Narrowing operators on template abstract domains

Gianluca Amato

Università di Chieti–Pescara
Pescara, Italy

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(joint work with S. Di Nardo Di Maio, M. C. Meo and F. Scozzari)

Overview

Context

Data-flow analysis
Abstract interpretation
Analysis of numerical properties
Interval domain (and other template domains)

Topic

Under which conditions *narrowing* may be avoided.

Plan of the talk

1. Two-phase (widening/narrowing based) analysis.
2. Narrowing with integer bounds.
3. Narrowing with rational bounds.
4. Conclusions and future work.

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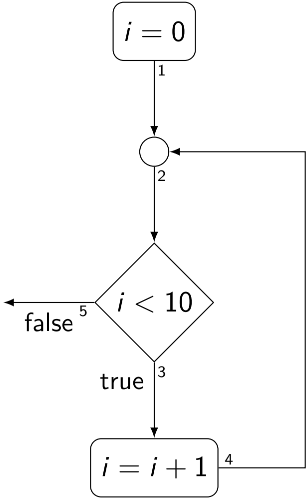
An example: interval analysis



$$x_1 = [0, 0]$$
$$x_2 = x_1 \vee x_4$$
$$x_3 = x_2 \wedge [-\infty, 9]$$
$$x_4 = x_3 + [1, 1]$$
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where

$[l, r]$ is an interval in $\mathbb{Z}$

$\vee$ is convex hull

$\wedge$ is intersection

$+$ is pointwise sum

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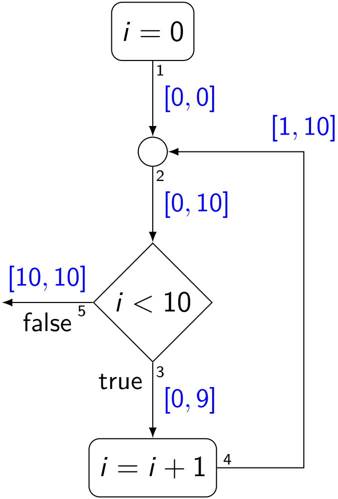

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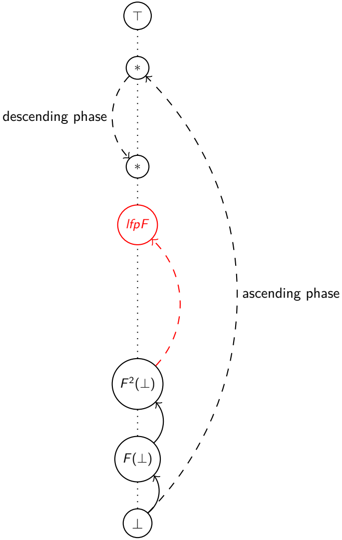
Two phase analysis



Widening

- A standard chaotic iteration might non terminate
- Introduce widening: accelerate convergence ensuring termination. Replace

$$x_i = \text{expr}$$

with

$$x_i = x_i \nabla \text{expr}$$

- Standard widening on intervals:
  - When a bound is increased, it goes to infinite
  - $\emptyset \nabla [l, r] = [l, r]$
  - $[1, 2] \nabla [1, 3] = [1, +\infty]$

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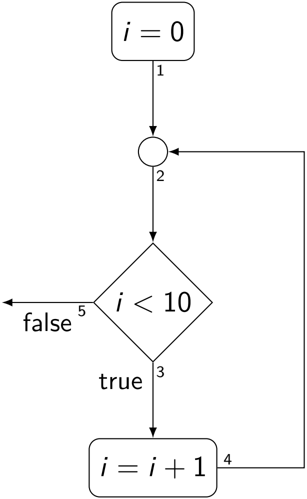

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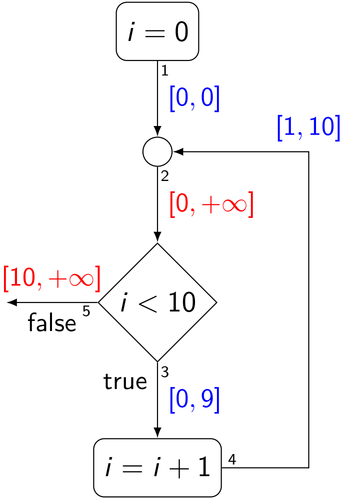

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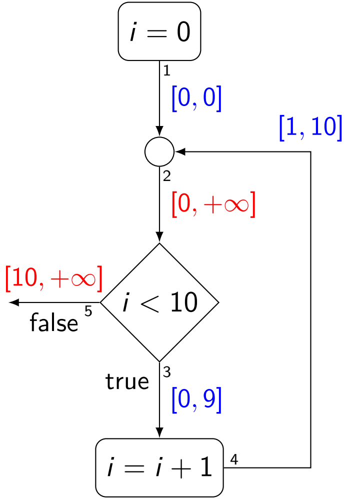

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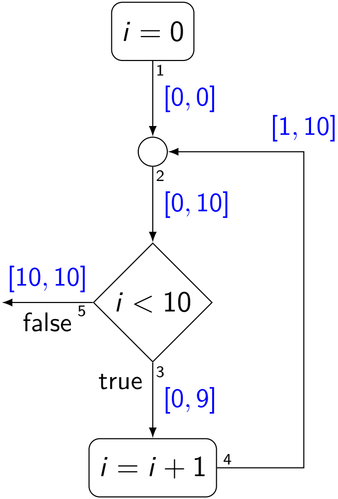

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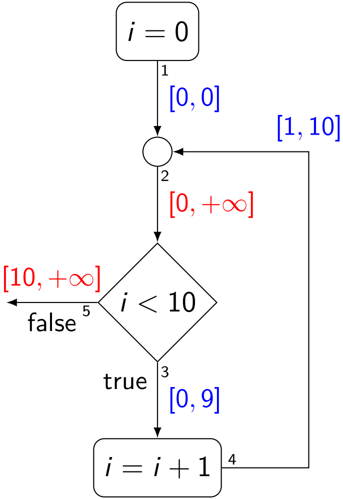

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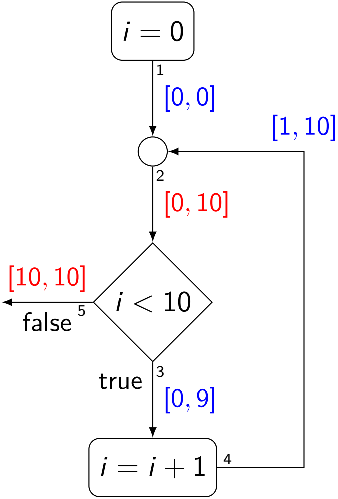

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Is narrowing really needed for termination?

- This is the classical example used to explain widening and narrowing
- ...nonetheless
  - narrowing is not needed in this case
  - descending chain terminates after a single iteration
- **how common is this scenario?**
- We try to answer this question using
  - *Alice benchmarks*
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  - *Jandom*
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Narrowing and the Alice benchmarks

- The two phase analysis with no narrowing:
  - terminates for all the models;
  - during the descending phase, each loop head is evaluated at most 3 times.
- The same is true for the Octagon domain ...
  - octagons are generalization of intervals with constraint of the kind $\pm x \pm y \leq c$ instead of $x \leq c$;
  - octagons and intervals are example of template domains.
- ... but not for the Polyhedra domain
  - the analysis of some models does not terminate;
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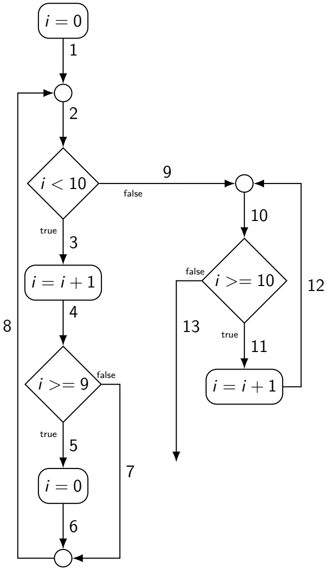
When narrowing is needed
Example program




```
i=0

while   (i<10) {

   i=i+1
   if (i>=9) i=0

}

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}
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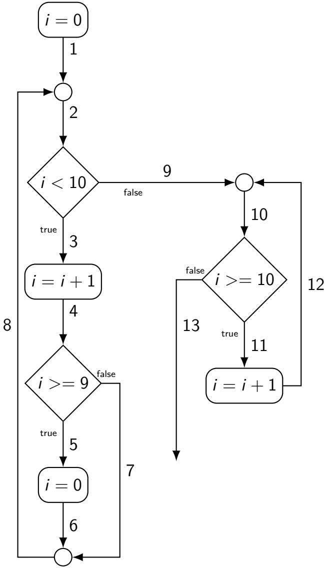
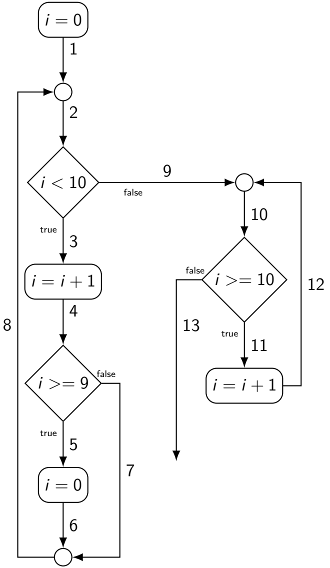



```
i=0
[x1]
while [x2] (i<10) {
   [x3]
   i=i+1
   if (i>=9) i=0
   [x8]
}
[x9]
while [x10] (i>=10) {
   [x11]
   i=i+1
   [x12]
}
```

When narrowing is needed

Ascending chain



i=0

$[x_1 \rightarrow i = 0]$

while $[x_2 \rightarrow 0 \leq i]$ (i<10) {

  $[x_3 \rightarrow 0 \leq i \leq 9]$

  i=i+1

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  $[x_8 \rightarrow 1 \leq i \leq 9]$

}

$[x_9 \rightarrow 10 \leq i]$

while $[x_{10} \rightarrow 10 \leq i\ ]$ (i>=10) {

  $[x_{11} \rightarrow 10 \leq i]$

  i=i+1

  $[x_{12} \rightarrow 11 \leq i]$

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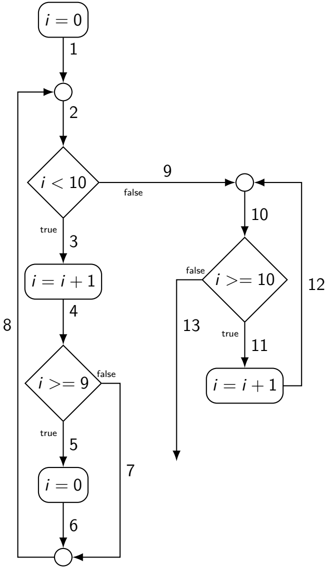
When narrowing is needed

Descending chain 1st loop



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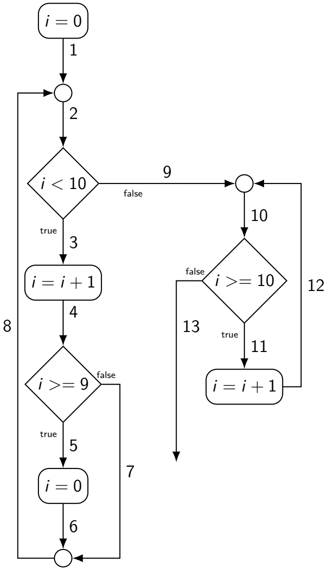

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With narrowing we would stop here

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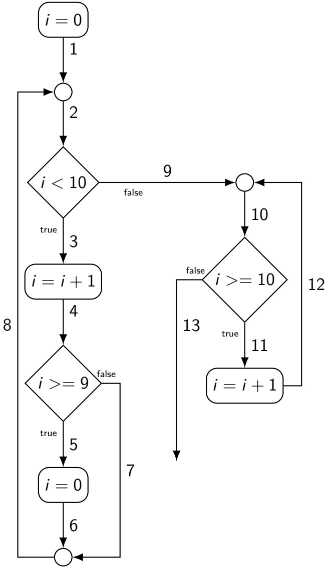
When narrowing is needed

Descending chain 2nd loop 1st iteration



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$[x_9 \rightarrow \emptyset]$
while $[x_{10} \rightarrow 11 \leq i]$ (i>=10) {
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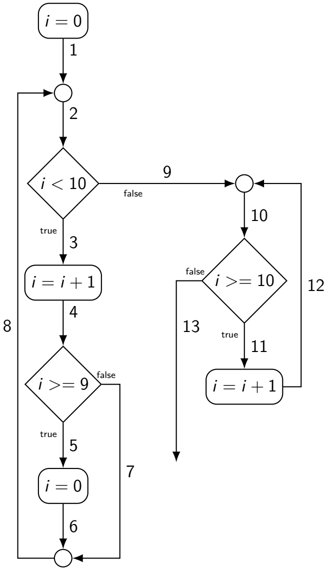
Descending chain 2nd loop 2nd iteration



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Narrowing and infinite descending chains

- x_{10} follows an infinite descending chain, whose limit is $\emptyset$:

$$[10, +\infty], [11, +\infty], [12, +\infty], \ldots$$

- there are only two kinds of infinite descending chains for intervals with integer bounds:

$$[n_0, +\infty], [n_1, +\infty], [n_2, +\infty], \ldots$$

$$[-\infty, -n_0], [-\infty, -n_1], [-\infty, -n_2], \ldots$$

  with $n_0 < n_1 < n2 < \ldots$

- infinite descending chains may only happen:
  - in presence of unreachable code;
  - when unreachability is detected during the descending phase.

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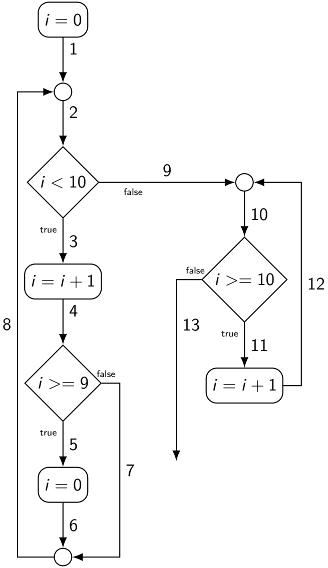
Detecting infinite descending chains



- edge 9 dominates the second loop
  - 9 unreachable $\rightarrow$ 10 unreachable
  - …but $x_{10} = x_9 \vee x_{12}$
  - …and $x_9 = \emptyset \not\rightarrow x_{10} = \emptyset$
- replace $\vee$ with a left-strict variant

$$l_1 \vee^{\emptyset} l_2 = \begin{cases} \emptyset & \text{if } l_1 = \emptyset, \\ l_1 \vee l_2 & \text{otherwise.} \end{cases}$$

- $x_{10} = x_9 \vee^{\emptyset} x_{12}$

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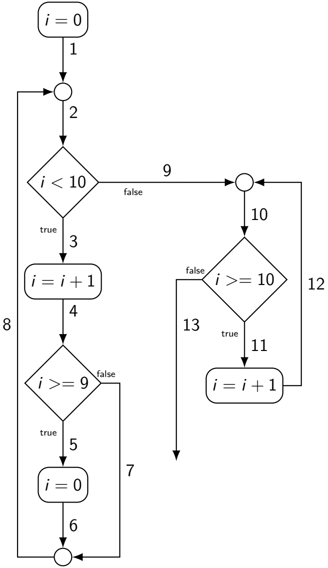

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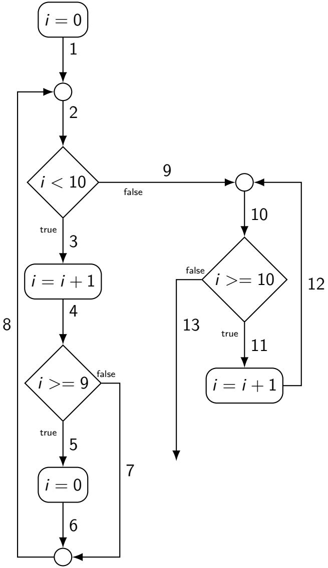

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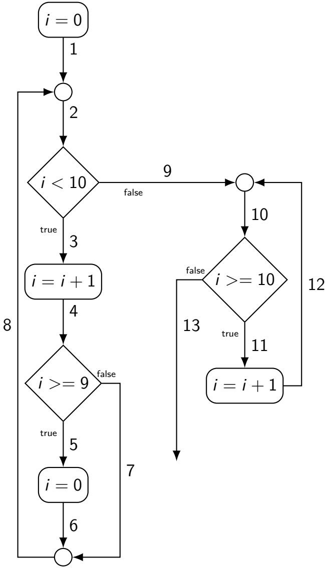

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$[x_3 \to 0 \leq i \leq 9]$

i=i+1

if (i>=9) i=0

$[x_8 \to 1 \leq i \leq 9]$

}

$[x_9 \to \emptyset]$

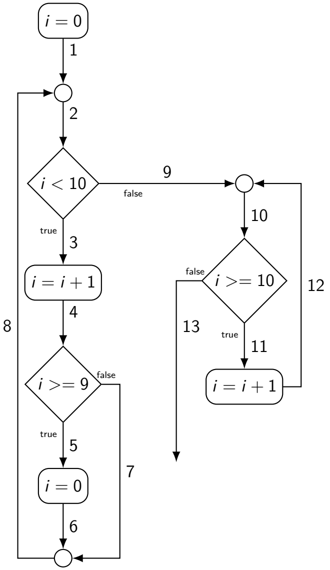
while $[x_{10} \to 10 \leq i\ ]$ (i>=10) {

$[x_{11} \to 10 \leq i]$

i=i+1

$[x_{12} \to 11 \leq i]$

}



i=0

$[x_1 \rightarrow i = 0]$

while $[x_2 \rightarrow 0 \leq i \leq 9]$ (i<10) {

  $[x_3 \rightarrow 0 \leq i \leq 9]$

  i=i+1

  if (i>=9) i=0

  $[x_8 \rightarrow 1 \leq i \leq 9]$

}

$[x_9 \rightarrow \emptyset]$

while $[x_{10} \rightarrow \emptyset]$ (i>=10) {

  $[x_{11} \rightarrow \emptyset]$

  i=i+1

  $[x_{12} \rightarrow \emptyset]$

}

Avoiding narrowing

Theorem

The set of data-flow equations corresponding to a reducible flow-chart may be analyzed without narrowing if we replace $\vee$ in join nodes with $\vee^{\emptyset}$.

- Are we proposing to replace narrowing for a non controlled descending chain?
- Not necessarily, because descending chains may be finite but very long
  - exponentially long in the size of the program

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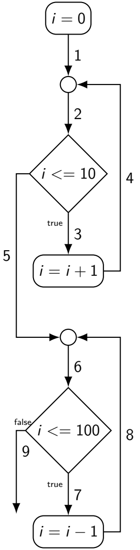
Very long descending chains

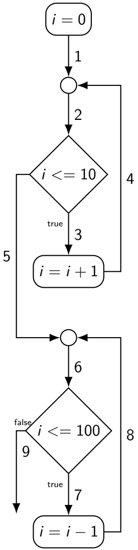
Example program



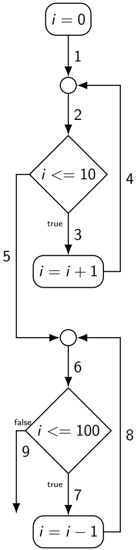

```
i=0
[x1]
while [x2] (i<=10) {
  [x3]
  i=i+1
  [x4]
}
[x5]
while [x6] (i<=100) {
  [x7]
  i=i-1
  [x8]
}
```

Very long descending chains

Ascending chain



i=0
$[x_1 \rightarrow i = 0]$
while $[x_2 \rightarrow 0 \leq i]$ (i<=10) {
  $[x_3 \rightarrow 0 \leq i \leq 10]$
  i=i+1
  $[x_4 \rightarrow 1 \leq i \leq 11]$
}
$[x_5 \rightarrow 11 \leq i]$
while $[x_6 \rightarrow \mathbb{R}]$ (i<=100) {
  $[x_7 \rightarrow i \leq 100]$
  i=i−1
  $[x_8 \rightarrow i \leq 99]$
}



i=0
$[x_1 \rightarrow i = 0]$
while $[x_2 \rightarrow 0 \leq i \leq 10]$ (i<=10) {
  $[x_3 \rightarrow 0 \leq i \leq 10]$
  i=i+1
  $[x_4 \rightarrow 1 \leq i \leq 11]$
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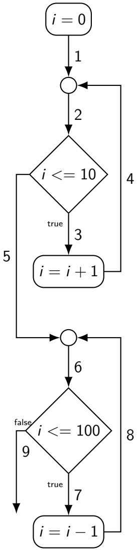
}

Very long descending chains

Descending chain 2nd loop 1st iteration



i=0

$[x_1 \to i = 0]$

while $[x_2 \to 0 \leq i \leq 10]$ (i<=10) {

  $[x_3 \to 0 \leq i \leq 10]$

  i=i+1

  $[x_4 \to 1 \leq i \leq 11]$

}

$[x_5 \to i = 11]$

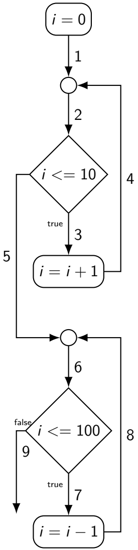
while $[x_6 \to i \leq 99]$ (i<=100) {

  $[x_7 \to i \leq 99]$

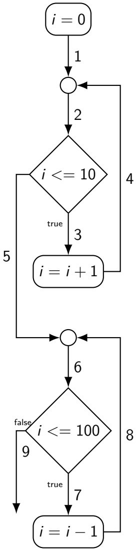
  i=i−1

  $[x_8 \to i \leq 98]$

}




```
i=0
[x1→ i = 0]
while [x2→ 0 ≤ i ≤ 10] (i<=10) {
  [x3→ 0 ≤ i ≤ 10]
  i=i+1
  [x4→ 1 ≤ i ≤ 11]
}
[x5→ i = 11]
while [x6→ i ≤ 98] (i<=100) {
  [x7→ i ≤ 98]
  i=i−1
  [x8→ i ≤ 97]
}
```




```
i=0
[x1→ i = 0]
while [x2→ 0 ≤ i ≤ 10] (i<=10) {
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}
[x5→ i = 11]
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  [x8→ i ≤ 10]
}
```

Rational bounds

When rational bounds are allowed, infinite descending chains may be generated in different ways.

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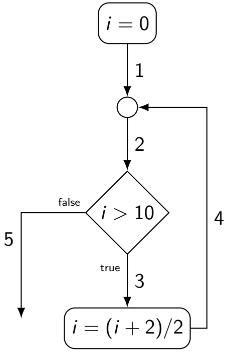
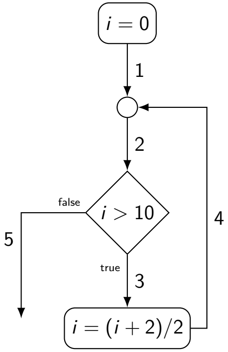
```
i=0
while(i <= 10) {
  i=(i+2)/2
}
```

Infinite descending chains with rational bounds

Example program



i=0
$[x_1]$
while $[x_2]$ (x<=10) {
    $[x_3]$
    i=(i+2)/2
    $[x_4]$
}




```
i=0
```
$[x_1 \rightarrow i = 0]$

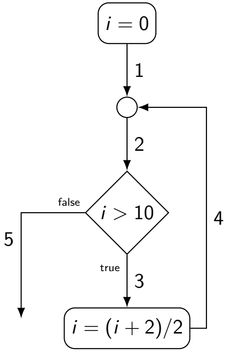
while $[x_2 \rightarrow 0 \leq i]$ (x<=10) {

  $[x_3 \rightarrow 0 \leq i \leq 10]$

  i=(i+2)/2

  $[x_4 \rightarrow 1 \leq i \leq 6]$

}



i=0

$[x_1 \rightarrow i = 0]$

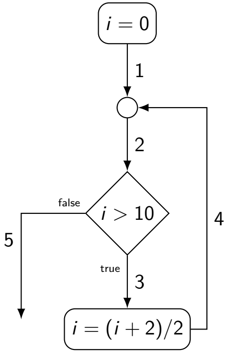
while $[x_2 \rightarrow 0 \leq i \leq 6]$ (x<=10) {

$[x_3 \rightarrow 0 \leq i \leq 6]$

i=(i+2)/2

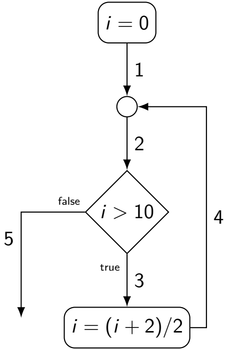
$[x_4 \rightarrow 1 \leq i \leq 4]$

}



i=0
$[x_1 \rightarrow i = 0]$
while $[x_2 \rightarrow 0 \leq i \leq 4]$ (x<=10) {
  $[x_3 \rightarrow 0 \leq i \leq 4]$
  i=(i+2)/2
  $[x_4 \rightarrow 1 \leq i \leq 3]$
}

Infinite descending chains with rational bounds



i=0

$[x_1 \to i = 0]$

while $[x_2 \to 0 \leq i \leq 4]$ (x<=10) {

$[x_3 \to 0 \leq i \leq 4]$

i=(i+2)/2

$[x_4 \to 1 \leq i \leq 3]$

}

x_2 descending chain:

$[0, 6], [0, 4], [0, 3], [0, 5/2], [0, 9/4], \ldots$

whose limit is $[0, 2]$

New narrowings on rationals

- We may define new narrowings which are much more precise by exploiting what we have seen for integer bounds.
- For example $\triangle^1$ is like intersection but we replace bounds with their integer values:

$$I \triangle^1 \emptyset = \emptyset$$

$$[l_1, u_1] \triangle^1 [l_2, u_2] = [l_1, u_1] \wedge [\lfloor l_2 \rfloor, \lceil r_2 \rceil]$$

- $[0, 2] \triangle^1 [0, 1.75] = [0, 2]$, but $[0, 2] \triangle^1 [0, 0.75] = [0, 1]$.
- With $\triangle^1$ narrowing, in the example before we stop at $[0, 3]$.
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Conclusions

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- infinite descending chains exist but are rare
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Future work

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Thank You!

Related Work

- performing descending chains without narrowing is common
  - termination ensured by fixing a limit on the number of descending steps;
  - we prove this limit may be removed most of the time.
- the same effect of $\vee^{\emptyset}$ may be realized using *localized narrowing with restart* [SAS '13]
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