Spin(9) structures and vector fields on spheres

Maurizio Parton\textsuperscript{1}  Paolo Piccinni\textsuperscript{2}  Victor Vuletescu\textsuperscript{3}

\textsuperscript{1}Università di Chieti-Pescara, Italy
\textsuperscript{2}Sapienza Università di Roma, Italy
\textsuperscript{3}University of Bucharest, Romania

New Trends in Differential Geometry
L’ Aquila, September 7th-9th, 2011
MP, Paolo Piccinni. 
Spin(9) and almost complex structures on 16-dimensional manifolds. 

MP, Paolo Piccinni. 
Spheres with more than 7 vector fields: all the fault of Spin(9). 

MP, Paolo Piccinni, Victor Vuletescu. 
16-dimensional manifolds with a locally conformal parallel Spin(9) metric. 
Work in progress.
1. $S^{15}$ and $\text{Spin}(9)$
   - $S^{15}$ is "more equal" among other spheres
   - $\text{Spin}(9)$ and Hopf fibrations

2. The $\text{Spin}(9)$ fundamental form
   - Quaternionic analogy
   - $\text{Spin}(9)$ and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\text{Spin}(9)}$

3. Vector fields on spheres
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$

4. Locally conformal parallel $\text{Spin}(9)$ manifolds
   - Definition and examples
   - Structure Theorem
First characterization: Hopf fibrations

\( S^{15} \) is the only sphere involved in three different Hopf fibrations.
$S^{15}$ is the only sphere involved in three different Hopf fibrations.
First characterization: Hopf fibrations

$S^{15}$ is the only sphere involved in three different Hopf fibrations.

[Diagram showing $S^{15}$ connected to $S^1$, $S^3$, $\mathbb{C}P^7$, and $\mathbb{H}P^3$.]
First characterization: Hopf fibrations

$S^{15}$ is the only sphere involved in three different Hopf fibrations.
$S^{15}$ is the only sphere involved in three different Hopf fibrations.

First characterization: Hopf fibrations

$S^{15}$ and Spin(9) $S^{15}$ is “more equal” among other spheres

Remark

The complex and quaternionic Hopf fibrations are not subfibrations of the octonionic one [Loo-Verjovsky, Topology 1992].
$S^{15}$ is the only sphere involved in three different Hopf fibrations.

**Remark**

The complex and quaternionic Hopf fibrations are not subfibrations of the octonionic one [Loo-Verjovsky, Topology 1992].
Second characterization: Einstein metrics

$S^{15}$ is the only sphere with three homogeneous Einstein metrics


$S^{15}$ and Spin(9) $S^{15}$ is “more equal” among other spheres
Second characterization: Einstein metrics

$S^{15}$ is the only sphere with three homogeneous Einstein metrics


- Round metric.
Second characterization: Einstein metrics

$S^{15}$ is the only sphere with three homogeneous Einstein metrics


- Round metric.
- Einstein metric on $\text{Sp}(4)/\text{Sp}(3)$ [Jensen, J. Diff. Geom. 1973].
Second characterization: Einstein metrics

$S^{15}$ and $\text{Spin}(9)$ is "more equal" among other spheres

$S^{15}$ is the only sphere with three homogeneous Einstein metrics


- Round metric.
- Einstein metric on $\text{Sp}(4)/\text{Sp}(3)$ [Jensen, J. Diff. Geom. 1973].
- Einstein metric on $\text{Spin}(9)/\text{Spin}(7)$
$S^{15}$ and $\text{Spin}(9)$

$S^{15}$ is “more equal” among other spheres

Third characterization: vector fields on spheres

$S^{15}$ is the lowest dimensional sphere admitting more than 7 vector fields

$S^{15}$ is the lowest dimensional sphere admitting more than 7 vector fields


- Number $\sigma(N)$ of linearly independent vector fields on $S^{N-1}$?
Third characterization: vector fields on spheres

$S^{15}$ is the lowest dimensional sphere admitting more than 7 vector fields


- Number $\sigma(N)$ of linearly independent vector fields on $S^{N-1}$?
- If $N = (2k + 1)2^p16^q$, with $0 \leq p \leq 3$, then

$$\sigma(N) = 8q + 2^p - 1$$
Third characterization: vector fields on spheres

$S^{15}$ is the lowest dimensional sphere admitting more than 7 vector fields


- Number $\sigma(N)$ of linearly independent vector fields on $S^{N-1}$?
- If $N = (2k + 1)2^p16^q$, with $0 \leq p \leq 3$, then

$$\sigma(N) = 8q + 2^p - 1$$

$\mathbb{C}, \mathbb{H}, \mathbb{O}$ contribution
Third characterization: vector fields on spheres

\(S^{15}\) is the lowest dimensional sphere admitting more than 7 vector fields


- Number \(\sigma(N)\) of linearly independent vector fields on \(S^{N-1}\)?
- If \(N = (2k + 1)2^p16^q\), with \(0 \leq p \leq 3\), then

\[
\sigma(N) = 8q + 2^p - 1
\]

\(\text{Spin}(9)\) contribution

\(\mathbb{C}, \mathbb{H}, \mathbb{O}\) contribution
1. **$S^{15}$ and Spin(9)**
   - $S^{15}$ is “more equal” among other spheres
   - Spin(9) and Hopf fibrations

2. **The Spin(9) fundamental form**
   - Quaternionic analogy
   - Spin(9) and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\text{Spin}(9)}$

3. **Vector fields on spheres**
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$

4. **Locally conformal parallel Spin(9) manifolds**
   - Definition and examples
   - Structure Theorem
Berger’s list and $\text{Spin}(9)$ refutation

$\text{Spin}(9)$ and Hopf fibrations

$\text{U}(n)$, $\text{SO}(n)$, $\text{SU}(n)$, $\text{Spin}(9)$, $\text{Spin}(7)$, $\text{Sp}(n) \cdot \text{Sp}(1)$, $\text{Sp}(n)$, $\text{G}_2$

$S^{15}$ and $\text{Spin}(9)$

Simply connected, complete, holonomy $\text{Spin}(9) \\ \Longleftrightarrow O\mathbb{P}^2 = F_4, R_{16}^{(\text{flat})}, O\mathbb{H}^2 = F_4^{(-20)} \text{Spin}(9) \left( s > 0 \right), R_{16}^{(\text{flat})}, O\mathbb{H}^2 = F_4^{(-20)} \text{Spin}(9) \left( s < 0 \right)$

Berger’s list and $\text{Spin}(9)$ refutation

Simply connected, complete, holonomy $\text{Spin}(9)$

$$\bigodot P^2 = \frac{F_4}{\text{Spin}(9)} (s > 0), \quad \mathbb{R}^{16} \text{ (flat)}, \quad \bigodot H^2 = \frac{F_4(-20)}{\text{Spin}(9)} (s < 0)$$

Berger’s list and $\text{Spin}(9)$ refutation

$\text{SU}(n)$
$\text{Sp}(n) \cdot \text{Sp}(1)$
$\text{Sp}(n)$

Simply connected, complete, holonomy $\text{Spin}(9)$

$\mathbb{O}P^2 = \frac{F_4}{\text{Spin}(9)}(s > 0)$,
$\mathbb{R}^{16}(\text{flat})$,
$\mathbb{O}H^2 = \frac{F_4(-20)}{\text{Spin}(9)}(s < 0)$

What is $\text{Spin}(9)$?

**Definition**

$\text{Spin}(9) \subset \text{SO}(16)$ is the group of symmetries of the Hopf fibration

$$\mathbb{O}^2 \supset S^{15} \xrightarrow{\nu} S^8 \cong \mathbb{O}P^1$$

What is Spin(9)?

**Definition**

Spin(9) ⊂ SO(16) is the group of symmetries of the Hopf fibration

\[ \mathbb{O}^2 \supset S^{15} \xrightarrow{\mathbb{S}^7} S^8 \cong \mathbb{O}P^1 \]  


- \( \Lambda^8(\mathbb{R}^{16})^{\text{Spin(9)}} = \Lambda^8_1 + \ldots \) [Friedrich, Asian Journ. Math 2001].
What is $\text{Spin}(9)$?

**Definition**

$\text{Spin}(9) \subset \text{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L’Enseignement Math. 1986].

- $\Lambda^8(\mathbb{R}^{16})^{\text{Spin}(9)} = \Lambda^8_1 + \ldots$ [Friedrich, Asian Journ. Math 2001].
- $\text{Spin}(9)$ is the stabilizer in $\text{SO}(16)$ of any element of $\Lambda^8_1$ [Brown-Gray, Diff. Geom. in honor of K. Yano 1972].
What is Spin(9)?

**Definition**

Spin(9) \(\subset SO(16)\) is the group of symmetries of the Hopf fibration

\[ \mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1 \]  


- \(\Lambda^8(\mathbb{R}^{16})^{\text{Spin}(9)} = \Lambda^8 + \ldots\) [Friedrich, Asian Journ. Math 2001].
- Spin(9) is the stabilizer in \(SO(16)\) of any element of \(\Lambda^8_1\)


**Definition**

Spin(9) is the stabilizer in \(SO(16)\) of the 8-form

\[
\Phi^{\text{Spin}(9)} \overset{\text{utc}}{=} \int_{\mathbb{O}P^1} p_1^* \nu_1 \, dl
\]

[Details]
What is $\text{Spin}(9)$?

**Definition**

$\text{Spin}(9) \subset \text{SO}(16)$ is the group of symmetries of the Hopf fibration $\mathbb{O}^2 \supset S^{15} \xrightarrow{S^7} S^8 \cong \mathbb{O}P^1$ [Gluck-Warner-Ziller, L’Enseignement Math. 1986].

- $\Lambda^8(\mathbb{R}^{16})^{\text{Spin}(9)} = \Lambda^8_1 + \ldots$ [Friedrich, Asian Journ. Math 2001].
- $\text{Spin}(9)$ is the stabilizer in $\text{SO}(16)$ of any element of $\Lambda^8_1$ [Brown-Gray, Diff. Geom. in honor of K. Yano 1972].

**Definition**

$\text{Spin}(9)$ is the stabilizer in $\text{SO}(16)$ of the 8-form

$$\Phi_{\text{Spin}(9)} \overset{\text{utc}}{=} \int_{\mathbb{O}P^1} p_1^* \nu_1 \, dl$$

Up to constants
1. $S^{15}$ and Spin(9)
   - $S^{15}$ is “more equal” among other spheres
   - Spin(9) and Hopf fibrations

2. The Spin(9) fundamental form
   - Quaternionic analogy
     - Spin(9) and Kähler forms on $\mathbb{R}^{16}$
     - An explicit formula for $\Phi_{\text{Spin}(9)}$

3. Vector fields on spheres
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$

4. Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem
A close relative: the quaternionic case

- \( \text{Sp}(2) \cdot \text{Sp}(1) \subset \text{SO}(8) \) is the group of symmetries of the Hopf fibration \( \mathbb{H}^2 \supset S^7 \xrightarrow{S^3} S^4 \cong \mathbb{H}P^1 \). [Gluck-Warner-Ziller, L’Enseignement Math. 1986].
A close relative: the quaternionic case

- $\text{Sp}(2) \cdot \text{Sp}(1) \subset \text{SO}(8)$ is the group of symmetries of the Hopf fibration $\mathbb{H}^2 \subset S^7 \xrightarrow{S^3} S^4 \cong \mathbb{H}P^1$ [Gluck-Warner-Ziller, L’Enseignement Math. 1986].

- $\text{Sp}(2) \cdot \text{Sp}(1)$ is the stabilizer in $\text{SO}(8)$ of the 4-form $\Phi_{\text{Sp}(2) \cdot \text{Sp}(1)}$ defined by

$$
\Phi_{\text{Sp}(2) \cdot \text{Sp}(1)} = \int_{\mathbb{H}P^1} p_1^* \nu_1 \, dl
$$

Consider in $Sp(2)$ the matrices

$$
\begin{pmatrix}
    r & R_{\bar{u}} \\
    R_u & -r
\end{pmatrix}
$$

where $(r, u) \in S^4 \subset \mathbb{R} \times \mathbb{H}$ and $\mathbb{H}^2 \cong \mathbb{R}^8$. 
Consider in $\text{Sp}(2)$ the matrices

$$
\begin{pmatrix}
    r & R_{\bar{u}} \\
    R_{u} & -r
\end{pmatrix}
$$

where $(r, u) \in S^4 \subset \mathbb{R} \times \mathbb{H}$ and $\mathbb{H}^2 \cong \mathbb{R}^8$.

The choice of $(r, u) = (1, 0), (0, 1), (0, i), (0, j), (0, k)$ gives

$I_1, \ldots, I_5 \in \text{SO}(8)$
Consider in $Sp(2)$ the matrices

\[
\begin{pmatrix}
  r & R_{\bar{u}} \\
  R_u & -r
\end{pmatrix}
\]

where $(r, u) \in S^4 \subset \mathbb{R} \times \mathbb{H}$ and $\mathbb{H}^2 \cong \mathbb{R}^8$.

The choice of $(r, u) = (1, 0), (0, 1), (0, i), (0, j), (0, k)$ gives $I_1, \ldots, I_5 \in SO(8)$.

$I_1, \ldots, I_5$ satisfy

\[
I_\alpha^2 = \text{Id}, \quad I_\alpha^* = I_\alpha, \quad I_\alpha \circ I_\beta = -I_\beta \circ I_\alpha
\]
Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 10 complex structures

$$J_{\alpha \beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for } \alpha < \beta$$
Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 10 complex structures

$$J_{\alpha \beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for } \alpha < \beta$$

The Kähler forms $\theta_{\alpha \beta}$ give rise to a $5 \times 5$ skew-symmetric matrix

$$\theta = (\theta_{\alpha \beta})$$
Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 10 complex structures

$$J_{\alpha \beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for} \quad \alpha < \beta$$

The Kähler forms $\theta_{\alpha \beta}$ give rise to a $5 \times 5$ skew-symmetric matrix

$$\theta = (\theta_{\alpha \beta})$$

**Remark**

Denote by $\tau_2(\theta)$ the second coefficient of the characteristic polynomial of $\theta = (\theta_{\alpha \beta})$. 
From involutions to Kähler forms

- Since $I_\alpha \circ I_\beta = -I_\beta \circ I_\alpha$, one gets 10 complex structures
  \[ J_{\alpha\beta} = I_\alpha \circ I_\beta \quad \text{for } \alpha < \beta \]

- The Kähler forms $\theta_{\alpha\beta}$ give rise to a $5 \times 5$ skew-symmetric matrix
  \[ \theta = (\theta_{\alpha\beta}) \]

Remark

Denote by $\tau_2(\theta)$ the second coefficient of the characteristic polynomial of $\theta = (\theta_{\alpha\beta})$. Then

\[ \Phi_{\text{Sp}(2) \cdot \text{Sp}(1)}^{\text{utc}} = \tau_2(\theta) \]
1. $S^{15}$ and Spin(9)
   - $S^{15}$ is “more equal” among other spheres
   - Spin(9) and Hopf fibrations

2. The Spin(9) fundamental form
   - Quaternionic analogy
   - Spin(9) and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\text{Spin}(9)}$

3. Vector fields on spheres
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$

4. Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem
Spin(9) is the subgroup of SO(16) generated by matrices

\[
\begin{pmatrix}
  r & R_{\bar{u}} \\
  R_u & -r
\end{pmatrix}
\]

where \((r, u) \in S^8 \subset \mathbb{R} \times \mathbb{O}\) and \(\mathbb{O}^2 \cong \mathbb{R}^{16}\)

[Harvey, Spinors and Calibrations 1990].
Spin(9) is the subgroup of SO(16) generated by matrices

\[
\begin{pmatrix}
  r & R_{\bar{u}} \\
  R_u & -r
\end{pmatrix}
\]

where \((r, u) \in S^8 \subset \mathbb{R} \times \mathbb{O}\) and \(\mathbb{O}^2 \cong \mathbb{R}^{16}\).

[Harvey, Spinors and Calibrations 1990].

The choice of \((r, u) = (1, 0), (0, 1), (0, i), (0, j), (0, k), (0, e), (0, f), (0, g), (0, h)\) gives

\[\mathcal{I}_1, \ldots, \mathcal{I}_9 \in \text{SO}(16)\]
Spin(9) is the subgroup of SO(16) generated by matrices

\[
\begin{pmatrix}
  r & R_{\bar{u}} \\
  R_u & -r
\end{pmatrix}
\]

where \((r, u) \in S^8 \subset \mathbb{R} \times \mathbb{O}\) and \(\mathbb{O}^2 \cong \mathbb{R}^{16}\)

[Harvey, Spinors and Calibrations 1990].

The choice of \((r, u) = (1, 0), (0, 1), (0, i), (0, j), (0, k), (0, e), (0, f), (0, g), (0, h)\) gives

\[\mathcal{I}_1, \ldots, \mathcal{I}_9 \in SO(16)\]

\[\mathcal{I}_1, \ldots, \mathcal{I}_9\] satisfy

\[\mathcal{I}_\alpha^2 = \text{Id}, \quad \mathcal{I}_\alpha^* = \mathcal{I}_\alpha, \quad \mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha\]
Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for } \alpha < \beta$$
From involutions to Kähler forms

- Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for } \alpha < \beta$$

$$\Lambda^2(\mathbb{R}^{16}) = \Lambda^2_{36} \oplus \Lambda^2_{84} = \text{spin}(9) \oplus \Lambda^2_{84}$$
Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for } \alpha < \beta$$

$$\Lambda^2(\mathbb{R}^{16}) = \Lambda^2_{36} \oplus \Lambda^2_{84} = \text{spin}(9) \oplus \Lambda^2_{84}$$

generated by $J_{\alpha\beta}$
From involutions to Kähler forms

Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for } \alpha < \beta$$

$$\Lambda^2(\mathbb{R}^{16}) = \Lambda^2_{36} \oplus \Lambda^2_{84} = \text{spin}(9) \oplus \Lambda^2_{84}$$

generated by $J_{\alpha\beta}$

generated by $J_{\alpha\beta\gamma}$
From involutions to Kähler forms

Since $\mathcal{I}_\alpha \circ \mathcal{I}_\beta = -\mathcal{I}_\beta \circ \mathcal{I}_\alpha$, one gets 36 complex structures

$$J_{\alpha\beta} = \mathcal{I}_\alpha \circ \mathcal{I}_\beta \quad \text{for } \alpha < \beta$$

$$\Lambda^2(\mathbb{R}^{16}) = \Lambda^2_{36} \oplus \Lambda^2_{84} = \text{spin}(9) \oplus \Lambda^2_{84}$$

- Their Kähler forms $\theta_{\alpha\beta}$ give rise to a $9 \times 9$ skew-symmetric matrix

$$\theta = (\theta_{\alpha\beta})$$
The fundamental form \( \text{Spin}(9) \) and Kähler forms on \( \mathbb{R}^{16} \)

From the Kähler forms to the \( \text{Spin}(9) \) form


Denote the characteristic polynomial of \( \theta \) by

\[
t^9 + \tau_2(\theta)t^7 + \tau_4(\theta)t^5 + \tau_6(\theta)t^3 + \tau_8(\theta)t
\]

Denote the characteristic polynomial of $\theta$ by

$$t^9 + \tau_2(\theta)t^7 + \tau_4(\theta)t^5 + \tau_6(\theta)t^3 + \tau_8(\theta)t$$

Then ($\tau_8(\theta) \overset{\text{utc}}{=} \text{volume form and}$)

$$\tau_2(\theta) = \tau_6(\theta) = 0 \quad \Phi_{\text{Spin}(9)} \overset{\text{utc}}{=} \tau_4(\theta)$$

Denote the characteristic polynomial of $\theta$ by

$$t^9 + \tau_2(\theta)t^7 + \tau_4(\theta)t^5 + \tau_6(\theta)t^3 + \tau_8(\theta)t$$

Then ($\tau_8(\theta) \overset{\text{utc}}{=} \text{volume form}$ and)

$$\tau_2(\theta) = \tau_6(\theta) = 0$$

$\Phi_{\text{Spin}(9)} \overset{\text{utc}}{=} \tau_4(\theta)$

Explicit formulas?
1. $S^{15}$ and Spin(9)
   - $S^{15}$ is “more equal” among other spheres
   - Spin(9) and Hopf fibrations

2. The Spin(9) fundamental form
   - Quaternionic analogy
   - Spin(9) and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\text{Spin}(9)}$

3. Vector fields on spheres
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$

4. Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem
The fundamental form of $\Phi_{\text{Spin}(9)}$

An explicit formula for $\Phi_{\text{Spin}(9)}$ is given by

$$\Phi_{\text{Spin}(9)} = \int_{\mathbb{P}^1} p^*_l \nu_l \, dl$$

This involves $\binom{16}{8} = 12870$ integrals of $\Phi_{\text{Spin}(9)}$

which can be computed with the help of Mathematica.

Previous work for $\Phi_{\text{Spin}(9)}$ includes:

- [Abe-Matsubara, Korea Japan Conf. Transf. Groups 1997],
- [Friedrich, Asian J. Math. 2001],
The $\binom{16}{8} = 12870$ integrals of

$$\Phi_{\text{Spin}(9)} = \int_{\mathbb{P}^1} p_i^* \nu_i \, dl$$

can be computed with the help of *Mathematica*.

1 \( S^{15} \) and Spin(9)
   - \( S^{15} \) is “more equal” among other spheres
   - Spin(9) and Hopf fibrations

2 The Spin(9) fundamental form
   - Quaternionic analogy
   - Spin(9) and Kähler forms on \( \mathbb{R}^{16} \)
   - An explicit formula for \( \Phi_{\text{Spin}(9)} \)

3 Vector fields on spheres
   - Maximal number and examples
   - Any \( S^{N-1} \subset \mathbb{R}^{N} \)

4 Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem
How many vector fields on spheres?

Spheres $S^{N-1} \subset \mathbb{R}^N$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1, 2$ or $3$ in

$$N = (2k + 1)2^p$$
How many vector fields on spheres?

- Spheres $S^{N-1} \subset \mathbb{R}^N$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1, 2$ or 3 in

$$N = (2k + 1)2^p$$

- In the general case

$$N = (2k + 1)2^p16^q \quad \text{with } q \geq 1 \quad \text{and} \quad p = 1, 2, 3$$

Spin, $\mathbb{H}, \mathbb{O}$ contribution

The lowest dimensional sphere with more than 7 vector field is $S^{15}$ [Hurwitz, Math. Ann. 1922], [Radon, Abh. Math. Hamburg 1923], [Adams, Ann. of Math. 1962].
Spheres $S^{N-1} \subset \mathbb{R}^N$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1, 2$ or 3 in

$$N = (2k + 1)2^p$$

In the general case

$$N = (2k + 1)2^p16^q$$ with $q \geq 1$ and $p = 1, 2, 3$

the maximal number of vector fields is

$$\sigma(N) = + 2^p - 1$$
How many vector fields on spheres?

- Spheres $S^{N-1} \subset \mathbb{R}^N$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1, 2$ or 3 in

$$N = (2k + 1)2^p$$

- In the general case

$$N = (2k + 1)2^p16^q \quad \text{with } q \geq 1 \quad \text{and} \quad p = 1, 2, 3$$

the maximal number of vector fields is

$$\sigma(N) = 8q + 2^p - 1$$

Spin(9) contribution

C, H, O contribution
How many vector fields on spheres?

- Spheres $S^{N-1} \subset \mathbb{R}^N$ admit 1, 3 or 7 linearly independent vector fields according to whether $p = 1, 2$ or 3 in

$$N = (2k + 1)2^p$$

- In the general case

$$N = (2k + 1)2^p16^q$$ with $q \geq 1$ and $p = 1, 2, 3$

the maximal number of vector fields is

$$\sigma(N) = 8q + 2^p - 1$$

The lowest dimensional sphere with more than 7 vector field is $S^{15}$

The lowest dimension: $S^{15}$

- Coordinates on $S^{15}$:

$$B = (x, y) = (x_1, \ldots, x_8, y_1, \ldots, y_8)$$

unit normal vector field
The lowest dimension: $S^{15}$

- Coordinates on $S^{15}$:
  \[ B = (x, y) = (x_1, \ldots, x_8, y_1, \ldots, y_8) \]  
  unit normal vector field

- $J_{\alpha\beta} =$ complex structures on $\mathbb{R}^{16}$ associated to the Spin(9) structure.
The lowest dimension: $S^{15}$

- Coordinates on $S^{15}$:
  \[ B = (x, y) = (x_1, \ldots, x_8, y_1, \ldots, y_8) \]  
  unit normal vector field

- $J_{\alpha\beta}$ = complex structures on $\mathbb{R}^{16}$ associated to the Spin(9) structure.

**Proposition**

A maximal system of 8 orthonormal vector fields on $S^{15}$ is given by

\[ J_{19}(B), J_{29}(B), J_{39}(B), J_{49}(B), J_{59}(B), J_{69}(B), J_{79}(B), J_{89}(B) \]
The lowest dimension: $S^{15}$

- Coordinates on $S^{15}$:
  
  \[ B = (x, y) = (x_1, \ldots, x_8, y_1, \ldots, y_8) \text{ unit normal vector field} \]

- \( J_{\alpha\beta} = \) complex structures on \( \mathbb{R}^{16} \) associated to the \( \text{Spin}(9) \) structure.

**Proposition**

A maximal system of 8 orthonormal vector fields on $S^{15}$ is given by

\[ J_{19}(B), J_{29}(B), J_{39}(B), J_{49}(B), J_{59}(B), J_{69}(B), J_{79}(B), J_{89}(B) \]

**Remark**

The frame \( \{ J_{19}(B), \ldots, J_{89}(B) \} \) has *nothing to do* with Hopf fibrations.
### Other spheres with $\sigma(N) > 7$

<table>
<thead>
<tr>
<th>Sphere</th>
<th>$\sigma(N)$</th>
<th>Vector fields</th>
<th>Notations</th>
<th>Involved structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^{15}$</td>
<td>8</td>
<td>$J_{19}B, \ldots, J_{89}B$</td>
<td>$B = s = (x, y)$</td>
<td>Spin(9)</td>
</tr>
<tr>
<td>$S^{31}$</td>
<td>9</td>
<td>$J_{19}B, \ldots, J_{89}B$ $\star \mathcal{L}_i B$</td>
<td>$B = s^1 + is^2, \mathcal{L}_i B = -s^2 + is^1$ $\star : (x, y) \mapsto (x, -y)$</td>
<td>Spin(9) + $\mathbb{C}$</td>
</tr>
<tr>
<td>$S^{63}$</td>
<td>11</td>
<td>$J_{19}B, \ldots, J_{89}B$ $\star \mathcal{L}_i B, \star \mathcal{L}_j B, \star \mathcal{L}_k B$</td>
<td>$B = s^1 + is^2 + js^3 + ks^4$ $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$ and $\star$ as above</td>
<td>Spin(9) + $\mathbb{H}$</td>
</tr>
<tr>
<td>$S^{127}$</td>
<td>15</td>
<td>$J_{19}B, \ldots, J_{89}B$ $\star \mathcal{L}_i B, \ldots, \star \mathcal{L}_h B$</td>
<td>$B = s^1 + is^2 + js^3 + ks^4 + es^5 + fs^6 + gs^7 + hs^8$ $\mathcal{L}_i, \ldots, \mathcal{L}_h$ and $\star$ as above</td>
<td>Spin(9) + $\mathbb{O}$</td>
</tr>
<tr>
<td>$S^{255}$</td>
<td>16</td>
<td>$J_{19}B, \ldots, J_{89}B$ $\star J_{19}^1 B, \ldots, \star J_{89}^1 B$</td>
<td>See explanations below</td>
<td>$(\text{Spin}(9))^2$</td>
</tr>
</tbody>
</table>
## Other spheres with $\sigma(N) > 7$

<table>
<thead>
<tr>
<th>Sphere</th>
<th>$\sigma(N)$</th>
<th>Vector fields</th>
<th>Notations</th>
<th>Involved structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^{255}$</td>
<td>16</td>
<td>$J_{19}B, \ldots, J_{89}B$</td>
<td>See explanations below</td>
<td>$(\text{Spin}(9))^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\star J_{19}^1B, \ldots, \star J_{89}^1B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S^{15}$</td>
<td>8</td>
<td>$J_{19}B, \ldots, J_{89}B$</td>
<td>$B = s = (x, y)$</td>
<td>Spin(9)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\star L_i B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S^{31}$</td>
<td>9</td>
<td>$J_{19}B, \ldots, J_{89}B$</td>
<td>$B = s^1 + is^2, L_i B = -s^2 + is^1$</td>
<td>Spin(9)+C</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\star : (x, y) \rightarrow (x, -y)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\star L_i B, \star L_j B, \star L_k B$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S^{63}$</td>
<td>11</td>
<td>$J_{19}B, \ldots, J_{89}B$</td>
<td>$B = s^1 + is^2 + js^3 + ks^4$</td>
<td>Spin(9)+\mathbb{H}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\star L_i B, \star L_j B, \star L_k B$</td>
<td>$L_i, L_j, L_k$ and $\star$ as above</td>
<td></td>
</tr>
<tr>
<td>$S^{127}$</td>
<td>15</td>
<td>$J_{19}B, \ldots, J_{89}B$</td>
<td>$B = s^1 + is^2 + js^3 + ks^4 + es^5 + fs^6 + gs^7 + hs^8$</td>
<td>Spin(9)+Ω</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\star L_i B, \ldots, \star L_h B$</td>
<td>$L_i, \ldots, L_h$ and $\star$ as above</td>
<td></td>
</tr>
<tr>
<td>$S^{255}$</td>
<td>16</td>
<td>$J_{19}B, \ldots, J_{89}B$</td>
<td>See explanations below</td>
<td>$(\text{Spin}(9))^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\star J_{19}^1B, \ldots, \star J_{89}^1B$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Other spheres with $\sigma(N) > 7$

<table>
<thead>
<tr>
<th>Sphere</th>
<th>$\sigma(N)$</th>
<th>Vector fields</th>
<th>Notations</th>
<th>Involved structures</th>
</tr>
</thead>
</table>
| $S^{255}$ | 16 | $J_{19}B, \ldots, J_{89}B$  
$\star J_{19}^1B, \ldots, \star J_{89}^1B$ | See explanations below | $(\text{Spin}(9))^2$ |

For $S^{255} \subset \mathbb{R}^{256}$ the $J_{19}, \ldots, J_{89}$ are defined on the 16-dimensional components of

$$\mathbb{R}^{256} = \mathbb{R}^{16} \oplus \ldots \oplus \mathbb{R}^{16}$$

The $J_{19}^1, \ldots, J_{89}^1$ are defined by the same $16 \times 16$ real matrices as $J_{19}, \ldots, J_{89}$, but acting formally on 16-ples of sedenions

$$B = (s^{1}, \ldots, s^{16}) \in \mathbb{R}^{256}$$
1. $S^{15}$ and Spin(9)
   - $S^{15}$ is “more equal” among other spheres
   - Spin(9) and Hopf fibrations

2. The Spin(9) fundamental form
   - Quaternionic analogy
   - Spin(9) and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\text{Spin}(9)}$

3. Vector fields on spheres
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^{N}$

4. Locally conformal parallel Spin(9) manifolds
   - Definition and examples
   - Structure Theorem
A $\ast$-algebra $\mathcal{A}$ is a real algebra equipped with a conjugation, namely a linear map $\ast : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$
a** = a, \quad (ab)^* = b^* a^*
$$
A $\ast$-algebra $\mathcal{A}$ is a real algebra equipped with a conjugation, namely a linear map $\ast : \mathcal{A} \to \mathcal{A}$ such that

$$a^{**} = a, \quad (ab)^* = b^*a^*$$

A new $\ast$-algebra structure can be defined on $\mathcal{A} \times \mathcal{A}$ by

$$(a, b)(c, d) = (ac - d^*b, da + bc^*) \quad \text{and} \quad (a, b)^* = (a^*, -b)$$
A $\ast$-algebra $\mathcal{A}$ is a real algebra equipped with a conjugation, namely a linear map $\ast : \mathcal{A} \to \mathcal{A}$ such that

$$a^{**} = a, \quad (ab)^* = b^*a^*$$

A new $\ast$-algebra structure can be defined on $\mathcal{A} \times \mathcal{A}$ by

$$(a, b)(c, d) = (ac - d^*b, da + bc^*) \quad \text{and} \quad (a, b)^* = (a^*, -b)$$

This produces

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to S \to \ldots$$
### Multiplication table for sedenions

<table>
<thead>
<tr>
<th></th>
<th>e₁</th>
<th>e₂</th>
<th>e₃</th>
<th>e₄</th>
<th>e₅</th>
<th>e₆</th>
<th>e₇</th>
<th>e₈</th>
<th>e₉</th>
<th>e₁₀</th>
<th>e₁₁</th>
<th>e₁₂</th>
<th>e₁₃</th>
<th>e₁₄</th>
<th>e₁₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>e₁</td>
<td>-1</td>
<td>e₃</td>
<td>-e₂</td>
<td>e₅</td>
<td>-e₄</td>
<td>-e₇</td>
<td>e₆</td>
<td>e₉</td>
<td>-e₈</td>
<td>-e₁₁</td>
<td>e₁₀</td>
<td>-e₁₃</td>
<td>e₁₂</td>
<td>e₁₅</td>
<td>-e₁₄</td>
</tr>
<tr>
<td>e₂</td>
<td>-e₃</td>
<td>-1</td>
<td>e₁</td>
<td>e₆</td>
<td>e₇</td>
<td>-e₄</td>
<td>-e₅</td>
<td>e₁₀</td>
<td>e₁₁</td>
<td>-e₈</td>
<td>-e₉</td>
<td>-e₁₄</td>
<td>-e₁₅</td>
<td>e₁₂</td>
<td>e₁₃</td>
</tr>
<tr>
<td>e₃</td>
<td>e₂</td>
<td>-e₁</td>
<td>-1</td>
<td>e₇</td>
<td>-e₆</td>
<td>e₅</td>
<td>-e₄</td>
<td>e₁₁</td>
<td>-e₁₀</td>
<td>e₉</td>
<td>-e₈</td>
<td>-e₁₅</td>
<td>e₁₄</td>
<td>-e₁₃</td>
<td>e₁₂</td>
</tr>
<tr>
<td>e₄</td>
<td>-e₅</td>
<td>-e₆</td>
<td>-e₇</td>
<td>-1</td>
<td>e₁</td>
<td>e₂</td>
<td>e₃</td>
<td>e₁₂</td>
<td>e₁₃</td>
<td>e₁₄</td>
<td>e₁₅</td>
<td>-e₈</td>
<td>-e₉</td>
<td>-e₁₀</td>
<td>-e₁₁</td>
</tr>
<tr>
<td>e₅</td>
<td>e₄</td>
<td>-e₇</td>
<td>e₆</td>
<td>-e₁</td>
<td>-1</td>
<td>e₃</td>
<td>e₂</td>
<td>e₁₃</td>
<td>-e₁₂</td>
<td>e₁₅</td>
<td>-e₁₄</td>
<td>e₉</td>
<td>-e₈</td>
<td>e₁₁</td>
<td>-e₁₀</td>
</tr>
<tr>
<td>e₆</td>
<td>e₇</td>
<td>e₄</td>
<td>-e₅</td>
<td>-e₂</td>
<td>e₃</td>
<td>-1</td>
<td>-e₁</td>
<td>e₁₄</td>
<td>-e₁₅</td>
<td>-e₁₂</td>
<td>e₁₃</td>
<td>e₁₀</td>
<td>-e₁₁</td>
<td>-e₈</td>
<td>e₉</td>
</tr>
<tr>
<td>e₇</td>
<td>-e₆</td>
<td>e₅</td>
<td>e₄</td>
<td>-e₃</td>
<td>-e₂</td>
<td>e₁</td>
<td>-1</td>
<td>e₁₅</td>
<td>e₁₄</td>
<td>-e₁₃</td>
<td>-e₁₂</td>
<td>e₁₁</td>
<td>e₁₀</td>
<td>-e₉</td>
<td>-e₈</td>
</tr>
<tr>
<td>e₈</td>
<td>-e₉</td>
<td>-e₁₀</td>
<td>-e₁₁</td>
<td>-e₁₂</td>
<td>-e₁₃</td>
<td>-e₁₄</td>
<td>-e₁₅</td>
<td>-1</td>
<td>e₁</td>
<td>e₂</td>
<td>e₃</td>
<td>e₄</td>
<td>e₅</td>
<td>e₆</td>
<td>e₇</td>
</tr>
<tr>
<td>e₉</td>
<td>e₈</td>
<td>-e₁₁</td>
<td>e₁₀</td>
<td>-e₁₃</td>
<td>e₁₂</td>
<td>e₁₅</td>
<td>-e₁₄</td>
<td>-e₁</td>
<td>-1</td>
<td>-e₃</td>
<td>e₂</td>
<td>-e₅</td>
<td>e₄</td>
<td>e₇</td>
<td>-e₆</td>
</tr>
<tr>
<td>e₁₀</td>
<td>e₁₁</td>
<td>e₈</td>
<td>-e₉</td>
<td>-e₁₄</td>
<td>-e₁₅</td>
<td>e₁₂</td>
<td>e₁₃</td>
<td>-e₂</td>
<td>e₃</td>
<td>-1</td>
<td>-e₁</td>
<td>-e₆</td>
<td>-e₇</td>
<td>e₄</td>
<td>e₅</td>
</tr>
<tr>
<td>e₁₁</td>
<td>-e₁₀</td>
<td>e₉</td>
<td>e₈</td>
<td>-e₁₅</td>
<td>e₁₄</td>
<td>-e₁₃</td>
<td>e₁₂</td>
<td>-e₃</td>
<td>-e₂</td>
<td>e₁</td>
<td>-1</td>
<td>-e₇</td>
<td>e₆</td>
<td>-e₅</td>
<td>e₄</td>
</tr>
<tr>
<td>e₁₂</td>
<td>e₁₃</td>
<td>e₁₄</td>
<td>e₁₅</td>
<td>e₈</td>
<td>-e₉</td>
<td>-e₁₀</td>
<td>-e₁₁</td>
<td>-e₄</td>
<td>e₅</td>
<td>e₆</td>
<td>e₇</td>
<td>-1</td>
<td>-e₁</td>
<td>-e₂</td>
<td>-e₃</td>
</tr>
<tr>
<td>e₁₃</td>
<td>-e₁₂</td>
<td>e₁₅</td>
<td>-e₁₄</td>
<td>e₉</td>
<td>e₈</td>
<td>e₁₁</td>
<td>-e₁₀</td>
<td>-e₅</td>
<td>-e₄</td>
<td>e₇</td>
<td>-e₆</td>
<td>e₁</td>
<td>-1</td>
<td>e₃</td>
<td>-e₂</td>
</tr>
<tr>
<td>e₁₄</td>
<td>-e₁₅</td>
<td>-e₁₂</td>
<td>e₁₃</td>
<td>e₁₀</td>
<td>-e₁₁</td>
<td>e₈</td>
<td>e₉</td>
<td>e₆</td>
<td>-e₇</td>
<td>-e₄</td>
<td>e₅</td>
<td>e₂</td>
<td>-e₃</td>
<td>-1</td>
<td>e₁</td>
</tr>
<tr>
<td>e₁₅</td>
<td>e₁₄</td>
<td>-e₁₂</td>
<td>-e₁₁</td>
<td>e₁₀</td>
<td>e₉</td>
<td>e₇</td>
<td>e₆</td>
<td>-e₅</td>
<td>-e₄</td>
<td>e₃</td>
<td>e₂</td>
<td>-e₁</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Vector fields in the general case


\[ \sigma(N) > 7? \quad \text{All the fault of Spin}(9) \]
### Vector fields in the general case

**Theorem [P-Piccinni, arXiv: 1107.0462, 2011]**

\[ \sigma(N) > 7? \quad \text{All the fault of Spin}(9) \]

<table>
<thead>
<tr>
<th>((k, p, q))</th>
<th>Sphere (S^{(2k+1)16^q-1})</th>
<th>(\sigma(N))</th>
<th>Vector fields</th>
<th>Involved structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>((k, 0, q))</td>
<td>(S^{(2k+1)16^q-1}) 8q</td>
<td>(J_{19}B, \ldots, J_{89}B) (\ast J_{19}^1B, \ldots, \ast J_{89}^1B) (\ast J_{19}^{q-1}B, \ldots, \ast J_{89}^{q-1}B)</td>
<td>(\text{(Spin}(9))^q)</td>
<td></td>
</tr>
<tr>
<td>((k, 1, q))</td>
<td>(S^{2(2k+1)16^q-1}) 8q + 1</td>
<td>(J_{19}B, \ldots, J_{89}B) (\ast J_{19}^1B, \ldots, \ast J_{89}^1B) (\ast J_{19}^{q-1}B, \ldots, \ast J_{89}^{q-1}B) (\ast L_iB)</td>
<td>(\text{(Spin}(9))^q + \mathbb{C})</td>
<td></td>
</tr>
<tr>
<td>((k, 2, q))</td>
<td>(S^{4(2k+1)16^q-1}) 8q + 3</td>
<td>(J_{19}B, \ldots, J_{89}B) (\ast J_{19}^1B, \ldots, \ast J_{89}^1B) (\ast J_{19}^{q-1}B, \ldots, \ast J_{89}^{q-1}B) (\ast L_iB, \ast L_jB, \ast L_kB)</td>
<td>(\text{(Spin}(9))^q + \mathbb{H})</td>
<td></td>
</tr>
<tr>
<td>((k, 3, q))</td>
<td>(S^{8(2k+1)16^q-1}) 8q + 7</td>
<td>(J_{19}B, \ldots, J_{89}B) (\ast J_{19}^1B, \ldots, \ast J_{89}^1B) (\ast J_{19}^{q-1}B, \ldots, \ast J_{89}^{q-1}B) (\ast L_iB \cdots \ast L_hB)</td>
<td>(\text{(Spin}(9))^q + \mathbb{O})</td>
<td></td>
</tr>
</tbody>
</table>

**Sketch of proof**
1 **$S^{15}$ and Spin(9)**
   - $S^{15}$ is “more equal” among other spheres
   - Spin(9) and Hopf fibrations

2 **The Spin(9) fundamental form**
   - Quaternionic analogy
   - Spin(9) and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\text{Spin}(9)}$

3 **Vector fields on spheres**
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$

4 **Locally conformal parallel Spin(9) manifolds**
   - Definition and examples
   - Structure Theorem
Definition

A locally conformal parallel Spin(9) manifold is a 16-dimensional Spin(9) manifold whose induced metric is locally conformal to metrics with holonomy contained in Spin(9).
A locally conformal parallel $\text{Spin}(9)$ manifold is a 16-dimensional $\text{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\text{Spin}(9)$. 

$(M, g)$ with a $\text{Spin}(9)$-structure
A locally conformal parallel $\text{Spin}(9)$ manifold is a 16-dimensional $\text{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\text{Spin}(9)$. 

$$(M, g) \text{ with a Spin}(9)\text{-structure}$$
A locally conformal parallel $\text{Spin}(9)$ manifold is a 16-dimensional $\text{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\text{Spin}(9)$. 

$$(M, g)$$ with a $\text{Spin}(9)$-structure
A locally conformal parallel $\text{Spin}(9)$ manifold is a 16-dimensional $\text{Spin}(9)$ manifold whose induced metric is locally conformal to metrics with holonomy contained in $\text{Spin}(9)$.

\[(M, g)\] with a $\text{Spin}(9)$-structure

\[g|_{U_\alpha} = e^{f_\alpha} g_\alpha \text{ where } g_\alpha \text{ has holonomy contained in } \text{Spin}(9)\]
The product $S^{15} \times S^1 = \mathbb{S}^{\mathbb{Z}^2-0} = \text{cone over } S^{15}$ with the (conformal class) of the flat metric.
Examples

The product $S^{15} \times S^1 = \mathbb{D}^2/\mathbb{Z} = \text{cone over } S^{15}$ with the (conformal class) of the flat metric.

The trivial $S^1$-bundle $\mathbb{R}P^{15} \times S^1$, with the metric induced by the flat cone $C(S^{15})$. 
Examples

The product $S^{15} \times S^1 = \mathbb{O}^2/\mathbb{Z} = $ cone over $S^{15}$ with the (conformal class) of the flat metric.

The trivial $S^1$-bundle $\mathbb{R}P^{15} \times S^1$, with the metric induced by the flat cone $C(S^{15})$.

The non-trivial $S^1$-bundle over $\mathbb{R}P^{15}$, with the metric induced by the flat cone $C(S^{15})$. 
1. $S^{15}$ and $\text{Spin}(9)$
   - $S^{15}$ is “more equal” among other spheres
   - $\text{Spin}(9)$ and Hopf fibrations

2. The $\text{Spin}(9)$ fundamental form
   - Quaternionic analogy
   - $\text{Spin}(9)$ and Kähler forms on $\mathbb{R}^{16}$
   - An explicit formula for $\Phi_{\text{Spin}(9)}$

3. Vector fields on spheres
   - Maximal number and examples
   - Any $S^{N-1} \subset \mathbb{R}^N$

4. Locally conformal parallel $\text{Spin}(9)$ manifolds
   - Definition and examples
   - Structure Theorem
Structure of compact locally conformal parallel $\text{Spin}(9)$ manifolds

**Theorem [P-Piccinni-Vuletescu]**

Let $(M, g)$ be a compact, locally conformal but not globally conformal parallel $\text{Spin}(9)$ manifold. Then

$$M = C(N)/\mathbb{Z}$$

where $C(N)$ is a flat cone over a compact 15-dimensional manifold $N$ with finite fundamental group.
Proof

1. On each $U_\alpha$ it is defined a $\nabla^\alpha$-parallel 8-form $\Phi_\alpha$. 
1. On each $U_\alpha$ it is defined a $\nabla^\alpha$-parallel 8-form $\Phi_\alpha$.
2. There is a 8-form $\Phi$ on $M$ locally given by $e^{4f_\alpha} \Phi_\alpha$. 
On each $U_\alpha$ it is defined a $\nabla^\alpha$-parallel 8-form $\Phi_\alpha$.

There is a 8-form $\Phi$ on $M$ locally given by $e^{4f_\alpha} \Phi_\alpha$.

There is a closed 1-form $\omega$ (the Lee form) on $M$, locally given by $4df_\alpha$, such that $d\Phi = \omega \wedge \Phi$. 

Proof
Proof

1. On each $U_\alpha$ it is defined a $\nabla^\alpha$-parallel 8-form $\Phi_\alpha$.
2. There is a 8-form $\Phi$ on $M$ locally given by $e^{4f_\alpha} \Phi_\alpha$.
3. There is a closed 1-form $\omega$ (the Lee form) on $M$, locally given by $4df_\alpha$, such that $d\Phi = \omega \wedge \Phi$.
4. The 1-form $\omega$ defines a closed Weyl connection $D$ on $M$ by $Dg = \omega \otimes g$. 
Proof

1. On each $U_\alpha$ it is defined a $\nabla^\alpha$-parallel 8-form $\Phi_\alpha$.

2. There is a 8-form $\Phi$ on $M$ locally given by $e^{4f_\alpha} \Phi_\alpha$.

3. There is a closed 1-form $\omega$ (the Lee form) on $M$, locally given by $4 df_\alpha$, such that $d\Phi = \omega \wedge \Phi$.

4. The 1-form $\omega$ defines a closed Weyl connection $D$ on $M$ by $Dg = \omega \otimes g$.

5. Since the local metrics $g_\alpha$ are Einstein, $D$ is Einstein-Weyl.
Let $g$ be the Gauduchon metric, so that $\nabla \omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$. 
Let $g$ be the Gauduchon metric, so that $\nabla \omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_\tilde{N})$, for a compact simply connected $\tilde{N}$.

On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f} \tilde{g})$ is the metric cone $C(\tilde{N})$. 
Let $g$ be the Gauduchon metric, so that $\nabla \omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.

On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f} \tilde{g})$ is the metric cone $C(\tilde{N})$.

The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.
Let $g$ be the Gauduchon metric, so that $\nabla \omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_{\tilde{N}})$, for a compact simply connected $\tilde{N}$.

On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f} \tilde{g})$ is the metric cone $C(\tilde{N})$.

The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.

Ricci-flat + holonomy $\text{Spin}(9) \Rightarrow$ flat.
Let $g$ be the Gauduchon metric, so that $\nabla \omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.

On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f} \tilde{g})$ is the metric cone $C(\tilde{N})$.

The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.

Ricci-flat + holonomy $\text{Spin}(9) \Rightarrow$ flat.

Since $\pi_1(M)$ acts by homotheties on $C(\tilde{N})$, and $\tilde{N}$ is compact, $\pi_1(M)$ contains a finite normal subgroup $I$ of isometries.
Let $g$ be the Gauduchon metric, so that $\nabla \omega = 0$. Then the universal covering $(\tilde{M}, \tilde{g})$ is reducible: $(\tilde{M}, \tilde{g}) = (\mathbb{R}, ds) \times (\tilde{N}, g_N)$, for a compact simply connected $\tilde{N}$.

On $\tilde{M}$ we have $\tilde{\omega} = df$, and $(\tilde{M}, e^{-f} \tilde{g})$ is the metric cone $C(\tilde{N})$.

The local metrics are Ricci-flat, that is, $C(\tilde{N})$ is Ricci-flat.

Ricci-flat + holonomy $\text{Spin}(9) \Rightarrow$ flat.

Since $\pi_1(M)$ acts by homotheties on $C(\tilde{N})$, and $\tilde{N}$ is compact, $\pi_1(M)$ contains a finite normal subgroup $I$ of isometries.

We obtain $\pi_1(M) = I \rtimes \mathbb{Z}$, and $M = C(\tilde{N}/I)/\mathbb{Z}$. 
Surprise: end of talk!
Details for $\Phi_{\text{Spin}(9)} = \int_{\mathbb{O}P^1} p_l^* \nu_l \, dl$

- $\nu_l = \text{volume form on the octonionic lines } l = \{(x, mx)\} \text{ or } l = \{(0, y)\} \text{ in } \mathbb{O}^2$.
- $p_l : \mathbb{O}^2 \to l = \text{projection on } l$.
- $p_l^* \nu_l = \text{8-form in } \mathbb{O}^2 = \mathbb{R}^{16}$.
- The integral over $\mathbb{O}P^1$ can be computed over $\mathbb{O}$ with polar coordinates.
- The formula arise from distinguished 8-planes in the Spin(9)-geometry $\to$ (forthcoming) calibrations.
The five involutions of $\text{Sp}(2) \cdot \text{Sp}(1)$ as $8 \times 8$ matrices

- $\mathcal{I}_1 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}$
- $\mathcal{I}_2 = \begin{pmatrix} 0 & -R_i^\mathbb{H} \\ R_i^\mathbb{H} & 0 \end{pmatrix}$
- $\mathcal{I}_3 = \begin{pmatrix} 0 & -R_j^\mathbb{H} \\ R_j^\mathbb{H} & 0 \end{pmatrix}$
- $\mathcal{I}_4 = \begin{pmatrix} 0 & -R_k^\mathbb{H} \\ R_k^\mathbb{H} & 0 \end{pmatrix}$
- $\mathcal{I}_5 = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}$
The nine involutions of $\text{Spin}(9)$ as $16 \times 16$ matrices:

- $\mathcal{I}_1 = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}$
- $\mathcal{I}_3 = \begin{pmatrix} 0 & -R_j \\ R_j & 0 \end{pmatrix}$
- $\mathcal{I}_4 = \begin{pmatrix} 0 & -R_k \\ R_k & 0 \end{pmatrix}$
- $\mathcal{I}_5 = \begin{pmatrix} 0 & -R_e \\ R_e & 0 \end{pmatrix}$
- $\mathcal{I}_6 = \begin{pmatrix} 0 & -R_f \\ R_f & 0 \end{pmatrix}$
- $\mathcal{I}_7 = \begin{pmatrix} 0 & -R_g \\ R_g & 0 \end{pmatrix}$
- $\mathcal{I}_8 = \begin{pmatrix} 0 & -R_h \\ R_h & 0 \end{pmatrix}$
- $\mathcal{I}_9 = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}$
Explicit formula for $\Phi_{G_2}$

Denote by $x_1, \ldots, x_7$ the coordinates in $\mathbb{R}^7$. Then $G_2 = \text{stabilizer in } SO(7)$ of

$$
\Phi_{G_2} = dx_1 \wedge dx_2 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5 + dx_3 \wedge dx_4 \wedge dx_6 \\
+ dx_4 \wedge dx_5 \wedge dx_7 + dx_5 \wedge dx_6 \wedge dx_1 + dx_6 \wedge dx_7 \wedge dx_2 \\
+ dx_7 \wedge dx_1 \wedge dx_3
$$

As a shortcut, we could write

$$
\Phi_{G_2} = 124 + 235 + 346 + 457 + 561 + 672 + 713
$$
<table>
<thead>
<tr>
<th>12345678</th>
<th>-14</th>
</tr>
</thead>
<tbody>
<tr>
<td>123457</td>
<td>1'3'</td>
</tr>
<tr>
<td>123458</td>
<td>2'3'</td>
</tr>
<tr>
<td>123467</td>
<td>5'8'</td>
</tr>
<tr>
<td>123468</td>
<td>6'9'</td>
</tr>
<tr>
<td>123467</td>
<td>1'2'3'4'5'6'7'8'</td>
</tr>
<tr>
<td>123467</td>
<td>5'6'7'8'</td>
</tr>
<tr>
<td>123468</td>
<td>2'3'4'5'6'7'8'</td>
</tr>
<tr>
<td>123457</td>
<td>5'6'7'8'</td>
</tr>
<tr>
<td>123458</td>
<td>2'3'4'5'6'7'8'</td>
</tr>
<tr>
<td>123457</td>
<td>5'6'7'8'</td>
</tr>
<tr>
<td>123458</td>
<td>2'3'4'5'6'7'8'</td>
</tr>
<tr>
<td>123457</td>
<td>5'6'7'8'</td>
</tr>
<tr>
<td>123458</td>
<td>2'3'4'5'6'7'8'</td>
</tr>
<tr>
<td>123457</td>
<td>5'6'7'8'</td>
</tr>
<tr>
<td>123458</td>
<td>2'3'4'5'6'7'8'</td>
</tr>
<tr>
<td>123457</td>
<td>5'6'7'8'</td>
</tr>
<tr>
<td>123458</td>
<td>2'3'4'5'6'7'8'</td>
</tr>
<tr>
<td>123457</td>
<td>5'6'7'8'</td>
</tr>
<tr>
<td>123458</td>
<td>2'3'4'5'6'7'8'</td>
</tr>
</tbody>
</table>

351 terms of $\Phi$ of $\text{Spin}(9)$ manifolds
Locally conformal parallel Spin(9) manifolds

70 terms of $\Phi_{\text{Spin}(9)}$

<p>| | | | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>12345678</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td>123456</td>
<td></td>
</tr>
<tr>
<td>-14</td>
<td>1'3'</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td></td>
</tr>
<tr>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td>123457</td>
<td></td>
</tr>
<tr>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td></td>
</tr>
<tr>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td>123458</td>
<td></td>
</tr>
<tr>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td>2  '</td>
<td></td>
</tr>
<tr>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td>123467</td>
<td></td>
</tr>
<tr>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td>5  '</td>
<td></td>
</tr>
<tr>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td>123468</td>
<td></td>
</tr>
<tr>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td>6  '</td>
<td></td>
</tr>
<tr>
<td>1234</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td>1'2'3'4'</td>
<td></td>
</tr>
<tr>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td>1235</td>
<td></td>
</tr>
<tr>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td>1'7'</td>
<td></td>
</tr>
<tr>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td>12351</td>
<td></td>
</tr>
<tr>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td>1'3'4'7'</td>
<td></td>
</tr>
</tbody>
</table>

- $\{1, 2, 3, 4, 5, 6, 7, 8, 1', 2', 3', 4', 5', 6', 7', 8'\}$ are (indexes of) coordinates in $\mathbb{R}^{16}$.
- A table entry $||123578 \ 1'7' \ -2||$ means that $\Phi_{\text{Spin}(9)} = \cdots - 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 \wedge dx_7 \wedge dx_8 \wedge dx'_1 \wedge dx'_7 + \cdots$
- Table obtained from Berger’s definition of $\Phi_{\text{Spin}(9)}$ with the help of Mathematica.
- The coefficients are normalized in such a way that they are all integers with gcd = 1.
In progress.
Locally conformal parallel Spin(9) manifolds

Structure Theorem

The inductive argument

- Reduce to the case $S^{16q-1} \subset \mathbb{R}^{16q}$, and use induction on $q$.
- Assume that there are $8(q-1)$ independent vector fields $B_1, \ldots, B_{8(q-1)}$ on $S^{16q-1} \subset \mathbb{R}^{16q-1}$.
- Look at $\mathbb{R}^{16q}$ as 16 copies of $\mathbb{R}^{16q-1}$:

  $\mathbb{R}^{16q} = \{(s_1, \ldots, s_{16}) | s_1, \ldots, s_{16} \in \mathbb{R}^{16q-1}\}$

- Define complex structures $J'_1, \ldots, J'_{89}$ on $\mathbb{R}^{16q}$ by the same matrices defining $J_1, \ldots, J_{89}$ but acting formally on the 16-ples $(s_1, \ldots, s_{16})$ of elements in $\mathbb{R}^{16q-1}$.
- Let $B$ be the radial vector field on $S^{16q-1} \subset \mathbb{R}^{16q}$. Prove that 

  $\{B_1, \ldots, B_{8(q-1)}, J'_1 B, \ldots, J'_{89} B\}$ is an orthonormal frame on $S^{16q-1}$.
The inductive argument

- Reduce to the case $S^{16q-1} \subset \mathbb{R}^{16q}$, and use induction on $q$.
- Assume that there are $8(q-1)$ independent vector fields $B_1, \ldots, B_{8(q-1)}$ on $S^{16q-1-1} \subset \mathbb{R}^{16q-1}$.
- Look at $\mathbb{R}^{16q}$ as 16 copies of $\mathbb{R}^{16q-1}$:

$$\mathbb{R}^{16q} = \{(s_1, \ldots, s_{16}) | s_1, \ldots, s_{16} \in \mathbb{R}^{16q-1}\}$$

- Define complex structures $J'_1, \ldots, J'_{89}$ on $\mathbb{R}^{16q}$ by the same matrices defining $J_1, \ldots, J_{89}$ but acting formally on the 16-ples $(s_1, \ldots, s_{16})$ of elements in $\mathbb{R}^{16q-1}$.
- Let $B$ be the radial vector field on $S^{16q-1} \subset \mathbb{R}^{16q}$. Prove that $\{B_1, \ldots, B_{8(q-1)}, J'_1 B, \ldots, J'_{89} B\}$ is an orthonormal frame on $S^{16q-1}$.
Lemma

The properties $(ab)^* = b^*a^*$, $\Re([a, b, c]) = 0$ and $<a, b> = \Re(ab^*)$ hold in any Cayley-Dickson algebra.
Locally conformal parallel Spin(9) manifolds

A more explicit $\Phi_{\text{Spin}(9)}$

$$\Phi_{\text{Spin}(9)} \overset{\text{utc}}{=} \sum_{1 \leq \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leq 9} \left( \psi_{\alpha_1 \alpha_2} \wedge \psi_{\alpha_3 \alpha_4} - \psi_{\alpha_1 \alpha_3} \wedge \psi_{\alpha_2 \alpha_4} + \psi_{\alpha_1 \alpha_4} \wedge \psi_{\alpha_2 \alpha_3} \right)^2$$

$$\begin{align*}
\psi_{12} &= (-12 + 34 + 56 - 78) - ( )' \\
\psi_{15} &= (-15 - 26 - 37 - 48) - ( )' \\
\psi_{18} &= (-18 - 27 + 36 + 45) - ( )' \\
\psi_{25} &= (-16 + 25 + 38 - 47) + ( )' \\
\psi_{28} &= (-17 + 28 - 35 + 46) + ( )' \\
\psi_{36} &= (-18 + 27 + 36 - 45) + ( )' \\
\psi_{45} &= (-18 + 27 - 36 + 45) + ( )' \\
\psi_{48} &= (15 - 26 - 37 + 48) + ( )' \\
\psi_{58} &= (-14 - 23 + 58 + 67) + ( )' \\
\psi_{78} &= (12 + 34 + 56 + 78) + ( )' \\
\psi_{13} &= (-13 - 24 + 57 + 68) - ( )' \\
\psi_{16} &= (-16 + 25 - 38 + 47) - ( )' \\
\psi_{23} &= (-14 + 23 - 58 + 67) + ( )' \\
\psi_{26} &= (15 + 26 - 37 - 48) + ( )' \\
\psi_{34} &= (-12 + 34 - 56 + 78) + ( )' \\
\psi_{37} &= (+15 - 26 + 37 - 48) + ( )' \\
\psi_{46} &= (17 + 28 + 35 + 46) + ( )' \\
\psi_{56} &= (-12 - 34 + 56 + 78) + ( )' \\
\psi_{67} &= (14 + 23 + 58 + 67) + ( )' \\
\psi_{14} &= (-14 + 23 + 58 - 67) - ( )' \\
\psi_{17} &= (-17 + 28 + 35 - 46) - ( )' \\
\psi_{24} &= (13 + 24 + 57 + 68) + ( )' \\
\psi_{27} &= (18 + 27 + 36 + 45) + ( )' \\
\psi_{35} &= (-17 - 28 + 35 + 46) + ( )' \\
\psi_{38} &= (16 + 25 + 38 + 47) + ( )' \\
\psi_{47} &= (-16 - 25 + 38 + 47) + ( )' \\
\psi_{57} &= (-13 + 24 + 57 - 68) + ( )' \\
\psi_{68} &= (-13 + 24 - 57 + 68) + ( )' \\
\psi_{19} &= -11' - 22' - 33' - 44' - 55' - 66' - 77' - 88' \\
\psi_{39} &= -13' - 24' + 31' + 42' + 57' + 68' - 75' - 86' \\
\psi_{59} &= -15' - 26' - 37' - 48' + 51' + 62' + 73' + 84' \\
\psi_{79} &= -17' + 28' + 35' - 46' - 53' + 64' + 71' - 82' \\
\psi_{29} &= -12' + 21' + 34' - 43' + 56' - 65' - 78' + 87' \\
\psi_{49} &= -14' + 23' - 32' + 41' + 58' - 67' + 76' - 85' \\
\psi_{69} &= -16' + 25' - 38' + 47' - 52' + 61' - 74' + 83' \\
\psi_{89} &= -18' - 27' + 36' + 45' - 54' - 63' + 72' + 81'
\end{align*}$$