **M** is a critical point for the area functional under variations **preserving the volume**, $\int_{\mathbf{M}} \phi = 0$ $\iff$ $\mathbf{H} \equiv$ constant.

CMC surfaces in nature

Soap films are minimal surfaces



Soap bubbles are nonzero CMC surfaces

Example (Graph of a function)

- $\mathbf{H} = \frac{1}{2}\mathrm{div}\frac{\nabla \mathbf{u}}{\sqrt{1+|\nabla \mathbf{u}|^2}}$ Quasi-linear elliptic PDE
- $\frac{|Hess(u)|^2}{(1+|\nabla u|^2)^2} \leq |\mathbf{A}|^2 \leq 2\frac{|Hess(u)|^2}{1+|\nabla u|^2}$

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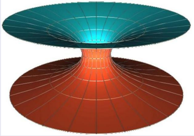
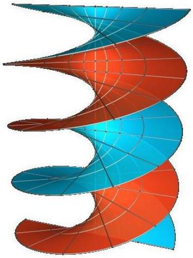


- Catenoid



- Helicoid

Introduction to the theory of CMC surfaces

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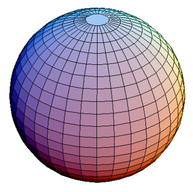
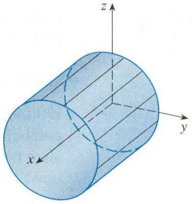
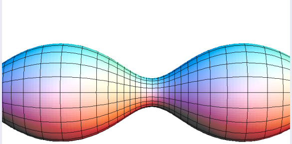


- Sphere



- Cylinder



- Delaunay surfaces

Historical perspective

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- If **M** is **embedded**, then it is a round sphere (1956, **Alexandrov**).

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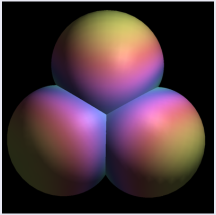
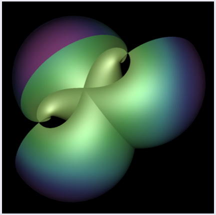
- ...
- If **M** is **stable**, then it is a round sphere (1983, **Barbosa-Do Carmo**).

Historical perspective

- Existence of immersed CMC Tori (1984, **Wente**).





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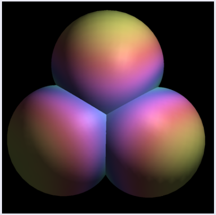
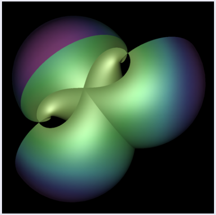
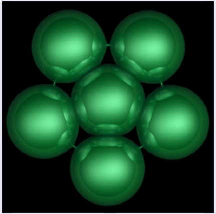




- Many examples of closed CMC surfaces (1994, **Kapouleas**; **Mazzeo-Pacard**, **Mazzeo-Pacard-Pollack**, et al.)

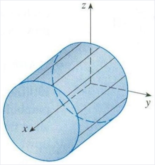


New uniqueness results for CMC surfaces

Question

Is the round sphere the only complete simply connected surface **embedded** in $\mathbf{R}^3$ with nonzero constant mean curvature?

NOT simply connected



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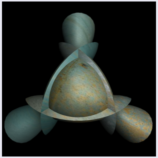

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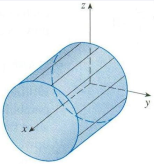

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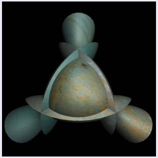

- Smyth surface

Answer (Meeks-T.)

Yes.

New uniqueness results for CMC surfaces

Theorem (Meeks-T.)

Round spheres are the only complete simply connected surfaces **embedded** in $\mathbb{R}^3$ with nonzero constant mean curvature.

1997, **Meeks** for **properly embedded**.

New uniqueness results for CMC surfaces

Theorem (Meeks-T.)

Round spheres are the only complete simply connected surfaces **embedded** in $\mathbf{R}^3$ with nonzero constant mean curvature.

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Let $\mathbf{M}$ be a complete and simply-connected CMC surface **embedded** in $\mathbf{R}^3$, then it is either

a plane, a helicoid or a round sphere.

(2008, **Colding-Minicozzi** and **Meeks-Rosenberg** for $\mathbf{H} = 0$)

Main Results

Definition

A **1**-disk is a simply-connected surface (possibly with boundary) **embedded** in $\mathbf{R}^3$ with constant mean curvature **1**.

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A $\mathbf{1}$-disk is a simply-connected surface (possibly with boundary) **embedded** in $\mathbf{R^3}$ with constant mean curvature $\mathbf{1}$.

Radius Estimate

There exists a universal constant $\mathbf{R}$ such that:
If $\mathbf{M}$ is a $\mathbf{1}$-disk, then $\mathbf{M}$ has radius less than $\mathbf{R}$,
$\text{dist}_{\mathbf{M}}(p, \partial\mathbf{M}) < \mathbf{R}$.

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There exists a universal constant **R** such that:
If **M** is a **1**-disk, then **M** has radius less than **R**, $\text{dist}_{\mathbf{M}}(p, \partial\mathbf{M}) < \mathbf{R}$.

In particular, if **M** is a complete **1**-disk then
Radius Estimate $\Longrightarrow$ **M** is compact $\Longrightarrow$ **M** is an embedded sphere $\Longrightarrow$ **M** is a round sphere.

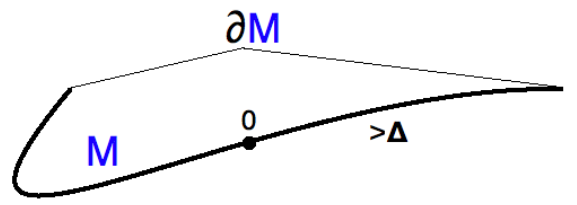
Main Results

The Radius Estimate is a non-trivial consequence of the following Intrinsic Curvature Estimate.

Intrinsic Curvature Estimate

Given $\mathbf{\Delta} > \mathbf{0}$ there exists $\mathbf{C} = \mathbf{C}(\mathbf{\Delta})$ such that:
If $\mathbf{M}$ is a $\mathbf{1}$-disk with $0 \in \mathbf{M}$ and $\text{dist}_{\mathbf{M}}(0, \partial \mathbf{M}) > \mathbf{\Delta}$, then

$$|\mathbf{A}|(0) \leq \mathbf{C}.$$



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- In general, a neighborhood of a point $p \in \mathbf{M}$ is always a graph over $T_p\mathbf{M}$. However, the size of such neighborhood depends on p.

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What does a uniform bound on $|\mathbf{A}|$ imply?

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- If $\sup_{\mathbf{M}} |\mathbf{A}| = \sup_{\mathbf{M}} |d\xi| \leq \mathbf{C}$ then the size of such neighborhood only depends on $\mathbf{C}$ and NOT on p.

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- If $\sup_{\mathbf{M}} |\mathbf{A}| = \sup_{\mathbf{M}} |d\xi| \leq \mathbf{C}$ then the size of such neighborhood only depends on $\mathbf{C}$ and NOT on p.
- Let the surface be CMC and $\mathbf{u}$ be such graph then
  - $\|\mathbf{u}\|_{C^2} \leq 10\mathbf{C}$
  - $\mathrm{div} \frac{\nabla \mathbf{u}}{\sqrt{1+|\nabla \mathbf{u}|^2}} = 2H$

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  - $\operatorname{div} \frac{\nabla \mathbf{u}}{\sqrt{1+|\nabla \mathbf{u}|^2}} = 2H$
  - then, $\|\mathbf{u}\|_{C^{2,\alpha}}$ is uniformly bounded independently of p.

The Intrinsic Curvature Estimate

Intrinsic Curvature Estimate

Given $\boldsymbol{\Delta}$ there exists $\mathbf{C} = \mathbf{C}(\boldsymbol{\Delta})$ such that:
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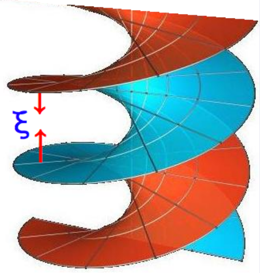
Note

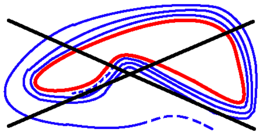
The local estimate on $|\mathbf{A}|$ fails in the minimal case; counterexamples being rescaled helicoids.



A global result

Theorem (Meeks-T.)

Let **M** be a complete, nonzero CMC surface **embedded** in $\mathbb{R}^3$ with finite topology. Then:



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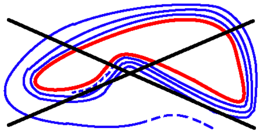

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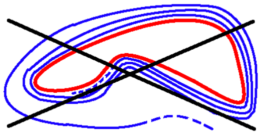


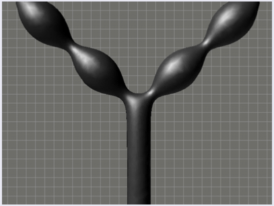
- **M** has bounded curvature and is **properly embedded**.
- **M** has more than one end or it is a round sphere.

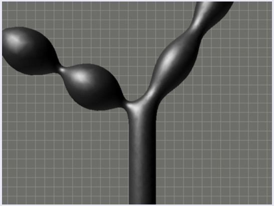
- If **M** has exactly two ends then it is a Delaunay surface.
- If **M** has more than one end then each end is asymptotic to a Delaunay surface.

For **properly embedded**: 1997, **Meeks**; 1998, **Korevaar-Kusner-Solomon**.

Examples of finite topology nonzero CMC surfaces

Genus 0 and 3 ends.





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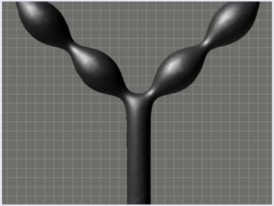
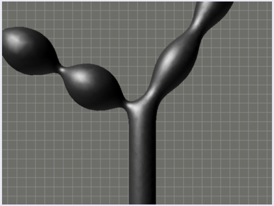
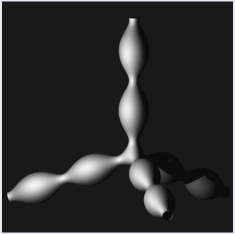
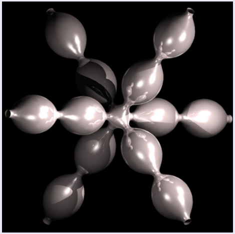



Genus 0 and 4, 6 ends.

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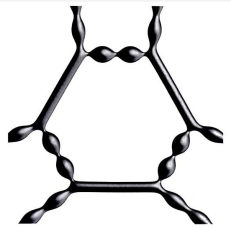

The proof of the Intrinsic Curvature Estimate

Intrinsic Curvature Estimate

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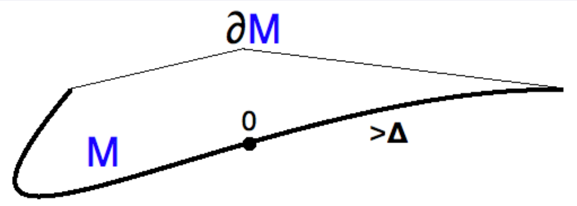
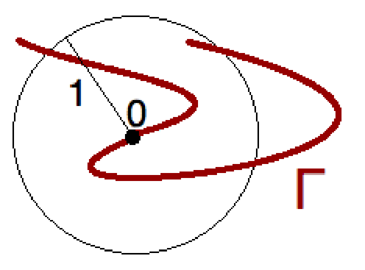


The proof of the Intrinsic Curvature Estimate

Step 1: Cord-arc Bound (**Colding-Minicozzi** for $\mathbf{H} = \mathbf{0}$)

There exists a universal constant $\mathbf{\Omega}$ such that:
If $\mathbf{M}$ is a $\mathbf{1}$-disk with $0 \in \mathbf{M}$, $\text{dist}_{\mathbf{M}}(0, \partial \mathbf{M}) > \mathbf{r\Omega}$, $\mathbf{r} > \mathbf{0}$, and $\mathbf{\Gamma}$ is a geodesic starting at the origin with length $> \mathbf{r\Omega}$, then

$$\mathbf{\Gamma} \cap \partial \mathbf{B}(\mathbf{r}) \neq \emptyset.$$



The proof of the Intrinsic Curvature Estimate

Step 2: Extrinsic Curvature Estimate

Given $\mathbf{\Lambda} > \mathbf{0}$ there exists a constant $\mathbf{C} = \mathbf{C}(\mathbf{\Lambda})$ such that:
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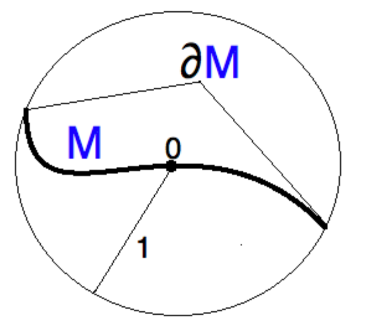
$$|\mathbf{A}|(0) \leq \mathbf{C}.$$



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$$|\mathbf{A}|(0) \leq \mathbf{C}.$$

Proof

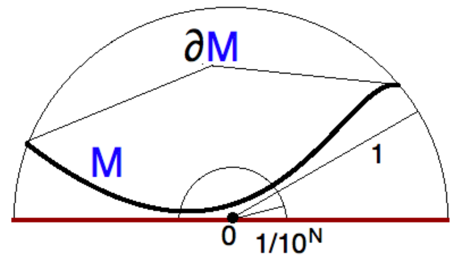
- Cord-arc Bound says that the connected component of $\mathbf{M} \cap \mathbf{B}(\frac{\mathbf{\Delta}}{\mathbf{\Omega}})$ containing the origin is a $\mathbf{1}$-disk with boundary in $\partial \mathbf{B}(\frac{\mathbf{\Delta}}{\mathbf{\Omega}})$.
- Apply the Extrinsic Curvature Estimate to such $\mathbf{1}$-disk.

The proof of the Chord-arc Bound

Key ingredient: One-sided Curvature Estimate (Colding-Minicozzi for $\mathbf{H} = \mathbf{0}$)

There exist universal constants $\mathbf{K}$ and $\mathbf{N}$ such that:
If $\mathbf{M}$ is an $\mathbf{H}$-disk with $|\mathbf{H}| \leq 1$, $\partial \mathbf{M} \subset \partial \mathbf{B}(1)$ and $\mathbf{M} \subset \{x_3 > 0\}$, then

$$\sup_{\mathbf{M} \cap \mathbf{B}(\frac{1}{10^N})} |\mathbf{A}| \leq \mathbf{K}.$$



Main Results

- Characterisation of the round sphere

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$\Uparrow$

- Radius Estimate

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- Intrinsic Curvature Estimate
  - Chord-arc Bound
    - One-sided Curvature Estimate
  - Extrinsic Curvature Estimate

Future Directions

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What can be said about the geometry of a complete nonzero CMC surface **M** embedded in a complete 3-manifolds **N**?

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M has positive injectivity radius + **N** has bounded sectional curvatures $\Longrightarrow$ curvature estimates $\Longrightarrow$ proper when the scalar curvature$> \varepsilon > 0$

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Question

M has finite topology + **N** is homogeneous $\Longrightarrow$
M has locally bounded second fundamental form

Future Directions

Question

Let $\mathbf{M}$ be a complete nonzero CMC surface embedded in $\mathbf{H}^3$ with $\mathbf{H} \geq 1$ and finite topology. Then:

- $\mathbf{M}$ has bounded curvature and is properly embedded.

- If $\mathbf{H} = 1$, then each annular end of $\mathbf{M}$ is asymptotic to a horosphere or a catenoid. Furthermore, if $\mathbf{M}$ has one end, then $\mathbf{M}$ is a horosphere.
- If $\mathbf{H} > 1$, then each annular end of $\mathbf{M}$ is asymptotic to the end of a Hsiang surface.

For **properly embedded + H=1**: 2001, **Collin-Hauswirth-Rosenberg**.
For **properly embedded + H>1**: 1992, **Korevaar-Kusner-Meeks-Solomon**.

Future Directions

Question

What can be said about the geometry of a surface $\mathbf{M}$ embedded in $\mathbf{R}^3$ with bounded mean curvature in $\mathbf{L}^p$?

Future Directions

Theorem (Bourni-T.)

Let $\mathbf{M}$ be a surface embedded in $\mathbf{R}^3$ containing the origin with $\mathbf{Inj_M(0)} \geq \mathbf{s} > \mathbf{0}$,

$$\int_{\mathbf{B_M(s)}} |\mathbf{A}|^2 \leq \mathbf{C_1}$$

and either

i. $\|\mathbf{H}\|^*_{\mathbf{W^{2,2}(B_M(s))}} \leq \mathbf{\Lambda_2(C_1)}$, if $\mathbf{p} = \mathbf{2}$ or

ii. $\|\mathbf{H}\|^*_{\mathbf{W^{1,p}(B_M(s))}} \leq \mathbf{\Lambda_p(C_1)}$, if $\mathbf{p} > \mathbf{2}$,

then

$$|\mathbf{A}|^2(\mathbf{0}) \leq \mathbf{C_2(p, C_1)s^{-2}}.$$

For $\mathbf{H=0}$: 2004, **Colding-Minicozzi**.

The proof of the Radius Estimate

Radius Estimate

There exists a universal constant $\mathbf{R}$ such that:
If $\mathbf{M}$ is a $\mathbf{1}$-disk, then $\mathbf{M}$ has radius less than $\mathbf{R}$.

The proof of the Radius Estimate

Sketch of the proof

- Arguing by contradiction let $\mathbf{M_n}$ be a sequence of $\mathbf{1}$-disks with radii $>\mathbf{n}$ and $|\mathbf{A}|$ uniformly bounded.

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- Arguing by contradiction let $\mathbf{M_n}$ be a sequence of $\mathbf{1}$-disks with radii $>\mathbf{n}$ and $|\mathbf{A}|$ uniformly bounded.
- A subsequence of $\mathbf{M_n}$ converges C^2 to a complete "**embedded**" CMC$=\mathbf{1}$ surface $\mathbf{M}$ with bounded curvature and genus zero.

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- $\mathbf{M}$ is **proper**. If NOT then the universal cover of $\overline{\mathbf{M}} - \mathbf{M}$ would be a (strongly) stable, complete surface with CMC$=\mathbf{1}$ but there is none.

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- $\mathbf{M}$ contains a Delaunay surface at "infinity" (**Meeks-T.**; 1998, **Korevaar-Kusner-Solomon** for **properly embedded + finite topology**).

The proof of the Radius Estimate

Sketch of the proof

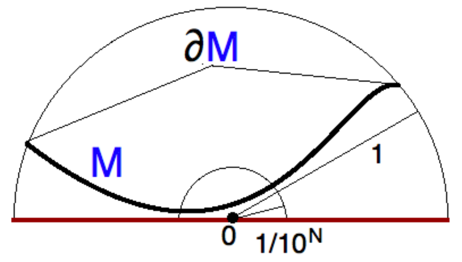
- Arguing by contradiction let $\mathbf{M_n}$ be a sequence of **1**-disks with radii $>\mathbf{n}$ and $|\mathbf{A}|$ uniformly bounded.
- A subsequence of $\mathbf{M_n}$ converges C^2 to a complete "**embedded**" CMC=**1** surface **M** with bounded curvature and genus zero.
- **M** is **proper**. If NOT then the universal cover of $\overline{\mathbf{M}} - \mathbf{M}$ would be a (strongly) stable, complete surface with CMC=**1** but there is none.
- **M** contains a Delaunay surface at "infinity" (**Meeks-T.**; 1998, **Korevaar-Kusner-Solomon** for **properly embedded + finite topology**).
- A Delaunay surface cannot be a limit of **1**-disks. Contradiction!

The proof of the One-sided Curvature Estimate

One-sided Curvature Estimate (Colding-Minicozzi for $\mathbf{H = 0}$)

There exist universal constants **K** and **N** such that:
If **M** is an **H**-disk with $|\mathbf{H}| \leq 1$, $\partial\mathbf{M} \subset \partial\mathbf{B}(1)$ and $\mathbf{M} \subset \{x_3 > 0\}$, then

$$\sup_{\mathbf{M}\cap\mathbf{B}(\frac{1}{10^N})} |\mathbf{A}| \leq \mathbf{K}.$$



The proof of the One-sided Curvature Estimate

Sketch of the proof

- Arguing by contradiction, let $p_n \in \mathbf{M}_n$ be a sequence of points converging to the origin where $|\mathbf{A}_n|$ is arbitrarily large.

The proof of the One-sided Curvature Estimate

Sketch of the proof

- Arguing by contradiction, let $p_n \in \mathbf{M}_n$ be a sequence of points converging to the origin where $|\mathbf{A}_n|$ is arbitrarily large.
- Around p_n, $\mathbf{M}_n$ looks like a vertical helicoid and thus the tangent plane at p_n is vertical.

The proof of the One-sided Curvature Estimate

Sketch of the proof

- Arguing by contradiction, let $p_n \in \mathbf{M}_n$ be a sequence of points converging to the origin where $|\mathbf{A}_n|$ is arbitrarily large.
- Around p_n, $\mathbf{M}_n$ looks like a vertical helicoid and thus the tangent plane at p_n is vertical.
- Let $\mathbf{\Gamma_n}$ be connected component of the pre-image of the equator via the Gauss map containing p_n (tangent plane is vertical).

The proof of the One-sided Curvature Estimate

Sketch of the proof

- Arguing by contradiction, let $p_n \in \mathbf{M}_n$ be a sequence of points converging to the origin where $|\mathbf{A}_n|$ is arbitrarily large.
- Around p_n, $\mathbf{M}_n$ looks like a vertical helicoid and thus the tangent plane at p_n is vertical.
- Let $\mathbf{\Gamma_n}$ be connected component of the pre-image of the equator via the Gauss map containing p_n (tangent plane is vertical).
- Around each point $p \in \mathbf{\Gamma_n}$, $\mathbf{M}_n$ looks like a vertical helicoid and thus the curve $\mathbf{\Gamma_n}$ cannot be contained in a half-space.

The proof of the One-sided Curvature Estimate

Key ingredients

- **Colding-Minicozzi** Theory.

The proof of the One-sided Curvature Estimate

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- **Colding-Minicozzi** Theory.
- Uniqueness of the helicoid (**Meeks-Rosenberg**).

The proof of the One-sided Curvature Estimate

Key ingredients

- **Colding-Minicozzi** Theory.
- Uniqueness of the helicoid (**Meeks-Rosenberg**).
- Geometry of minimal and CMC laminations (**Meeks-Perez-Ros**).

Thanks