Spin(9) structures and vector fields on spheres

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New Trends in Differential Geometry
L'Aquila, September 7th-9th, 2011

MP, Paolo Piccinni.
$\mathrm{Spin}(9)$ and almost complex structures on 16-dimensional manifolds.
arXiv: 1105.5318, to appear in Ann. Gl. An. Geom.

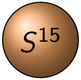
MP, Paolo Piccinni.
Spheres with more than 7 vector fields: all the fault of $\mathrm{Spin}(9)$.
arXiv: 1107.0462.

MP, Paolo Piccinni, Victor Vuletescu.
16-dimensional manifolds with a locally conformal parallel $\mathrm{Spin}(9)$ metric.
Work in progress.

1. S^{15} and $\mathrm{Spin}(9)$
   - S^{15} is "more equal" among other spheres
   - $\mathrm{Spin}(9)$ and Hopf fibrations
2. The $\mathrm{Spin}(9)$ fundamental form
   - Quaternionic analogy
   - $\mathrm{Spin}(9)$ and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\mathrm{Spin}(9)}$
3. Vector fields on spheres
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$
4. Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem

First characterization: Hopf fibrations

S^{15} is the only sphere involved in three different Hopf fibrations.



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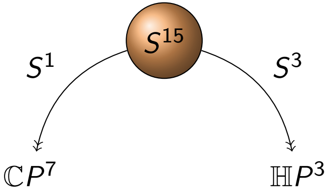

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\begin{tikzcd}
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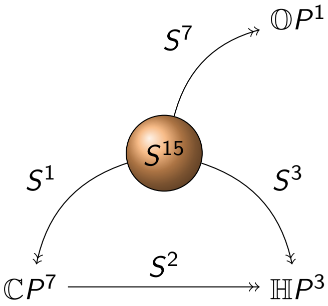

Remark

The complex and quaternionic Hopf fibrations are not subfibrations of the octonionic one [Loo-Verjovsky, Topology 1992].

Second characterization: Einstein metrics

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Berger's list and $\mathrm{Spin}(9)$ refutation

$\mathrm{U}(n)$

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What is $\mathrm{Spin}(9)$?

Definition

$\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].

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A close relative: the quaternionic case

- $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(8)$ is the group of symmetries of the Hopf fibration $\mathbb{H}^2 \supset S^7 \xrightarrow{S^3} S^4 \cong \mathbb{H}P^1$ [Gluck-Warner-Ziller, L'Enseignement Math. 1986].

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Five involutions for $\mathrm{Spin}(5)$

- Consider in $\mathrm{Sp}(2)$ the matrices

$$\begin{pmatrix} r & R_{\overline{u}} \\ R_u & -r \end{pmatrix}$$

where $(r, u) \in S^4 \subset \mathbb{R} \times \mathbb{H}$ and $\mathbb{H}^2 \cong \mathbb{R}^8$.

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From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 10 complex structures

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From the Kähler forms to the $\mathrm{Spin}(9)$ form

Theorem [P-Piccinni, Ann. Gl. An. Geom. 2011]

Denote the characteristic polynomial of θ by

$$t^9 + \tau_2(\theta)t^7 + \tau_4(\theta)t^5 + \tau_6(\theta)t^3 + \tau_8(\theta)t$$

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▸ Explicit formulas?

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An explicit formula for $\Phi_{\mathrm{Spin}(9)}$

- The $\binom{16}{8} = 12870$ integrals of

$$\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$$

can be computed with the help of *Mathematica*.

▸ Show all 702 terms ▸ Show only 70 terms ▸ Show source code

Previous work for $\Phi_{\mathrm{Spin}(9)}$ in [Abe-Matsubara, Korea Japan Conf. Transf. Groups 1997], [Friedrich, Asian J. Math. 2001], [C. Lopez-Gadea-Mykytyuk, int. J. Geom. Methods 2010].

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$$N = (2k+1)2^p 16^q \qquad \text{with } q \geq 1 \quad \text{and} \quad p = 1, 2, 3$$

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$$N = (2k+1)2^p 16^q \qquad \text{with } q \geq 1 \quad \text{and} \quad p = 1, 2, 3$$

the maximal number of vector fields is

$$\sigma(N) = \qquad + \boxed{2^p - 1}$$

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Spin(9) contribution

$\mathbb{C}, \mathbb{H}, \mathbb{O}$ contribution

The lowest dimensional sphere with more than 7 vector field is S^{15}

[Hurwitz, Math. Ann. 1922], [Radon, Abh. Math. Hamburg 1923], [Adams, Ann. of Math. 1962].

The lowest dimension: S^{15}

- Coordinates on S^{15}:

$$B = (x, y) = (x_1, \dots, x_8, y_1, \dots, y_8) \qquad \text{unit normal vector field}$$

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Proposition

A maximal system of 8 orthonormal vector fields on S^{15} is given by

$$J_{19}(B), J_{29}(B), J_{39}(B), J_{49}(B), J_{59}(B), J_{69}(B), J_{79}(B), J_{89}(B)$$

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Remark

The frame $\{J_{19}(B), \dots, J_{89}(B)\}$ has *nothing to do* with Hopf fibrations.

Other spheres with $\sigma(N) > 7$

Sphere	$\sigma(N)$	Vector fields	Notations	Involved structures
S^{15}	8	$J_{19}B, \dots, J_{89}B$	$B = s = (x, y)$	$\mathrm{Spin}(9)$
S^{31}	9	$J_{19}B, \dots, J_{89}B$ $\star\mathcal{L}_i B$	$B = s^1 + is^2, \mathcal{L}_i B = -s^2 + is^1$ $\star : (x, y) \to (x, -y)$	$\mathrm{Spin}(9)+\mathbb{C}$
S^{63}	11	$J_{19}B, \dots, J_{89}B$ $\star\mathcal{L}_i B, \star\mathcal{L}_j B, \star\mathcal{L}_k B$	$B = s^1 + is^2 + js^3 + ks^4$ $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$ and $\star$ as above	$\mathrm{Spin}(9)+\mathbb{H}$
S^{127}	15	$J_{19}B, \dots, J_{89}B$ $\star\mathcal{L}_i B, \dots, \star\mathcal{L}_h B$	$B = s^1 + is^2 + js^3 + ks^4 + es^5 + fs^6 + gs^7 + hs^8$ $\mathcal{L}_i, \dots, \mathcal{L}_h$ and $\star$ as above	$\mathrm{Spin}(9)+\mathbb{O}$
S^{255}	16	$J_{19}B, \dots, J_{89}B$ $\star J^1_{19}B, \dots, \star J^1_{89}B$	See explanations below	$(\mathrm{Spin}(9))^2$

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For $S^{255} \subset \mathbb{R}^{256}$ the $J_{19}, \dots, J_{89}$ are defined on the 16-dimensional components of

$$\mathbb{R}^{256} = \mathbb{R}^{16} \oplus \cdots \oplus \mathbb{R}^{16}$$

The $J^1_{19}, \dots, J^1_{89}$ are defined by the same 16×16 real matrices as $J_{19}, \dots, J_{89}$, but acting formally on 16-ples of sedenions

$$B = (s^1, \dots, s^{16}) \in \mathbb{R}^{256}$$

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Cayley-Dickson process

- A $*$-algebra $\mathcal{A}$ is a real algebra equipped with a conjugation, namely a linear map $* : \mathcal{A} \to \mathcal{A}$ such that

$$a^{**} = a, \qquad (ab)^* = b^* a^*$$

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- A new $*$-algebra structure can be defined on $\mathcal{A} \times \mathcal{A}$ by

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- This produces

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \dots$$

Multiplication table for sedenions

1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Vector fields in the general case

Theorem [P-Piccinni, arXiv: 1107.0462, 2011]

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(k,p,q)	Sphere	$\sigma(N)$	Vector fields	Involved structures
$(k,0,q)$	$S^{(2k+1)16^q-1}$	$8q$	$J_{19}B,\dots,J_{89}B$ $\star J^1_{19}B,\dots,\star J^1_{89}B$ $\dots$ $\star J^{q-1}_{19}B,\dots,\star J^{q-1}_{89}B$	$(\mathrm{Spin}(9))^q$
$(k,1,q)$	$S^{2(2k+1)16^q-1}$	$8q+1$	$J_{19}B,\dots,J_{89}B$ $\star J^1_{19}B,\dots,\star J^1_{89}B$ $\dots$ $\star J^{q-1}_{19}B,\dots,\star J^{q-1}_{89}B$ $\star\mathcal{L}_iB$	$(\mathrm{Spin}(9))^q + \mathbb{C}$
$(k,2,q)$	$S^{4(2k+1)16^q-1}$	$8q+3$	$J_{19}B,\dots,J_{89}B$ $\star J^1_{19}B,\dots,\star J^1_{89}B$ $\dots$ $\star J^{q-1}_{19}B,\dots,\star J^{q-1}_{89}B$ $\star\mathcal{L}_iB,\ \star\mathcal{L}_jB,\ \star\mathcal{L}_kB$	$(\mathrm{Spin}(9))^q + \mathbb{H}$
$(k,3,q)$	$S^{8(2k+1)16^q-1}$	$8q+7$	$J_{19}B,\dots,J_{89}B$ $\star J^1_{19}B,\dots,\star J^1_{89}B$ $\dots$ $\star J^{q-1}_{19}B,\dots,\star J^{q-1}_{89}B$ $\star\mathcal{L}_iB\cdots\star\mathcal{L}_hB$	$(\mathrm{Spin}(9))^q + \mathbb{O}$

Sketch of proof

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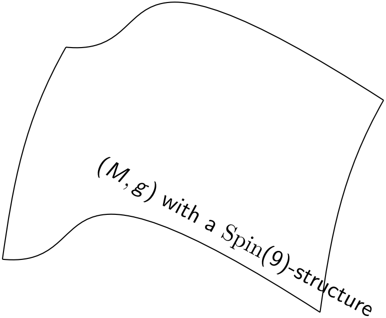

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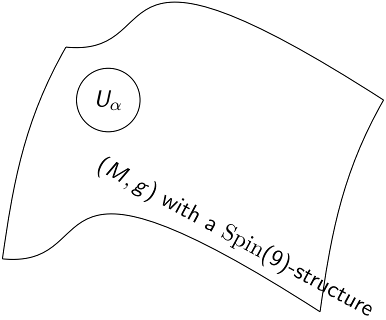

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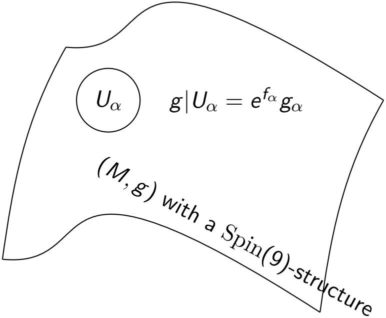

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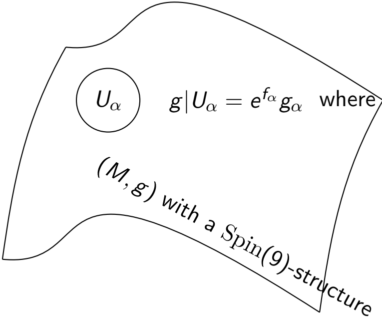

$g|U_\alpha = e^{f_\alpha} g_\alpha$ where g_α has holonomy contained in $\mathrm{Spin}(9)$

Examples

The product $S^{15} \times S^1 = \frac{\mathbb{O}^2 - 0}{\mathbb{Z}}$ = cone over S^{15} with the (conformal class) of the flat metric.

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The non-trivial S^1-bundle over $\mathbb{R}P^{15}$, with the metric induced by the flat cone $C(S^{15})$.

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Structure of compact locally conformal parallel $\mathrm{Spin}(9)$ manifolds

Theorem [P-Piccinni-Vuletescu]

Let (M, g) be a compact, locally conformal but not globally conformal parallel $\mathrm{Spin}(9)$ manifold. Then

$$M = C(N)/\mathbb{Z}$$

where $C(N)$ is a flat cone over a compact 15-dimensional manifold N with finite fundamental group.

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5. Since the local metrics g_α are Einstein, D is Einstein-Weyl.

Proof, on the universal covering

6. Let g be the Gauduchon metric, so that $\nabla\omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.

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10. Since $\pi_1(M)$ acts by homotheties on $C(\tilde{N})$, and $\tilde{N}$ is compact, $\pi_1(M)$ contains a finite normal subgroup I of isometries.
11. We obtain $\pi_1(M) = I \rtimes \mathbb{Z}$, and $M = C(\tilde{N}/I)/\mathbb{Z}$.

Surprise: end of talk!

Details for $\Phi_{\mathrm{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$

- ν_l = volume form on the octonionic lines $l = \{(x, mx)\}$ or $l = \{(0, y)\}$ in $\mathbb{O}^2$.
- $p_l : \mathbb{O}^2 \to l$ = projection on l.
- $p_l^* \nu_l$ = 8-form in $\mathbb{O}^2 = \mathbb{R}^{16}$.
- The integral over $\mathbb{O}P^1$ can be computed over $\mathbb{O}$ with polar coordinates.
- The formula arise from distinguished 8-planes in the $\mathrm{Spin}(9)$-geometry $\to$ (forthcoming) calibrations.

Go back

The five involutions of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ as 8×8 matrices

$$\mathcal{I}_1 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right)$$

$$\mathcal{I}_2 = \left(\begin{array}{c|c} 0 & -R_i^{\mathbb{H}} \\ \hline R_i^{\mathbb{H}} & 0 \end{array}\right)$$

Go back

$$\mathcal{I}_5 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right)$$

$$\mathcal{I}_3 = \left(\begin{array}{c|c} 0 & -R_j^{\mathbb{H}} \\ \hline R_j^{\mathbb{H}} & 0 \end{array}\right)$$

$$\mathcal{I}_4 = \left(\begin{array}{c|c} 0 & -R_k^{\mathbb{H}} \\ \hline R_k^{\mathbb{H}} & 0 \end{array}\right)$$

The nine involutions of $\mathrm{Spin}(9)$ as 16×16 matrices

$$= \left(\begin{array}{c|c} 0 & -R_i \\ \hline R_i & 0 \end{array}\right)$$

$$\mathcal{I}_1 = \left(\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline \mathrm{Id} & 0 \end{array}\right)$$

$$\mathcal{I}_3 = \left(\begin{array}{c|c} 0 & -R_j \\ \hline R_j & 0 \end{array}\right)$$

$$\mathcal{I}_4 = \left(\begin{array}{c|c} 0 & -R_k \\ \hline R_k & 0 \end{array}\right)$$

$$\mathcal{I}_5 = \left(\begin{array}{c|c} 0 & -R_e \\ \hline R_e & 0 \end{array}\right)$$

$$\mathcal{I}_6 = \left(\begin{array}{c|c} 0 & -R_f \\ \hline R_f & 0 \end{array}\right)$$

$$\mathcal{I}_7 = \left(\begin{array}{c|c} 0 & -R_g \\ \hline R_g & 0 \end{array}\right)$$

$$\mathcal{I}_8 = \left(\begin{array}{c|c} 0 & -R_h \\ \hline R_h & 0 \end{array}\right)$$

$$\mathcal{I}_9 = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array}\right)$$

Go back

Explicit formula for Φ_{G_2}

Denote by $x_1, \ldots, x_7$ the coordinates in $\mathbb{R}^7$. Then $\mathrm{G}_2 =$ stabilizer in $\mathrm{SO}(7)$ of

$$\begin{aligned}\Phi_{\mathrm{G}_2} = & dx_1 \wedge dx_2 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5 + dx_3 \wedge dx_4 \wedge dx_6 \\ & + dx_4 \wedge dx_5 \wedge dx_7 + dx_5 \wedge dx_6 \wedge dx_1 + dx_6 \wedge dx_7 \wedge dx_2 \\ & + dx_7 \wedge dx_1 \wedge dx_3\end{aligned}$$

As a shortcut, we could write

$$\Phi_{\mathrm{G}_2} = \mathbf{124} + \mathbf{235} + \mathbf{346} + \mathbf{457} + \mathbf{561} + \mathbf{672} + \mathbf{713}$$

Go back

351 terms of $\Phi_{\mathrm{Spin}(9)}$

Go back

12345678		-14	123456	1′2′	2	123456	3′4′	-2	123456	5′6′	-2	123456	7′8′	-2
123457	1′3′	2	123457	2′4′	2	123457	5′7′	-2	123457	6′8′	2	123458	1′4′	2
123458	2′3′	-2	123458	5′8′	-2	123458	6′7′	-2	123467	1′4′	-2	123467	2′3′	2
123467	5′8′	-2	123467	6′7′	-2	123468	1′3′	2	123468	2′4′	2	123468	5′7′	2
123468	6′8′	-2	123478	1′2′	-2	123478	3′4′	2	123478	5′6′	-2	123478	7′8′	-2
1234	1′2′3′4′	-2	1234	5′6′7′8′	-2	123567	1′5′	-2	123567	2′6′	-2	123567	3′7′	-2
123567	4′8′	2	123568	1′6′	-2	123568	2′5′	2	123568	3′8′	-2	123568	4′7′	-2
123578	1′7′	-2	123578	2′8′	2	123578	3′5′	2	123578	4′6′	2	1235	1′2′3′5′	-1
1235	1′2′4′6′	-1	1235	1′3′4′7′	-1	1235	1′5′6′7′	-1	1235	2′3′4′8′	1	1235	2′5′6′8′	1
1235	3′5′7′8′	1	1235	4′6′7′8′	1	123678	1′8′	-2	123678	2′7′	-2	123678	3′6′	2
123678	4′5′	-2	1236	1′2′3′6′	-1	1236	1′2′4′5′	1	1236	1′3′4′8′	-1	1236	1′5′6′8′	-1
1236	2′3′4′7′	-1	1236	2′5′6′7′	-1	1236	3′6′7′8′	1	1236	4′5′7′8′	-1	1237	1′2′3′7′	-1
1237	1′2′4′8′	1	1237	1′3′4′5′	1	1237	1′5′7′8′	-1	1237	2′3′4′6′	1	1237	2′6′7′8′	-1
1237	3′5′6′7′	-1	1237	4′5′6′8′	1	1238	1′2′3′8′	-1	1238	1′2′4′7′	-1	1238	1′3′4′6′	1
1238	1′6′7′8′	-1	1238	2′3′4′5′	-1	1238	2′5′7′8′	1	1238	3′5′6′8′	-1	1238	4′5′6′7′	-1
124567	1′6′	2	124567	2′5′	-2	124567	3′8′	-2	124567	4′7′	-2	124568	1′5′	-2
124568	2′6′	-2	124568	3′7′	2	124568	4′8′	-2	124578	1′8′	-2	124578	2′7′	-2
124578	3′6′	-2	124578	4′5′	2	1245	1′2′3′6′	1	1245	1′2′4′5′	-1	1245	1′3′4′8′	-1
1245	1′5′6′8′	-1	1245	2′3′4′7′	-1	1245	2′5′6′7′	-1	1245	3′6′7′8′	-1	1245	4′5′7′8′	1
124678	1′7′	2	124678	2′8′	-2	124678	3′5′	2	124678	4′6′	2	1246	1′2′3′5′	-1
1246	1′2′4′6′	-1	1246	1′3′4′7′	1	1246	1′5′6′7′	1	1246	2′3′4′8′	-1	1246	2′5′6′8′	-1
1246	3′5′7′8′	1	1246	4′6′7′8′	1	1247	1′2′3′8′	-1	1247	1′2′4′7′	-1	1247	1′3′4′6′	-1
1247	1′6′7′8′	1	1247	2′3′4′5′	1	1247	2′5′7′8′	-1	1247	3′5′6′8′	-1	1247	4′5′6′7′	-1
1248	1′2′3′7′	1	1248	1′2′4′8′	-1	1248	1′3′4′5′	1	1248	1′5′7′8′	-1	1248	2′3′4′6′	1
1248	2′6′7′8′	-1	1248	3′5′6′7′	1	1248	4′5′6′8′	-1	125678	1′2′	-2	125678	3′4′	-2
125678	5′6′	2	125678	7′8′	-2	1256	1′2′5′6′	-2	1256	3′4′7′8′	-2	1257	1′2′5′7′	-1
1257	1′2′6′8′	1	1257	1′3′5′6′	-1	1257	1′3′7′8′	1	1257	2′4′5′6′	-1	1257	2′4′7′8′	1
1257	3′4′5′7′	-1	1257	3′4′6′8′	1	1258	1′2′5′8′	-1	1258	1′2′6′7′	-1	1258	1′4′5′6′	-1
1258	1′4′7′8′	1	1258	2′3′5′6′	1	1258	2′3′7′8′	-1	1258	3′4′5′8′	-1	1258	3′4′6′7′	-1
1267	1′2′5′8′	-1	1267	1′2′6′7′	-1	1267	1′4′5′6′	1	1267	1′4′7′8′	-1	1267	2′3′5′6′	-1
1267	2′3′7′8′	1	1267	3′4′5′8′	-1	1267	3′4′6′7′	-1	1268	1′2′5′7′	1	1268	1′2′6′8′	-1
1268	1′3′5′6′	-1	1268	1′3′7′8′	1	1268	2′4′5′6′	-1	1268	2′4′7′8′	1	1268	3′4′5′7′	1
1268	3′4′6′8′	-1	1278	1′2′7′8′	-2	1278	3′4′5′6′	-2	12	1′2′3′4′5′6′	2	12	1′2′3′4′7′8′	-2
12	1′2′5′6′7′8′	-2	12	3′4′5′6′7′8′	-2	134567	1′7′	2	134567	2′8′	2	134567	3′5′	-2
134567	4′6′	2	134568	1′8′	2	134568	2′7′	-2	134568	3′6′	-2	134568	4′5′	-2
134578	1′5′	-2	134578	2′6′	2	134578	3′7′	-2	134578	4′8′	-2	1345	1′2′3′7′	1
1345	1′2′4′8′	1	1345	1′3′4′5′	-1	1345	1′5′7′8′	-1	1345	2′3′4′6′	1	1345	2′6′7′8′	1
1345	3′5′6′7′	-1	1345	4′5′6′8′	-1	134678	1′6′	-2	134678	2′5′	-2	134678	3′8′	-2
134678	4′7′	2	1346	1′2′3′8′	1	1346	1′2′4′7′	-1	1346	1′3′4′6′	-1	1346	1′6′7′8′	-1
1346	2′3′4′5′	-1	1346	2′5′7′8′	-1	1346	3′5′6′8′	-1	1346	4′5′6′7′	1	1347	1′2′3′5′	-1
1347	1′2′4′6′	1	1347	1′3′4′7′	-1	1347	1′5′6′7′	1	1347	2′3′4′8′	-1	1347	2′5′6′8′	1
1347	3′5′7′8′	-1	1347	4′6′7′8′	1	1348	1′2′3′6′	-1	1348	1′2′4′5′	-1	1348	1′3′4′8′	-1
1348	1′5′6′8′	1	1348	2′3′4′7′	1	1348	2′5′6′7′	-1	1348	3′6′7′8′	-1	1348	4′5′7′8′	-1
135678	1′3′	-2	135678	2′4′	2	135678	5′7′	2	135678	6′8′	2	1356	1′2′5′7′	-1
1356	1′2′6′8′	-1	1356	1′3′5′6′	-1	1356	1′3′7′8′	-1	1356	2′4′5′6′	1	1356	2′4′7′8′	1
1356	3′4′5′7′	1	1356	3′4′6′8′	1	1357	1′3′5′7′	-2	1357	2′4′6′8′	-2	1358	1′3′5′8′	-1
1358	1′3′6′7′	-1	1358	1′4′5′7′	-1	1358	1′4′6′8′	-1	1358	2′3′5′7′	1	1358	2′3′6′8′	1
1358	2′4′5′8′	1	1358	2′4′6′7′	1	1367	1′3′5′8′	-1	1367	1′3′6′7′	-1	1367	1′4′5′7′	1
1367	1′4′6′8′	1	1367	2′3′5′7′	-1	1367	2′3′6′8′	-1	1367	2′4′5′8′	1	1367	2′4′6′7′	1
1368	1′3′6′8′	-2	1368	2′4′5′7′	-2	1378	1′2′5′7′	1	1378	1′2′6′8′	1	1378	1′3′5′6′	-1
1378	1′3′7′8′	-1	1378	2′4′5′6′	1	1378	2′4′7′8′	1	1378	3′4′5′7′	-1	1378	3′4′6′8′	-1
13	1′2′3′4′5′7′	2	13	1′2′3′4′6′8′	2	13	1′3′5′6′7′8′	-2	13	2′4′5′6′7′8′	2	145678	1′4′	-2
145678	2′3′	-2	145678	5′8′	2	145678	6′7′	-2	1456	1′2′5′8′	-1	1456	1′2′6′7′	1
1456	1′4′5′6′	-1	1456	1′4′7′8′	-1	1456	2′3′5′6′	-1	1456	2′3′7′8′	-1	1456	3′4′5′8′	1
1456	3′4′6′7′	-1	1457	1′3′5′8′	-1	1457	1′3′6′7′	1	1457	1′4′5′7′	-1	1457	1′4′6′8′	1
1457	2′3′5′7′	-1	1457	2′3′6′8′	1	1457	2′4′5′8′	-1	1457	2′4′6′7′	1	1458	1′4′5′8′	-2
1458	2′3′6′7′	-2	1467	1′4′6′7′	-2	1467	2′3′5′8′	-2	1468	1′3′5′8′	-1	1468	1′3′6′7′	1
1468	1′4′5′7′	1	1468	1′4′6′8′	-1	1468	2′3′5′7′	1	1468	2′3′6′8′	-1	1468	2′4′5′8′	-1
1468	2′4′6′7′	1	1478	1′2′5′8′	1	1478	1′2′6′7′	-1	1478	1′4′5′6′	-1	1478	1′4′7′8′	-1
1478	2′3′5′6′	-1	1478	2′3′7′8′	-1	1478	3′4′5′8′	-1	1478	3′4′6′7′	1	14	1′2′3′4′5′8′	2
14	1′2′3′4′6′7′	-2	14	1′4′5′6′7′8′	-2	14	2′3′5′6′7′8′	-2	1567	1′2′3′5′	-1	1567	1′2′4′6′	1
1567	1′3′4′7′	1	1567	1′5′6′7′	-1	1567	2′3′4′8′	1	1567	2′5′6′8′	-1	1567	3′5′7′8′	-1
1567	4′6′7′8′	1	1568	1′2′3′6′	-1	1568	1′2′4′5′	-1	1568	1′3′4′8′	1	1568	1′5′6′8′	-1
1568	2′3′4′7′	-1	1568	2′5′6′7′	1	1568	3′6′7′8′	-1	1568	4′5′7′8′	-1	1578	1′2′3′7′	-1
1578	1′2′4′8′	-1	1578	1′3′4′5′	-1	1578	1′5′7′8′	-1	1578	2′3′4′6′	1	1578	2′6′7′8′	1
1578	3′5′6′7′	1	1578	4′5′6′8′	1	15	1′2′3′5′6′7′	-2	15	1′2′4′5′6′8′	-2	15	1′3′4′5′7′8′	-2
15	2′3′4′6′7′8′	2	1678	1′2′3′8′	-1	1678	1′2′4′7′	1	1678	1′3′4′6′	-1	1678	1′6′7′8′	-1
1678	2′3′4′5′	-1	1678	2′5′7′8′	-1	1678	3′5′6′8′	1	1678	4′5′6′7′	-1	16	1′2′3′5′6′8′	-2
16	1′2′4′5′6′7′	2	16	1′3′4′6′7′8′	-2	16	2′3′4′5′7′8′	-2	17	1′2′3′5′7′8′	-2	17	1′2′4′6′7′8′	2
17	1′3′4′5′6′7′	2	17	2′3′4′5′6′8′	2	18	1′2′3′6′7′8′	-2	18	1′2′4′5′7′8′	-2	18	1′3′4′5′6′8′	2
18	2′3′4′5′6′7′	-2												

70 terms of $\Phi_{\mathrm{Spin}(9)}$

12345678		-14	123456	1′2′	2	123456	3′4′	-2	123456	5′6′	-2	123456	7′8′	-2
123457	1′3′	2	123457	2′4′	2	123457	5′7′	-2	123457	6′8′	2	123458	1′4′	2
123458	2′3′	-2	123458	5′8′	-2	123458	6′7′	-2	123467	1′4′	-2	123467	2′3′	2
123467	5′8′	-2	123467	6′7′	-2	123468	1′3′	2	123468	2′4′	2	123468	5′7′	2
123468	6′8′	-2	123478	1′2′	-2	123478	3′4′	2	123478	5′6′	-2	123478	7′8′	-2
1234	1′2′3′4′	-2	1234	5′6′7′8′	-2	123567	1′5′	-2	123567	2′6′	-2	123567	3′7′	-2
123567	4′8′	2	123568	1′6′	-2	123568	2′5′	2	123568	3′8′	-2	123568	4′7′	-2
123578	1′7′	-2	123578	2′8′	2	123578	3′5′	2	123578	4′6′	2	1235	1′2′3′5′	-1
1235	1′2′4′6′	-1	1235	1′3′4′7′	-1	1235	1′5′6′7′	-1	1235	2′3′4′8′	1	1235	2′5′6′8′	1
1235	3′5′7′8′	1	1235	4′6′7′8′	1	123678	1′8′	-2	123678	2′7′	-2	123678	3′6′	2
123678	4′5′	-2	1236	1′2′3′6′	-1	1236	1′2′4′5′	1	1236	1′3′4′8′	-1	1236	1′5′6′8′	-1
1236	2′3′4′7′	-1	1236	2′5′6′7′	-1	1236	3′6′7′8′	1	1236	4′5′7′8′	-1	1237	1′2′3′7′	-1
1237	1′2′4′8′	1	1237	1′3′4′5′	1	1237	1′5′7′8′	-1	1237	2′3′4′6′	1	1237	2′6′7′8′	-1
1237	3′5′6′7′	-1	1237	4′5′6′8′	1	1238	1′2′3′8′	-1	1238	1′2′4′7′	-1	1238	1′3′4′6′	1

- $\{1,2,3,4,5,6,7,8,1',2',3',4',5',6',7',8'\}$ are (indexes of) coordinates in $\mathbb{R}^{16}$.
- A table entry $||$**123578 1′7′** $-2||$ means that
  $\Phi_{\mathrm{Spin}(9)} = \dots - 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 \wedge dx_7 \wedge dx_8 \wedge dx_1' \wedge dx_7' + \dots$
- Table obtained from Berger's definition of $\Phi_{\mathrm{Spin}(9)}$ with the help of *Mathematica*.
- The coefficients are normalized in such a way that they are all integers with $\gcd = 1$.

Go back

Pieces of source code for $\Phi_{\mathrm{Spin}(9)}$ computation

In progress.

Go back

The inductive argument

- Reduce to the case $S^{16^q-1} \subset \mathbb{R}^{16^q}$, and use induction on q.
- Assume that there are $8(q-1)$ independent vector fields $B_1, \dots, B_{8(q-1)}$ on $S^{16^{q-1}-1} \subset \mathbb{R}^{16^{q-1}}$.
- Look at $\mathbb{R}^{16^q}$ as 16 copies of $\mathbb{R}^{16^{q-1}}$:

$$\mathbb{R}^{16^q} = \{(s_1, \dots, s_{16}) | s_1, \dots, s_{16} \in \mathbb{R}^{16^{q-1}}\}$$

- Define complex structures $J'_{19}, \dots, J'_{89}$ on $\mathbb{R}^{16^q}$ by the same matrices defining $J_{19}, \dots, J_{89}$ but acting formally on the 16-ples $(s_1, \dots, s_{16})$ of elements in $\mathbb{R}^{16^{q-1}}$.
- Let B be the radial vector field on $S^{16^q-1} \subset \mathbb{R}^{16^q}$. Prove that $\{B_1, \dots, B_{8(q-1)}, J'_{19}B, \dots, J'_{89}B\}$ is an orthonormal frame on S^{16^q-1}.

Go back

The inductive argument

- Reduce to the case $S^{16^q-1} \subset \mathbb{R}^{16^q}$, and use induction on q.
- Assume that there are $8(q-1)$ independent vector fields $B_1, \dots, B_{8(q-1)}$ on $S^{16^{q-1}-1} \subset \mathbb{R}^{16^{q-1}}$.
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- Let B be the radial vector field on $S^{16^q-1} \subset \mathbb{R}^{16^q}$. Prove that $\{B_1, \dots, B_{8(q-1)}, J'_{19}B, \dots, J'_{89}B\}$ is an orthonormal frame on S^{16^q-1}.

Lemma needed!

Lemma

Lemma

The properties $(ab)^* = b^* a^*$, $\Re([a,b,c]) = 0$ and $< a, b >= \Re(ab^*)$ hold in any Cayley-Dickson algebra.

Go back 2 levels

Go back 1 level

A more explicit $\Phi_{\mathrm{Spin}(9)}$

$$\Phi_{\mathrm{Spin}(9)} \overset{\text{utc}}{=} \sum_{1\le\alpha_1<\alpha_2<\alpha_3<\alpha_4\le 9} (\psi_{\alpha_1\alpha_2}\wedge\psi_{\alpha_3\alpha_4}-\psi_{\alpha_1\alpha_3}\wedge\psi_{\alpha_2\alpha_4}+\psi_{\alpha_1\alpha_4}\wedge\psi_{\alpha_2\alpha_3})^2$$

$\psi_{12} = (-12+34+56-78)-(\ )'$ $\quad$ $\psi_{13} = (-13-24+57+68)-(\ )'$ $\quad$ $\psi_{14} = (-14+23+58-67)-(\ )'$

$\psi_{15} = (-15-26-37-48)-(\ )'$ $\quad$ $\psi_{16} = (-16+25-38+47)-(\ )'$ $\quad$ $\psi_{17} = (-17+28+35-46)-(\ )'$

$\psi_{18} = (-18-27+36+45)-(\ )'$ $\quad$ $\psi_{23} = (-14+23-58+67)+(\ )'$ $\quad$ $\psi_{24} = (13+24+57+68)+(\ )'$

$\psi_{25} = (-16+25+38-47)+(\ )'$ $\quad$ $\psi_{26} = (15+26-37-48)+(\ )'$ $\quad$ $\psi_{27} = (18+27+36+45)+(\ )'$

$\psi_{28} = (-17+28-35+46)+(\ )'$ $\quad$ $\psi_{34} = (-12+34-56+78)+(\ )'$ $\quad$ $\psi_{35} = (-17-28+35+46)+(\ )'$

$\psi_{36} = (-18+27+36-45)+(\ )'$ $\quad$ $\psi_{37} = (+15-26+37-48)+(\ )'$ $\quad$ $\psi_{38} = (16+25+38+47)+(\ )'$

$\psi_{45} = (-18+27-36+45)+(\ )'$ $\quad$ $\psi_{46} = (17+28+35+46)+(\ )'$ $\quad$ $\psi_{47} = (-16-25+38+47)+(\ )'$

$\psi_{48} = (15-26-37+48)+(\ )'$ $\quad$ $\psi_{56} = (-12-34+56+78)+(\ )'$ $\quad$ $\psi_{57} = (-13+24+57-68)+(\ )'$

$\psi_{58} = (-14-23+58+67)+(\ )'$ $\quad$ $\psi_{67} = (14+23+58+67)+(\ )'$ $\quad$ $\psi_{68} = (-13+24-57+68)+(\ )'$

$\psi_{78} = (12+34+56+78)+(\ )'$

$\psi_{19} = -11'-22'-33'-44'-55'-66'-77'-88'$ $\quad$ $\psi_{29} = -12'+21'+34'-43'+56'-65'-78'+87'$

$\psi_{39} = -13'-24'+31'+42'+57'+68'-75'-86'$ $\quad$ $\psi_{49} = -14'+23'-32'+41'+58'-67'+76'-85'$

$\psi_{59} = -15'-26'-37'-48'+51'+62'+73'+84'$ $\quad$ $\psi_{69} = -16'+25'-38'+47'-52'+61'-74'+83'$

$\psi_{79} = -17'+28'+35'-46'-53'+64'+71'-82'$ $\quad$ $\psi_{89} = -18'-27'+36'+45'-54'-63'+72'+81'$

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