The geometry of constant mean curvature surfaces embedded in $\mathbb{R}^3$.
(joint work with Meeks)

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Outline:

- Introduction to the theory of constant mean curvature (CMC) surfaces.
- Historical perspective
- Main results.
- Future directions.
Let $M$ be an oriented surface in $\mathbb{R}^3$, let $\xi$ be the unit vector field normal to $M$:

$$A = -d\xi : T_pM \to T_{\xi(p)}S^2 \cong T_pM$$

is the shape operator of $M$ (second fundamental form).
Introduction to the theory of CMC surfaces

Definition

- The eigenvalues $k_1, k_2$ of $A_p$ are the **principal curvatures** of $M$ at $p$.
- $H = \frac{1}{2} \text{tr}(A) = \frac{k_1 + k_2}{2}$ is the **mean curvature**.
- $|A| = \sqrt{k_1^2 + k_2^2}$ is the **norm of the second fundamental form**.

Gauss equation

$$4H^2 = |A|^2 + 2K_G \quad (K_G = \text{Gaussian curvature})$$
Introduction to the theory of CMC surfaces

First Variation Formula

\[ M_t = \{ p + t\phi(p)\xi(p) \mid p \in M \}, \quad \phi \in C_0^\infty(M) \]

\[ \frac{d}{dt} \text{Area}(M_t) \bigg|_{t=0} = -2 \int_M H\phi \]

Definition

M is a **minimal surface** \iff M is a critical point for the area functional \iff H \equiv 0.

Definition

M is a **CMC surface** \iff M is a critical point for the area functional under variations preserving the volume, \( \int_M \phi = 0 \) \iff H \equiv \text{constant}. 
Soap films are minimal surfaces

Soap bubbles are nonzero CMC surfaces
Introduction to the theory of CMC surfaces

Example (Graph of a function)

\[ H = \frac{1}{2} \text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \]

Quasi-linear elliptic PDE

\[ \frac{|\text{Hess}(u)|^2}{(1 + |\nabla u|^2)^2} \leq |A|^2 \leq 2 \frac{|\text{Hess}(u)|^2}{1 + |\nabla u|^2} \]
Definition

\( M \) is a **minimal surface** \( \iff \) \( M \) is a critical point for the area functional \( \iff H \equiv 0. \)

- Catenoid
- Helicoid
Definition

**M** is a **CMC surface** $\iff H \equiv \text{constant} \iff M$ is a critical point for the area functional under variations preserving the volume.

- **Sphere**
- **Cylinder**
- **Delaunay surfaces**
Let $M$ be a closed (compact without boundary) CMC surface in $\mathbb{R}^3$:
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- If $M$ is **embedded**, then it is a round sphere (1956, Alexandrov).
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- If $M$ is **embedded**, then it is a round sphere (1956, Alexandrov).
- . . .
- If $M$ is **stable**, then it is a round sphere (1983, Barbosa-Do Carmo).
Historical perspective


Many examples of closed CMC surfaces (1994, Kapouleas; Mazzeo-Pacard, Mazzeo-Pacard-Pollack, et al.)
Question

Is the round sphere the only complete simply connected surface \textbf{embedded} in $\mathbb{R}^3$ with nonzero constant mean curvature?

\[ \begin{tabular}{ll}
\textbf{NOT simply connected} & \textbf{NOT embedded} \\
\includegraphics[width=0.3\textwidth]{cylinder.png} & \includegraphics[width=0.3\textwidth]{smyth_surface.png} \\
\textbullet Cylinder & \textbullet Smyth surface
\end{tabular} \]

Answer (Meeks-T.): Yes.
New uniqueness results for CMC surfaces

**Question**
Is the round sphere the only complete simply connected surface embedded in $\mathbb{R}^3$ with nonzero constant mean curvature?

**Answer (Meeks-T.)**
Yes.

- **NOT simply connected**
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- **NOT embedded**
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New uniqueness results for CMC surfaces

<table>
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1997, Meeks for properly embedded.
New uniqueness results for CMC surfaces

**Theorem (Meeks-T.)**

Round spheres are the only complete simply connected surfaces embedded in $\mathbb{R}^3$ with nonzero constant mean curvature.

1997, Meeks for properly embedded.

Let $M$ be a complete and simply-connected CMC surface embedded in $\mathbb{R}^3$, then it is either

- a plane,
- a helicoid, or
- a round sphere.

(2008, Colding-Minicozzi and Meeks-Rosenberg for $H = 0$)
Main Results

Definition

A 1-disk is a simply-connected surface (possibly with boundary) embedded in $\mathbb{R}^3$ with constant mean curvature 1.
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Radius Estimate

There exists a universal constant $R$ such that:
If $M$ is a 1-disk, then $M$ has radius less than $R$, $\text{dist}_M(p, \partial M) < R$. 

In particular, if $M$ is a complete 1-disk then Radius Estimate $\Rightarrow M$ is compact $\Rightarrow M$ is an embedded sphere $\Rightarrow M$ is a round sphere.
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Radius Estimate $\implies$ $M$ is compact $\implies$ $M$ is an embedded sphere $\implies$ $M$ is a round sphere.
The Radius Estimate is a non-trivial consequence of the following Intrinsic Curvature Estimate.

**Intrinsic Curvature Estimate**

Given \( \Delta > 0 \) there exists \( C = C(\Delta) \) such that:

If \( M \) is a 1-disk with \( 0 \in M \) and \( \text{dist}_M(0, \partial M) > \Delta \), then

\[
|A|(0) \leq C.
\]
Main Results

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What does a uniform bound on $|A|$ imply?
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- In general, a neighborhood of a point $p \in M$ is always a graph over $T_pM$. However, the size of such neighborhood depends on $p$. 

$$\text{sup}_M |A| \leq C$$

Let the surface be CMC and $u$ be such graph then

$$\|u\|_{C^2} \leq 10C \text{div}\nabla u \sqrt{1 + |\nabla u|^2} = 2H$$

then, $\|u\|_{C^2,\alpha}$ is uniformly bounded independently of $p$. 

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Intrinsic Curvature Estimate

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Note

The local estimate on $|A|$ fails in the minimal case; counterexamples being rescaled helicoids.
A global result

Theorem (Meeks-T.)

Let $M$ be a complete, nonzero CMC surface embedded in $\mathbb{R}^3$ with finite topology. Then:

- $M$ has bounded curvature and is properly embedded.
- $M$ has more than one end or it is a round sphere.
- If $M$ has exactly two ends then it is a Delaunay surface.
- If $M$ has more than one end then each end is asymptotic to a Delaunay surface.

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Examples of finite topology nonzero CMC surfaces

Genus 0 and 3 ends.
Examples of finite topology nonzero CMC surfaces

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Intrinsic Curvature Estimate

Given $\Delta$ there exists $C = C(\Delta)$ such that:
If $M$ is a 1-disk with $0 \in M$ and $\text{dist}_M(0, \partial M) > \Delta$, then

$$|A|(0) \leq C.$$
Step 1: Cord-arc Bound (Colding-Minicozzi for $H = 0$)

There exists a universal constant $\Omega$ such that:

If $M$ is a 1-disk with $0 \in M$, $\text{dist}_M(0, \partial M) > r\Omega$, $r > 0$, and $\Gamma$ is a geodesic starting at the origin with length $> r\Omega$, then

$$\Gamma \cap \partial B(r) \neq \emptyset.$$
The proof of the Intrinsic Curvature Estimate

Step 2: Extrinsic Curvature Estimate

Given $\Lambda > 0$ there exists a constant $C = C(\Lambda)$ such that:

If $M$ is a 1-disk with $0 \in M$ and $\partial M \subset \partial B(\Lambda)$, then

$$|A|(0) \leq C.$$
Intrinsic Curvature Estimate

Given $\Delta > 0$ there exists $C = C(\Delta)$ such that:

If $M$ is a 1-disk with $0 \in M$ and $\text{dist}_M(0, \partial M) > \Delta$, then

$$|A|(0) \leq C.$$  

Proof

- Cord-arc Bound says that the connected component of $M \cap B\left(\frac{A}{\Omega}\right)$ containing the origin is a 1-disk with boundary in $\partial B\left(\frac{A}{\Omega}\right)$.
- Apply the Extrinsic Curvature Estimate to such 1-disk.
The proof of the Chord-arc Bound

Key ingredient: One-sided Curvature Estimate
(Colding-Minicozzi for $H = 0$)

There exist universal constants $K$ and $N$ such that:
If $M$ is an $H$-disk with $|H| \leq 1$, $\partial M \subset \partial B(1)$ and $M \subset \{x_3 > 0\}$, then

\[
\sup_{M \cap B \left(\frac{1}{10^N}\right)} |A| \leq K.
\]
Main Results

- Characterisation of the round sphere
Main Results

- Characterisation of the round sphere
  - Radius Estimate

- Intrinsic Curvature Estimate

- Chord-arc Bound

- Extrinsic Curvature Estimate
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Future Directions

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What can be said about the geometry of a complete nonzero CMC surface $M$ embedded in a complete 3-manifolds $N$?
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Question
$M$ has positive injectivity radius $+$ $N$ has bounded sectional curvatures $\implies$ curvature estimates $\implies$ proper when the scalar curvature $> \varepsilon > 0$

Question
$M$ has finite topology $+$ $N$ is homogeneous $\implies$ $M$ has locally bounded second fundamental form
Future Directions

**Question**

Let $M$ be a complete nonzero CMC surface embedded in $H^3$ with $H \geq 1$ and finite topology. Then:

- $M$ has bounded curvature and is properly embedded.

- If $H = 1$, then each annular end of $M$ is asymptotic to a horosphere or a catenoid. Furthermore, if $M$ has one end, then $M$ is a horosphere.

- If $H > 1$, then each annular end of $M$ is asymptotic to the end of a Hsiang surface.


For properly embedded $+ H>1$: 1992, Korevaar-Kusner-Meeks-Solomon.
Future Directions

Question
What can be said about the geometry of a surface $M$ embedded in $\mathbb{R}^3$ with bounded mean curvature in $L^p$?
Theorem (Bourni-T.)

Let $M$ be a surface embedded in $\mathbb{R}^3$ containing the origin with $\text{Inj}_M(0) \geq s > 0$,

$$\int_{B_M(s)} |A|^2 \leq C_1$$

and either

i. $\|H\|_{W^{2,2}(B_M(s))}^* \leq \Lambda_2(C_1)$, if $p = 2$ or

ii. $\|H\|_{W^{1,p}(B_M(s))}^* \leq \Lambda_p(C_1)$, if $p > 2$,

then

$$|A|^2(0) \leq C_2(p, C_1)s^{-2}.$$

For $H=0$: 2004, Colding-Minicozzi.
The proof of the Radius Estimate

Radius Estimate

There exists a universal constant $R$ such that:
If $M$ is a 1-disk, then $M$ has radius less than $R$. 
Arguing by contradiction let $M_n$ be a sequence of 1-disks with radii $>n$ and $|A|$ uniformly bounded.
Sketch of the proof

- Arguing by contradiction let $M_n$ be a sequence of 1-disks with radii $>n$ and $|A|$ uniformly bounded.
- A subsequence of $M_n$ converges $C^2$ to a complete "embedded" CMC=1 surface $M$ with bounded curvature and genus zero.
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- $M$ is proper. If NOT then the universal cover of $\overline{M} - M$ would be a (strongly) stable, complete surface with $\text{CMC}=1$ but there is none.
The proof of the Radius Estimate

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- $M$ is proper. If NOT then the universal cover of $\overline{M} - M$ would be a (strongly) stable, complete surface with CMC$=1$ but there is none.
- A Delaunay surface cannot be a limit of $1$-disks. Contradiction!
The proof of the One-sided Curvature Estimate

One-sided Curvature Estimate
(Colding-Minicozzi for $H = 0$)

There exist universal constants $K$ and $N$ such that:
If $M$ is an $H$-disk with $|H| \leq 1$, $\partial M \subset \partial B(1)$ and $M \subset \{x_3 > 0\}$, then

$$\sup_{M \cap B\left(\frac{1}{10N}\right)} |A| \leq K.$$
Arguing by contradiction, let $p_n \in \mathbf{M}_n$ be a sequence of points converging to the origin where $|A_n|$ is arbitrarily large.
Sketch of the proof

- Arguing by contradiction, let $p_n \in M_n$ be a sequence of points converging to the origin where $|A_n|$ is arbitrarily large.

- Around $p_n$, $M_n$ looks like a vertical helicoid and thus the tangent plane at $p_n$ is vertical.
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- Let $\Gamma_n$ be connected component of the pre-image of the equator via the Gauss map containing $p_n$ (tangent plane is vertical).
Sketch of the proof

- Arguing by contradiction, let \( p_n \in \mathcal{M}_n \) be a sequence of points converging to the origin where \( |A_n| \) is arbitrarily large.

- Around \( p_n \), \( \mathcal{M}_n \) looks like a vertical helicoid and thus the tangent plane at \( p_n \) is vertical.

- Let \( \Gamma_n \) be connected component of the pre-image of the equator via the Gauss map containing \( p_n \) (tangent plane is vertical).

- Around each point \( p \in \Gamma_n \), \( \mathcal{M}_n \) looks like a vertical helicoid and thus the curve \( \Gamma_n \) cannot be contained in a half-space.
The proof of the One-sided Curvature Estimate

Key ingredients

- Colding-Minicozzi Theory.
The proof of the One-sided Curvature Estimate

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- **Colding-Minicozzi** Theory.
- Uniqueness of the helicoid (**Meeks-Rosenberg**).
The proof of the One-sided Curvature Estimate

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- **Colding-Minicozzi** Theory.
- Uniqueness of the helicoid (*Meeks-Rosenberg*).
- Geometry of minimal and CMC laminations (*Meeks-Perez-Ros*).
Thanks