A survey on biharmonic submanifolds in space forms

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Most of the results presented are in collaboration with:
A. Balmus and C. Oniciuc

Chen definition

Let

$$\mathbf{i} : M \hookrightarrow \mathbb{R}^n$$

be the canonical inclusion and $\mathbf{H} = (H_1, \ldots, H_n)$ the mean curvature vector field.

<u>Definition</u> (B-Y. Chen) A submanifold $M \subset \mathbb{R}^n$ is *biharmonic* iff

$$\boxed{\Delta\mathbf{H} = (\Delta H_1, \ldots, \Delta H_n) = 0}$$

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- CMC submanifolds, $|\mathbf{H}| = \mathrm{constant}$, are not necessarily biharmonic.

Biharmonic submanifolds in $\mathbb{E}^n(c)$

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$$\boxed{\Delta^{\mathbf{i}}\,\mathbf{H} = m\,c\,\mathbf{H}}$$

where

- $\mathbf{H} \in C(\mathbf{i}^{-1}(T\mathbb{E}^n(c)))$ denotes the mean curvature vector field of M in $\mathbb{E}^n(c)$
- $\Delta^{\mathbf{i}}$ is the rough Laplacian on $\mathbf{i}^{-1}(T\mathbb{E}^n(c))$

Remark

If $\mathbb{E}^n(1) = \mathbb{S}^n$, then one can consider $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and the inclusion

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where does this come from?

The bienergy Functional

Biharmonic maps $\varphi : (M, g) \to (N, h)$ are critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \, v_g$$

(Eells–Lemaire)

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Biharmonic maps are solutions of the Euler-Lagrange equation:

$$\boxed{\tau_2(\varphi) = -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0}$$

where R^N is the curvature operator on N. (Jiang)

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- So far we only know examples of biharmonic maps $\mathbb{T}^2 \to \mathbb{S}^2$ whose image is a curve.

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- Let $f(x_1, ..., x_n) = \sum_{i=1}^n a_i x_i$, $a_i \in \mathbb{R}$, then

$$g(x) = |x|^{2-n} f(x)$$

is proper biharmonic (M–Impera)

Examples of proper biharmonic maps

- The generalized Kelvin transformation

$$\varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \quad \varphi(p) = \frac{p}{|p|^\ell}$$

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- The quaternionic multiplication

$$\mathbb{H} \to \mathbb{H}, \quad q \mapsto q^n$$

is biharmonic for any $n \in \mathbb{N}$ (Fueter, 1935)

Lets go back to biharmonic submanifolds

If $\varphi : M \to \mathbb{E}^n(c)$ is an isometric immersion then

$$\tau(\varphi) = m\mathbf{H}, \qquad \tau_2(\varphi) = -m\Delta^{\varphi}\mathbf{H} + c\,m^2\,\mathbf{H}$$

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Now, choosing $\varphi = \mathbf{i} : M^m \hookrightarrow \mathbb{E}^n(c)$ to be the inclusion we get the biharmonic condition we have started with

Geometric conditions for biharmonic submanifolds

Biharmonic $\Leftrightarrow$ $\Delta^{\mathbf{i}}\mathbf{H} = m\,c\,\mathbf{H}$

$\Updownarrow$ decomposing

$$\begin{cases} -\Delta^{\perp}\mathbf{H} - \operatorname{trace} B(\cdot, A_{\mathbf{H}}\cdot) + m\,c\,\mathbf{H} = 0 & \text{(normal)} \\ 2\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}\mathbf{H}}(\cdot) + \frac{m}{2}\operatorname{grad}(|\mathbf{H}|^2) = 0 & \text{(tangent)} \end{cases}$$

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For hypersurfaces $\boxed{\mathbf{H} = f\,\eta}$ η unit normal

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$$f = \text{constant} \neq 0 \quad \Rightarrow \quad |A|^2 = m\,c \quad \Rightarrow \quad c > 0$$

Chen's Conjecture

Proposition [Chen ($c = 0$), Caddeo–M–Oniciuc ($c \leq 0$)]
If $c \leq 0$, there exists no proper biharmonic surfaces $M^2 \subset \mathbb{E}^3(c)$.

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Conjecture

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Partial solutions of the conjecture are known for:

- curves of $\mathbb{R}^n$ (Dimitric)
- submanifolds of finite type in $\mathbb{R}^n$ (Dimitric)
- hypersurfaces with at most two principal curvatures (B–M–O)
- pseudo-umbilical submanifolds $M^m \subset \mathbb{E}^n(c)$, $c \leq 0$, $m \neq 4$, (Caddeo–M–O, Dimitric)
- hypersurfaces of $\mathbb{E}^4(c), c \leq 0$ (Hasanis–Vlachos, B–M–O)
- spherical submanifolds of $\mathbb{R}^n$ (Chen)
- submanifolds of bounded geometry (Ichiyama–Inoguchi–Urakawa)

Biharmonic submanifolds of $\mathbb{S}^n$

All the non existence results described in the previous section do not hold for submanifolds in the sphere.

Problem:

Classify all biharmonic submanifolds of $\mathbb{S}^n$

Main examples of biharmonic submanifolds in $\mathbb{S}^n$

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B1 The small hypersphere

$$\mathbb{S}^m\left(\frac{1}{\sqrt{2}}\right) \xrightarrow{\text{biharmonic}} \mathbb{S}^{m+1}$$

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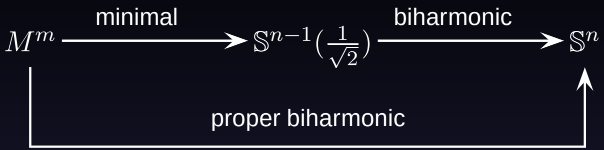
B2 The standard products of spheres

$$\mathbb{S}^{m_1}\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{S}^{m_2}\left(\frac{1}{\sqrt{2}}\right) \xrightarrow{\text{biharmonic}} \mathbb{S}^{m+1}$$

$m_1 + m_2 = m$ and $m_1 \neq m_2$.

Main examples of biharmonic submanifolds in $\mathbb{S}^n$

B3 Composition property



Main examples of biharmonic submanifolds in $\mathbb{S}^n$

B3 Composition property

$$M^m \xrightarrow{\text{minimal}} \mathbb{S}^{n-1}\left(\tfrac{1}{\sqrt{2}}\right) \xrightarrow{\text{biharmonic}} \mathbb{S}^n$$

$$M^m \xrightarrow{\text{proper biharmonic}} \mathbb{S}^n$$

B4 Product composition property

$$M_1^{m_1} \times M_2^{m_2} \xrightarrow{\text{minimal}} \mathbb{S}^{n_1}\left(\tfrac{1}{\sqrt{2}}\right) \times \mathbb{S}^{n_2}\left(\tfrac{1}{\sqrt{2}}\right) \longrightarrow \mathbb{S}^n$$

$$M_1^{m_1} \times M_2^{m_2} \xrightarrow{\text{proper biharmonic}} \mathbb{S}^n$$

$$n_1 + n_2 = n - 1,\ m_1 \neq m_2$$

Biharmonic hypersurfaces in $\mathbb{S}^n$

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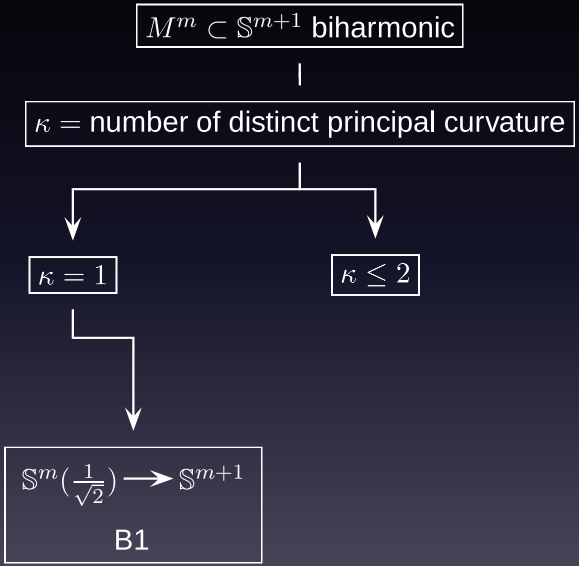

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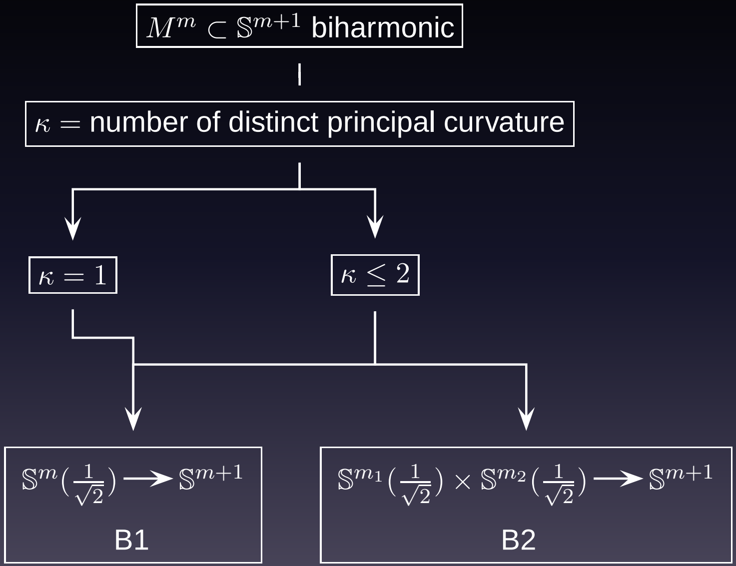

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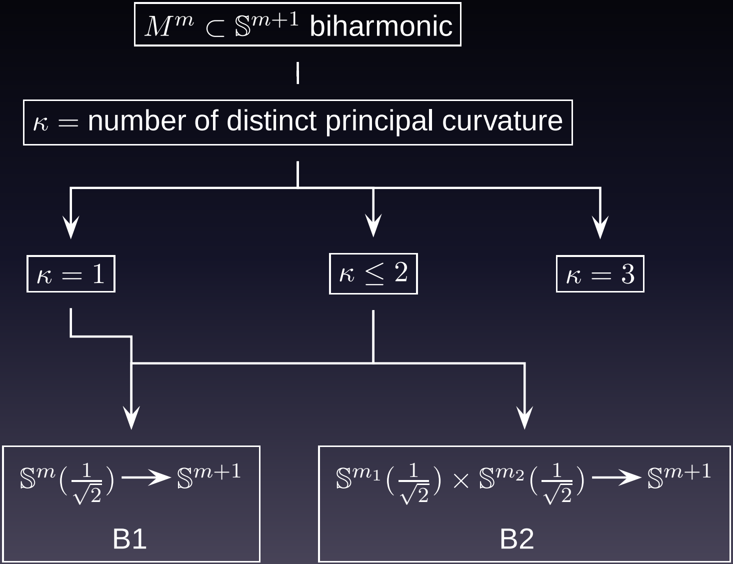

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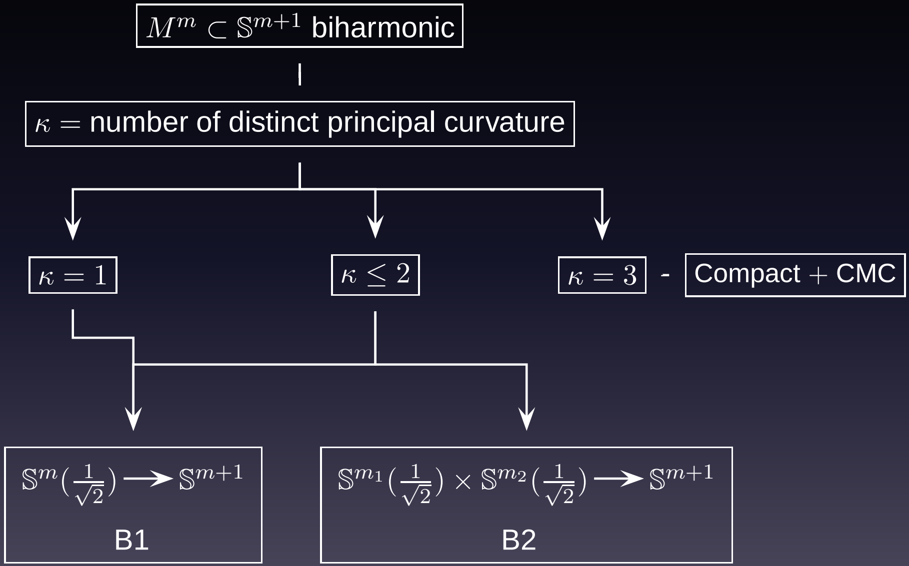

Biharmonic hypersurfaces in $\mathbb{S}^n$

$M^m \subset \mathbb{S}^{m+1}$ biharmonic

$\kappa =$ number of distinct principal curvature

$\kappa = 1$

$\kappa \leq 2$

$\kappa = 3$ - Compact $+$ CMC

Non Existence

$\mathbb{S}^m(\frac{1}{\sqrt{2}}) \longrightarrow \mathbb{S}^{m+1}$

B1

$\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}}) \longrightarrow \mathbb{S}^{m+1}$

B2

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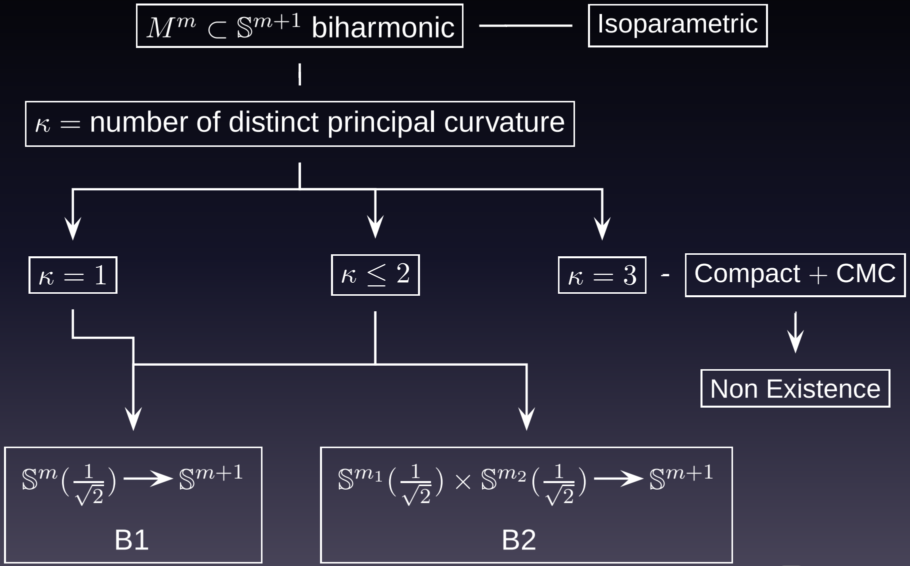

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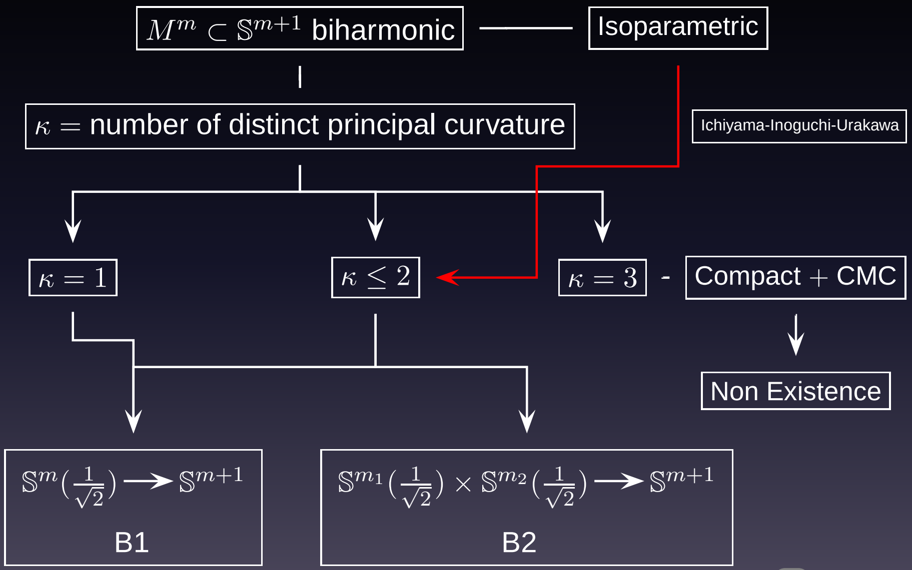

CMC Biharmonic submanifolds in $\mathbb{S}^n$

$$\boxed{M^m \subset \mathbb{S}^n \text{ biharmonic} + \text{CMC}}$$

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$\boxed{M^3 \subset \mathbb{S}^4}$ $\longrightarrow$ $\boxed{M^m \subset \mathbb{S}^n \text{ biharmonic} + \text{CMC}}$

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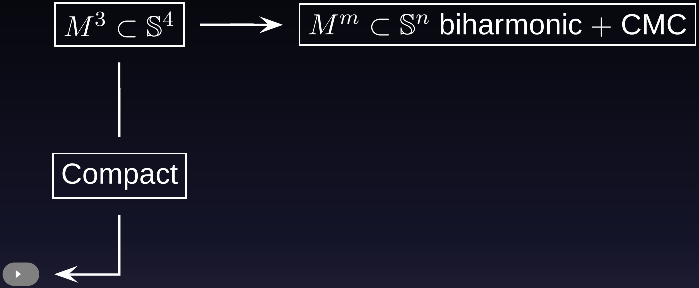

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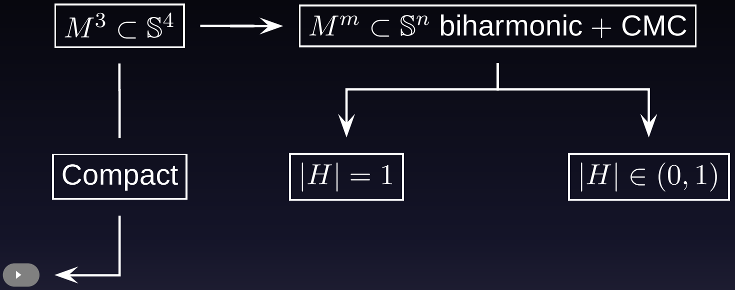

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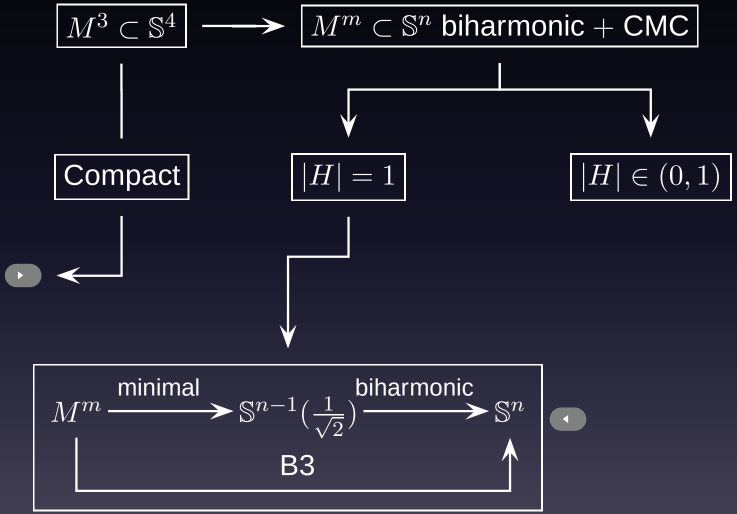

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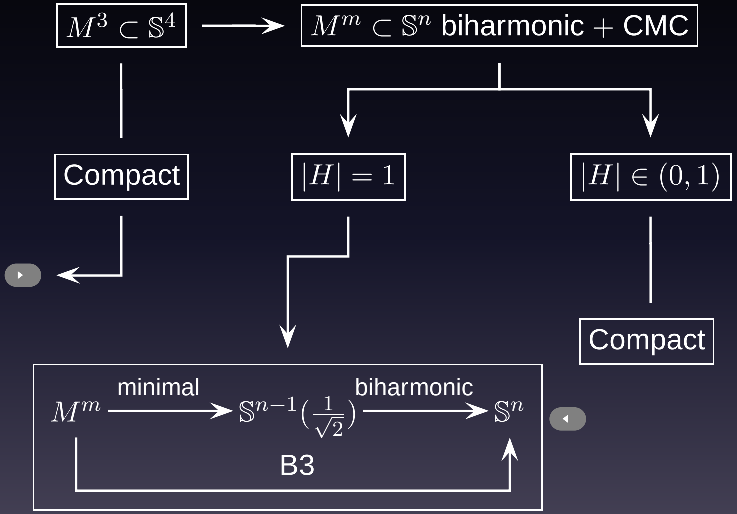

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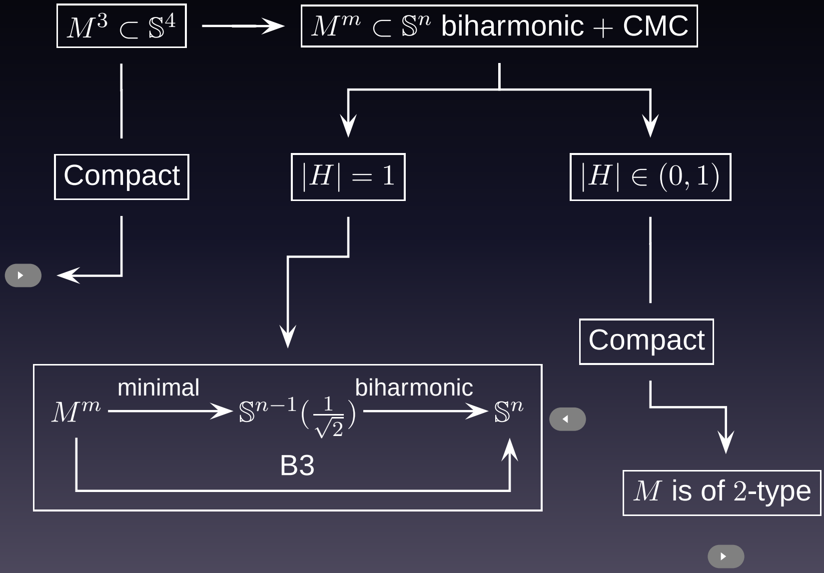

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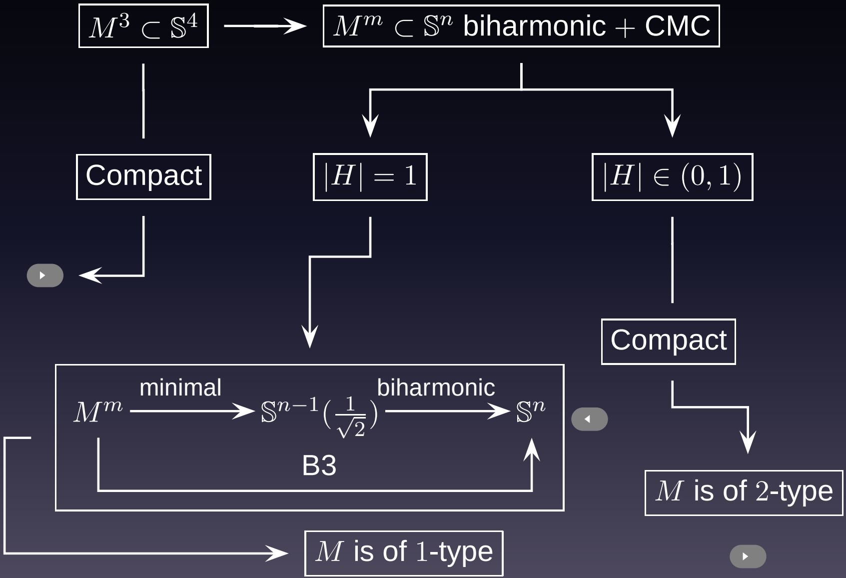

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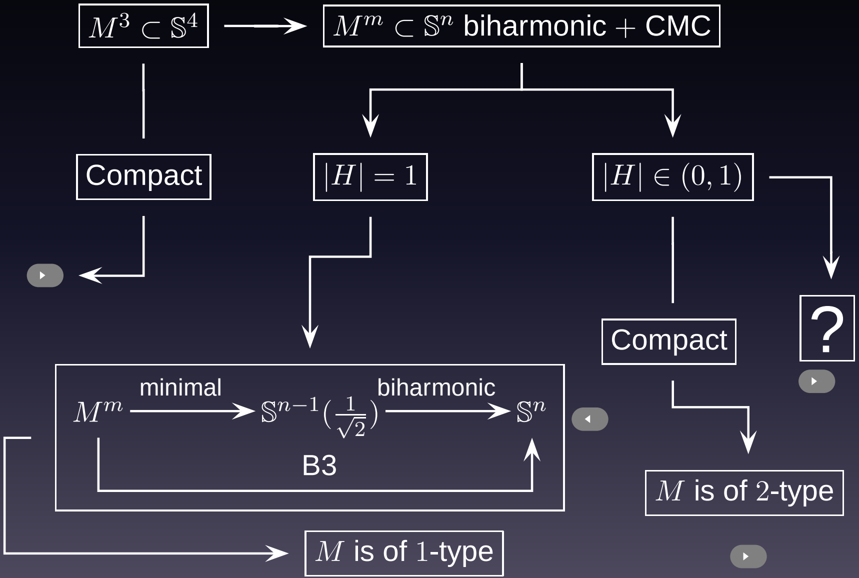

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$\longrightarrow$

$|H| = 1$ or $|H| \in (0, \frac{m-2}{m}]$

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$|H| = 1$ or $|H| \in (0, \frac{m-2}{m}]$

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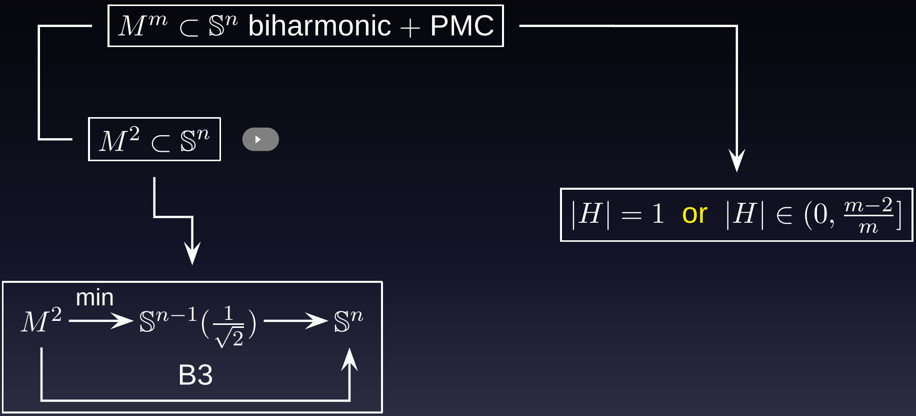

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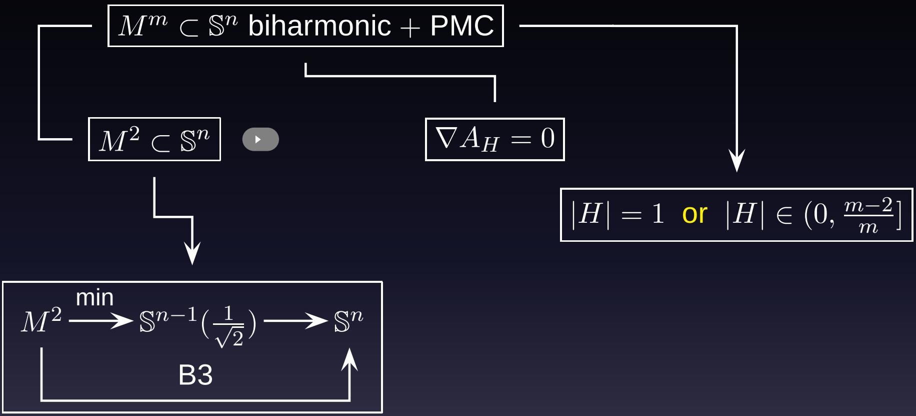

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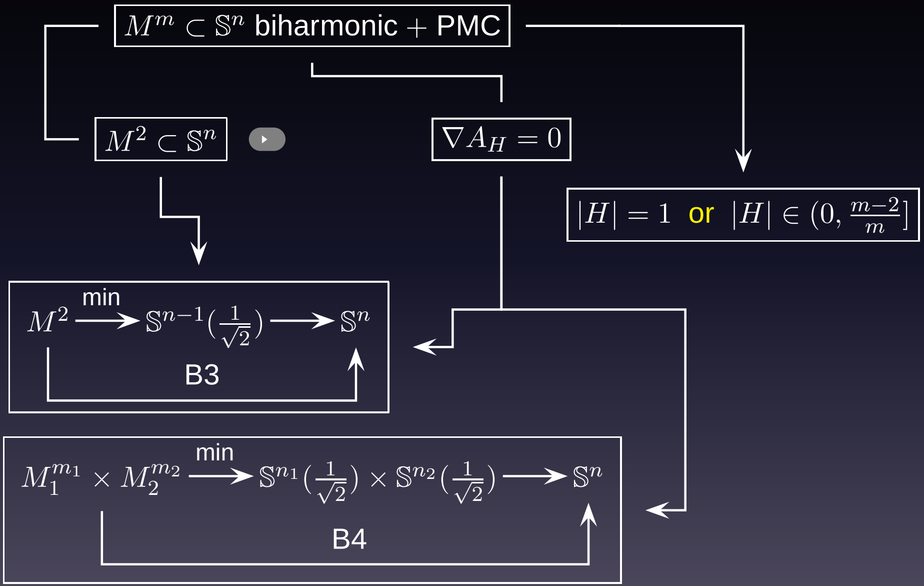

Pseudo-umbilical biharmonic submanifolds in $\mathbb{S}^n$

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Theorem Let $\varphi : M^3 \to \mathbb{S}^5$ be a proper biharmonic anti-invariant immersion. Then the position vector field $x_0 = x_0(u, v, w)$ in $\mathbb{R}^6$ is given by

$$x_0(u, v, w) = e^{iw}(e^{iu}, ie^{-iu} \sin \sqrt{2}v, ie^{-iu} \cos \sqrt{2}v)$$

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Theorem Let $\phi : M^2 \to \mathbb{S}^5$ be a proper biharmonic Legendre immersion. Then the position vector field $x_0 = x_0(u, v)$ of M in $\mathbb{R}^6$ is given by:

$$x_0(u, v) = \frac{1}{\sqrt{2}}\big(\cos u, \sin u \sin(\sqrt{2}v), -\sin u \cos(\sqrt{2}v), \sin u, \cos u \sin(\sqrt{2}v), -\cos u \cos(\sqrt{2}v)\big).$$

The immersion ϕ is NOT PMC and NOT pseudo-umbilical

Open Problems

Conjecture

The only proper biharmonic hypersurfaces in $\mathbb{S}^n$ are B1 or B2.

Conjecture

Any biharmonic submanifold in $\mathbb{S}^n$ has constant mean curvature.

skip-strees

Remark

The $\mathbf{i} : M^m \hookrightarrow \mathbb{E}^n(c)$ is biharmonic iff

$$\begin{cases} -\Delta^{\perp}\mathbf{H} - \operatorname{trace} B(\cdot, A_{\mathbf{H}}\cdot) + mc\mathbf{H} = 0 & \text{(normal)} \\ 2\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}\mathbf{H}}(\cdot) + \frac{m}{2}\operatorname{grad}(|\mathbf{H}|^2) = 0 & \text{(tangent)} \end{cases}$$

Most of the classification results described depend only on the tangent part of τ_2.

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- For biharmonic maps the stress-energy tensor is

$$\begin{aligned} S_2(X,Y) &= \frac{1}{2}|\tau(\varphi)|^2 \langle X, Y \rangle + \langle d\varphi, \nabla \tau(\varphi) \rangle \langle X, Y \rangle \\ &\quad - \langle d\varphi(X), \nabla_Y \tau(\varphi) \rangle - \langle d\varphi(Y), \nabla_X \tau(\varphi) \rangle \end{aligned}$$

with

$$\boxed{\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi \rangle}$$

(Jiang, Loubeau–M–Oniciuc)

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Theorem

$$\delta(F(g_t)) = -\frac{1}{2} \int_M \langle S_2, \omega \rangle \; v_g,$$

The tensor S_2 vanishes precisely at critical points of the energy (bienergy) for variations of the domain metric, rather than for variations of the map.

(The harmonic case is of Sanini)

Isometric immersion

If $\varphi : (M, g) \to (N, h)$ is an isometric immersion from

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi \rangle$$

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Problem

Study isometric immersions in space forms with $\operatorname{div} S_2 = 0$

Biharmonic submanifolds in a Riemannian manifold

An isometric immersion

$$\varphi : (M, g) \to (N, h)$$

is biharmonic iff

$$\begin{cases} \Delta^{\perp}\mathbf{H} + \operatorname{trace} B(\cdot, A_{\mathbf{H}}\cdot) + \operatorname{trace}(R^N(\cdot, \mathbf{H})\cdot)^{\perp} = 0 \\ \\ \frac{m}{2} \operatorname{grad} |\mathbf{H}|^2 + 2 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}\mathbf{H}}(\cdot) + 2 \operatorname{trace}(R^N(\cdot, \mathbf{H})\cdot)^{\top} = 0 \end{cases}$$

Results for Bih. Sub. in non constant sec. curv. manifolds

- In three-dimensional homogeneous spaces (Thurston's geometries)

  (Inoguchi, Ou–Wang, Caddeo–Piu–M–O)

- There exists examples of proper biharmonic hypersurfaces in a space with negative non constant sectional curvature

  (Ou–Tang)

- It is initiated the study of biharmonic submanifolds in complex space forms

  (Ichiyama–Inoguchi–Urakawa, Fetcu–Loubeau–M–O, Sasahara)

- There are several works on biharmonic submanifolds in contact manifold and Sasakian space forms

  (Inoguchi, Fetcu–O, Sasahara)

In a Sasakian manifold

$$(N, \Phi, \xi, \eta, g)$$

a submanifold $M \subset N$ tangent to ξ is called *anti-invariant* if Φ maps any tangent vector to M, which is normal to ξ, to a vector which is normal to M.

Finite k-type submanifolds

An isometric immersion $\phi : M \to \mathbb{R}^{n+1}$ (M compact) is called of finite k-type if

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k$$

where

$$\Delta\phi_i = \lambda_i \phi_i, \quad i = 1, \ldots, k$$

and $\phi_0 \in \mathbb{R}^{n+1}$ is the center of mass

A submanifold $M \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is said to be of finite type if it is of finite type as a submanifold of $\mathbb{R}^{n+1}$.

A non null finite type submanifold in $\mathbb{S}^n$ is said to be mass-symmetric if the constant vector ϕ_0 of its spectral decomposition is the center of the hypersphere $\mathbb{S}^n$, i.e. $\phi_0 = 0$.