A survey on biharmonic submanifolds in space forms

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Most of the results presented are in collaboration with:
A. Balmus and C. Oniciuc
Chen definition

Let

\[ i : M \hookrightarrow \mathbb{R}^n \]

be the canonical inclusion and \( H = (H_1, \ldots, H_n) \) the mean curvature vector field.

**Definition** (B-Y. Chen) A submanifold \( M \subset \mathbb{R}^n \) is **biharmonic** iff

\[ \Delta H = (\Delta H_1, \ldots, \Delta H_n) = 0 \]

where \( \Delta \) is the Beltrami-Laplace operator on \( M \) w.r.t. the metric induced by \( i \).
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- Why biharmonic?

\[
m \Delta \textbf{H} = \Delta (-\Delta \textbf{i}) = -\Delta^2 \textbf{i}
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- Why biharmonic?

\[
m \Delta H = \Delta(-\Delta i) = -\Delta^2 i
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- CMC submanifolds, \( |H| = \text{constant} \), are not necessarily biharmonic.
Biharmonic submanifolds in $\mathbb{E}^n(c)$

Let

$$ i : M^m \hookrightarrow \mathbb{E}^n(c) $$

be the canonical inclusion of a submanifold $M$ in a constant sectional curvature $c$ manifold.
Biharmonic submanifolds in $\mathbb{E}^n(c)$

Let

$$i : M^m \hookrightarrow \mathbb{E}^n(c)$$

be the canonical inclusion of a submanifold $M$ in a constant sectional curvature $c$ manifold.

**Definition** $M$ is a **biharmonic** submanifold iff

$$\Delta^i H = m \cdot c \cdot H$$

where

- $H \in C(i^{-1}(T\mathbb{E}^n(c)))$ denotes the mean curvature vector field of $M$ in $\mathbb{E}^n(c)$

- $\Delta^i$ is the rough Laplacian on $i^{-1}(T\mathbb{E}^n(c))$
Remark

If $E^n(1) = S^n$, then one can consider $S^n \subset \mathbb{R}^{n+1}$ and the inclusion

$$i : M^m \hookrightarrow S^n \subset \mathbb{R}^{n+1}$$

can be seen as a map into $\mathbb{R}^{n+1}$. 
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Alternative problem (Alias, Barros, Ferrández)

$$\Delta H' = (\Delta H_1, \ldots, \Delta H_{n+1}) = m \mathbf{H}'$$

where $\mathbf{H}'$ is the mean curvature vector field of the inclusion as a map into $\mathbb{R}^{n+1}$. 
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where does this come from?
The bienergy Functional

**Biharmonic maps** \( \varphi : (M, g) \rightarrow (N, h) \) are critical points of the bienergy functional

\[
E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \, v_g
\]

(Eells–Lemaire)

where

\[
\tau(\varphi) = \text{trace}_g \nabla d\varphi
\]

is the tension field \((\tau(\varphi) = 0 \text{ means } \varphi \text{ harmonic})\)
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where

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($\tau(\varphi) = 0$ means $\varphi$ harmonic)

Biharmonic maps are solutions of the Euler-Lagrange equation:

$$\tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \text{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0$$

where $R^N$ is the curvature operator on $N$.  

(Jiang)
Remarks: \( \varphi : (M, g) \rightarrow (N, h) \)

- \( M \) compact and \( \text{Sec}^N \leq 0 \) then biharmonic \( \Rightarrow \) harmonic
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  \varphi : M \rightarrow N
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in each homotopy class (Eells–Sampson)
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- There exists \textbf{NO} harmonic map from
  \[
  \mathbb{T}^2 \to \mathbb{S}^2
  \]
  in the homotopy class of \textbf{Brower} degree \( \pm 1 \) (Eells–Wood)
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**Problem** Find biharmonic maps \( \mathbb{T}^2 \to \mathbb{S}^2 \) of degree \( \pm 1 \)

- So far we only know examples of biharmonic maps \( \mathbb{T}^2 \to \mathbb{S}^2 \) whose image is a curve.
Examples of proper biharmonic maps
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- Any polynomial map of degree 3 between Euclidean spaces
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**Property** (Almansi): \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be any harmonic function then

\[
g(x) = |x|^2 f(x)
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is proper biharmonic.
Examples of proper biharmonic maps

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- From the **Hopf** map $H : \mathbb{C}^2 \to \mathbb{R} \times \mathbb{C}$ we get the proper biharmonic map

$$\mathbb{C}^2 \to \mathbb{R} \times \mathbb{C}, \ (z, w) \mapsto (|z|^2 + |w|^2)(|z|^2 - |w|^2, 2z\bar{w})$$
Examples of proper biharmonic maps

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\]

- Let \( f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i, \ a_i \in \mathbb{R}, \) then

\[
g(x) = |x|^{2-n} f(x)
\]

is proper biharmonic (M–Impera)
Examples of proper biharmonic maps

- The generalized \textit{Kelvin} transformation

\[ \varphi : \mathbb{R}^m \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{0\}, \quad \varphi(p) = \frac{p}{|p|^\ell} \]

is proper biharmonic iff \( \ell = m - 2 \) \hspace{1cm} \text{(Balmus–M–Oniciuc)}
Examples of proper biharmonic maps

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- The quaternionic multiplication

\[ \mathbb{H} \to \mathbb{H}, \quad q \mapsto q^n \]

is biharmonic for any \( n \in \mathbb{N} \) \hfill (Fueter, 1935)
Let's go back to biharmonic submanifolds

If \( \varphi : M \to \mathbb{E}^n(c) \) is an isometric immersion then

\[
\tau(\varphi) = mH, \quad \tau_2(\varphi) = -m\Delta \varphi H + cm^2 H
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If \( \varphi : M \to \mathbb{E}^n(c) \) is an isometric immersion then

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thus \( \varphi \) is biharmonic (\( \tau_2 = 0 \)) iff

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Let's go back to biharmonic submanifolds

If \( \varphi : M \rightarrow \mathbb{E}^n(c) \) is an isometric immersion then

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thus \( \varphi \) is biharmonic \( (\tau_2 = 0) \) iff

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\]

Now, choosing \( \varphi = i : M^m \hookrightarrow \mathbb{E}^n(c) \) to be the inclusion we get the biharmonic condition we have started with.
Geometric conditions for biharmonic submanifolds

Biharmonic ⇔ $\Delta^i H = m c H$

$\Updownarrow$ decomposing

\[
\begin{cases}
-\Delta^\perp H - \text{trace } B(\cdot, A_{H} \cdot) + m c H = 0 \quad \text{(normal)} \\
2 \text{trace } A_{\nabla^\perp (\cdot)} H(\cdot) + \frac{m}{2} \text{grad}(|H|^2) = 0 \quad \text{(tangent)}
\end{cases}
\]
Geometric conditions for biharmonic submanifolds

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\end{align*} \]

For hypersurfaces \( H = f \eta \) \( \eta \) unit normal

\[ \begin{align*}
\Delta f - (m c - |A|^2) f &= 0 \\
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Geometric conditions for biharmonic submanifolds

Biharmonic \iff \Delta^i H = m c H

\uparrow \quad \text{decomposing}

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\end{align*}

\( f = \text{constant} \neq 0 \implies |A|^2 = m c \implies c > 0 \)
Chen’s Conjecture

**Proposition [Chen ($c = 0$), Caddeo–M–Oniciuc ($c \leq 0$)]**

If $c \leq 0$, there exists no proper biharmonic surfaces $M^2 \subset \mathbb{E}^3(c)$. 
Chen’s Conjecture

Proposition [Chen \((c = 0)\), Caddeo–M–Oniciuc \((c \leq 0)\)]

If \(c \leq 0\), there exists no proper biharmonic surfaces \(M^2 \subset \mathbb{E}^3(c)\).

Conjecture

\textit{Biharmonic submanifolds of } \(\mathbb{E}^n(c)\), \(c \leq 0\), \textit{are minimal}
Chen’s Conjecture

Proposition [Chen \((c = 0)\), Caddeo–M–Oniciuc \((c \leq 0)\)]

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Conjecture

*Biharmonic submanifolds of \(\mathbb{E}^n(c), c \leq 0\), are minimal*

Partial solutions of the conjecture are known for:

- curves of \(\mathbb{R}^n\) (Dimitric)
- submanifolds of finite type in \(\mathbb{R}^n\) (Dimitric)
- hypersurfaces with at most two principal curvatures (B–M–O)
- pseudo-umbilical submanifolds \(M^m \subset \mathbb{E}^n(c), c \leq 0, m \neq 4\), (Caddeo–M–O, Dimitric)
- hypersurfaces of \(\mathbb{E}^4(c), c \leq 0\) (Hasanis–Vlachos, B–M–O)
- spherical submanifolds of \(\mathbb{R}^n\) (Chen)
- submanifolds of bounded geometry (Ichiyama–Inoguchi–Urakawa)
Biharmonic submanifolds of $\mathbb{S}^n$

All the non existence results described in the previous section do not hold for submanifolds in the sphere.

**Problem:**

Classify all biharmonic submanifolds of $\mathbb{S}^n$
Main examples of biharmonic submanifolds in $\mathbb{S}^n$
Main examples of biharmonic submanifolds in $\mathbb{S}^n$

B1 The small hypersphere

$$\mathbb{S}^m \left( \frac{1}{\sqrt{2}} \right) \text{ biharmonic } \mathbb{S}^{m+1}$$
Main examples of biharmonic submanifolds in $S^n$

**B1** The small hypersphere

$$S^m\left(\frac{1}{\sqrt{2}}\right) \quad \text{biharmonic} \quad S^{m+1}$$

**B2** The standard products of spheres

$$S^{m_1}\left(\frac{1}{\sqrt{2}}\right) \times S^{m_2}\left(\frac{1}{\sqrt{2}}\right) \quad \text{biharmonic} \quad S^{m+1}$$

$m_1 + m_2 = m$ and $m_1 \neq m_2$. 
Main examples of biharmonic submanifolds in $S^n$

**B3** Composition property

$$M^m \xrightarrow{\text{minimal}} S^{n-1}\left(\frac{1}{\sqrt{2}}\right) \xrightarrow{\text{biharmonic}} S^n$$

proper biharmonic
Main examples of biharmonic submanifolds in $S^n$

**B3** Composition property

$$M^m \overset{\text{minimal}}{\longrightarrow} S^{n-1}(\frac{1}{\sqrt{2}}) \overset{\text{biharmonic}}{\longrightarrow} S^n$$

proper biharmonic

**B4** Product composition property

$$M_1^{m_1} \times M_2^{m_2} \overset{\text{minimal}}{\longrightarrow} S^{n_1}(\frac{1}{\sqrt{2}}) \times S^{n_2}(\frac{1}{\sqrt{2}}) \overset{\text{biharmonic}}{\longrightarrow} S^n$$

proper biharmonic

$$n_1 + n_2 = n - 1, \ m_1 \neq m_2$$
Biharmonic hypersurfaces in $\mathbb{S}^n$
Biharmonic hypersurfaces in $S^n$

\[ M^m \subset S^{m+1} \text{ biharmonic} \]

$\kappa = \text{number of distinct principal curvature}$
Biharmonic hypersurfaces in $S^n$

$M^m \subset S^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$
Biharmonic hypersurfaces in $\mathbb{S}^n$

$M^m \subset \mathbb{S}^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$

$\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right) \rightarrow \mathbb{S}^{m+1}$

B1
Biharmonic hypersurfaces in $S^n$

$M^m \subset S^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$

$\kappa \leq 2$

$S^m\left(\frac{1}{\sqrt{2}}\right) \rightarrow S^{m+1}$

B1
Biharmonic hypersurfaces in $S^n$

$M^m \subset S^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$

$\kappa \leq 2$

$S^m \left( \frac{1}{\sqrt{2}} \right) \rightarrow S^{m+1}$

B1

$S^{m_1} \left( \frac{1}{\sqrt{2}} \right) \times S^{m_2} \left( \frac{1}{\sqrt{2}} \right) \rightarrow S^{m+1}$

B2
Biharmonic hypersurfaces in $\mathbb{S}^n$

$M^m \subset \mathbb{S}^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$  

$\mathbb{S}^m(\frac{1}{\sqrt{2}}) \rightarrow \mathbb{S}^{m+1}$  
B1

$\kappa \leq 2$

$\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}}) \rightarrow \mathbb{S}^{m+1}$  
B2

$\kappa = 3$
Biharmonic hypersurfaces in $S^n$

$M^m \subset S^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

- $\kappa = 1$

  - $S^m(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$
    - B1

- $\kappa \leq 2$

- $\kappa = 3$ - Compact + CMC

  - $S^{m_1}(\frac{1}{\sqrt{2}}) \times S^{m_2}(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$
    - B2

$S^m(\frac{1}{\sqrt{2}}) = S^m \cap S^m(\frac{1}{\sqrt{2}})$

$S^m(\frac{1}{\sqrt{2}})$ is the unit sphere in $\mathbb{R}^m$ with radius $\frac{1}{\sqrt{2}}$.
Biharmonic hypersurfaces in $S^n$

$M^m \subset S^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$

$S^m(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$

B1

$\kappa \leq 2$

$S^{m_1}(\frac{1}{\sqrt{2}}) \times S^{m_2}(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$

B2

$\kappa = 3$ - Compact + CMC

Non Existence
Biharmonic hypersurfaces in $S^n$

\[ M^m \subset S^{m+1} \text{ biharmonic} \quad \text{Isoparametric} \]

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$

$\kappa = 3$ - Compact + CMC

Non Existence

$S^m(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$

B1

$S^{m_1}(\frac{1}{\sqrt{2}}) \times S^{m_2}(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$

B2
Biharmonic hypersurfaces in $S^n$

$M^m \subset S^{m+1}$ biharmonic

$k = \text{number of distinct principal curvature}$

$k = 1$

$k \leq 2$

$k = 3$

- Compact + CMC

Non Existence

$Ichiyama-Inoguchi-Urakawa$
CMC Biharmonic submanifolds in $S^n$

$M^m \subset S^n$ biharmonic + CMC
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4 \rightarrow M^m \subset S^n$ biharmonic + CMC
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4$ \quad \rightarrow \quad M^m \subset S^n$ biharmonic + CMC

Compact
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4 \quad \rightarrow \quad M^m \subset S^n \text{ biharmonic } + \text{ CMC}$

Compact
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4 \quad \rightarrow \quad M^m \subset S^n \text{ biharmonic } + \text{ CMC}$

- Compact
- $|H| = 1$
- $|H| \in (0, 1)$
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4 \quad \rightarrow \quad M^m \subset S^n$ biharmonic + CMC

Compact

$|H| = 1 \quad \rightarrow \quad |H| \in (0, 1)$

$M^m \subset S^n$ minimal biharmonic $\quad \rightarrow \quad S^{n-1}(\frac{1}{\sqrt{2}}) \quad \rightarrow \quad S^n$

B3
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4$  $\rightarrow$  $M^m \subset S^n$ biharmonic + CMC

Compact  $\rightarrow$  $|H| = 1$  $\rightarrow$  $|H| \in (0, 1)$

$M^m$ minimal $S^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic $S^n$

B3

Compact
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4$ \rightarrow $M^m \subset S^n$ biharmonic + CMC

Compact $\rightarrow |H| = 1$ \rightarrow $|H| \in (0, 1)$

Minimal $M^m \rightarrow S^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic $S^n$

B3 $\rightarrow \rightarrow \rightarrow$

$M$ is of 2-type
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4 \quad \rightarrow \quad M^m \subset S^n$ biharmonic + CMC

Compact

$|H| = 1$

$|H| \in (0, 1)$

$M^m$ minimal biharmonic $S^{n-1}(\frac{1}{\sqrt{2}}) \quad \rightarrow \quad S^n$

B3

$M$ is of 2-type

$M$ is of 1-type
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4 \rightarrow M^m \subset S^n$ biharmonic + CMC

- Compact
- $|H| = 1$
- $|H| \in (0, 1)$

$M^m$ minimal $\rightarrow S^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic $\rightarrow S^n$

- B3
- $M$ is of 2-type
- $M$ is of 1-type

Compact $\rightarrow$ $H| = 1$ $\rightarrow$ Compact

$M^m \subset S^n$ biharmonic + CMC $\rightarrow$ $H| \in (0, 1)$ $\rightarrow$ $M$ is of 2-type
Biharmonic submanifolds in $\mathbb{S}^n$ with $\nabla^\perp H = 0$ (PMC)

$M^m \subset \mathbb{S}^n$ biharmonic + PMC
Biharmonic submanifolds in $\mathbb{S}^n$ with $\nabla \perp H = 0$ (PMC)

$M^m \subset \mathbb{S}^n$ biharmonic + PMC

$|H| = 1$ or $|H| \in (0, \frac{m-2}{m}]$
Biharmonic submanifolds in $\mathbb{S}^n$ with $\nabla \perp H = 0$ (PMC)

$M^m \subset \mathbb{S}^n$ biharmonic + PMC

$M^2 \subset \mathbb{S}^n$

$|H| = 1$ or $|H| \in (0, \frac{m-2}{m}]$
Biharmonic submanifolds in $S^n$ with $\nabla^\perp H = 0$ (PMC)

$M^m \subset S^n$ biharmonic + PMC

$M^2 \subset S^n$

$|H| = 1 \text{ or } |H| \in (0, \frac{m-2}{m}]$

$M^2 \xrightarrow{\min} S^{n-1}(\frac{1}{\sqrt{2}}) \xrightarrow{} S^n$

B3
Biharmonic submanifolds in $S^n$ with $\nabla^\perp H = 0$ (PMC)

- $M^m \subset S^n$ biharmonic + PMC
- $M^2 \subset S^n$
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B3

B4
Pseudo-umbilical biharmonic submanifolds in $S^n$

CMC proper biharmonic submanifolds with $|H| = 1$ in $S^n$ are B3 and they are pseudo-umbilical:

$$A_H = |H|^2 \text{Id}$$
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compacts + CMC + pseudo-umbilical $\Rightarrow$ PMC  \hspace{1cm} (H. Li)
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The examples of Sasahara et al

**Theorem** Let $\varphi : M^3 \to S^5$ be a proper biharmonic anti-invariant immersion. Then the position vector field $x_0 = x_0(u, v, w)$ in $\mathbb{R}^6$ is given by

$$x_0(u, v, w) = e^{iw}(e^{iu}, ie^{-iu} \sin \sqrt{2}v, ie^{-iu} \cos \sqrt{2}v)$$

Moreover, $|H| = 1/3$.

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**Theorem** Let $\phi : M^2 \to S^5$ be a proper biharmonic Legendre immersion. Then the position vector field $x_0 = x_0(u, v)$ of $M$ in $\mathbb{R}^6$ is given by:

$$x_0(u, v) = \frac{1}{\sqrt{2}} \left( \cos u, \sin u \sin(\sqrt{2}v), - \sin u \cos(\sqrt{2}v), \sin u, \cos u \sin(\sqrt{2}v), - \cos u \cos(\sqrt{2}v) \right).$$

The immersion $\phi$ is NOT PMC and NOT pseudo-umbilical.
Open Problems

Conjecture
The only proper biharmonic hypersurfaces in $S^n$ are $B_1$ or $B_2$.

Conjecture
Any biharmonic submanifold in $S^n$ has constant mean curvature.
Remark

The \( \mathbf{i} : \mathbb{M}^m \leftrightarrow \mathbb{E}^n(c) \) is biharmonic iff

\[
\begin{array}{l}
-\Delta^\perp \mathbf{H} - \text{trace} \, B(\cdot, A_H \cdot) + mc\mathbf{H} = 0 \quad \text{(normal)} \\
2 \text{trace} \, A_{\nabla^\perp (\cdot)} \mathbf{H}(\cdot) + \frac{m}{2} \text{grad}(|\mathbf{H}|^2) = 0 \quad \text{(tangent)}
\end{array}
\]

Most of the classification results described depend only on the tangent part of \( \tau_2 \).
Remark

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Has the condition

$$\tau_2(\varphi)^\top = 0$$

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As described by Hilbert, the stress-energy tensor associated to a variational problem is a symmetric 2-covariant tensor field $S$ conservative at critical points, i.e. $\text{div } S = 0$ at these points.
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(Baird–Eells)
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- For biharmonic maps the stress-energy tensor is

$$S_2(X, Y) = \frac{1}{2} |\tau(\varphi)|^2 \langle X, Y \rangle + \langle d\varphi, \nabla \tau(\varphi) \rangle \langle X, Y \rangle$$

$$-\langle d\varphi(X), \nabla_Y \tau(\varphi) \rangle - \langle d\varphi(Y), \nabla_X \tau(\varphi) \rangle$$

with

$$\text{div } S_2 = -\langle \tau_2(\varphi), d\varphi \rangle$$

(Jiang, Loubeau–M–Oniciuc)
The meaning of $S_2 = 0$ (Loubeau–M–Oniciuc)
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A smooth map $\varphi : (M, g) \to (N, h)$ is biharmonic if it is a critical points of the bienergy w.r.t. variations of the map.
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$$F : G \rightarrow \mathbb{R}, \quad F(g) = E_2(\varphi),$$

where $G$ is the set of Riemannian metrics on $M$
The meaning of $S_2 = 0$ (Loubeau–M–Oniciuc)

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**Theorem**

$$\delta(F(g_t)) = -\frac{1}{2} \int_M \langle S_2, \omega \rangle \, \nu_g,$$

The tensor $S_2$ vanishes precisely at critical points of the energy (bienergy) for variations of the domain metric, rather than for variations of the map.

(The harmonic case is of Sanini)
Isometric immersion

If \( \varphi : (M, g) \rightarrow (N, h) \) is an isometric immersion from

\[
\begin{align*}
\text{div } S_2 &= \langle \tau_2(\varphi), d\varphi \rangle \\
\Downarrow \\
\text{div } S_2 &= -\tau_2(\varphi)^T
\end{align*}
\]
Isometric immersion

If \( \varphi : (M, g) \to (N, h) \) is an isometric immersion from

\[
\text{div } S_2 = -\langle \tau_2(\varphi), d\varphi \rangle
\]

\[
\downarrow
\]

\[
\text{div } S_2 = -\tau_2(\varphi)^\top
\]

Problem

Study isometric immersions in space forms with \( \text{div } S_2 = 0 \)
Biharmonic submanifolds in a Riemannian manifold

An isometric immersion

\[ \varphi : (M, g) \to (N, h) \]

is biharmonic iff

\[
\begin{align*}
\Delta H^\perp + \text{trace } B(\cdot, A_H \cdot) + \text{trace } (R^N(\cdot, H) \cdot)^\perp &= 0 \\
\frac{m}{2} \ \text{grad } |H|^2 + 2 \text{trace } A_{\nabla^\perp(\cdot) H}(\cdot) + 2 \text{trace } (R^N(\cdot, H) \cdot)^\top &= 0
\end{align*}
\]
Results for Bih. Sub. in non constant sec. curv. manifolds

- In three-dimensional homogeneous spaces (Thurston’s geometries) (Inoguchi, Ou–Wang, Caddeo–Piu–M–O)

- There exists examples of proper biharmonic hypersurfaces in a space with negative non constant sectional curvature (Ou–Tang)

- It is initiated the study of biharmonic submanifolds in complex space forms (Ichiyama–Inoguchi–Urakawa, Fetcu–Loubeau–M–O, Sasahara)

- There are several works on biharmonic submanifolds in contact manifold and Sasakian space forms (Inoguchi, Fetcu–O, Sasahara)
In a Sasakian manifold 

\[(N, \Phi, \xi, \eta, g)\]

a submanifold \( M \subset N \) tangent to \( \xi \) is called \textit{anti-invariant} if \( \Phi \) maps any tangent vector to \( M \), which is normal to \( \xi \), to a vector which is normal to \( M \).
Finite $k$-type submanifolds

An isometric immersion $\phi : M \rightarrow \mathbb{R}^{n+1}$ ($M$ compact) is called of finite $k$-type if

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k$$

where

$$\Delta \phi_i = \lambda_i \phi_i, \quad i = 1, \ldots, k$$

and $\phi_0 \in \mathbb{R}^{n+1}$ is the center of mass

A submanifold $M \subset S^n \subset \mathbb{R}^{n+1}$ is said to be of finite type if it is of finite type as a submanifold of $\mathbb{R}^{n+1}$.

A non null finite type submanifold in $S^n$ is said to be mass-symmetric if the constant vector $\phi_0$ of its spectral decomposition is the center of the hypersphere $S^n$, i.e. $\phi_0 = 0$. 
