Canonical Metrics on Sasaki Manifolds

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The objective

Given a closed manifold of Sasaki type, we want to provide it with a preferred shape.

Different but complementary perspectives of the study:

a) To provide a polarized manifold of Sasaki type with its best shape by finding optimizers of Riemannian functionals (functions on the space of metrics that are constant on orbits of the diffeomorphism group).

b) To solve a fully non-linear elliptic equation that is non-local (pseudo-differential).

These perspectives result naturally when we attempt to generalize the uniformization theorem for Riemann surfaces.

The canonical shape of a manifold

Let M a closed manifold of dimension n. We consider the set $\mathfrak{M}$ of all Riemannian metrics, an open subset of the space of symmetric two-tensors $\mathcal{S}^2(M)$ on the manifold. We denote by $\mathfrak{M}_1$ the space of all Riemannian metrics of volume 1.

Is there a canonical element of $\mathfrak{M}_1$?

A general approach

We let $\mathcal{S}^p(M)$ be the bundle of symmetric of symmetric p-tensors on M. Covariant differentiation defines a mapping

$$\nabla_g^p : \mathcal{S}^p(M) \to \Omega^1 M \otimes \mathcal{S}^p(M)\,,$$

that can be composed with the symmetrization operator

$$\delta_g^* : \mathcal{S}^p(M) \to \mathcal{S}^{p+1}(M)\,.$$

We then take the metric dual of δ_g^* to obtain the operator

$$\delta = \delta_g : \mathcal{S}^{p+1}(M) \to \mathcal{S}^p(M)\,.$$

Let us consider the path of metrics

$$g_t = g + th\,,$$

for a symmetric two-tensor h. The volume and scalar curvature of this path of metrics vary according to the expressions

$$\begin{aligned} d\mu_{g_t}\,|_{t=0} &= \tfrac{1}{2}\mathrm{trace}_g\, h\, d\mu_g\,, \\ \tfrac{d}{dt} s_{g+th}\,|_{t=0} &= \Delta_g(\mathrm{trace}_g\, h) + \delta_g(\delta_g h) - (r_g, h)_g\,. \end{aligned}$$

The g-trace of h is just the inner product $(g, h)_g$.

Since the first two terms in the variation of the scalar curvature are divergence, we easily obtain the well-known expression

$$\frac{d}{dt}\int s_{g+th} d\mu_{g+th}\,|_{t=0} = \int \left(\frac{s_g}{2} g - r_g, h\right)_g d\mu_g$$

for the variation of the Hilbert functional over $\mathfrak{M}$.

Instead of this Hilbert functional that yields Einstein metrics as its critical points over $\mathfrak{M}_1$, we now consider

$$\begin{array}{rcl} \mathfrak{M}_1 & \rightarrow & \mathbb{R} \\ g & \mapsto & \int_M s_g^2 d\mu_g \end{array} . \tag{1}$$

Proposition. **A riemannian metric g is a critical point of (1) if, and only if, we have that**

$$\left(2\Delta_g s_g + \frac{s_g^2}{2}\right) g + 2\nabla_g ds_g - 2s_g r_g = \lambda_g g\,,$$

where

$$\lambda_g = \frac{n-4}{2n}\int s_g^2 d\mu_g\,.$$

$\square$

Theorem. **If g is a critical point of (1), then g must have constant scalar curvature, and a metric of constant scalar curvature is a critical point if, and only if, it is either Einstein or scalar-flat.** $\square$

Why do we do this?

Example 1. **The standard metric g on the sphere $\mathbb{S}^n$ is Einstein, with scalar curvature $n(n-1)$. All geodesics are minimizing before distance π, and if we fix an open ball B_π of radius π centered at the origin of $T_p\mathbb{S}^n$ for some point p, we have that $\exp_p(B_\pi) = \mathbb{S}^n \setminus \{-p\}$, and the cut locus $\mathrm{Cut}(p)$ is just the point $\{-p\}$. This yields the canonical CW decomposition of the sphere into cells of dimen-**

sion n and 0, respectively.

Relative to the standard metric g, the sphere has volume $\mu_g(\mathbb{S}^n) = (n+1)\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+3}{2})$, where Γ is the *gamma function*. The results above do not change if we scale g to a volume one metric. $\square$

Example 2. Let us view $(\mathbb{RP}^n, g)$ as the base of a Riemannian submersion $pr : \mathbb{S}^n \to \mathbb{RP}^n$, where the sphere is endowed with its standard metric. The geodesics in $\mathbb{RP}^n$ are the projections under pr of the geodesics in $\mathbb{S}^n$, and so, when parametrized by arc-length, these are periodic curves of period π. All geodesics minimize before distance $\pi/2$. Given a point $p \in \mathbb{RP}^n$, if $B_{\pi/2}$ is the ball of radius $\pi/2$ centered at the origin of $T_p\mathbb{RP}^n$, we have that $\exp_p(B_{\pi/2}) = \mathbb{RP}^n \setminus \mathbb{RP}^{n-1}$, and $\mathrm{Cut}(p)$ is the submanifold $\mathbb{RP}^{n-1}$. Since $\mathrm{Cut}(p)$ is a geodesic submanifold, by iteration of this argument we derive the canonical CW decomposition of the real projective space, with one cell in each dimension between 0 and n, respectively. $\square$

Sasaki Manifolds

The geometry of Sasaki manifolds lies naturally in between two Kähler geometries: Sasaki manifolds are the bases of metric cones that are Kähler, and any Sasaki manifold is contact, and the one-dimensional foliation associated to the characteristic Reeb vector field is transversally Kähler.

Almost contact structure on M: it is given by a triple (ξ, η, Φ), where ξ is a vector field, η is a one form, and Φ is a tensor of type $(1,1)$, subject to the relations

$$\eta(\xi) = 1\,, \quad \Phi^2 = -\mathbb{1} + \xi \otimes \eta\,.$$

Characteristic foliation on M: a decomposition of the manifold into lines, the integral curves of ξ. The kernel of η defines the codimension one sub-bundle $\mathcal{D}$. This yields a canonical splitting

$$TM = \mathcal{D} \oplus L_\xi\,, \tag{2}$$

where L_ξ is the trivial line bundle generated by ξ.

Compatible metrics with (ξ, η, Φ)**: These are Riemannian metrics** g **such that**

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y)$$

for any pair of vector fields X, Y**. These metrics induce** *metrics* **on** $\mathcal{D}$**.**

Contact metric structure (ξ, η, Φ, g)**: it is an almost contact structure with compatible metric such that**

$$g(\Phi X, Y) = d\eta(X, Y)\,.$$

In this case, the measure induced by the metric is given by

$$d\mu_g = \frac{1}{n!}\eta \wedge (d\eta)^n\,. \tag{3}$$

The *metric cones* **result as follows: on** $C(M) = M \times \mathbb{R}^+$**, we introduce the metric**

$$g_C = dr^2 + r^2 g\,.$$

The radial vector field $r\partial_r$ **satisfies**

$$\pounds_{r\partial_r} g_C = 2g_C\,,$$

and we may define an almost complex structure I on $C(M)$ by

$$I(Y) = \Phi(Y) + \eta(Y) r\partial_r \, , \quad I(r\partial_r) = -\xi \, .$$

The almost contact manifold (M, ξ, η, Φ) is said to be *normal* if the pair $(C(M), I)$ is a complex manifold. In that case, the induced almost complex structure J on $\mathcal{D}$ is integrable.

Definition. We say that a contact metric structure (ξ, η, Φ, g) is a *Sasaki structure* on M if (ξ, η, Φ) is normal. $\square$

We thus obtain the two Kähler geometries alluded to above, the Kähler cone $C(M)$ for which the integrability of I implies that ξ generates a group of isometries, and a codimension one integrable strictly pseudo-convex CR structure $(\mathcal{D}, J)$, where $\mathcal{D} = \ker \eta$ is the contact bundle and $J = \Phi|_{\mathcal{D}}$, and where the restriction of g to $\mathcal{D}$ defines a symmetric form on $(\mathcal{D}, J)$ that, abusing terminology a bit, we refer to as the transverse Kähler metric g^T.

In many interesting situations, the orbits of the Reeb vector field are all closed, the quasi-regular case. Compact quasi-regular Sasakian manifolds have the structure of an orbifold circle bundle over a compact Kähler orbifold, which must be algebraic and which has at most cyclic quotient singularities. Much of the study of compact Kähler manifolds and extremal metrics can be extended to the orbifold case. This is often both, necessary, and then done in a fairly straightforward way. Thus, it is not surprising that we can then "translate" statements involving compact Kähler orbifolds to conclude parallel statements regarding quasi-regular Sasakian structures, approach that has been spectacularly successful in constructing new quasi-regular Sasaki-Einstein metrics on various contact manifolds of odd dimension greater than 3.

We make a case in favor of a variational formulation of special Sasaki metrics, analogous to the notion of extremal metric in Kähler geometry. The quasi-regularity property should no longer be a key factor, and all Sasaki-Einstein metrics should indeed appear as minima of a suitable Riemannian functional. This would put on equal footing the analysis of all Sasakian structures, the quasi-regular or the irregular ones. Thus, we should be able to study the existence and uniqueness of these canonical Sasakian metrics in a way analogous to what has been done in Kähler geometry.

Until now, this way of finding canonical Sasakian structures has not been pursued, perhaps due to the lack of evidence that the orbibundle approach would be insufficient.

Polarized Sasaki manifolds

We fix a Sasaki structure (ξ, η, Φ, g) on M. It is reasonable to consider two possible sets of deformations of this structure, those where the Reeb vector field remains fixed while the underlying CR structure changes by a diffeomorphism, and those where the underlying CR structure stays the same while the Reeb vector field varies. We study the first here.

We consider the set

$$\mathcal{S}(\xi) = \{\text{Sasaki structure } (\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g}) \mid \tilde{\xi} = \xi\} \,,$$

and provide it with the C^∞ compact-open topology as sections of vector bundles. For any element $(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ in this set, the 1-form $\zeta = \tilde{\eta} - \eta$ is basic, and so $[d\tilde{\eta}]_B = [d\eta]_B$. Thus, all of these structures correspond to the same basic cohomology class.

$\mathcal{S}(\xi)$ is the *the space of Sasakian structures compatible with* ξ. We say that ξ *polarizes* the Sasakian manifold M.

Transversal normal bundle: Let $\mathcal{F}_\xi$ be the characteristic foliation. We let $\nu(\mathcal{F}_\xi)$ be the vector bundle whose fiber at a point $p \in M$ is the quotient space T_pM/L_ξ, and let $\pi_\nu : TM \to \nu(\mathcal{F}_\xi)$ be the natural projection. The background structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ induces a complex structure $\bar{J}$ on $\nu(\mathcal{F}_\xi)$. This is defined by $\bar{J}\bar{X} := \overline{\Phi(X)}$, where X is any vector field in M such that $\pi(X) = \bar{X}$. We say that $(M, \xi, \bar{J})$ is a polarized Sasaki manifold.

We define $\mathcal{S}(\xi, \bar{J})$ to be the subset of all structures

$(\tilde{\xi}, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ in $\mathcal{S}(\xi)$ such that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{\tilde{\Phi}} & TM \\ \downarrow_{\pi_\nu} & & \downarrow_{\pi_\nu} \\ \nu(\mathcal{F}_\xi) & \xrightarrow{\bar{J}} & \nu(\mathcal{F}_\xi) \end{array}$$

commutes. This is the set of elements of $\mathcal{S}(\xi)$ with the same complex normal bundle $(\nu(\mathcal{F}_\xi), \bar{J})$.

The relevant metrics: The space $\mathcal{S}(\xi, \bar{J})$ is an affine space modeled on the basic functions $C_B^\infty(M) \times C_B^\infty(M)$:

$$\begin{aligned} \tilde{\eta} &= \eta + d^c\varphi + d\psi\,, \\ \tilde{\Phi} &= \Phi - (\xi \otimes (\tilde{\eta} - \eta)) \circ \Phi\,, \\ \tilde{g} &= d\tilde{\eta} \circ (\mathbb{1} \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}\,, \end{aligned}$$

where $d^c = \frac{i}{2}(\overline{\partial} - \partial)$. In particular, $d\tilde{\eta} = d\eta + i\partial\overline{\partial}\varphi$.

We denote by $\mathfrak{M}(\xi, \bar{J})$ the set of all compatible Sasakian metrics arising from structures in $\mathcal{S}(\xi, \bar{J})$. These metrics all give M the same volume, but are fewer in number than the space of all Riemannian metrics of this volume on M. So the functional

$$\begin{array}{rcl} \mathfrak{M}(\xi, \bar{J}) & \xrightarrow{E} & \mathbb{R}\,, \\ g & \mapsto & \displaystyle\int_M s_g^2 d\mu_g\,, \end{array} \qquad (4)$$

will have more critical points than those of (1).

Proposition. The variation of the functional E at $g \in \mathfrak{M}(\xi, \bar{J})$ in the direction of (φ, ψ) is given by

$$\frac{d}{dt}E(g_t) \mid_{t=0} = -4\int_M (s^T - 2n)\left((\bar{\partial}\partial^{\#})^*\bar{\partial}\partial^{\#}\varphi\right)\, d\mu\,.$$

In the statement above,

$$L_g^B\varphi := (\bar{\partial}\partial^{\#})^*\bar{\partial}\partial^{\#}\varphi = \frac{1}{4}(\Delta_B^2\varphi + 4(\rho^T, i\partial\overline{\partial}\varphi) + 2(\partial s^T)\lrcorner\partial^{\#}\varphi)\,,$$

acting on basic functions. Transversally constant functions are always in the kernel of this operator, and of these, only the constants are basic. Thus, the kernel of L_g^B is always of dimension at least 1.

Theorem. The Sasaki metric $g \in \mathfrak{M}(\xi, \bar{J})$ is a critical point of (4) if, and only if, the basic vector field $\partial_g^{\#}s_g^T = \partial_g^{\#}s_g$ is transversally holomorphic.

A Sasaki-Futaki invariant

Given a choice of (ξ, η, Φ, g) in $\mathcal{S}(\xi, \bar{J})$, the bundle $(\nu(\mathcal{F}_\xi), \bar{J})$ is identified with the underlying CR structure $(\mathcal{D}, J)$. We use the characteristice foliation to write any vector field as

$$X = X_{\mathcal{D}} + c(X)\xi\,, \tag{5}$$

which defines the component function $X \mapsto c(X) := \eta(X)$, and the class $\bar{X}$ on $\nu(\mathcal{F}_\xi)$ defined by the vector field X is represented by $X_{\mathcal{D}}$.

Definition. We say that a vector field X is *transversally holomorphic* if given any section $\bar{Y}$ of $\nu(\mathcal{F}_\xi)$, we have that

$$[\bar{X}, \bar{J}\bar{Y}] = \bar{J}\,\overline{[X,Y]}\,.$$

The set of all these fields is denoted by $\mathfrak{h}^T(\xi, \bar{J})$. □

We may define the transversal Futaki invariant $\mathfrak{F} = \mathfrak{F}_{(\xi,J)}$ of the polarized Sasakian manifold $(M, \xi, \bar{J})$ to be the functional

$$\begin{gathered} \mathfrak{F} : \mathfrak{h}^T(\xi, \bar{J}) \longrightarrow \mathbb{C} \\ \mathfrak{F}(X) = \int_M X(\psi_g) d\mu_g = -\int_M X(G_g^T s_g^T) d\mu_g\,, \end{gathered} \tag{6}$$

where g is any metric in $\mathfrak{M}(\xi, \bar{J})$, and ψ_g is the Ricci potential of the transversal Kähler metric, defined by the Hodge decomposition of the transversal Ricci form ρ^T

given by

$$\rho^T = \rho_h^T + i\partial\overline{\partial}\psi_g \,,$$

where ρ_h^T is the g_h^T-harmonic representative of the foliated cohomology class represented by ρ^T.

Theorem. Let (ξ, η, Φ, g) be a canonical Sasaki representative of $\mathcal{S}(\xi, \bar{J})$. Then, the metric g has constant scalar curvature if, and only if, $\mathfrak{F}(\,\cdot\,) = 0$. $\square$

The Sasaki cone

Let $(\mathcal{D}, J)$ be a CR structure of Sasaki type on M, and let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a contact metric structure whose underlying CR structure is $(\mathcal{D}, J)$. Then $\mathcal{S}$ is a Sasakian structure if, and only if, $\xi \in \mathfrak{cr}(\mathcal{D}, J)$. Here,

$$\mathfrak{cr}(\mathcal{D}, J) = \{X \in \mathfrak{con}(M, \mathcal{D}) \mid \pounds_X J = 0\} \,,$$

where

$$\mathfrak{con}(M, \mathcal{D}) = \{X \in \mathfrak{X}(M) \mid \pounds_X \eta = a(X)\eta \,,\ a(X) \in C^\infty(M)\} \,.$$

The defining condition for elements of $\mathfrak{cr}(\mathcal{D}, J)$ makes sense even though J is not a tensor field on M. This follows from the fact that these vector fields leaves $\mathcal{D}$ invariant.

We fix a strictly pseudoconvex CR structure $(\mathcal{D}, J)$ on M, and define the set

$$\mathcal{S}(\mathcal{D}, J) = \left\{ \begin{array}{c} \mathcal{S} = (\xi, \eta, \Phi, g) : \ \mathcal{S} \text{ a Sasakian structure} \\ (\ker \eta, \Phi \mid_{\ker \eta}) = (\mathcal{D}, J) \end{array} \right\} . \quad (7)$$

Let $(\mathcal{D}, J)$ be a CR structure of Sasaki type, and let $\mathcal{S}_0 = (\xi_0, \eta_0, \Phi_0, g_0) \in \mathcal{S}(\mathcal{D}, J)$. If $\mathcal{S} = (\xi, \eta, \Phi, g) \in \mathcal{S}(\mathcal{D}, J)$, we have that $\eta_0(\xi) > 0$, and $\eta = \dfrac{\eta_0}{\eta_0(\xi)}$.

Let $(\mathcal{D}, J)$ be a strictly pseudoconvex CR structure of Sasaki type. We say that a vector field $X \in \mathfrak{cr}(\mathcal{D}, J)$ is *positive* if $\eta(X) > 0$ for any $\mathcal{S} = (\xi, \eta, \Phi, g) \in \mathcal{S}(\mathcal{D}, J)$. We denote by $\mathfrak{cr}^+(\mathcal{D}, J)$ the subset of all positive elements of $\mathfrak{cr}(\mathcal{D}, J)$.

We consider the mapping ι defined by projection,

$$\begin{array}{ccc} \mathcal{S}(\mathcal{D}, J) & \stackrel{\iota}{\rightarrow} & \mathfrak{cr}^+(\mathcal{D}, J) \\ \mathcal{S} & \mapsto & \xi \end{array} . \quad (8)$$

Lemma. **Let $(\mathcal{D}, J)$ be a strictly pseudoconvex CR struc-**

ture of Sasaki type. Then

1. $\mathfrak{cr}^+(\mathcal{D}, J)$ is naturally identified with $\mathcal{S}(\mathcal{D}, J)$.

2. $\mathfrak{cr}^+(\mathcal{D}, J)$ is an open convex cone in $\mathfrak{cr}(\mathcal{D}, J)$.

3. The subset $\mathfrak{cr}^+(\mathcal{D}, J)$ is invariant under the adjoint action of the Lie group of CR transformations of $(\mathcal{D}, J)$.

Theorem. Let M be a closed manifold of dimension $2n+1$, and let $(\mathcal{D}, J)$ be a CR structure of Sasaki type on it. Then the Lie algebra $\mathfrak{cr}(\mathcal{D}, J)$ decomposes as $\mathfrak{cr}(\mathcal{D}, J) = \mathfrak{t}_k + \mathfrak{p}$, where $\mathfrak{t}_k$ is the Lie algebra of a maximal torus T_k of dimension k, $1 \leq k \leq n+1$, and $\mathfrak{p}$ is a completely reducible T_k-module. Furthermore, every $X \in \mathfrak{cr}^+(\mathcal{D}, J)$ is conjugate to a positive element in the Lie algebra $\mathfrak{t}_k$.

Definition. Let $(\mathcal{D}, J)$ be a CR structure of Sasaki type on M. We define the Sasaki cone $\kappa(\mathcal{D}, J)$ to be the moduli space of Sasakian structures compatible with $(\mathcal{D}, J)$,

$$\kappa(\mathcal{D}, J) = \mathcal{S}(\mathcal{D}, J)/\mathfrak{CR}(\mathcal{D}, J)\,.$$

$\square$

The theorem above together with the mapping (8) says that each orbit can be represented by choosing a positive element in the Lie algebra $\mathfrak{t}_k$ of a maximal torus T_k. We denote the subset of positive elements by $\mathfrak{t}_k^+$, so we have an identification $\mathfrak{t}_k^+ = \mathfrak{t}_k \cap \mathfrak{cr}^+(\mathcal{D}, J) \approx \kappa(\mathcal{D}, J)$.

Definition. We say that $(\xi, \eta, \Phi, g) \in \kappa(\mathcal{D}, J)$ is a *canonical* element of the Sasaki cone if the space $\mathcal{S}(\xi, \bar{J})$ admits a canonical representative. We denote by $\mathfrak{e}(\mathcal{D}, J)$ the set of all canonical elements of the Sasaki cone, and refer to it as the *canonical Sasaki cone* of the CR structure $(\mathcal{D}, J)$.

$\square$

Theorem. Let $(\mathcal{D}, J)$ be a CR structure of of Sasaki type on M. Then the canonical cone $\mathfrak{e}(\mathcal{D}, J)$ is an open subset of the Sasaki cone $\kappa(\mathcal{D}, J)$.