Geometry of Kaluza-Klein metrics on the sphere $\mathbb{S}^3$

Giovanni Calvaruso[1]

[1]University of Salento, Lecce, Italy. Joint work with D. Perrone.

L'Aquila, 7-9 Settembre 2011

Outline

The well known Berger metrics are defined as the canonical variation g_λ, $\lambda > 0$, of the standard metric g_0 of constant sectional curvature on $\mathbb{S}^3$, obtained deforming g_0 along the fibres of the Hopf fibration:

The well known Berger metrics are defined as the canonical variation g_λ, $\lambda > 0$, of the standard metric g_0 of constant sectional curvature on $\mathbb{S}^3$, obtained deforming g_0 along the fibres of the Hopf fibration:

$$g_\lambda|_{\xi_1^\perp} = g_0|_{\xi_1^\perp}, \qquad g_\lambda(\xi_1, \cdot) = \lambda g(\xi_1, \cdot), \qquad g_\lambda(\xi_1, \xi_1^\perp) = 0,$$

where ξ_1 denotes the standard Hopf vector field on $\mathbb{S}^3$.

The well known Berger metrics are defined as the canonical variation g_λ, $\lambda > 0$, of the standard metric g_0 of constant sectional curvature on $\mathbb{S}^3$, obtained deforming g_0 along the fibres of the Hopf fibration:

$$g_\lambda|_{\xi_1^\perp} = g_0|_{\xi_1^\perp}, \qquad g_\lambda(\xi_1, \cdot) = \lambda g(\xi_1, \cdot), \qquad g_\lambda(\xi_1, \xi_1^\perp) = 0,$$

where ξ_1 denotes the standard Hopf vector field on $\mathbb{S}^3$.

ξ_1, ξ_2, ξ_3: unit vector fields on $\mathbb{S}^3$ corresponding to the standard complex structures I, J, K,
$\theta^1, \theta^2, \theta^3$: 1-forms dual to ξ_1, ξ_2, ξ_3 with respect to g_0. Then,

$$g_\lambda = \lambda \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3.$$

The well known Berger metrics are defined as the canonical variation g_λ, $\lambda > 0$, of the standard metric g_0 of constant sectional curvature on $\mathbb{S}^3$, obtained deforming g_0 along the fibres of the Hopf fibration:

$$g_\lambda|_{\xi_1^\perp} = g_0|_{\xi_1^\perp}, \qquad g_\lambda(\xi_1, \cdot) = \lambda\, g(\xi_1, \cdot), \qquad g_\lambda(\xi_1, \xi_1^\perp) = 0,$$

where ξ_1 denotes the standard Hopf vector field on $\mathbb{S}^3$.

ξ_1, ξ_2, ξ_3: unit vector fields on $\mathbb{S}^3$ corresponding to the standard complex structures I, J, K,
$\theta^1, \theta^2, \theta^3$: 1-forms dual to ξ_1, ξ_2, ξ_3 with respect to g_0. Then,

$$g_\lambda = \lambda\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3.$$

Generalizing, one can allow deformations of g_0 also in the direction of ξ_2 and ξ_3, considering metrics of the form

$$\tilde{g}_{\lambda\mu\nu} = \lambda\, \theta^1 \otimes \theta^1 + \mu\, \theta^2 \otimes \theta^2 + \nu\, \theta^3 \otimes \theta^3, \quad \lambda, \mu, \nu > 0. \tag{1}$$

(M, g): Riemannian manifold, TM: tangent bundle of M.

(M,g): Riemannian manifold, TM: tangent bundle of M. At any point $(x,u) \in TM$,

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

where $\mathcal{V}_{(x,u)}$ is the kernel of $d\pi_{(x,u)}$ and $\mathcal{H}_{(x,u)}$ is the kernel of the connection map at (x,u).

(M,g): Riemannian manifold, TM: tangent bundle of M. At any point $(x,u) \in TM$,

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

where $\mathcal{V}_{(x,u)}$ is the kernel of $d\pi_{(x,u)}$ and $\mathcal{H}_{(x,u)}$ is the kernel of the connection map at (x,u).

The horizontal lift of a vector $X \in M_x$ is $X^h \in \mathcal{H}_{(x,u)}$, such that $d\pi X^h = X$. The vertical lift is $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M.

The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$, the map $X \to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$.

Riemannian g-natural metrics are the image of g under first order natural operators $D : S^2_+ T^* \rightsquigarrow (S^2 T^*)T$, which transform Riemannian metrics on manifolds into metrics on their tangent bundles.

Riemannian g-natural metrics are the image of g under first order natural operators $D : S^2_+ T^* \rightsquigarrow (S^2 T^*)T$, which transform Riemannian metrics on manifolds into metrics on their tangent bundles.
These metrics depend on six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$. Explicitly,

$$\begin{aligned}
G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) \\
&\quad + (\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\
G_{(x,u)}(X^h, Y^v) &= G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2) g_x(X, Y) \\
&\quad + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\
G_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u),
\end{aligned}$$

for every $u, X, Y \in M_x$, where $r^2 = g_x(u, u)$.
G is Riemannian under some restrictions on α_i, β_i.

Some well known Riemannian metrics on TM are g-natural:

Some well known Riemannian metrics on TM are g-natural:

- the Sasaki metric g_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.

Some well known Riemannian metrics on TM are g-natural:

- the Sasaki metric g_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.
- the Cheeger–Gromoll metric g_{GC} is obtained when $\alpha_2 = \beta_2 = 0$, $\alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}$ and $\alpha_3(t) = \frac{t}{1+t}$.

Some well known Riemannian metrics on TM are g-natural:

- the Sasaki metric g_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.
- the Cheeger–Gromoll metric g_{GC} is obtained when $\alpha_2 = \beta_2 = 0$, $\alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}$ and $\alpha_3(t) = \frac{t}{1+t}$.
- metrics of Cheeger-Gromoll type $h_{m,r}$ are obtained for $\alpha_1(t) = \frac{1}{(1+t)^m}$, $\alpha_3 = 1 - \alpha_1$, $\alpha_2 = \beta_2 = 0$, $\beta_1(t) = -\beta_3(t) = \frac{r}{(1+t)^m}$.

Some well known Riemannian metrics on TM are g-natural:

- the Sasaki metric g_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.
- the Cheeger–Gromoll metric g_{GC} is obtained when $\alpha_2 = \beta_2 = 0$, $\alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}$ and $\alpha_3(t) = \frac{t}{1+t}$.
- metrics of Cheeger-Gromoll type $h_{m,r}$ are obtained for $\alpha_1(t) = \frac{1}{(1+t)^m}$, $\alpha_3 = 1 - \alpha_1$, $\alpha_2 = \beta_2 = 0$, $\beta_1(t) = -\beta_3(t) = \frac{r}{(1+t)^m}$.
- Kaluza–Klein metrics, as commonly defined on principal bundles, are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$.

Some well known Riemannian metrics on TM are g-natural:

- the Sasaki metric g_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.
- the Cheeger–Gromoll metric g_{GC} is obtained when $\alpha_2 = \beta_2 = 0$, $\alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}$ and $\alpha_3(t) = \frac{t}{1+t}$.
- metrics of Cheeger-Gromoll type $h_{m,r}$ are obtained for $\alpha_1(t) = \frac{1}{(1+t)^m}$, $\alpha_3 = 1 - \alpha_1$, $\alpha_2 = \beta_2 = 0$, $\beta_1(t) = -\beta_3(t) = \frac{r}{(1+t)^m}$.
- Kaluza–Klein metrics, as commonly defined on principal bundles, are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$.
- The class of metrics of Kaluza–Klein type, which includes all examples above, is defined by the geometric condition of orthogonality between horizontal and vertical distributions ($\alpha_2 = \beta_2 = 0$).

The unit tangent sphere bundle over (M, g) is the hypersurface $T_1M = \{(x, u) \in TM : g_x(u, u) = 1\}$.

The unit tangent sphere bundle over (M, g) is the hypersurface $T_1M = \{(x, u) \in TM : g_x(u, u) = 1\}$. By definition, g-natural metrics on T_1M are the restrictions of g-natural metrics of TM to T_1M.

The unit tangent sphere bundle over (M, g) is the hypersurface $T_1M = \{(x, u) \in TM : g_x(u, u) = 1\}$. By definition, g-natural metrics on T_1M are the restrictions of g-natural metrics of TM to T_1M. Such a metric $\tilde{G}$ on T_1M is completely determined by

$$\begin{aligned}
\tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)\, g_x(X, Y) + d\, g_x(X, u) g_x(Y, u), \\
\tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= \tilde{G}_{(x,u)}(X^{t_G}, Y^h) = b\, g_x(X, Y), \\
\tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= a\, g_x(X, Y) - \tfrac{\phi}{a+c+d}\, g_x(X, u) g_x(Y, u),
\end{aligned}$$

where $X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\, N^G_{(x,u)}$ is the *tangential lift* of $X \in M_x$ to $(x, u) \in T_1M$.

The unit tangent sphere bundle over (M,g) is the hypersurface $T_1M = \{(x,u) \in TM : g_x(u,u) = 1\}$. By definition, g-natural metrics on T_1M are the restrictions of g-natural metrics of TM to T_1M. Such a metric $\tilde{G}$ on T_1M is completely determined by

$$\begin{aligned}
\tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)\, g_x(X,Y) + d\, g_x(X,u) g_x(Y,u), \\
\tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= \tilde{G}_{(x,u)}(X^{t_G}, Y^h) = b\, g_x(X,Y), \\
\tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= a\, g_x(X,Y) - \tfrac{\phi}{a+c+d}\, g_x(X,u) g_x(Y,u),
\end{aligned}$$

where $X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\, N^G_{(x,u)}$ is the *tangential lift* of $X \in M_x$ to $(x,u) \in T_1M$. $\tilde{G}$ is Riemannian if and only if

$$a > 0, \; \alpha := a(a+c) - b^2 > 0 \text{ and } \phi := a(a+c+d) - b^2 > 0.$$

The unit tangent sphere bundle over (M, g) is the hypersurface $T_1M = \{(x, u) \in TM : g_x(u, u) = 1\}$. By definition, g-natural metrics on T_1M are the restrictions of g-natural metrics of TM to T_1M. Such a metric $\tilde{G}$ on T_1M is completely determined by

$$\begin{aligned}
\tilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)\, g_x(X, Y) + d\, g_x(X, u) g_x(Y, u), \\
\tilde{G}_{(x,u)}(X^h, Y^{t_G}) &= \tilde{G}_{(x,u)}(X^{t_G}, Y^h) = b\, g_x(X, Y), \\
\tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= a\, g_x(X, Y) - \tfrac{\phi}{a+c+d}\, g_x(X, u) g_x(Y, u),
\end{aligned}$$

where $X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\, N^G_{(x,u)}$ is the *tangential lift* of $X \in M_x$ to $(x, u) \in T_1M$. $\tilde{G}$ is Riemannian if and only if

$$a > 0,\ \alpha := a(a+c) - b^2 > 0 \text{ and } \phi := a(a+c+d) - b^2 > 0.$$

$\tilde{G}$ is of Kaluza-Klein type if $b = 0$, a Kaluza-Klein metric if $b = d = 0$.

The Hopf map

$\mathbb{H} = \{q = a_1 + a_2 i + a_3 j + a_4 k \,:\, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ quaternions algebra, $\mathbb{S}^3(1) = \{q \in \mathbb{H} : ||q|| = 1\}$.

The Hopf map

$\mathbb{H} = \{q = a_1 + a_2 i + a_3 j + a_4 k : a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ quaternions algebra, $\mathbb{S}^3(1) = \{q \in \mathbb{H} : ||q|| = 1\}$.
For any $q \in \mathbb{S}^3(1)$, the map $\varphi_q(z) := \bar{q} z q$ defines an orthogonal transformation of $\mathbb{H}$, which leaves invariant $\mathbb{R}^3 = \{q \in \mathbb{H} : q = a_2 i + a_3 j + a_4 k\}$.

The Hopf map

$\mathbb{H} = \{q = a_1 + a_2 i + a_3 j + a_4 k : a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ quaternions algebra, $\mathbb{S}^3(1) = \{q \in \mathbb{H} : ||q|| = 1\}$.
For any $q \in \mathbb{S}^3(1)$, the map $\varphi_q(z) := \bar{q}zq$ defines an orthogonal transformation of $\mathbb{H}$, which leaves invariant $\mathbb{R}^3 = \{q \in \mathbb{H} : q = a_2 i + a_3 j + a_4 k\}$. The map

$$\Phi : \mathbb{S}^3(1) \to SO(3), \; q \mapsto \varphi_q$$

describes $\mathbb{S}^3(1)$ as universal covering of $SO(3)$.

The Hopf map

$\mathbb{H} = \{q = a_1 + a_2 i + a_3 j + a_4 k : a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ quaternions algebra, $\mathbb{S}^3(1) = \{q \in \mathbb{H} : ||q|| = 1\}$.
For any $q \in \mathbb{S}^3(1)$, the map $\varphi_q(z) := \bar{q}zq$ defines an orthogonal transformation of $\mathbb{H}$, which leaves invariant $\mathbb{R}^3 = \{q \in \mathbb{H} : q = a_2 i + a_3 j + a_4 k\}$. The map

$$\Phi : \mathbb{S}^3(1) \to SO(3),\ q \mapsto \varphi_q$$

describes $\mathbb{S}^3(1)$ as universal covering of $SO(3)$.

The Hopf map

$\mathbb{H} = \{q = a_1 + a_2 i + a_3 j + a_4 k : a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ quaternions algebra, $\mathbb{S}^3(1) = \{q \in \mathbb{H} : ||q|| = 1\}$.
For any $q \in \mathbb{S}^3(1)$, the map $\varphi_q(z) := \bar{q}zq$ defines an orthogonal transformation of $\mathbb{H}$, which leaves invariant $\mathbb{R}^3 = \{q \in \mathbb{H} : q = a_2 i + a_3 j + a_4 k\}$. The map

$$\Phi : \mathbb{S}^3(1) \to SO(3),\ q \mapsto \varphi_q$$

describes $\mathbb{S}^3(1)$ as universal covering of $SO(3)$.
Starting from the three-sphere $\mathbb{S}^3(\kappa/4)$, the Hopf map is given by

$$\tilde{h} : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^2(\kappa),\ p \mapsto \frac{1}{\sqrt{\kappa}}\,\varphi_q(i), \text{ where } q = \frac{\sqrt{\kappa}}{2} p \in \mathbb{S}^3(1).$$

As $T_1\mathbb{S}^2(\kappa) = \{(x,u) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \in \mathbb{S}^2(\kappa),\ u \perp x,\ ||u|| = 1\}$, we can consider the diffeomorphism

$$\psi : T_1\mathbb{S}^2(\kappa) \to SO(3),\ \ (x,u) \mapsto (\sqrt{\kappa}\,x, \sqrt{\kappa}\,u \wedge x, u)$$

As $T_1\mathbb{S}^2(\kappa) = \{(x,u) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \in \mathbb{S}^2(\kappa),\, u \perp x,\, ||u|| = 1\}$, we can consider the diffeomorphism

$$\psi : T_1\mathbb{S}^2(\kappa) \to SO(3), \;\; (x,u) \mapsto (\sqrt{\kappa}\, x, \sqrt{\kappa}\, u \wedge x, u)$$

Then, introducing the homothety

$$\tau : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^3(1), \qquad p \mapsto \frac{\sqrt{\kappa}}{2} p,$$

As $T_1\mathbb{S}^2(\kappa) = \{(x,u) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \in \mathbb{S}^2(\kappa),\ u \perp x,\ ||u|| = 1\}$, we can consider the diffeomorphism

$$\psi : T_1\mathbb{S}^2(\kappa) \to SO(3),\ \ (x,u) \mapsto (\sqrt{\kappa}\, x, \sqrt{\kappa}\, u \wedge x, u)$$

Then, introducing the homothety

$$\tau : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^3(1), \qquad p \mapsto \frac{\sqrt{\kappa}}{2} p,$$

we have the covering map

$$F = \psi^{-1} \circ \Phi \circ \tau : \quad \mathbb{S}^3(\kappa/4) \to T_1\mathbb{S}^2(\kappa),$$
$$p = \frac{2}{\sqrt{\kappa}} q \mapsto \left(\frac{1}{\sqrt{\kappa}} \varphi_q(i), \varphi_q(k) \right).$$

Explicitly, if $p = (z_1, z_2)$, with $z_1, z_2 \in \mathbb{C}$, then

$$F(z_1, z_2) = \left(\frac{\sqrt{\kappa}}{4} \left(|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2 \right), \frac{\kappa}{4} \left(-2\mathrm{Re}(z_1 z_2), z_1^2 - \bar{z}_2^2 \right) \right).$$

Explicitly, if $p=(z_1,z_2)$, with $z_1,z_2\in\mathbb{C}$, then

$$F(z_1,z_2)=\left(\frac{\sqrt{\kappa}}{4}\left(|z_1|^2-|z_2|^2,2z_1\bar{z}_2\right),\frac{\kappa}{4}\left(-2\mathrm{Re}(z_1z_2),z_1^2-\bar{z}_2^2\right)\right).$$

Consider now the unit vector fields $\{\xi_1,\xi_2,\xi_3\}$ on $\mathbb{S}^3(\kappa/4)$ corresponding to the standard complex structures I,J,K:

$$\begin{cases}\xi_1(p)=\frac{\sqrt{\kappa}}{2}ip &= \frac{\sqrt{\kappa}}{2}(-x_2,x_1,-x_4,x_3),\\ \xi_2(p)=\frac{\sqrt{\kappa}}{2}jp &= \frac{\sqrt{\kappa}}{2}(-x_3,x_4,x_1,-x_2),\\ \xi_3(p)=\frac{\sqrt{\kappa}}{2}kp &= \frac{\sqrt{\kappa}}{2}(-x_4,-x_3,x_2,x_1).\end{cases}$$

Explicitly, if $p = (z_1, z_2)$, with $z_1, z_2 \in \mathbb{C}$, then

$$F(z_1, z_2) = \left(\frac{\sqrt{\kappa}}{4} \left(|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2 \right), \frac{\kappa}{4} \left(-2\mathrm{Re}(z_1 z_2), z_1^2 - \bar{z}_2^2 \right) \right).$$

Consider now the unit vector fields $\{\xi_1, \xi_2, \xi_3\}$ on $\mathbb{S}^3(\kappa/4)$ corresponding to the standard complex structures I, J, K:

$$\begin{cases} \xi_1(p) = \frac{\sqrt{\kappa}}{2} ip = \frac{\sqrt{\kappa}}{2}(-x_2, x_1, -x_4, x_3), \\ \xi_2(p) = \frac{\sqrt{\kappa}}{2} jp = \frac{\sqrt{\kappa}}{2}(-x_3, x_4, x_1, -x_2), \\ \xi_3(p) = \frac{\sqrt{\kappa}}{2} kp = \frac{\sqrt{\kappa}}{2}(-x_4, -x_3, x_2, x_1). \end{cases}$$

Then, $F_*\xi_1 = -\sqrt{\kappa}\,(\tilde{J}u)^v, \quad F_*\xi_2 = u^h, \quad F_*\xi_3 = (\tilde{J}u)^h$, where $u = \varphi_q(k)$.

Explicitly, if $p=(z_1,z_2)$, with $z_1,z_2\in\mathbb{C}$, then

$$F(z_1,z_2)=\left(\frac{\sqrt{\kappa}}{4}\left(|z_1|^2-|z_2|^2,2z_1\bar{z}_2\right),\frac{\kappa}{4}\left(-2\mathrm{Re}(z_1z_2),z_1^2-\bar{z}_2^2\right)\right).$$

Consider now the unit vector fields $\{\xi_1,\xi_2,\xi_3\}$ on $\mathbb{S}^3(\kappa/4)$ corresponding to the standard complex structures I,J,K:

$$\begin{cases}\xi_1(p)=\frac{\sqrt{\kappa}}{2}ip\ =\frac{\sqrt{\kappa}}{2}(-x_2,x_1,-x_4,x_3),\\ \xi_2(p)=\frac{\sqrt{\kappa}}{2}jp\ =\frac{\sqrt{\kappa}}{2}(-x_3,x_4,x_1,-x_2),\\ \xi_3(p)=\frac{\sqrt{\kappa}}{2}kp=\frac{\sqrt{\kappa}}{2}(-x_4,-x_3,x_2,x_1).\end{cases}$$

Then, $F_*\xi_1=-\sqrt{\kappa}\,(\tilde{J}u)^v$, $\quad F_*\xi_2=u^h$, $\quad F_*\xi_3=(\tilde{J}u)^h$, where $u=\varphi_q(k)$. Denoted by $\pi_1:T_1\mathbb{S}^2(\kappa)\to\mathbb{S}^2(\kappa)$ the canonical projection, the Hopf map is explicitly given by

$$\begin{aligned}\tilde{h}=\pi_1\circ F:\mathbb{S}^3(\kappa/4)\ &\to\ \mathbb{S}^2(\kappa)\\ (z_1,z_2)\ &\mapsto\ \frac{\sqrt{\kappa}}{4}\left(2z_1\bar{z}_2,|z_1|^2-|z_2|^2\right)\end{aligned}$$

Let $\tilde{G}$ be a Riemannian g-natural metric of Kaluza-Klein type on $T_1\mathbb{S}^2(\kappa)$, determined by three real parameters a, c, d, satisfying $a, a+c, a+c+d > 0$.

Let $\tilde{G}$ be a Riemannian g-natural metric of Kaluza-Klein type on $T_1\mathbb{S}^2(\kappa)$, determined by three real parameters a, c, d, satisfying $a, a+c, a+c+d > 0$. If we put

$$X_1 = (\tilde{J}u)^v, \qquad X_2 = (\tilde{J}u)^h, \qquad X_3 = u^h,$$

then $\{X_1, X_2, X_3\}$ is a $\tilde{G}$-orthogonal frame field on $T_1\mathbb{S}^2(\kappa)$, with

$$||X_1||^2_{\tilde{G}} = a, \qquad ||X_2||^2_{\tilde{G}} = a+c, \qquad ||X_3||^2_{\tilde{G}} = a+c+d.$$

Let $\tilde{G}$ be a Riemannian g-natural metric of Kaluza-Klein type on $T_1\mathbb{S}^2(\kappa)$, determined by three real parameters a, c, d, satisfying $a, a+c, a+c+d > 0$. If we put

$$X_1 = (\tilde{J}u)^v, \qquad X_2 = (\tilde{J}u)^h, \qquad X_3 = u^h,$$

then $\{X_1, X_2, X_3\}$ is a $\tilde{G}$-orthogonal frame field on $T_1\mathbb{S}^2(\kappa)$, with

$$||X_1||^2_{\tilde{G}} = a, \qquad ||X_2||^2_{\tilde{G}} = a+c, \qquad ||X_3||^2_{\tilde{G}} = a+c+d.$$

Let now η^i denote the 1-forms $\tilde{G}$-dual of X_i, $i = 1, 2, 3$. Then,

$$\tilde{G} = \frac{1}{a}\,\eta^1 \otimes \eta^1 + \frac{1}{a+c}\,\eta^2 \otimes \eta^2 + \frac{1}{a+c+d}\,\eta^3 \otimes \eta^3$$

and the corresponding metric $\tilde{g} := F^*\tilde{G}$ on $\mathbb{S}^3$ is given by

$$\tilde{g} = \frac{1}{a}\,(F^*\eta^1) \otimes (F^*\eta^1) + \frac{1}{a+c}\,(F^*\eta^2) \otimes (F^*\eta^2) + \frac{1}{a+c+d}\,(F^*\eta^3) \otimes (F^*\eta^3).$$

As $F^*\eta^1 = -\sqrt{k}\, a\, \theta^1,\ F^*\eta^2 = (a+c)\,\theta^3,\ F^*\eta^3 = (a+c+d)\,\theta^2$, we have

$$\tilde{g} = ka\,\theta^1 \otimes \theta^1 + (a+c+d)\,\theta^2 \otimes \theta^2 + (a+c)\,\theta^3 \otimes \theta^3.$$

As $F^*\eta^1 = -\sqrt{k}\, a\, \theta^1,\ F^*\eta^2 = (a+c)\,\theta^3,\ F^*\eta^3 = (a+c+d)\,\theta^2$, we have

$$\tilde{g} = ka\,\theta^1 \otimes \theta^1 + (a+c+d)\,\theta^2 \otimes \theta^2 + (a+c)\,\theta^3 \otimes \theta^3.$$

Thus, $\tilde{g}$ is exactly of the form (1), with $\lambda = ka$, $\mu = a+c+d$, $\nu = a+c$. So, we have

Theorem

The covering map $F : \mathbb{S}^3(\kappa/4) \to T_1\mathbb{S}^2(\kappa)$ establishes a one-to-one correspondence between Riemannian metrics on $\mathbb{S}^3$ of the form (1) and metrics of Kaluza-Klein type on $T_1\mathbb{S}^2(\kappa)$.

As $F^*\eta^1 = -\sqrt{k}\, a\, \theta^1$, $F^*\eta^2 = (a+c)\,\theta^3$, $F^*\eta^3 = (a+c+d)\,\theta^2$, we have

$$\tilde{g} = ka\,\theta^1 \otimes \theta^1 + (a+c+d)\,\theta^2 \otimes \theta^2 + (a+c)\,\theta^3 \otimes \theta^3.$$

Thus, $\tilde{g}$ is exactly of the form (1), with $\lambda = ka$, $\mu = a+c+d$, $\nu = a+c$. So, we have

Theorem

The covering map $F : \mathbb{S}^3(\kappa/4) \to T_1\mathbb{S}^2(\kappa)$ establishes a one-to-one correspondence between Riemannian metrics on $\mathbb{S}^3$ of the form (1) and metrics of Kaluza-Klein type on $T_1\mathbb{S}^2(\kappa)$.

Definition

Riemannian metrics on $\mathbb{S}^3$ of the form (1) are called metrics of Kaluza-Klein type (in particular, Kaluza-Klein metrics are defined by the condition $\mu = \nu$).

A (smooth) map $f : (M', g') \to (M, g)$ between two Riemannian manifolds is harmonic if f is a critical point of the energy functional $\mathcal{E}(f, \Omega) := \frac{1}{2}\int_\Omega ||df||^2 dv_{g'}$, for any compact domain $\Omega \subset M'$.

A (smooth) map $f : (M', g') \to (M, g)$ between two Riemannian manifolds is harmonic if f is a critical point of the energy functional $\mathcal{E}(f, \Omega) := \frac{1}{2} \int_\Omega ||df||^2 dv_{g'}$, for any compact domain $\Omega \subset M'$.

A map $\varphi : (M', g') \to (M, g)$ is a harmonic morphism if it pulls back (local) harmonic functions to harmonic functions, that is, for any open set U of M with $\varphi^{-1}(U) \neq \emptyset$ and any harmonic function f on $(U, g|_U)$, the map $f \circ \varphi$ is a harmonic function on $(\varphi^{-1}(U), g'|_{\varphi^{-1}(U)})$.

A (smooth) map $f : (M', g') \to (M, g)$ between two Riemannian manifolds is harmonic if f is a critical point of the energy functional $\mathcal{E}(f, \Omega) := \frac{1}{2} \int_\Omega ||df||^2 dv_{g'}$, for any compact domain $\Omega \subset M'$.

A map $\varphi : (M', g') \to (M, g)$ is a harmonic morphism if it pulls back (local) harmonic functions to harmonic functions, that is, for any open set U of M with $\varphi^{-1}(U) \neq \emptyset$ and any harmonic function f on $(U, g|_U)$, the map $f \circ \varphi$ is a harmonic function on $(\varphi^{-1}(U), g'|_{\varphi^{-1}(U)})$.

A smooth map is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal.

"Bernstein theorem" by Baird and Wood

After an orthogonal change of coordinates,

"Bernstein theorem" by Baird and Wood

After an orthogonal change of coordinates, any harmonic morphism from $\mathbb{S}^3$ to a Riemannian surface M^2 coincides with the standard Hopf fibration $h : \mathbb{S}^3(1) \to \mathbb{S}^2(4)$.

"Bernstein theorem" by Baird and Wood

After an orthogonal change of coordinates, any harmonic morphism from $\mathbb{S}^3$ to a Riemannian surface M^2 coincides with the standard Hopf fibration $h : \mathbb{S}^3(1) \to \mathbb{S}^2(4)$.

So, a harmonic morphism from the standard three-sphere to a Riemannian surface is just the simplest possible one, as it essentially coincides with the standard Hopf fibration.

“Bernstein theorem” by Baird and Wood

After an orthogonal change of coordinates, any harmonic morphism from $\mathbb{S}^3$ to a Riemannian surface M^2 coincides with the standard Hopf fibration $h : \mathbb{S}^3(1) \to \mathbb{S}^2(4)$.

So, a harmonic morphism from the standard three-sphere to a Riemannian surface is just the simplest possible one, as it essentially coincides with the standard Hopf fibration.

It is then natural to ask what happens when we deform the standard metric of the three-sphere.

Theorem

The canonical projection $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is a harmonic morphism if and only if $\tilde{G}$ is a Kaluza-Klein metric, that is, when $b = d = 0$.

Theorem

The canonical projection $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is a harmonic morphism if and only if $\tilde{G}$ is a Kaluza-Klein metric, that is, when $b = d = 0$.

Consequence

Let $\tilde{g}_{ac}$ be any Kaluza-Klein metric on $\mathbb{S}^3$. Then, the maps

$$\tilde{h}_{ac} : (\mathbb{S}^3, \tilde{g}_{ac}) \to (\mathbb{S}^2(\kappa), g_0)$$

give a two-parameter family of harmonic morphisms.

Theorem

The canonical projection $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is a harmonic morphism if and only if $\tilde{G}$ is a Kaluza-Klein metric, that is, when $b = d = 0$.

Consequence

Let $\tilde{g}_{ac}$ be any Kaluza-Klein metric on $\mathbb{S}^3$. Then, the maps

$$\tilde{h}_{ac} : (\mathbb{S}^3, \tilde{g}_{ac}) \to (\mathbb{S}^2(\kappa), g_0)$$

give a two-parameter family of harmonic morphisms.

(i) for $a = 1/\kappa$ and $c = 1 - 1/\kappa$ we get the Hopf map $\tilde{h} : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^2(\kappa)$;

Theorem

The canonical projection $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is a harmonic morphism if and only if $\tilde{G}$ is a Kaluza-Klein metric, that is, when $b = d = 0$.

Consequence

Let $\tilde{g}_{ac}$ be any Kaluza-Klein metric on $\mathbb{S}^3$. Then, the maps

$$\tilde{h}_{ac} : (\mathbb{S}^3, \tilde{g}_{ac}) \to (\mathbb{S}^2(\kappa), g_0)$$

give a two-parameter family of harmonic morphisms.

(i) for $a = 1/\kappa$ and $c = 1 - 1/\kappa$ we get the Hopf map $\tilde{h} : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^2(\kappa)$;

(ii) for $a = (\lambda/\kappa) > 0$ and $c = 1 - (\lambda/\kappa)$, $\tilde{h}_\lambda$ are harmonic morphisms defined on Berger spheres.

Remark

The correspondence between Riemannian metrics (1) on $\mathbb{S}^3$ and Riemannian g-natural metrics of $T_1\mathbb{S}^2(\kappa)$

Remark

The correspondence between Riemannian metrics (1) on $\mathbb{S}^3$ and Riemannian g-natural metrics of $T_1\mathbb{S}^2(\kappa)$

- gives a natural geometric interpretation of metrics (1);

Remark

The correspondence between Riemannian metrics (1) on $\mathbb{S}^3$ and Riemannian g-natural metrics of $T_1\mathbb{S}^2(\kappa)$

- gives a natural geometric interpretation of metrics (1);
- permits to describe the Levi Civita connection and curvature of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ using the corresponding results on $(T_1\mathbb{S}^2(\kappa), \tilde{G}_{acd})$.

By (1), we have that

$$e_1 := \frac{1}{\sqrt{\lambda}}\xi_1, \quad e_2 := \frac{1}{\sqrt{\mu}}\xi_2, \quad e_3 := \frac{1}{\sqrt{\nu}}\xi_3$$

is a global frame field on $\mathbb{S}^3$, orthonormal with respect to $\tilde{g}_{\lambda\mu\nu}$.

By (1), we have that

$$e_1 := \frac{1}{\sqrt{\lambda}}\,\xi_1, \quad e_2 := \frac{1}{\sqrt{\mu}}\,\xi_2, \quad e_3 := \frac{1}{\sqrt{\nu}}\,\xi_3$$

is a global frame field on $\mathbb{S}^3$, orthonormal with respect to $\tilde{g}_{\lambda\mu\nu}$.

Vector fields e_i (and so, ξ_i) are Ricci eigenvectors for $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, corresponding to the Ricci eigenvalues

$$\varrho_{11} = \frac{\kappa\big[\lambda^2-(\mu-\nu)^2\big]}{2\lambda\mu\nu}, \quad \varrho_{22} = \frac{\kappa\big[\mu^2-(\lambda-\nu)^2\big]}{2\lambda\mu\nu}, \quad \varrho_{33} = \frac{\kappa\big[\nu^2-(\lambda-\mu)^2\big]}{2\lambda\mu\nu}.$$

Proposition

An arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type

Proposition

An arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type

(i) has three distinct Ricci eigenvalues if and only if λ, μ, ν satisfy $\lambda \neq \mu \neq \nu \neq \lambda$, $\lambda \neq \mu + \nu$, $\mu \neq \lambda + \nu$, $\nu \neq \lambda + \mu$;

Proposition

An arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type

(i) has three distinct Ricci eigenvalues if and only if λ, μ, ν satisfy $\lambda \neq \mu \neq \nu \neq \lambda$, $\lambda \neq \mu + \nu$, $\mu \neq \lambda + \nu$, $\nu \neq \lambda + \mu$;

(ii) has two distinct Ricci eigenvalues if and only if either
(a) exactly two of λ, μ, ν coincide, or
(b) one of them is the sum of the remaining two;

Proposition

An arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type

(i) has three distinct Ricci eigenvalues if and only if λ, μ, ν satisfy $\lambda \neq \mu \neq \nu \neq \lambda$, $\lambda \neq \mu + \nu$, $\mu \neq \lambda + \nu$, $\nu \neq \lambda + \mu$;

(ii) has two distinct Ricci eigenvalues if and only if either
(a) exactly two of λ, μ, ν coincide, or
(b) one of them is the sum of the remaining two;

(iii) has constant sectional curvature if and only if $\lambda = \mu = \nu$.

Proposition

An arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type

(i) has three distinct Ricci eigenvalues if and only if λ, μ, ν satisfy $\lambda \neq \mu \neq \nu \neq \lambda$, $\lambda \neq \mu + \nu$, $\mu \neq \lambda + \nu$, $\nu \neq \lambda + \mu$;

(ii) has two distinct Ricci eigenvalues if and only if either
(a) exactly two of λ, μ, ν coincide, or
(b) one of them is the sum of the remaining two;

(iii) has constant sectional curvature if and only if $\lambda = \mu = \nu$.

Berger metrics are included in case *(ii)*.

Proposition

An arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type

(i) has three distinct Ricci eigenvalues if and only if λ, μ, ν satisfy $\lambda \neq \mu \neq \nu \neq \lambda$, $\lambda \neq \mu + \nu$, $\mu \neq \lambda + \nu$, $\nu \neq \lambda + \mu$;

(ii) has two distinct Ricci eigenvalues if and only if either
(a) exactly two of λ, μ, ν coincide, or
(b) one of them is the sum of the remaining two;

(iii) has constant sectional curvature if and only if $\lambda = \mu = \nu$.

Berger metrics are included in case *(ii)*. The remaining case listed in case *(ii)* has a special geometrical meaning. In fact, if either $\lambda = \mu + \nu$, $\mu = \lambda + \nu$ or $\nu = \lambda + \mu$, then two Ricci eigenvalues vanish.

A Riemannian manifold (M, g) is said to be an Ivanov-Petrova manifold (shortly, an *IP* manifold) if its skew-symmetric curvature operator

$$R(\pi) = |g(X,X)g(Y,Y) - g(X,Y)^2|^{-1/2}R(X,Y),$$

where $\pi = \mathrm{Span}(X,Y)$, has constant eigenvalues on $G^+(2,n)$, the Grassmannian of all oriented 2-planes.

A Riemannian manifold (M, g) is said to be an Ivanov-Petrova manifold (shortly, an *IP* manifold) if its skew-symmetric curvature operator

$$R(\pi) = |g(X,X)g(Y,Y) - g(X,Y)^2|^{-1/2}R(X,Y),$$

where $\pi = \mathrm{Span}(X,Y)$, has constant eigenvalues on $G^+(2,n)$, the Grassmannian of all oriented 2-planes. *IP* manifolds of dimension $n \geq 4$ are completely classified. A three-dimensional Riemannian manifold is *IP* if and only if either it has constant curvature, or its Ricci tensor has rank 1.

A Riemannian manifold (M, g) is said to be an Ivanov-Petrova manifold (shortly, an *IP* manifold) if its skew-symmetric curvature operator

$$R(\pi) = |g(X,X)g(Y,Y) - g(X,Y)^2|^{-1/2} R(X,Y),$$

where $\pi = \mathrm{Span}(X, Y)$, has constant eigenvalues on $G^+(2, n)$, the Grassmannian of all oriented 2-planes. *IP* manifolds of dimension $n \geq 4$ are completely classified. A three-dimensional Riemannian manifold is *IP* if and only if either it has constant curvature, or its Ricci tensor has rank 1.

Theorem

The sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type is an *IP* manifold if and only if one among λ, μ, ν is the sum of the remaining two.

We first remark that, independently from $g_{\lambda\mu\nu}$,

$$[\xi_1, \xi_2] = -\sqrt{\kappa}\,\xi_3, \qquad [\xi_2, \xi_3] = -\sqrt{\kappa}\,\xi_1, \qquad [\xi_3, \xi_1] = -\sqrt{\kappa}\,\xi_2.$$

We first remark that, independently from $g_{\lambda\mu\nu}$,

$$[\xi_1, \xi_2] = -\sqrt{\kappa}\,\xi_3, \qquad [\xi_2, \xi_3] = -\sqrt{\kappa}\,\xi_1, \qquad [\xi_3, \xi_1] = -\sqrt{\kappa}\,\xi_2.$$

Hence,

Theorem

Any sphere of Kaluza-Klein type $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ has a Lie group structure, unique up to isomorphisms, such that the vector fields ξ_1, ξ_2, ξ_3 are left invariant.

We first remark that, independently from $g_{\lambda\mu\nu}$,

$$[\xi_1, \xi_2] = -\sqrt{\kappa}\,\xi_3, \qquad [\xi_2, \xi_3] = -\sqrt{\kappa}\,\xi_1, \qquad [\xi_3, \xi_1] = -\sqrt{\kappa}\,\xi_2.$$

Hence,

Theorem

Any sphere of Kaluza-Klein type $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ has a Lie group structure, unique up to isomorphisms, such that the vector fields ξ_1, ξ_2, ξ_3 are left invariant.

More precisely, the signs of the coefficients in the Lie brackets above, together with Milnor's classification of 3D Riemannian Lie groups, yield that $\tilde{g}_{\lambda\mu\nu}$ corresponds to a left-invariant Riemannian metric on $SU(2)$, as it could be expected.

We first remark that, independently from $g_{\lambda\mu\nu}$,

$$[\xi_1, \xi_2] = -\sqrt{\kappa}\,\xi_3, \qquad [\xi_2, \xi_3] = -\sqrt{\kappa}\,\xi_1, \qquad [\xi_3, \xi_1] = -\sqrt{\kappa}\,\xi_2.$$

Hence,

Theorem

Any sphere of Kaluza-Klein type $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ has a Lie group structure, unique up to isomorphisms, such that the vector fields ξ_1, ξ_2, ξ_3 are left invariant.

More precisely, the signs of the coefficients in the Lie brackets above, together with Milnor's classification of 3D Riemannian Lie groups, yield that $\tilde{g}_{\lambda\mu\nu}$ corresponds to a left-invariant Riemannian metric on $SU(2)$, as it could be expected.

COROLLARY

$(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is a homogeneous space.

Definition

Let (M, g) be a Riemannian manifold. A homogeneous structure on (M, g) is a tensor field T of type $(1, 2)$ on M, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies the Ambrose-Singer equations

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} T = 0.$$

Definition

Let (M, g) be a Riemannian manifold. A homogeneous structure on (M, g) is a tensor field T of type $(1, 2)$ on M, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies the Ambrose-Singer equations

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} T = 0.$$

The geometric meaning of the existence of a homogeneous structure is explained by the

Theorem of Ambrose and Singer

A connected, simply connected and complete Riemannian manifold (M, g) is homogeneous if and only if it admits a homogeneous structure.

Each homogeneous structure gives a representation of (M, g) as quotient space G/H, where G is a Lie group of isometries acting transitively on M ($\tilde{\nabla} = \nabla - T$ is its canonical connection).

Each homogeneous structure gives a representation of (M, g) as quotient space G/H, where G is a Lie group of isometries acting transitively on M ($\tilde{\nabla} = \nabla - T$ is its canonical connection).

Two different homogeneous structures T_1 and T_2 on (M, g) give either different decompositions $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{h}_1 = \mathfrak{m}_2 \oplus \mathfrak{h}_2$ of the same Lie algebra $\mathfrak{g}$, or representations of (M, g) by two non-isomorphic Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

Each homogeneous structure gives a representation of (M, g) as quotient space G/H, where G is a Lie group of isometries acting transitively on M ($\tilde{\nabla} = \nabla - T$ is its canonical connection).

Two different homogeneous structures T_1 and T_2 on (M, g) give either different decompositions $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{h}_1 = \mathfrak{m}_2 \oplus \mathfrak{h}_2$ of the same Lie algebra $\mathfrak{g}$, or representations of (M, g) by two non-isomorphic Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

In particular, suppose that (M, g) admits a global orthonormal frame field $\{e_1, .., e_n\}$ and some real constants γ_{ij}^k, satisfying $\nabla_{e_i} e_j = \sum_k \gamma_{ij}^k e_k$ for all i, j.

Each homogeneous structure gives a representation of (M, g) as quotient space G/H, where G is a Lie group of isometries acting transitively on M ($\tilde{\nabla} = \nabla - T$ is its canonical connection).

Two different homogeneous structures T_1 and T_2 on (M, g) give either different decompositions $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{h}_1 = \mathfrak{m}_2 \oplus \mathfrak{h}_2$ of the same Lie algebra $\mathfrak{g}$, or representations of (M, g) by two non-isomorphic Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

In particular, suppose that (M, g) admits a global orthonormal frame field $\{e_1, .., e_n\}$ and some real constants γ_{ij}^k, satisfying $\nabla_{e_i} e_j = \sum_k \gamma_{ij}^k e_k$ for all i, j. Then, M has a Lie group structure, unique up to isomorphisms, such that e_i are left-invariant vector fields and g is left-invariant, corresponding to the special homogeneous structure

$$T_{e_i} := \frac{1}{2} \sum_{jk} \gamma_{ij}^k \, e_j \wedge e_k \, ,$$

Theorem

Let $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ denote an arbitrary sphere of Kaluza-Klein type.

Theorem

Let $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ denote an arbitrary sphere of Kaluza-Klein type.

(i) If the Ricci eigenvalues are all distinct, then $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ only admits one homogeneous structure, given by

$$T = \gamma_{12}^3\, \theta^1 \otimes (\theta^2 \wedge \theta^3) + \gamma_{21}^3\, \theta^2 \otimes (\theta^1 \wedge \theta^3) + \gamma_{31}^2\, \theta^3 \otimes (\theta^1 \wedge \theta^2),$$

corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

Theorem

Let $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ denote an arbitrary sphere of Kaluza-Klein type.

(i) If the Ricci eigenvalues are all distinct, then $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ only admits one homogeneous structure, given by

$$T = \gamma^3_{12}\,\theta^1 \otimes (\theta^2 \wedge \theta^3) + \gamma^3_{21}\,\theta^2 \otimes (\theta^1 \wedge \theta^3) + \gamma^2_{31}\,\theta^3 \otimes (\theta^1 \wedge \theta^2),$$

corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

(ii) When $\lambda = \mu + \nu$ ($\mu = \lambda + \nu$, $\nu = \lambda + \mu$), the sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ admits a one-parameter family of homogeneous structures, given by

$$T = t\,\theta^1 \otimes (\theta^2 \wedge \theta^3) + \sqrt{\tfrac{k}{\mu\nu(\mu+\nu)}}\left(\nu\,\theta^2 \otimes (\theta^1 \wedge \theta^3) - \mu\,\theta^3 \otimes (\theta^1 \wedge \theta^2)\right),$$

$t \in \mathbb{R}$. For $t = 0$, T gives the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

(iii) When $\lambda \neq \mu = \nu$ ($\mu \neq \lambda = \nu$, $\nu \neq \lambda = \mu$), the homogeneous structures on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ are given by

$$T = t\,\theta^1 \otimes (\theta^2 \wedge \theta^3) + \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}\,\lambda\,(\theta^2 \otimes (\theta^1 \wedge \theta^3) - \theta^3 \otimes (\theta^1 \wedge \theta^2)),$$

For $t = \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}(\lambda - 2\mu)$, T is the homogeneous structure corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

(iii) When $\lambda \neq \mu = \nu$ ($\mu \neq \lambda = \nu$, $\nu \neq \lambda = \mu$), the homogeneous structures on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ are given by

$$T = t\,\theta^1 \otimes (\theta^2 \wedge \theta^3) + \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}\,\lambda\left(\theta^2 \otimes (\theta^1 \wedge \theta^3) - \theta^3 \otimes (\theta^1 \wedge \theta^2)\right),$$

For $t = \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}(\lambda - 2\mu)$, T is the homogeneous structure corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

(iv) When $\lambda = \mu = \nu$, one gets the three-parameter family of homogeneous structures on the standard sphere.

(iii) When $\lambda \neq \mu = \nu$ ($\mu \neq \lambda = \nu$, $\nu \neq \lambda = \mu$), the homogeneous structures on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ are given by

$$T = t\,\theta^1 \otimes (\theta^2 \wedge \theta^3) + \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}\,\lambda\left(\theta^2 \otimes (\theta^1 \wedge \theta^3) - \theta^3 \otimes (\theta^1 \wedge \theta^2)\right),$$

For $t = \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}(\lambda - 2\mu)$, T is the homogeneous structure corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

(iv) When $\lambda = \mu = \nu$, one gets the three-parameter family of homogeneous structures on the standard sphere.

Remark

Each Berger sphere admits a one-parameter family of homogeneous structures [Gadea and Oubiña]. This property extends to Kaluza-Klein metrics, but not to general spheres of Kaluza-Klein type.

A homogeneous space (M, g) is naturally reductive if it admits a reductive decomposition $M = G/H$, such that

$$< [x, y]_{\mathfrak{m}}, z > + < [x, z]_{\mathfrak{m}}, y >= 0, \quad x, y, z \in \mathfrak{m}.$$

A homogeneous space (M,g) is naturally reductive if it admits a reductive decomposition $M = G/H$, such that

$$< [x,y]_{\mathfrak{m}}, z > + < [x,z]_{\mathfrak{m}}, y >= 0, \quad x,y,z \in \mathfrak{m}.$$

In dimension three, a homogeneous Riemannian manifold is naturally reductive if and only if its Ricci tensor is cyclic-parallel, that is, when

$$(\nabla_X \varrho)(Y,Z) + (\nabla_Y \varrho)(Z,X) + (\nabla_Z \varrho)(X,Y) = 0,$$

for all vector fields X, Y, Z tangent to M.

A homogeneous space (M, g) is naturally reductive if it admits a reductive decomposition $M = G/H$, such that

$$< [x, y]_{\mathfrak{m}}, z > + < [x, z]_{\mathfrak{m}}, y >= 0, \quad x, y, z \in \mathfrak{m}.$$

In dimension three, a homogeneous Riemannian manifold is naturally reductive if and only if its Ricci tensor is cyclic-parallel, that is, when

$$(\nabla_X \varrho)(Y, Z) + (\nabla_Y \varrho)(Z, X) + (\nabla_Z \varrho)(X, Y) = 0,$$

for all vector fields X, Y, Z tangent to M.

Theorem

A sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type is naturally reductive if and only if at least two among parameters λ, μ, ν coincide. In particular, if $\tilde{g}_{\lambda\mu\nu}$ is a Kaluza-Klein metric, then $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is naturally reductive.

An almost contact structure on a $(2n+1)$-dimensional smooth manifold M is a triple (φ, ξ, η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\eta(\xi) = 1, \qquad \varphi^2 = -Id + \eta \otimes \xi.$$

An almost contact structure on a $(2n+1)$-dimensional smooth manifold M is a triple (φ, ξ, η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\eta(\xi) = 1, \qquad \varphi^2 = -Id + \eta \otimes \xi.$$

Then, $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ and φ has rank $2n$. The one-form η is said to be a contact form if $\eta \wedge (d\eta)^n \neq 0$.

An almost contact structure on a $(2n+1)$-dimensional smooth manifold M is a triple (φ, ξ, η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\eta(\xi) = 1, \qquad \varphi^2 = -Id + \eta \otimes \xi.$$

Then, $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ and φ has rank $2n$. The one-form η is said to be a contact form if $\eta \wedge (d\eta)^n \neq 0$. A Riemannian metric g on M is called compatible with the almost contact structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

In this case, $\eta(X) = g(\xi, X)$.

An almost contact structure on a $(2n+1)$-dimensional smooth manifold M is a triple (φ, ξ, η), where φ is a $(1,1)$-tensor, ξ a global vector field and η a 1-form, such that

$$\eta(\xi) = 1, \qquad \varphi^2 = -Id + \eta \otimes \xi.$$

Then, $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ and φ has rank $2n$. The one-form η is said to be a contact form if $\eta \wedge (d\eta)^n \neq 0$. A Riemannian metric g on M is called compatible with the almost contact structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

In this case, $\eta(X) = g(\xi, X)$. If the compatible metric g satisfies

$$(d\eta)(X, Y) = \Phi(X, Y) := g(X, \varphi Y),$$

then η is a contact form on M, ξ the associated Reeb vector field, g an associated metric, and (M, η, g) (or $(M, \varphi, \xi, \eta, g)$) is called a contact metric manifold.

Given an almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$, one considers on $M^{2n+1} \times \mathbb{R}$ the almost complex structure

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right).$$

Given an almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$, one considers on $M^{2n+1} \times \mathbb{R}$ the almost complex structure

$$J\big(X, f\frac{d}{dt}\big) = \big(\varphi X - f\xi, \eta(X)\frac{d}{dt}\big).$$

The structure (φ, ξ, η) is said to be normal if J is integrable. Necessary and sufficient condition for integrability of J is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0.$$

Given an almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$, one considers on $M^{2n+1} \times \mathbb{R}$ the almost complex structure

$$J\big(X, f\frac{d}{dt}\big) = \big(\varphi X - f\xi, \eta(X)\frac{d}{dt}\big).$$

The structure (φ, ξ, η) is said to be normal if J is integrable. Necessary and sufficient condition for integrability of J is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0.$$

A contact metric manifold is said to be K-contact if ξ is a Killing vector field, or equivalently, the tensor $h := \mathcal{L}_\xi \varphi$ vanishes.

Given an almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$, one considers on $M^{2n+1} \times \mathbb{R}$ the almost complex structure

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right).$$

The structure (φ, ξ, η) is said to be normal if J is integrable. Necessary and sufficient condition for integrability of J is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0.$$

A contact metric manifold is said to be K-contact if ξ is a Killing vector field, or equivalently, the tensor $h := \mathcal{L}_\xi \varphi$ vanishes.
A Sasakian manifold is a normal contact metric manifold. Any Sasakian manifold is K-contact, and the converse also holds in dimension three.

Given an almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$, one considers on $M^{2n+1} \times \mathbb{R}$ the almost complex structure

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}).$$

The structure (φ, ξ, η) is said to be normal if J is integrable. Necessary and sufficient condition for integrability of J is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0.$$

A contact metric manifold is said to be K-contact if ξ is a Killing vector field, or equivalently, the tensor $h := \mathcal{L}_\xi \varphi$ vanishes. A Sasakian manifold is a normal contact metric manifold. Any Sasakian manifold is K-contact, and the converse also holds in dimension three. An almost contact metric manifold is called quasi Sasakian if $d\Phi = 0$.

Consider an arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, $\{e_i\}$ its global orthonormal frame field, $\{\bar{\theta}^i\}$ the basis of one-forms dual to $\{e_i\}$ with respect to $\tilde{g}_{\lambda\mu\nu}$.

Consider an arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, $\{e_i\}$ its global orthonormal frame field, $\{\bar{\theta}^i\}$ the basis of one-forms dual to $\{e_i\}$ with respect to $\tilde{g}_{\lambda\mu\nu}$. We put

$$\eta := \bar{\theta}^1, \qquad \xi := e_1, \qquad \varphi := \bar{\theta}^3 \otimes e_2 - \bar{\theta}^2 \otimes e_3. \tag{2}$$

Consider an arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, $\{e_i\}$ its global orthonormal frame field, $\{\bar{\theta}^i\}$ the basis of one-forms dual to $\{e_i\}$ with respect to $\tilde{g}_{\lambda\mu\nu}$. We put

$$\eta := \bar{\theta}^1, \qquad \xi := e_1, \qquad \varphi := \bar{\theta}^3 \otimes e_2 - \bar{\theta}^2 \otimes e_3. \tag{2}$$

Then, (2) defines an almost contact structure on $\mathbb{S}^3$. Moreover, $\tilde{g}_{\lambda\mu\nu}$ is compatible with this almost contact structure.

Consider an arbitrary sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, $\{e_i\}$ its global orthonormal frame field, $\{\bar{\theta}^i\}$ the basis of one-forms dual to $\{e_i\}$ with respect to $\tilde{g}_{\lambda\mu\nu}$. We put

$$\eta := \bar{\theta}^1, \qquad \xi := e_1, \qquad \varphi := \bar{\theta}^3 \otimes e_2 - \bar{\theta}^2 \otimes e_3. \tag{2}$$

Then, (2) defines an almost contact structure on $\mathbb{S}^3$. Moreover, $\tilde{g}_{\lambda\mu\nu}$ is compatible with this almost contact structure. So, we have the following.

Theorem

Any sphere of Kaluza-Klein type $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is an almost contact metric manifold, whose almost contact structure (φ, ξ, η) is described by (2).

We calculate $d\eta$ and we find

$$d\eta(\xi,\cdot) = 0, \qquad d\eta(e_2,e_3) = -d\eta(e_3,e_2) = \tfrac{1}{2}\sqrt{\tfrac{\kappa\lambda}{\mu\nu}} > 0,$$

which easily implies $\eta \wedge d\eta \neq 0$ and

$$(d\eta)(X,Y) = \tfrac{1}{2}\sqrt{\tfrac{\kappa\lambda}{\mu\nu}}\,\Phi(X,Y),$$

for all X, Y tangent to M.

We calculate $d\eta$ and we find

$$d\eta(\xi,\cdot)=0, \qquad d\eta(e_2,e_3)=-d\eta(e_3,e_2)=\frac{1}{2}\sqrt{\frac{\kappa\lambda}{\mu\nu}}>0,$$

which easily implies $\eta\wedge d\eta\neq 0$ and

$$(d\eta)(X,Y)=\frac{1}{2}\sqrt{\frac{\kappa\lambda}{\mu\nu}}\,\Phi(X,Y),$$

for all X,Y tangent to M. Thus, $d\Phi=0$, and g is an associated metric if and only if $\kappa\lambda=4\mu\nu$.

We calculate $d\eta$ and we find

$$d\eta(\xi,\cdot)=0, \qquad d\eta(e_2,e_3)=-d\eta(e_3,e_2)=\tfrac{1}{2}\sqrt{\tfrac{\kappa\lambda}{\mu\nu}}>0,$$

which easily implies $\eta\wedge d\eta\neq 0$ and

$$(d\eta)(X,Y)=\tfrac{1}{2}\sqrt{\tfrac{\kappa\lambda}{\mu\nu}}\,\Phi(X,Y),$$

for all X,Y tangent to M. Thus, $d\Phi=0$, and g is an associated metric if and only if $\kappa\lambda=4\mu\nu$.

Theorem

The one-form η of $(\mathbb{S}^3,\varphi,\xi,\eta,\tilde{g}_{\lambda\mu\nu})$ is always a contact form. Moreover, $(\mathbb{S}^3,\varphi,\xi,\eta,\tilde{g}_{\lambda\mu\nu})$ is quasi Sasakian. In particular, it is a contact metric manifold if and only if $\kappa\lambda=4\mu\nu$.

H-contact manifolds are contact metric manifolds whose Reeb vector field ξ is a critical point for the energy functional restricted to the space of all unit vector fields.

H-contact manifolds are contact metric manifolds whose Reeb vector field ξ is a critical point for the energy functional restricted to the space of all unit vector fields.
They are characterized by the fact that ξ is a Ricci eigenvector (so, Sasakian and K-contact manifolds are H-contact).

H-contact manifolds are contact metric manifolds whose Reeb vector field ξ is a critical point for the energy functional restricted to the space of all unit vector fields.
They are characterized by the fact that ξ is a Ricci eigenvector (so, Sasakian and K-contact manifolds are H-contact).
Taking into account the description of the Ricci tensor of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, we have

$$Ric(\xi) = \frac{\kappa\left[\lambda^2-(\mu-\nu)^2\right]}{2\lambda\mu\nu}\xi$$

H-contact manifolds are contact metric manifolds whose Reeb vector field ξ is a critical point for the energy functional restricted to the space of all unit vector fields.
They are characterized by the fact that ξ is a Ricci eigenvector (so, Sasakian and K-contact manifolds are H-contact).
Taking into account the description of the Ricci tensor of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, we have

$$Ric(\xi) = \frac{\kappa\left[\lambda^2 - (\mu-\nu)^2\right]}{2\lambda\mu\nu}\xi$$

Corollary

Whenever $\kappa\lambda = 4\mu\nu$, $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is H-contact. So, for any real constant $\kappa > 0$, there exists a two-parameter family of H-contact structures of Kaluza-Klein type on $\mathbb{S}^3$.

An α-Sasakian manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X, Y)\xi - \eta(Y)X\big),$$

where $\alpha \in \mathbb{R}$.

An α-Sasakian manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X, Y)\xi - \eta(Y)X\big),$$

where $\alpha \in \mathbb{R}$. 1-Sasakian manifolds are Sasakian manifolds.

An α-Sasakian manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big),$$

where $\alpha \in \mathbb{R}$. 1-Sasakian manifolds are Sasakian manifolds.

Theorem

For the almost contact structure (φ, ξ, η) on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, described in (2), the following properties are equivalent:

An α-Sasakian manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X, Y)\xi - \eta(Y)X\big),$$

where $\alpha \in \mathbb{R}$. 1-Sasakian manifolds are Sasakian manifolds.

Theorem

For the almost contact structure (φ, ξ, η) on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, described in (2), the following properties are equivalent:

(i) (φ, ξ, η) is normal;

An α-Sasakian manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big),$$

where $\alpha \in \mathbb{R}$. 1-Sasakian manifolds are Sasakian manifolds.

Theorem

For the almost contact structure (φ, ξ, η) on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, described in (2), the following properties are equivalent:

- **(i)** (φ, ξ, η) is normal;
- **(ii)** (φ, ξ, η) is α-Sasakian. In this case, $\alpha = \frac{\sqrt{\kappa\lambda}}{2\mu}$.

An α-Sasakian manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X, Y)\xi - \eta(Y)X\big),$$

where $\alpha \in \mathbb{R}$. 1-Sasakian manifolds are Sasakian manifolds.

Theorem

For the almost contact structure (φ, ξ, η) on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, described in (2), the following properties are equivalent:

(i) (φ, ξ, η) is normal;

(ii) (φ, ξ, η) is α-Sasakian. In this case, $\alpha = \frac{\sqrt{\kappa\lambda}}{2\mu}$.

(iii) $\mu = \nu$, that is, $\tilde{g}_{\lambda\mu\nu}$ is a Kaluza-Klein metric.

An α-Sasakian manifold is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X,Y)\xi - \eta(Y)X\big),$$

where $\alpha \in \mathbb{R}$. 1-Sasakian manifolds are Sasakian manifolds.

Theorem

For the almost contact structure (φ, ξ, η) on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, described in (2), the following properties are equivalent:

(i) (φ, ξ, η) is normal;

(ii) (φ, ξ, η) is α-Sasakian. In this case, $\alpha = \frac{\sqrt{\kappa\lambda}}{2\mu}$.

(iii) $\mu = \nu$, that is, $\tilde{g}_{\lambda\mu\nu}$ is a Kaluza-Klein metric.

Corollary

$(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is Sasakian if and only if $\mu = \nu$ and $\kappa\lambda = 4\mu^2$.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is homogeneous if there exists a Lie group G of isometries acting transitively on M and leaving η invariant.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is homogeneous if there exists a Lie group G of isometries acting transitively on M and leaving η invariant.

This is equivalent to requiring that there exists a homogeneous structure T on (M, g), such that $\tilde{\nabla} = \nabla - T$ satisfies $\tilde{\nabla}\varphi = 0$.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is homogeneous if there exists a Lie group G of isometries acting transitively on M and leaving η invariant.

This is equivalent to requiring that there exists a homogeneous structure T on (M, g), such that $\tilde{\nabla} = \nabla - T$ satisfies $\tilde{\nabla}\varphi = 0$.

In the case of $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$, such condition is satisfied by the homogeneous structure corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

Theorem

Every sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, equipped with the almost contact structure (φ, ξ, η) described in (2), is a homogeneous almost contact metric manifold.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is homogeneous if there exists a Lie group G of isometries acting transitively on M and leaving η invariant.

This is equivalent to requiring that there exists a homogeneous structure T on (M, g), such that $\tilde{\nabla} = \nabla - T$ satisfies $\tilde{\nabla}\varphi = 0$.

In the case of $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$, such condition is satisfied by the homogeneous structure corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

Theorem

Every sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, equipped with the almost contact structure (φ, ξ, η) described in (2), is a homogeneous almost contact metric manifold.
If $\kappa\lambda = 4\mu\nu$, then $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is a homogeneous contact metric manifold.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be η-Einstein if its Ricci tensor ϱ is of the form

$$\varrho = Ag + B\eta \otimes \eta,$$

where A, B are two smooth functions on M.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be η-Einstein if its Ricci tensor ϱ is of the form

$$\varrho = Ag + B\eta \otimes \eta,$$

where A, B are two smooth functions on M.

Theorem

A sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, equipped with the natural almost contact structure (φ, ξ, η) described in (2), is η-Einstein if and only if one of the following conditions holds:

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be η-Einstein if its Ricci tensor ϱ is of the form

$$\varrho = Ag + B\eta \otimes \eta,$$

where A, B are two smooth functions on M.

Theorem

A sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, equipped with the natural almost contact structure (φ, ξ, η) described in (2), is η-Einstein if and only if one of the following conditions holds:

(i) $\lambda = \mu + \nu$. In this case, $\varrho = \frac{2\kappa}{\lambda}\eta \otimes \eta$.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be η-Einstein if its Ricci tensor ϱ is of the form

$$\varrho = Ag + B\eta \otimes \eta,$$

where A, B are two smooth functions on M.

Theorem

A sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, equipped with the natural almost contact structure (φ, ξ, η) described in (2), is η-Einstein if and only if one of the following conditions holds:

- **(i)** $\lambda = \mu + \nu$. In this case, $\varrho = \frac{2\kappa}{\lambda} \eta \otimes \eta$.
- **(ii)** $\mu = \nu$. In this case, $\varrho = \frac{\kappa(2\mu - \lambda)}{2\mu^2} g + \frac{\kappa(\lambda - \mu)}{2\mu^2} \eta \otimes \eta$.

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be η-Einstein if its Ricci tensor ϱ is of the form

$$\varrho = Ag + B\eta \otimes \eta,$$

where A, B are two smooth functions on M.

Theorem

A sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza-Klein type, equipped with the natural almost contact structure (φ, ξ, η) described in (2), is η-Einstein if and only if one of the following conditions holds:

(i) $\lambda = \mu + \nu$. In this case, $\varrho = \frac{2\kappa}{\lambda}\eta \otimes \eta$.

(ii) $\mu = \nu$. In this case, $\varrho = \frac{\kappa(2\mu-\lambda)}{2\mu^2} g + \frac{\kappa(\lambda-\mu)}{2\mu^2}\eta \otimes \eta$.

In particular, all Berger spheres, equipped with their natural almost contact structures, are η-Einstein.

Work in progress...

Work in progress...

- Harmonicity properties of Hopf vector fields ξ_1, ξ_2, ξ_3 with respect to metrics of Kaluza-Klein type.

Work in progress...

- Harmonicity properties of Hopf vector fields ξ_1, ξ_2, ξ_3 with respect to metrics of Kaluza-Klein type.
- Lorentzian metrics of Kaluza-Klein type on $\mathbb{S}^3$ (allowing λ, μ, ν to be negative)

Work in progress...

- Harmonicity properties of Hopf vector fields ξ_1, ξ_2, ξ_3 with respect to metrics of Kaluza-Klein type.
- Lorentzian metrics of Kaluza-Klein type on $\mathbb{S}^3$ (allowing λ, μ, ν to be negative)

THANK YOU FOR YOUR KIND ATTENTION!