New Trends in Differential Geometry

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Generalized pseudohermitian manifolds

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1. Generalized pseudohermitian structures.

(M, HM, J) almost CR manifold of type (n, k), $n \geq 1$, $k \geq 0$:

M is a $\mathcal{C}^\infty$ manifold of dimension $2n + k$,

HM a vector subbundle of TM of rank $2n$,

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- The structure is called partially integrable if

$$[X,Y] - [JX,JY] \in \Gamma HM \quad \forall X,Y \in \Gamma HM.$$

- If in addition

$$N_J(X,Y) := [JX,Y] + [X,JY] - J([X,Y] - [JX,JY]) = 0$$

for every $X,Y \in \Gamma HM$, the structure is integrable and (M,HM,J) is termed a CR manifold.

Let (M, HM, J) be an almost CR manifold.

A generalized pseudohermitian structure on M is defined as a pair (h, P) where:

- h is Hermitian fiber metric on HM:

$$h(JX, JY) = h(X, Y) \quad \forall X, Y \in \Gamma HM$$

- $P : TM \to TM$ is a smooth projector such that:

$$Im(P) = HM.$$

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If $k = 0$, then $P = Id$ and (M, h, J) is an almost Hermitian manifold.

Let

$$f : (M, HM, J, h, P) \to (M', HM', J', h', P')$$

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f will be called pseudohermitian map if:

$$||f_* X||_{h'} \leq ||X||_h \quad \forall X \in H_x M, \tag{1}$$

$$Im(f_* \circ P_x - P'_{f(x)} \circ f_*) \subset f_*(H_x M)^\perp \subset H_{f(x)} M', \tag{2}$$

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In this case $\dim_{CR} M \leq \dim_{CR} M'$.

If f is a diffeomorphism such that both f and f^{-1} are pseudohermitian maps, f is called an equivalence or an isomorphism.

Let (M, HM, J, h, P) be a generalized pseudohermitian manifold.

We can define an operator

$$\Gamma : \Gamma HM \times \Gamma HM \to \Gamma HM$$

as follows:

$$\Gamma_X Y := P(\nabla^g_X Y)$$

where

- g is an arbitrary Riemannian metric extending h and such that $Ker(P) = HM^\perp$
- ∇^g is the Levi-Civita connection of g.

Γ does not depend on the choice of g but only on the pair (h, P).

Γ is invariant under equivalence.

It will be called the Koszul operator of M.

For every $X \in \Gamma HM$ define

$$\Gamma_X(J) : \Gamma HM \to \Gamma HM \qquad \Gamma_X(J)Y := \Gamma_X(JY) - J(\Gamma_X Y).$$

$\Gamma_X(J)$ is skew-symmetric w.r. to h and anticommutes with J.

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$$\alpha(X, Y, Z) := h(\Gamma_X(J)Y, Z).$$

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For each point $x \in M$, $\alpha_x : H_xM \times H_xM \times H_xM \to \mathbb{R}$ belongs to the Gray-Hervella space [1]

$$W = \{\alpha \in V^* \otimes V^* \otimes V^* | \alpha(X, Y, Z) = -\alpha(X, Z, Y) = \alpha(X, JY, JZ)\}$$

where $V = (H_xM, J_x, h_x)$.

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[1] A. Gray, L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., 1980.

Generalized pseudohermitian structures fall into sixteen classes, according to the decomposition

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$$

of W into irriducible components w.r. to the natural $U(n)$ action.

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In particular, one can consider geneneralized pseudohermitian manifolds

- of Kähler type if $\alpha = 0$
  $(\Gamma_X(J)Y = 0)$
- of nearly Kähler type if for each $x \in M \quad \alpha_x \in W_1$
  $(\Gamma_X(J)X = 0)$
- of almost Kähler type if for each $x \in M \ \ \alpha_x \in W_2$
  $(\mathcal{S}_{XYZ} h(\Gamma_X(J)Y, Z) = 0)$
- of quasi Kähler type if for each $x \in M \ \ \alpha_x \in W_1 \oplus W_2$
  $(\Gamma_X(J)Y + \Gamma_{JX}(J)JY = 0)$

Example

Let (M, HM, J) be an almost CR manifold endowed with a Riemannian metric g whose restriction h to HM is Hermitian.

Let $P_g : TM \to TM$ be the orthogonal projection onto HM.

Then (h, P_g) is a generalized pseudohermitian structure.

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Let $(\bar{M}, J, \bar{g})$ be an almost Hermitian manifold,

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Then M is endowed with a generalized pseudohermitian structure.

If $(\bar{M}, J, \bar{g})$ belongs to some Gray-Hervella class $\mathcal{U}$, then the generalized pseudohermitian structure belongs to the class of the same type.

Example

Let (M, HM, J) be a strongly pseudoconvex CR manifold of CR codimension $k \geq 1$.

Define a generalized pseudohermitian structure as follows:

$h =$ fixed positive definite Levi form $\mathcal{L}_\eta$, $\eta \in \Gamma H^0 M$,

$P =$ projection onto HM relative to the decomposition

$$TM = HM \oplus V^\eta$$

where V^η is the rank k subbundle of TM whose fiber at $x \in M$ is

$$V_x^\eta := \{\xi \in T_x M \,|\, d_x\eta(X, \xi) = 0 \ \ \forall X \in H_x M\}.$$

This structure is of Kähler type.

In the above example, if $k = 1$ then M is of hypersurface type.

V^η is the rank 1 vector bundle spanned by the Reeb vector field.

(M, HM, J, η) is a pseudohermitian manifold in the sense of Webster.[2]

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Webster showed that the equivalence problem for pseudohermitian manifolds can be canonically reduced to the equivalence of absolute parallelisms in spaces of dimension $(n+1)^2$.

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This is gained by attaching to each pseudohermitian manifold a canonical linear connection, the Tanaka-Webster connection, which parallelizes J and the Webster metric g_η.[2,3]

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[2] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom., 1978.

[3] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math., 1976.

Theorem

Let (M, HM, J, P, h) be a generalized pseudohermitian manifold. Then there exists a unique connection D on HM such that:

1. *D is compatible with the metric h and J is D-parallel.*
2. *For each $X \in \Gamma HM$, the operator*
$$\Lambda(X) := D_X - \Gamma_X : \Gamma HM \to \Gamma HM$$
anticommutes with J.
3. *For each $\xi \in \Gamma Ker(P)$ the skew-symmetric part of the tensor*
$$\tau_\xi : \Gamma HM \to \Gamma HM \qquad \tau_\xi(X) := D_\xi X - P[\xi, X]$$
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D is invariant under equivalence.

D will be called the canonical connection of the generalized pseudohermitian manifold M.

We denote by R the curvature tensor of D and the curvature tensor of type $(0, 4)$ defined by

$$R(X, Y, Z, W) := h(R(Z, W)Y, X)$$

for any $X, Y, Z, W \in \Gamma HM$.

Let $\sigma \subset H_x M$ be a holomorphic 2-plane at $x \in M$, that is $J\sigma = \sigma$, $\{X, JX\}$ an orthonormal basis of σ.

The pseudoholomorphic sectional curvature of σ is defined by

$$K(\sigma) := R_x(X, JX, X, JX),$$

also denoted by $K(X)$.

M is flat if $K(X) = 0$ for any unit holomorphic vector X.

2. Structures of kind 2.

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Let M be a smooth manifold and HM a smooth distribution of constant rank.

For each point $x \in M$ the Tanaka algebra of M at x is a fundamental graded Lie algebra $\mathfrak{m}(x) = \bigoplus_{p<0} \mathfrak{m}_p(x)$ constructed in a natural manner.[4]

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A kind 2 distribution is characterized by the requirement that

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in which case $\mathfrak{m}(x) = \mathfrak{m}_{-1}(x) \oplus \mathfrak{m}_{-2}(x)$,

$$\mathfrak{m}_{-1}(x) = H_x M, \quad \mathfrak{m}_{-2}(x) = T_x M / H_x M$$

and the Lie product is obtained by passing to the quotient from the Lie bracket of vector fields.

[4] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math., 1976.

Let (M, HM, J, h, P) be a generalized pseudohermitian manifold such that HM is a kind 2 distribution.

The Tanaka algebra of M at x is

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where the non trivial Lie bracket is the bilinear map:

$$L_x : H_x M \times H_x M \to Ker(P_x)$$

determined by the Levi-Tanaka form $L : \Gamma HM \times \Gamma HM \to \Gamma Ker(P)$

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The Levi-Tanaka form determines a *surjective* vector bundle homomorphism

$$L : \Gamma HM \wedge \Gamma HM \to \Gamma Ker(P)$$

which allows us to extend the sub-Riemannian metric h.

Proposition

Let (M, HM, J, h, P) *be a generalized pseudohermitian manifold of type* (n, k) *such that* HM *is a kind* 2 *distribution. Then,*

- h *extends canonically to a Riemannian metric* g *with respect to which* $TM = HM \oplus Ker(P)$ *is an orthogonal decomposition.*

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- *h extends canonically to a Riemannian metric g with respect to which $TM = HM \oplus Ker(P)$ is an orthogonal decomposition.*
- *There exists a unique linear connection ∇ extending the canonical connection D to TM and such that:*
  - a) *∇ is compatible with the Riemannian metric g,*
  - b) *the torsion T of ∇ satisfies*
  $$QT(\xi, \xi') = 0$$
  for every $\xi, \xi' \in \Gamma Ker(P)$, where $Q = Id - P$,
  - c) *for each $X \in \Gamma HM$, the bundle homomorphism*
  $$F_X : Ker(P) \to Ker(P), \qquad F_X(\xi) = QT(\xi, X),$$
  is symmetric with respect to g.

Remark

Let (M, HM, J, η) be a pseudohermitian manifold (in the sense of Webster).

HM is a kind 2 distribution.

- g coincides with the Webster metric g_η,
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Remark

For a generalized pseudohermitian manifold (M, HM, J, h, P), the distribution HM is of kind 1 if and only if $k = 0$, that is M is an almost Hermitian manifold.

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Tanno proved that if the automorphism group of an almost Hermitian $2n$-dimensional manifold M has maximum dimension $n^2 + 2n$, then M is Kähler and has constant holomorphic curvature.[5]

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[5] S. Tanno, *The automorphism groups of almost Hermitian manifolds*, Trans. Amer. Math. Soc., 1969.

Theorem

The equivalence problem for generalized pseudohermitian manifolds of type (n, k) and having kind ≤ 2 reduces in a natural way to the equivalence of complete parallelisms in spaces of dimension $N = n^2 + 2n + k$.

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Actually, we have a correspondence

$$(M, HM, J, h, P) \mapsto (P(M), \gamma)$$

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$\gamma = \omega + \theta \colon TP(M) \to \mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}^k$ is a complete parallelism

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The correspondence is compatible with the respective isomorphisms.

Theorem

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If equality holds:

- $Psh(M)$ *is transitive, i.e.* M *is homogeneous*
- M *is of Kähler type*
- M *has constant pseudoholomorphic curvature.*
- *The Tanaka algebra* $\mathfrak{m}(x)$ *at an arbitrary point must be isomorphic to one of the following models*

$$\mathfrak{m} = \mathbb{R}^{2n} \oplus W^*, \qquad [X,Y](A) = {}^tXAY,$$

where W is one of the following $\mathbb{R}$-linear subspaces of $\mathfrak{so}(2n)$:

	W	k
type I	$\{0\}$	0
	$\langle J_o \rangle$	1
	$\mathfrak{su}(n)$	n^2-1
	$\mathfrak{u}(n)$	n^2

	W	k
type II	$\mathfrak{p}$	n^2-n
	$\mathfrak{p} \oplus \langle J_o \rangle$	n^2-n+1
	$\mathfrak{p} \oplus \mathfrak{su}(n)$	$2n^2-n-1$
	$\mathfrak{p} \oplus \mathfrak{u}(n)$	$2n^2-n$
	$\mathfrak{so}(2n)$	$2n^2-n$

where $\mathfrak{p} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \middle| X_1, X_2 \in \mathfrak{so}(n) \right\}$ and $J_o = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

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Moreover, J is integrable if and only if $\mathfrak{m}$ is of type I.

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Moreover, J is integrable if and only if $\mathfrak{m}$ is of type I.
For each model $\mathfrak{m}$, there exists a simply connected, flat generalized pseudohermitian manifold whose Tanaka algebra at each point is isomorphic to $\mathfrak{m}$, and whose automorphism group $Psh(M)$ has the maximum dimension.

Construction of flat models:

$$\mathfrak{m} = \mathbb{R}^{2n} \oplus W^* \tag{3}$$

J_o standard complex structure on $\mathbb{R}^{2n}$

$\langle \cdot, \cdot \rangle$ standard inner product on $\mathbb{R}^{2n}$

$p : \mathfrak{m} \to \mathfrak{m}$ linear projection onto $\mathbb{R}^{2n}$ relative to decomposition (3)

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The data $(\mathbb{R}^{2n}, J_o, \langle\ ,\ \rangle, p)$ give rise canonically to a left invariant generalized pseudohermitian structure $(HM(\mathfrak{m}), J, h, P)$ on $M(\mathfrak{m})$.

Construction of flat models:

$$\mathfrak{m} = \mathbb{R}^{2n} \oplus W^* \tag{3}$$

J_o standard complex structure on $\mathbb{R}^{2n}$

$\langle \cdot, \cdot \rangle$ standard inner product on $\mathbb{R}^{2n}$

$p : \mathfrak{m} \to \mathfrak{m}$ linear projection onto $\mathbb{R}^{2n}$ relative to decomposition (3)

$M(\mathfrak{m})$ the simply connected Lie group with Lie algebra $\mathfrak{m}$.

The data $(\mathbb{R}^{2n}, J_o, \langle\ ,\ \rangle, p)$ give rise canonically to a left invariant generalized pseudohermitian structure $(HM(\mathfrak{m}), J, h, P)$ on $M(\mathfrak{m})$.

The canonical connection is the unique connection D on $HM(\mathfrak{m})$ such that $D_Z X = 0$ for any left invariant vector fields $Z \in \mathfrak{m}$, $X \in \mathbb{R}^{2n}$.

The pseudoholomorphic sectional curvature vanishes.

$Psh(M(\mathfrak{m}))$ has the maximum dimension $n^2 + 2n + k$.

Example: 3-Sasakian manifolds.

A 3-Sasakian manifold is a $(4n+3)$-dimensional manifold M endowed with three Sasakian structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$, with the same compatible metric g, such that for any even permutation (α, β, γ) of $\{1, 2, 3\}$:

$$\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta,$$

$$\xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \qquad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha.$$

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M carries three canonical generalized pseudohermitian structures (HM, J_α, h, P):

- $HM = \bigcap_{\alpha=1}^{3} Ker\,(\eta_\alpha)$ (distribution of kind 2)
- $J_\alpha :=$ restriction of ϕ_α to HM
- $h :=$ restriction of g to HM
- $P : TM \to TM$ orthogonal projection onto HM

Each structure is of Kähler type and not partially integrable.

Theorem

Let $(M, \varphi_\delta, \xi_\delta, \eta_\delta, g)$ *be a* 3*-Sasakian manifold. Suppose that* M *has constant pseudoholomorphic sectional curvature* c *with respect to one of the three canonical generalized pseudohermitian structures* (HM, J_α, h, P)*,* $\alpha = 1, 2, 3$*. Then, each generalized pseudohermitian structure has constant pseudoholomorphic sectional curvature* $c = 4$*. Furthermore, the Riemannian metric* g *has constant curvature* 1*.*

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Theorem

Let $(M, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-Sasakian manifold. If $Psh(M)$ has the maximum dimension, then M has dimension 7, constant pseudoholomorphic sectional curvature 4 and constant Riemannian curvature 1. The Tanaka algebra is $\mathfrak{m} = \mathbb{R}^4 \oplus \mathfrak{p} \oplus \langle J_o \rangle$.

Theorem

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The 7-dimensional sphere $S^7 \subset \mathbb{H}^2$ with the standard 3-Sasakian structure provides a non flat example for which $Psh(S^7)$ has actually the maximum dimension 11.

3. Isopseudohermitian immersions.

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Let

$$f : (M, HM, J, h, P) \to (M', HM', J', h', P')$$

be an isopseudohermitian immersion.

Define the real subbundle $HM^\perp$ of the pullback $f^*(HM')$

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$$HM'|_M = HM \oplus HM^{\perp}.$$

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D canonical connection on HM

D' canonical connection on HM' and the induced covariant differentiation on $HM'|_M$

Theorem

For each $X \in \mathfrak{X}(M)$, $Y \in \Gamma HM$, $\zeta \in \Gamma HM^\perp$ we have

$$D'_X Y = \underbrace{D_X Y + \beta_{P'QX}(Y)}_{\text{tangent}} + \underbrace{\alpha(X,Y)}_{\text{normal}} \qquad \text{(Gauss like formula)}$$

$$D'_X \zeta = \underbrace{-A_\zeta X}_{\text{tangent}} + \underbrace{D'^{\perp}_X \zeta}_{\text{normal}} \qquad \text{(Weingarten like formula)}$$

- $\beta_\zeta : HM \to HM$ is the bundle homomorphism defined by:

$$4h(\beta_\zeta Y, Z) = -h'(P'Q([Y,Z] + [JY,JZ]), \zeta).$$

  β_ζ is skew-symmetric and commutes with J.

- $A_\zeta : TM \to HM$ is a bundle homomorphism such that:

$$h(A_\zeta X, Y) = h'(\alpha(X,Y), \zeta) \quad \forall X \in \mathfrak{X}(M),\ Y \in \Gamma HM.$$

  In general, $A_\zeta : HM \to HM$ fails to be symmetric.

Let (M, HM, J, h, P) be a generalized pseudohermitian manifold.

Levi-Tanaka form:

$$L : \Gamma HM \times \Gamma HM \to \Gamma Ker(P) \qquad L(X,Y) := Q[X,Y],$$

Levi form:

- $\mathcal{C}^\infty(M)$-bilinear map $\mathcal{L} : \Gamma HM \times \Gamma HM \to \Gamma Ker(P)$

  $$2\mathcal{L}(X,Y) := L(X, JY) + L(Y, JX)$$

  which is Hermitian symmetric.

  If (M, HM, J, η) is a pseudohermitian manifold, then $\mathcal{L}$ coincides with the Levi form $\mathcal{L}_\eta$ corresponding to η.

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  If (M, HM, J, η) is a pseudohermitian manifold, then $\mathcal{L}$ coincides with the Levi form $\mathcal{L}_\eta$ corresponding to η.

- the vector valued quadratic form at each point $x \in M$

  $$\mathcal{L} : H_xM \to T_xM, \qquad \mathcal{L}(X) := Q([\tilde{X}, J\tilde{X}]_x)$$

  where $\tilde{X} \in \Gamma HM$ is an arbitrary extension of $X \in T_xM$.

Let $f : (M, HM, J, h, P) \to (M', HM', J', h', P')$ be an isopseudohermitian immersion.

Proposition

The following are equivalent:

a) $\beta_\zeta = 0 \qquad \forall \zeta \in HM^\perp$

b) $\mathcal{L}(X) = \mathcal{L}'(X) \qquad \forall X \in HM.$

Isopseudohermitian immersions with $\beta \equiv 0$ will be called Levi form preserving.

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The pseudohermitian immersions between strongly pseudoconvex CR manifolds introduced by Dragomir are Levi-form preserving.[6]

[6] S. Dragomir, *On pseudo-Hermitian immersions between strictly pseudoconvex CR manifolds*, Amer. J. Math., 1995.

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Proposition

If M' is of quasi-Kähler type, then a) and b) are equivalent to

c) $\alpha : HM \times HM \to HM^\perp$ *is symmetric.*

---

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Theorem

Let $f : (M, HM, J, h, P) \to (M', HM', J', h', P')$ be a Levi form preserving isopseudohermitian immersion between generalized pseudohermitian manifolds. Assume that M' is of quasi-Kähler type. Then for any holomorphic 2-plane σ of M

$$K(\sigma) \leq K'(\sigma).$$

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Remark

As particular cases the above result gives the known results concerning Kähler submanifolds and pseudohermitian immersions between pseudohermitian manifolds.[7]

[7] E. Barletta, *On the pseudohermitian sectional curvature of a strictly pseudoconvex CR manifold*, Differential Geom. Appl., 2007.

Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a semisimple Levi-Tanaka algebra

G the simply connected Lie group whose Lie algebra is $\mathfrak{g}$

G_+ the analytic subgroup corresponding to $\mathfrak{g}_+ = \bigoplus_{p \geq 0} \mathfrak{g}_p$

The standard (compact homogeneous) CR manifold [8]

$$S(\mathfrak{g}) := G/G_+$$

carries a generalized pseudohermitian structure of Kähler type which is not flat, with non-negative pseudoholomorphic curvature.

[8] C. Medori, M. Nacinovich, *Levi-Tanaka algebras and homogeneous CR manifolds*, Compos. Math., 1997.

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Theorem

There is no Levi form preserving isopseudohermitian immersion $f : S(\mathfrak{g}) \to M$ from a compact standard homogeneous pseudohermitian manifold into a generalized pseudohermitian manifold M of quasi Kähler type having non positive pseudoholomorphic curvature.

[8] C. Medori, M. Nacinovich, *Levi-Tanaka algebras and homogeneous CR manifolds*, Compos. Math., 1997.

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Standard homogeneous CR manifolds.

A Levi-Tanaka algebra is a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ such that

- the subalgebra $\mathfrak{m} := \oplus_{p<0}\mathfrak{g}_p$ is
  - fundamental: $\mathfrak{g}_{-1}$ generates $\mathfrak{m}$,
  - pseudocomplex: there exists a complex structure $J : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ satisfying

$$[JX, JY] = [X, Y] \quad \forall X, Y \in \mathfrak{g}_{-1}$$

- $ad : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{m})$ yields an isomorphism between $\mathfrak{g}_0$ and the algebra of 0-degree derivations of $\mathfrak{m}$ whose restriction to $\mathfrak{g}_{-1}$ commutes with J
- $\mathfrak{g}$ is the maximal transitive prolongation of the graded Lie algebra $\mathfrak{m} \oplus \mathfrak{g}_0$:

  $[X, \mathfrak{g}_{-1}] \neq 0$ for $X \in \bigoplus_{p\geq 0} \mathfrak{g}_p$, $X \neq 0$.

Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a semisimple Levi-Tanaka algebra

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$S(\mathfrak{g}) := G/G_+$ carries a G-invariant CR structure $(HS(\mathfrak{g}), J)$.

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Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an adapted Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{k} = \bigoplus_{j=0}^{\mu} \mathfrak{k}_{|j|}, \quad \mathfrak{p} = \bigoplus_{j=0}^{\mu} \mathfrak{p}_{|j|}, \quad \mathfrak{k}_{|j|} = \mathfrak{k} \cap (\mathfrak{g}_j \oplus \mathfrak{g}_{-j}), \quad \mathfrak{p}_{|j|} = \mathfrak{p} \cap (\mathfrak{g}_j \oplus \mathfrak{g}_{-j}).$$

$\mathfrak{k}$ is a maximal Lie subalgebra of $\mathfrak{g}$ on which the Killing form is negative definite

[9] C. Medori, M. Nacinovich, *Levi-Tanaka algebras and homogeneous CR manifolds*, Compos. Math., 1997.

The analytic subgroup $K \subset G$ corresponding to $\mathfrak{k}$ acts transitively on $S(\mathfrak{g})$

$S(\mathfrak{g}) = K/K_o$ is a reductive homogeneous space in the sense of Nomizu, with reductive decomposition

$$\mathfrak{k} = \mathfrak{k}_{|0|} \oplus \mathfrak{n} \qquad \mathfrak{n} := \mathfrak{k}_{|1|} \oplus \bigoplus_{p=2}^{\mu} \mathfrak{k}_{|p|}$$

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Canonical identifications:

$$T_o S(\mathfrak{g}) \cong \mathfrak{n} \qquad H_o S(\mathfrak{g}) \cong \mathfrak{k}_{|1|}$$

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Canonical identifications:

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Define

- $h :=$ K-invariant Hermitian metric on $HS(\mathfrak{g})$ determined by the opposite of the Killing form
- $P :=$ K-invariant tensor field corresponding to the linear projection $P_o : \mathfrak{n} \to \mathfrak{n}$ onto $\mathfrak{k}_{|1|}$

(h, P) is a generalized pseudohermitian structure of Kähler type.

$PS(\mathfrak{g}) := U(n)$ reduction of the frame bundle $\mathcal{F}(HS(\mathfrak{g}))$ of $HS(\mathfrak{g})$

Wang's theorem:

$$\{\text{K-Invariant connection on } PS(\mathfrak{g})\} \leftrightarrow \{\text{Equivariant } \ \Lambda : \mathfrak{n} \to \mathfrak{u}(n)\}$$

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Wang's theorem:

$\{$K-Invariant connection on $PS(\mathfrak{g})\} \leftrightarrow \{$Equivariant $\Lambda : \mathfrak{n} \to \mathfrak{u}(n)\}$

Theorem

The canonical connection of $S(\mathfrak{g})$ is the K-invariant connection on $PS(\mathfrak{g})$ corresponding to the linear map $\Lambda : \mathfrak{n} \to \mathfrak{u}(n)$ defined by

$$\Lambda(Z)(X) = [Z, X]_{|1|} \qquad X \in \mathfrak{k}_{|1|}.$$

Moreover, $S(\mathfrak{g})$ is not flat and the pseudoholomorphic curvature is non-negative, namely for each unit vector $X \in H_o S(\mathfrak{g}) \cong \mathfrak{k}_{|1|}$,

$$K(X) = ||[X, JX]||^2.$$

Let $P(M)$ be the canonical reduction od $\mathcal{F}(HM)$ to $U(n)$.

The connection D reduces to a principal connection on $P(M)$

Consider the connection form:

$$\omega : TP(M) \to \mathfrak{u}(n)$$

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kind 1 case (k=0)

Define the 1-form

$$\theta : TP(M) \to \mathbb{C}^n$$

$$\theta_u(Z) := u^{-1}(\pi_* Z), \quad Z \in T_u P(M)$$

- $\pi : P(M) \to M$ is the bunble projection
- $u \in P(M)$ is an $\mathbb{R}$-linear isomorphism $u : \mathbb{C}^n \to T_x M$.

Then we have a complete parallelism on $P(M)$:

$$\gamma := \omega + \theta : TP(M) \to \mathfrak{u}(n) \oplus \mathbb{C}^n$$

kind 2 case

Consider the surjective vector bundle homomorphism determined by the Levi-Tanaka form:

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- $\{u_1, \ldots, u_{2n}\}$ basis of $H_x M$.

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- $\{u_1, \dots, u_{2n}\}$ basis of H_xM.
- $\{w_1, \dots, w_N\} = \{u_i \wedge u_j | \, i < j\}$ basis of $H_xM \wedge H_xM$ ordered in a natural manner

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- $\{L(w_{j_1}), \ldots, L(w_{j_k})\}$ basis of $Ker(P_x)$ extracted from $\{L(w_1), \ldots, L(w_N)\}$ where $j_1 < j_2 < \cdots < j_k$ are uniquely determined according to

  $j_1 = min\{j | \, L(w_j) \neq 0\}$

  $j_s = min\{j | \, \dim_{\mathbb{R}} \langle L(w_{j_1}), \ldots, L(w_{j_{s-1}}), L(w_j) \rangle = s\}, s > 1$

Define the 1-form

$$\theta : TP(M) \to \mathbb{C}^n \oplus \mathbb{R}^k$$

such that

$$\theta_u(Z) := \bar{u}^{-1}(\pi_* Z), \quad Z \in T_u P(M)$$

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where for each frame $u \in P(M)$,

$$\bar{u} := u + \Psi(u) : \mathbb{C}^n \oplus \mathbb{R}^k \to T_x M, \quad x = \pi(u),$$

- $u \in P(M)$ is an $\mathbb{R}$-linear isomorphism $u : \mathbb{C}^n \to H_x M$
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Then we have a complete parallelism on $P(M)$:

$$\gamma := \omega + \theta : TP(M) \to \mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}^k$$

$$\mathfrak{m} = \mathbb{R}^{2n} \oplus W^*, \qquad [X, Y](A) = {}^t X A Y,$$

where W is one of the following $\mathbb{R}$-linear subspaces of $\mathfrak{so}(2n)$:

	W	$k = 3$
type I	$\{0\}$	0
	$\langle J_o \rangle$	1
	$\mathfrak{su}(n)$	$n^2 - 1$
	$\mathfrak{u}(n)$	n^2

	W	$k = 3$
type II	$\mathfrak{p}$	$n^2 - n$
	$\mathfrak{p} \oplus \langle J_o \rangle$	$n^2 - n + 1$
	$\mathfrak{p} \oplus \mathfrak{su}(n)$	$2n^2 - n - 1$
	$\mathfrak{p} \oplus \mathfrak{u}(n)$	$2n^2 - n$
	$\mathfrak{so}(2n)$	$2n^2 - n$

where $\mathfrak{p} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \middle| X_1, X_2 \in \mathfrak{so}(n) \right\}$ and $J_o = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

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	$\mathfrak{su}(n)$	n^2-1
	$\mathfrak{u}(n)$	n^2

	W	$k=3$
type II	$\mathfrak{p}$	n^2-n
	$\mathfrak{p} \oplus \langle J_o \rangle$	n^2-n+1
	$\mathfrak{p} \oplus \mathfrak{su}(n)$	$2n^2-n-1$
	$\mathfrak{p} \oplus \mathfrak{u}(n)$	$2n^2-n$
	$\mathfrak{so}(2n)$	$2n^2-n$

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Moreover, J is integrable if and only if $\mathfrak{m}$ is of type I.

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	$\mathfrak{su}(n)$	n^2-1
	$\mathfrak{u}(n)$	n^2

	W	$k=3$
type II	$\mathfrak{p}$	n^2-n
	$\mathfrak{p} \oplus \langle J_o \rangle$	n^2-n+1
	$\mathfrak{p} \oplus \mathfrak{su}(n)$	$2n^2-n-1$
	$\mathfrak{p} \oplus \mathfrak{u}(n)$	$2n^2-n$
	$\mathfrak{so}(2n)$	$2n^2-n$

where $\mathfrak{p} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \middle| X_1, X_2 \in \mathfrak{so}(n) \right\}$ and $J_o = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Moreover, J is integrable if and only if $\mathfrak{m}$ is of type I.
For each model $\mathfrak{m}$, there exists a simply connected, flat generalized pseudohermitian manifold whose Tanaka algebra at each point is isomorphic to $\mathfrak{m}$, and whose automorphism group $Psh(M)$ has the maximum dimension.