New Trends in Differential Geometry

Admissible metrics on contact manifolds

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$\xi :=$ Reeb vector field

$$\eta(\xi) = 1, \quad d\eta(X,\xi) = 0 \qquad \forall X \in \mathfrak{X}(M)$$

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Sasakian metric : J is integrable $+$ ξ is Killing

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Conjecture also confirmed in the (k, μ) class:

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad h := \frac{1}{2}\mathcal{L}_\xi \varphi.$$

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Let (M, η) be a contact manifold. A Riemannian metric g on M will be called admissible if $\xi = \eta^\sharp$ i.e. the Reeb vector field ξ is of unit lenght and orthogonal to the contact distribution $\mathfrak{D}$ with respect to g.

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Theorem

Let (M, η) be a homogeneous simply connected contact manifold of dimension $N \geq 5$. Then M does not admit any admissible homogeneous Riemannian metric g having nonpositive curvature.

Remarks about the proof

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The Claim is proved using the structure theory for $(\mathfrak{s}, \langle\ ,\ \rangle)$ (Azencott,Wilson, Alekseevskii, Wolter)

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For contact sub-Riemannian symmetric manifolds: connection used by Falbel & Gorodski (1995)

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Definition

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If g is an associated metric: Levi-parallel $\iff$ J is integrable

Reformulation of a result of Tanno (1989)

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Consequence of the parallelism:

The spectrum $\mathfrak{S}$ of $\varphi^2 : \mathfrak{D} \to \mathfrak{D}$ consists of negative constants

$\mathfrak{S}$ will be called the *spectrum* of (M,η,g)

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In this case ξ is Killing $(\tau = 0)$ (Sasakian type).

The Einstein case

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In particular, all ξ*-sectional curvatures are positive.*

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Corollary

Let (M, η, g) be a Levi-parallel contact Riemannian manifold. Assume $\dim(M) \geq 5$. If M has constant sectional curvature c, then

$$c > 0,\ \xi \text{ is Killing, and } \mathfrak{S} = \{-c\}.$$

Basic ingredients of the proof

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Two formulas hold *for any* Levi-parallel (M, η, g):

- Comparison of Ricci tensors:

$$s(X,Y) = Ric(X,Y) - 2g(\varphi^2 X, Y) + g((\nabla_\xi \tau)X, Y) \quad \forall X, Y \in \Gamma\mathfrak{D}$$

- For $Y \in \mathfrak{D}(\lambda)$ eigendistribution of φ^2, $\lambda \in \mathfrak{S}$

$$s(\varphi X, \varphi Y) = -\lambda s(X,Y) + 2(2\lambda - tr(\varphi^2))g(\tau\varphi X, Y) \quad \forall X \in \Gamma\mathfrak{D}$$

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$\pi : P \to B$ non trivial principal $\mathbb{S}^1$-bundle $B := M_1 \times \cdots \times M_k$

$$e(P) = \sum b_i \pi_i^* \alpha_i, \; b_i \in \mathbb{Z} \qquad c_1(M_i) = q_i \alpha_i, \; \alpha_i \in H^2(M_i, \mathbb{Z})$$

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Theorem (Wang-Ziller, 1990)

Up to scaling there exists a unique Einstein metric g *on* P *such that* $\pi : (P, g) \to (B, g_o)$ *is a Riemannian submersion with totally geodesic fibers and*

$$g_o = x_1 g_1 \perp \cdots \perp x_k g_k, \quad x_i > 0.$$

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Theorem (Wang-Ziller, 1990)

Up to scaling there exists a unique Einstein metric g on P such that $\pi : (P, g) \to (B, g_o)$ is a Riemannian submersion with totally geodesic fibers and

$$g_o = x_1 g_1 \perp \cdots \perp x_k g_k, \quad x_i > 0.$$

(P, η, g) is a Levi-parallel contact Riemannian manifold, where

$$\eta \text{ connection form with } d\eta = \pi^*\left(\sum b_i \pi_i^* \Omega_i\right).$$

$\mathfrak{S}$ depends on the constants $x_1, \dots, x_k, b_1, \dots, b_k$.

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Theorem (Dileo-L,2011)

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5-dimensional case solved by Blair, Sierra (1993) and Pastore (1998)

General case: Boeckx-Cho (2006) (computer aided).

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