Localizing Widening and Narrowing

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(joint work with Francesca Scozzari)
We will show two techniques to improve precision in AI-based static analysis on nested loops:

- **Localized widening** restricts the scope of application of widening
- **Localized narrowing** intertwines ascending and descending phases for nested loops

These techniques are:

- domain independent
- compatible with other precision-improving techniques
- effective (at least for localized widening)
- new (?)
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- new (?)
An idea similar to localized narrowing has been presented yesterday, at PLDI:

K. Apinis, H. Seidl, V. Vojdani

*How to Combine Widening and Narrowing for Non-monotonic Systems of Equation*
Abstract Interpretation and Equations

In abstract interpretation-based static analysis, the program to analyze is typically translated into a set of equations:

\[
\begin{align*}
  x_1 &= \Phi_1(x_1, \ldots, x_n) \\
  & \vdots \\
  x_n &= \Phi_n(x_1, \ldots, x_n)
\end{align*}
\]

- each \( x_i \)
  - is the member of an abstract domain
  - corresponds to a program point
- each \( \Phi_i \)
  - is an abstract transformer
  - corresponds to a node in the control flow graph
- we want a post-fixpoint (possibly the least fixpoint) of this system
How to get non-trivial post-fixpoints

Several techniques:
- standard Kleene iteration
  - may not terminate if the domain is infinite
  - may be slow for finite domains
- policy/strategy iteration
  - powerful technique but only for template domains
- acceleration
  - powerful technique but only for restricted linear assignments
- widening/narrowing

Widening is generally applicable:
- ascending phase using a widening operator which ensures convergence
- an optional descending phase to improve the result
How to get non-trivial post-fixpoints

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- acceleration
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Widening is generally applicable:
- ascending phase using a widening operator which ensures convergence
- an optional descending phase to improve the result
From programs to equations

\[ i = 0 \]
\[ \textbf{while} \ (i < 10) \{ \]
\[ j = 0 \]
\[ \textbf{while} \ (j < 10) \]
\[ j = j+1 \]
\[ i = i+1 \}

\[ x_1 = [0, 0] \times [-\infty, \infty] \]
\[ x_2 = x_1 \lor x_{10} \]
\[ x_3 = x_2 \land ([-\infty, 9] \times [-\infty, \infty]) \]
\[ x_4 = x_2 \land ([10, \infty] \times [-\infty, \infty]) \]
\[ x_5 = \text{first}(x_3) \times [0, 0] \]
\[ x_6 = x_5 \lor x_9 \]
\[ x_7 = x_6 \land ([-\infty, \infty] \times [-\infty, 9]) \]
\[ x_8 = x_6 \land ([-\infty, \infty] \times [10, \infty]) \]
\[ x_9 = x_7 + ([0, 0] \times [1, 1]) \]
\[ x_{10} = x_8 + ([1, 1] \times [0, 0]) \]
Introducing widening

```
i = 0
while (i < 10) {
    j = 0
    while (j < 10)
        j = j + 1
    i = i + 1
}
```

\[
x_1 = [0, 0] \times [-\infty, \infty]
\]
\[
x_2 = x_1 \lor x_{10}
\]
\[
x_3 = x_2 \land([-\infty, 9] \times [-\infty, \infty])
\]
\[
x_4 = x_2 \land([10, \infty] \times [-\infty, \infty])
\]
\[
x_5 = \text{first}(x_3) \times [0, 0]
\]
\[
x_6 = x_5 \lor x_9
\]
\[
x_7 = x_6 \land([-\infty, \infty] \times [-\infty, 9])
\]
\[
x_8 = x_6 \land([-\infty, \infty] \times [10, \infty])
\]
\[
x_9 = x_7 + ([0, 0] \times [1, 1])
\]
\[
x_{10} = x_8 + ([1, 1] \times [0, 0])
\]
Introducing widening

```plaintext
i = 0
while (i < 10) {
    j = 0
    while (j < 10)
        j = j + 1
    i = i + 1
}
```

\[
\begin{align*}
x_1 &= [0, 0] \times [-\infty, \infty] \\
x_2 &= x_2 \lor (x_1 \lor x_{10}) \\
x_3 &= x_2 \land ([-\infty, 9] \times [-\infty, \infty]) \\
x_4 &= x_2 \land ([10, \infty] \times [-\infty, \infty]) \\
x_5 &= \text{first}(x_3) \times [0, 0] \\
x_6 &= x_6 \lor (x_5 \lor x_9) \\
x_7 &= x_6 \land ([-\infty, \infty] \times [-\infty, 9]) \\
x_8 &= x_6 \land ([-\infty, \infty] \times [10, \infty]) \\
x_9 &= x_7 + ([0, 0] \times [1, 1]) \\
x_{10} &= x_8 + ([1, 1] \times [0, 0])
\end{align*}
\]
Problems with nested loops

Ascending chain

\( i = 0 \)

\( i < 10 \)

\( j = 0 \)

\( j = j + 1 \)

\( i = i + 1 \)

\( x_1 = \bot \)

\( x_2 = \bot \)

\( x_3 = \bot \)

\( x_4 = \bot \)

\( x_5 = \bot \)

\( x_6 = \bot \)

\( x_7 = \bot \)

\( x_8 = \bot \)

\( x_9 = \bot \)

\( x_{10} = \bot \)
Problems with nested loops

Ascendig chain

\[ i = 0 \]

\[ j = 0 \]

\[ j = j + 1 \]

\[ i = i + 1 \]

\[ x_1 = \{i = 0\} \]
\[ x_2 = \{i = 0\} \]
\[ x_3 = \{i = 0\} \]
\[ x_4 = \perp \]
\[ x_5 = \{i = 0, j = 0\} \]
\[ x_6 = \perp \]
\[ x_7 = \perp \]
\[ x_8 = \perp \]
\[ x_9 = \perp \]
\[ x_{10} = \perp \]
Problems with nested loops

Ascending chain

\[
\begin{align*}
  &i = 0 \\
  &j = 0 \\
  &j = j + 1 \\
  &i = i + 1 \\
\end{align*}
\]

\[
\begin{align*}
  x_1 &= \{i = 0\} \\
  x_2 &= \{i = 0\} \\
  x_3 &= \{i = 0\} \\
  x_4 &= \bot \\
  x_5 &= \{i = 0, j = 0\} \\
  x_6 &= \{i = 0, 0 \leq j\} \\
  x_7 &= \{i = 0, 0 \leq j \leq 9\} \\
  x_8 &= \{i = 0, 10 \leq j\} \\
  x_9 &= \{i = 0, 1 \leq j \leq 10\} \\
  x_{10} &= \bot
\end{align*}
\]
Problems with nested loops

Ascending chain

\[ i = 0 \]
\[ j = 0 \]
\[ j = j + 1 \]
\[ i = i + 1 \]

\[ i < 10 \]

\[ x_1 = \{ i = 0 \} \]
\[ x_2 = \{ 0 \leq i \} \]
\[ x_3 = \{ 0 \leq i \leq 9 \} \]
\[ x_4 = \{ 10 \leq i \} \]
\[ x_5 = \{ 0 \leq i \leq 9, j = 0 \} \]
\[ x_6 = \{ i = 0, 0 \leq j \} \]
\[ x_7 = \{ i = 0, 0 \leq j \leq 9 \} \]
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Problems with nested loops

Ascending chain

\begin{align*}
  x_1 &= \{i = 0\} \\
  x_2 &= \{0 \leq i\} \\
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  x_4 &= \{10 \leq i\} \\
  x_5 &= \{0 \leq i \leq 9, j = 0\} \\
  x_6 &= \{i = 0, 0 \leq j\} \\
  x_7 &= \{i = 0, 0 \leq j \leq 9\} \\
  x_8 &= \{i = 0, 10 \leq j\} \\
  x_9 &= \{i = 0, 1 \leq j \leq 10\} \\
  x_{10} &= \{i = 1, 10 \leq j\} \\
\end{align*}

\[ x_6 = x_6 \lor (x_5 \cup x_9) \]
Problems with nested loops

Ascending chain

\[ x_1 = \{i = 0\} \]
\[ x_2 = \{0 \leq i\} \]
\[ x_3 = \{0 \leq i \leq 9\} \]
\[ x_4 = \{10 \leq i\} \]
\[ x_5 = \{0 \leq i \leq 9, j = 0\} \]
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\[ x_9 = \{i = 0, 1 \leq j \leq 10\} \]
\[ x_{10} = \{i = 1, 10 \leq j\} \]

\[ x_6 = \{i = 0, 0 \leq j\} \lor \{0 \leq i \leq 9, 0 \leq j \leq 10\} \]
Problems with nested loops

Ascending chain

\begin{align*}
  x_1 &= \{i = 0\} \\
  x_2 &= \{0 \leq i\} \\
  x_3 &= \{0 \leq i \leq 9\} \\
  x_4 &= \{10 \leq i\} \\
  x_5 &= \{0 \leq i \leq 9, j = 0\} \\
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  x_8 &= \{i = 0, 10 \leq j\} \\
  x_9 &= \{i = 0, 1 \leq j \leq 10\} \\
  x_{10} &= \{i = 1, 10 \leq j\}
\end{align*}
Problems with nested loops

Result of ascending chain

\[ i = 0 \]

\[ i < 10 \]

\[ j = 0 \]

\[ j = j + 1 \]

\[ i = i + 1 \]

\[ x_1 = \{ i = 0 \} \]
\[ x_2 = \{ 0 \leq i \} \]
\[ x_3 = \{ 0 \leq i \leq 9 \} \]
\[ x_4 = \{ 10 \leq i \} \]
\[ x_5 = \{ 0 \leq i \leq 9, j = 0 \} \]
\[ x_6 = \{ 0 \leq i, 0 \leq j \} \]
\[ x_7 = \{ 0 \leq i, 0 \leq j \leq 9 \} \]
\[ x_8 = \{ 0 \leq i, 10 \leq j \} \]
\[ x_9 = \{ 0 \leq i, 1 \leq j \leq 10 \} \]
\[ x_{10} = \{ 1 \leq i, 10 \leq j \} \]
Problems with nested loops

Result of descending chain

\[
\begin{align*}
x_1 &= \{ i = 0 \} \\
x_2 &= \{ 0 \leq i \} \\
x_3 &= \{ 0 \leq i \leq 9 \} \\
x_4 &= \{ 10 \leq i \} \\
x_5 &= \{ 0 \leq i \leq 9, j = 0 \} \\
x_6 &= \{ 0 \leq i, 0 \leq j \leq 10 \} \\
x_7 &= \{ 0 \leq i, 0 \leq j \leq 9 \} \\
x_8 &= \{ 0 \leq i, 10 \leq j \leq 10 \} \\
x_9 &= \{ 0 \leq i, 1 \leq j \leq 10 \} \\
x_{10} &= \{ 1 \leq i, 10 \leq j \leq 10 \}
\end{align*}
\]

We cannot prove \( i = 10 \) in \( x_4 \)
Problems with nested loops

A step back

$$i = 0$$

$$i < 10$$

$$j = 0$$

$$j = j + 1$$

$$i = i + 1$$

- $$x_1 = \{ i = 0 \}$$
- $$x_2 = \{ 0 \leq i \}$$
- $$x_3 = \{ 0 \leq i \leq 9 \}$$
- $$x_4 = \{ 10 \leq i \}$$
- $$x_5 = \{ 0 \leq i \leq 9, j = 0 \}$$
- $$x_6 = \{ i = 0, 0 \leq j \}$$
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- $$x_9 = \{ i = 0, 1 \leq j \leq 10 \}$$
- $$x_{10} = \{ i = 1, 10 \leq j \}$$

$$x_6 = x_6 \vee (x_5 \cup x_9)$$
Localized widening

What is localized widening?

\[ x_1 = \{ i = 0 \} \]
\[ x_2 = \{ 0 \leq i \} \]
\[ x_3 = \{ 0 \leq i \leq 9 \} \]
\[ x_4 = \{ 10 \leq i \} \]
\[ x_5 = \{ 0 \leq i \leq 9, j = 0 \} \]
\[ x_6 = \{ i = 0, 0 \leq j \} \]
\[ x_7 = \{ i = 0, 0 \leq j \leq 9 \} \]
\[ x_8 = \{ i = 0, 10 \leq j \} \]
\[ x_9 = \{ i = 0, 1 \leq j \leq 10 \} \]
\[ x_{10} = \{ i = 1, 10 \leq j \} \]

\[ x_6 = x_6 \triangledown (x_5 \cup x_9) \quad \Rightarrow \quad x_6 = x_5 \cup (x_6 \triangledown x_9) \]
Localized widening

What is localized widening?

\[ i = 0 \]

- \[ x_1 = \{ i = 0 \} \]
- \[ x_2 = \{ 0 \leq i \} \]
- \[ x_3 = \{ 0 \leq i \leq 9 \} \]
- \[ x_4 = \{ 10 \leq i \} \]
- \[ x_5 = \{ 0 \leq i \leq 9, j = 0 \} \]
- \[ x_6 = \{ i = 0, 0 \leq j \} \]
- \[ x_7 = \{ i = 0, 0 \leq j \leq 9 \} \]
- \[ x_8 = \{ i = 0, 10 \leq j \} \]
- \[ x_9 = \{ i = 0, 1 \leq j \leq 10 \} \]
- \[ x_{10} = \{ i = 1, 10 \leq j \} \]

\[ x_6 = \{ 0 \leq i \leq 9, 0 \leq j \} \]
Localized widening

Result of ascending chain with localized widening

\[
\begin{align*}
\text{false} & \quad \text{false} \\
i < 10 & \quad i < 10 \\
\text{true} & \quad j = 0 \\
n & \quad j = j + 1 \\
i = i + 1 & \\
\end{align*}
\]

\[
\begin{align*}
x_1 &= \{i = 0\} \\
x_2 &= \{0 \leq i\} \\
x_3 &= \{0 \leq i \leq 9\} \\
x_4 &= \{10 \leq i\} \\
x_5 &= \{0 \leq i \leq 9, j = 0\} \\
x_6 &= \{0 \leq i \leq 9, 0 \leq j\} \\
x_7 &= \{0 \leq i \leq 9, 0 \leq j \leq 9\} \\
x_8 &= \{0 \leq i \leq 9, 10 \leq j\} \\
x_9 &= \{0 \leq i \leq 9, 1 \leq j \leq 10\} \\
x_{10} &= \{1 \leq i \leq 10, 10 \leq j\}
\end{align*}
\]
Localized widening

Result of descending chain with localized widening

\[
\begin{align*}
x_1 &= \{i = 0\} \\
x_2 &= \{0 \leq i \leq 10\} \\
x_3 &= \{0 \leq i \leq 9\} \\
x_4 &= \{10 \leq i \leq 10\} \\
x_5 &= \{0 \leq i \leq 9, j = 0\} \\
x_6 &= \{0 \leq i \leq 9, 0 \leq j \leq 10\} \\
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x_9 &= \{0 \leq i \leq 9, 1 \leq j \leq 10\} \\
x_{10} &= \{1 \leq i \leq 10, 10 \leq j \leq 10\}
\end{align*}
\]

We can prove \(i = 10\) in \(x_4\)
Localized Widening

**Definition (informal)**

We call *localized widening* the use of a widening in the form

\[ x_i = x_{in} \lor (x_i \triangledown x_{back}) \]

instead of

\[ x_i = x_i \triangledown (x_{in} \lor x_{back}) \]

where

- \( x_{in} \) is the value of the edge(s) coming from the outer component;
- \( x_{back} \) is the value of the back edge(s) of the loop.

**Theorem**

Using localized widening, every chaotic iteration sequence terminates on a post-fixpoint (if every loop head is a widening point).
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We call localized widening the use of a widening in the form

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Theorem

Using localized widening, every chaotic iteration sequence terminates on a post-fixpoint (if every loop head is a widening point).
Is it really new?

The optimization given by localized widening is very very simple. Is it really new?

We cannot be sure it has not been implemented before, but we think it has not been described before.

We are sure both INTERPROC and PAGAI do not implement localized widening.

We implemented it in RANDOM long before this paper.
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We are sure both INTERPROC and PAGAI do not implement localized widening.

We implemented it in RANDOM long before this paper.
Is it really worth to implement it? **Definitely yes.**

- It is very simple. Our implementation in Pagai, apart from stuff related to configuration and user interface, only requires to change a single line of code.

- Nested loops are common. Most programs benefits from the improved precision.

- Computational overhead should be negligible. Hard to say something more precise, due to non-monotonicity of widening.
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- Nested loops are common.  
  Most programs benefits from the improved precision.

- Computational overhead should be negligible.  
  Hard to say something more precise, due to non-monotonicity of widening.
We implemented localized widening in Pagai and compared it with standard widening.

- **Benefits of localized widening**
  - WCET Benchmarks
  - Improved program points

- **Total**:
  - 379 nodes, improvements in 164 nodes

- **Standard widening**:
  - 4.48 secs, 19522 asc. steps and 23415 desc. steps

- **Localized widening**:
  - 5.22 secs, 19363 asc. steps and 22528 desc. steps
Experiments
Comparison with Halbwachs & Henry (SAS ’12)

91 improved head nodes, 2 regressed, 2 incomparable

localized widening: 5.26 secs, 19363 asc. steps and 22528 desc. steps

hh narrowing: 5.88 secs, 22665 asc. steps and 27134 desc. steps
Loops as black boxes

\[\begin{align*}
&i = 0 \\
&j = 0 \\
&j = j + 1 \\
&i = i + 1 \\
&i < 10 \\
&i = 0 \\
&j = 0 \\
&j = j + 1 \\
&i = i + 1 \\
&i < 10
\end{align*}\]
Iteration with localized narrowing

\[ i = 0 \]

\[ i < 10 \]

\[ j = 0 \]

\[ j = j + 1 \]

\[ i = i + 1 \]

\[ x_1 = \bot \]
\[ x_2 = \bot \]
\[ x_3 = \bot \]
\[ x_4 = \bot \]
\[ x_5 = \bot \]
\[ x_6 = \bot \]
\[ x_7 = \bot \]
\[ x_8 = \bot \]
\[ x_9 = \bot \]
\[ x_{10} = \bot \]
Iteration with localized narrowing

\[
\begin{align*}
\text{x}_1 &= \{i = 0\} \\
\text{x}_2 &= \{i = 0\} \\
\text{x}_3 &= \{i = 0\} \\
\text{x}_4 &= \bot \\
\text{x}_5 &= \{i = 0, j = 0\} \\
\text{x}_6 &= \bot \\
\text{x}_7 &= \bot \\
\text{x}_8 &= \bot \\
\text{x}_9 &= \bot \\
\text{x}_{10} &= \bot
\end{align*}
\]
Iteration with localized narrowing

\[ i = 0 \]
\[ i < 10 \]
\[ j = 0 \]
\[ j = j + 1 \]
\[ i = i + 1 \]

\[ x_1 = \{ i = 0 \} \]
\[ x_2 = \{ i = 0 \} \]
\[ x_3 = \{ i = 0 \} \]
\[ x_4 = \bot \]
\[ x_5 = \{ i = 0, j = 0 \} \]
\[ x_6 = \{ i = 0, 0 \leq j \leq 10 \} \]
\[ x_7 = \{ i = 0, 0 \leq j \leq 9 \} \]
\[ x_8 = \{ i = 0, j = 10 \} \]
\[ x_9 = \{ i = 0, 1 \leq j \leq 10 \} \]
\[ x_{10} = \bot \]

Ascending and descending chain
Iteration with localized narrowing

\[

c_0 = 0 \\

\text{while } i < 10 \\

\text{let } j = 0 \\

\text{while } j < 10 \\

j = j + 1 \\

\text{if } i < 10 \\

\text{else } i = i + 1
\]

\[
x_1 = \{ i = 0 \} \\
x_2 = \{ 0 \leq i \} \\
x_3 = \{ 0 \leq i \leq 9 \} \\
x_4 = \{ 10 \leq i \} \\
x_5 = \{ 0 \leq i \leq 9, j = 0 \} \\
x_6 = \{ i = 0, 0 \leq j \leq 10 \} \\
x_7 = \{ i = 0, 0 \leq j \leq 9 \} \\
x_8 = \{ i = 0, j = 10 \} \\
x_9 = \{ i = 0, 1 \leq j \leq 10 \} \\
x_{10} = \{ i = 1, j = 10 \}
\]

What to do with \( x_6 - x_9 \)?
Iteration with localized narrowing

\[\begin{align*}
i &= 0 \\
&\xrightarrow{1} 2 \\
&\xleftarrow{\text{false}} 4 \\
\quad \text{false} \quad i < 10 \quad \text{true} \\
&\xrightarrow{3} 3 \\
&\xrightarrow{5} 5 \\
&\xrightarrow{6} 6 \\
&\xleftarrow{\text{false}} 9 \\
&\xrightarrow{9} 9 \\
\quad j = j + 1 \\
&\xrightarrow{8} 8 \\
&\xrightarrow{7} 7 \\
&\xleftarrow{\text{true}} 9 \\
&\xrightarrow{9} 9 \\
\end{align*}\]

\[\begin{align*}
x_1 &= \{i = 0\} \\
x_2 &= \{0 \leq i\} \\
x_3 &= \{0 \leq i \leq 9\} \\
x_4 &= \{10 \leq i\} \\
x_5 &= \{0 \leq i \leq 9, j = 0\} \\
x_6 &= \bot \\
x_7 &= \bot \\
x_8 &= \bot \\
x_9 &= \bot \\
x_{10} &= \{i = 1, j = 10\}
\]

Start with \(x_6 = x_7 = x_8 = x_9 = \bot \ldots\)
Iteration with localized narrowing

\[ i = 0 \]
\[ j = 0 \]

\[ i < 10 \]
\[ j < 10 \]

\[ x_1 = \{ i = 0 \} \]
\[ x_2 = \{ 0 \leq i \} \]
\[ x_3 = \{ 0 \leq i \leq 9 \} \]
\[ x_4 = \{ 10 \leq i \} \]
\[ x_5 = \{ 0 \leq i \leq 9, j = 0 \} \]
\[ x_6 = \{ 0 \leq i \leq 9, 0 \leq j \leq 10 \} \]
\[ x_7 = \{ 0 \leq i \leq 9, 0 \leq j \leq 9 \} \]
\[ x_8 = \{ 0 \leq i \leq 9, j = 10 \} \]
\[ x_9 = \{ 0 \leq i \leq 9, 1 \leq j \leq 10 \} \]
\[ x_{10} = \{ i = 1, j = 10 \} \]

...and re-analyze
Iteration with localized narrowing

\begin{align*}
  x_1 &= \{i = 0\} \\
  x_2 &= \{0 \leq i\} \\
  x_3 &= \{0 \leq i \leq 9\} \\
  x_4 &= \{10 \leq i\} \\
  x_5 &= \{0 \leq i \leq 9, j = 0\} \\
  x_6 &= \{0 \leq i \leq 9, 0 \leq j \leq 10\} \\
  x_7 &= \{0 \leq i \leq 9, 0 \leq j \leq 9\} \\
  x_8 &= \{0 \leq i \leq 9, j = 10\} \\
  x_9 &= \{0 \leq i \leq 9, 1 \leq j \leq 10\} \\
  x_{10} &= \{1 \leq i \leq 10, j = 10\}
\end{align*}

End of external ascending sequence
Iteration with localized narrowing

\[
i = 0
\]

\[
j = 0
\]

\[
i < 10
\]

\[
 j = j + 1
\]

\[
i = i + 1
\]

\[
 x_1 = \{i = 0\}
\]

\[
 x_2 = \{0 \leq i \leq 10\}
\]

\[
 x_3 = \{0 \leq i \leq 9\}
\]

\[
 x_4 = \{i = 10\}
\]

\[
 x_5 = \{0 \leq i \leq 9, j = 0\}
\]

\[
 x_6 = \{0 \leq i \leq 9, 0 \leq j \leq 10\}
\]

\[
 x_7 = \{0 \leq i \leq 9, 0 \leq j \leq 9\}
\]

\[
 x_8 = \{0 \leq i \leq 9, j = 10\}
\]

\[
 x_9 = \{0 \leq i \leq 9, 1 \leq j \leq 10\}
\]

\[
 x_{10} = \{1 \leq i \leq 10, j = 10\}
\]

End of external descending sequence
In this case, both techniques give the same result, but in general? Due to non-monotonicity, it is difficult to actually prove something, but in general, localized narrowing is more precise.

```plaintext
i = 0
while (TRUE) {
  i = i + 1
  j = 0
  while (j < 10) {
    // Inv: 0 ≤ i ≤ 10
    j = j + 1
  }
  if (i > 9) i = 0
}
```

*Localized narrowing* proves the required invariant

*Localized widening* doesn’t because the check for the upper bound of $i$ is after the inner loop.
Initialization policies strategies

When we restart the analysis of an inner loop, which values to choose for the data flow variables?

- **RESTART** policy: start from $\bot$ (the one we used in the example)
- **CONTINUE** policy: start from last values
- **HYBRID** policy:
  - behave as **CONTINUE** if outer loop is in ascending phase;
  - behave as **RESTART** if outer loop is in descending phase
Problems with CONTINUE policy

The CONTINUE policy is faster, but less precise.

\[
\begin{align*}
&i = 0 \\
&\textbf{while} \ (\text{TRUE}) \ {\{} \\
&\quad i = i + 1 \\
&\quad j = 0 \\
&\quad \textbf{while} \ (j < 10) \ {\{} \\
&\quad\quad // \ \text{Inv: } 0 \leq i \leq 10 \\
&\quad\quad j = j + 1 \\
&\quad {\}} \\
&\quad \textbf{if} \ (i > 9) \ i = 0 \\
&{\}}
\end{align*}
\]

\textit{RESTART policy} proves the required invariant

\textit{CONTINUE policy} doesn’t because the check for the upper bound of \(i\) is after the inner loop.
When we restart the analysis of an inner loop, which values to choose for the data flow variables?

- **RESTART** policy: start from $\bot$ (the one we used in the example)
- **CONTINUE** policy: start from last values
- **HYBRID** policy:
  - behave as **CONTINUE** if outer loop is in ascending phase;
  - behave as **RESTART** if outer loop is in descending phase
Localized widening works for any iteration strategy.

Localized narrowing is based on a recursive iteration strategy as defined in Burdoncle ’93 but:
- different loops may be in different phases
- for two nested loops, all possible combinations of ascending and descending phases are possible
We have implemented localized narrowing in our analyzer \texttt{Jandom}.

\texttt{Jandom} is an abstract interpreter for Java bytecode and other languages (https://github.com/jandom-devel/Jandom).

... but it is not yet ready to analyze real code.

Localized narrowing has not been implemented in \texttt{Pagai}.

... hence, we have only limited qualitative comparison of precision with other techniques.
Four programs

### nested

```haskell
i = 0
while (i < 10) {
  j = 0
  while (j < 10)
    j = j + 1
  i = i + 1
}
// Inv: i = 10
```

### hybrid

```haskell
i = 0
while (TRUE) {
  i = i + 1
  j = 0
  while (j < 10)
    // Inv: 0 ≤ i ≤ 10
    j = j + 1
  if (i > 9) i = 0
}
```

### hh (Halbwachs & Henry)

```haskell
i = 0
while (i < 4) {
  j = 0
  while (j < 4)
    // Inv: i ≤ j + 3
    i = i + 1
    j = j + 1
  i = i − j + 1
}
```

### nested2

```haskell
i = 0
while (TRUE) {
  // Inv: i ≥ 0
  j = 0
  while (j < 10)
    j = j + 1
  i = i + 11 − j
}
```
Results on the four programs

<table>
<thead>
<tr>
<th>program</th>
<th>loc widening</th>
<th>continue</th>
<th>hybrid</th>
<th>guided</th>
<th>lookahead</th>
<th>HH</th>
</tr>
</thead>
<tbody>
<tr>
<td>nested</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>nested2</td>
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<td>yes</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
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<tr>
<td>hybrid</td>
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<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>hh</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
</tbody>
</table>

loc widening localized widening
continue localized narrowing with CONTINUE policy
hybrid localized narrowing with HYBRID policy
guided guided static analysis
lookahead lookahead widening

HH optimized narrowing in Halbwachs & Henry 2012
Conclusions

1. **localized widening**: do not see any reason not to implement it
   - easy to implement
   - good precision improvement
   - low overhead

2. **localized narrowing**: more work needed to establish its usefulness
   - implementation from easy (for AST-based analyzers) to medium difficulty
   - better than localized widening, but how much better?
   - what overhead?

Results in PLDI ’13 paper by Apinis, Seidl, Vojdani are encouraging.
Future work

1. Evaluate benefits of localized narrowing
   - Implement localized narrowing in PAGAI
   - Finish implementation of Java bytecode analysis in JANDOM

2. Merge ideas from this paper and the PLDI ’13 paper by Apinis, Seidl, Vojdani.