Università di Pisa
Dipartimento di Informatica

Technical Report: TR-05-08

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28 February 2005

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On abstract unification for variable aliasing

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Abstract. In the context of logic program static analysis, we face the problem of devising optimal unification operators for domains of abstract substitutions. We are interested in abstract domains for sharing and linearity properties, which are widely used in logic program analysis. We propose a new (infinite) domain $\mathtt{ShLin}^{\omega}$ which can be thought of as a general framework from which other domains can be easily derived by abstraction. The advantage is that abstract unification in $\mathtt{ShLin}^{\omega}$ is simple and elegant, and it is based on a new concept of *sharing graph* which plays the same role of alternating paths for pair sharing analysis. We also provide an alternative, purely algebraic description of sharing graphs. Exploiting the results for $\mathtt{ShLin}^{\omega}$, we derive optimal abstract operators for two well-known domains which combine sharing and linearity: the classic reduced product $\mathtt{Sharing} \times \mathtt{Lin}$ and the recent, more precise domain $\mathtt{ShLin}^2$ proposed by Andy King.

1 Introduction

In the field of logic program static analysis, the theory of abstract interpretation [9, 10] has been widely used to design new analysis and to improve existing ones. An abstract interpretation is specified by a concrete domain and a concrete semantics which describe the program behavior. Once fixed the concrete domain and semantics, a static analysis is specified by providing an abstract domain (a collection of abstract objects) which encodes the property to analyze, and an abstract semantics, which computes the approximate semantics over the abstract objects. The concrete and abstract domains are related by means of abstraction and concretization maps, which allow each abstract object to be described in terms of concrete objects. In most cases, the abstract semantics is given by a set of abstract operators, which are the counterparts, in the abstract domain, of the concrete ones. For example, in the case of logic programs, one can individuate in the concrete semantics the main operations (unification, projection, union), and an abstract semantics can be specified by giving the abstract unification, abstract projection and abstract union operations. The theory of abstract interpretation assures us that, for any concrete operator, there exists a best abstract operator, called the optimal operator. It computes the most precise result among all possible correct operators, on a given abstract domain. Designing the optimal abstract counterpart of each concrete operator is often a very difficult task. In fact, even if the definition of the optimal operator for any abstract domain is known from the theory of abstract interpretation (as a composition of the concrete operator and the abstraction map), the hard task is to devise an algorithm on the abstract domain which computes that function.

The context. In this article, we investigate the interaction between sharing and linearity properties. The property of sharing has been the object of many works [16, 3, 23, 8, 12] both on the theoretical and practical point of view. Typical applications of sharing analysis are in the fields of optimization of unification [24] and parallelization of logic programs [13]. The goal of (set) sharing analysis is to detect sets of variables which share a common variable in the answer substitutions. For instance, consider the substitution $\{x/f(u,v), y/g(u,u,u), z/v\}$. We say that x and y share the variable u, while x and z share the variable v, and no variable is shared by x, y, z altogether. Many domains concerning sharing properties also consider linearity properties in order to improve the precision of the analysis. We say that a term is linear if it does not contains multiple occurrences of the same variable. For instance, the term $f(x, f(y,z))$ is linear, while $f(x, f(y,x))$ is not, due to the double occurrence of the variable x.

The problem. It is now widely recognized that the original domain proposed for sharing analysis, namely `Sharing` [20, 16] by Jacobs and Langen, is not very precise, so that it is often combined with other domains for treating freeness, linearity, groundness or structural information (see [4] for a comparative evaluation). In particular, adding some kind of linearity information seems to be very profitable, both for the gain in precision and speed which can be obtained, and for the fact that it can be easily and elegantly embedded inside the sharing groups (see [17]). In the literature, many authors have proposed abstract unification operators (e.g. [17, 12, 23, 6]) for domains of sharing properties, encoding different amounts of linearity information. However, optimal operators for combined analysis of sharing and linearity have never been devised, neither for the domain $\texttt{ShLin}^2$ [17], nor for the more broadly adopted reduced product $\texttt{Sharing} \times \texttt{Lin}$ [12, 23] or `ASub` [6].

The lack of optimal operators brings two kinds of disadvantages: first, the analysis obviously looses in precision when using sub-optimal abstract operators; second, computing approximated abstract objects can lead to a speed-down of the analysis. The latter is typical of sharing analysis, where abstract domains are usually defined in such a way that, the less information we have, the more the abstract objects are complex. This is not the case for other kind of analyses, such as groundness analysis, where the complexity of abstract objects may grow accordingly to the amount of groundness information they encode.

The lack of optimal operators is due to the fact that the role played by linearity in the unification process has never been fully clarified. The traditional domains which combine sharing and linearity information are too abstract to capture in a clean way the effect of repeated occurrences of a variable in a term and most of the effects of (non-)linearity are obscured by the abstraction process.

The solution. We propose an abstract domain $\mathtt{ShLin}^\omega$ which is able to encode the *amount* of non-linearity, i.e., which keeps track of the exact number of occurrences of the same variable in a term. Consider again the substitution $\theta = \{x/f(u,v), y/g(u,u,u), z/v\}$. Intuitively, to each variable in the range of the substitution, we associate the multiset of domain variables which contains it, and call it a ω-sharing group. For instance, we associate, to the variable u, the ω-sharing group $\{x,y,y,y\}$, to denote that u appears once in $\theta(x)$ and three times in $\theta(y)$. To the variable v, we associate the ω-sharing group $\{x,z\}$, to denote that v appears once in $\theta(x)$ and in $\theta(z)$. Then we consider the collection of all the multisets so obtained $\{\{x,y,y,y\},\{x,z\}\}$, which describes both the sharing property and the exact amount of non-linearity in the given substitution. The domain we obtain is conceptually simple and elegant, but cannot be directly used for static analysis, at least without resorting to widening operators, since it contains infinite ascending chains. However, in this domain the role played by (non-)linearity is manifest, and the optimal abstract operator for unification has a very clean form. The cornerstone of the abstract unification is the concept of *sharing graph* which plays the same role of alternating paths [24, 18] for pair sharing analysis. A sharing graph is a graph theoretic notion to figure out the ω-sharing groups which are combined during the unification process to obtain a new ω-sharing group. The use of sharing graphs offers a new perspective to look at single variables in the process of unification, and simplify the proofs of correctness and optimality of the abstract operators. We also provide a purely algebraic characterization of the results, which can help in implementing the domain by making use of widening operators and in devising abstract operators for further abstractions of $\mathtt{ShLin}^\omega$.

In addition, we provide a parallel unification operator, able to compute the abstract unification over $\mathtt{ShLin}^\omega$ by considering all the bindings at the same time. Usually, both concrete and abstract unification are computed by considering one binding at a time. For instance, the unification of a substitution θ with $\{x_1/t_1, x_2/t_2, \ldots, x_n/t_n\}$ is performed by first computing the unification of θ with $\{x_1/t_1\}$, and then unifying the result with $\{x_2/t_2, \ldots, x_n/t_n\}$. We show that our parallel unification operator and the standard (sequential) one do coincide over $\mathtt{ShLin}^\omega$. We will show that this is not the case for many domains in the literature for sharing analysis, including the reduce product $\mathtt{Sharing} \times \mathtt{Lin}$.

The result. Finally, we consider two well-known domains for sharing properties, namely the classic reduced product $\mathtt{Sharing} \times \mathtt{Lin}$ and the more precise domain $\mathtt{ShLin}^2$ by Andy King, and show that they can be immediately obtained as abstractions of $\mathtt{ShLin}^\omega$. By exploiting the optimal abstract unification operator on $\mathtt{ShLin}^\omega$, we are able to provide the optimal abstract unification operators, in the case of single binding substitutions, for both domains. We also show that unification between an abstract object and a substitution cannot be computed one binding at a time while remaining optimal. This means that the classical schema of computing unification iteratively on the number of bindings in the substitution cannot be used when looking for optimality, at least with these domains.

Structure of the article. The next section recalls some basic notions and the notations about substitution, multiset and abstract interpretation. In Section 3 we briefly recall the domain of existential substitutions and its operators, which will be used throughout the article. In Section 4 we define the domain $\mathtt{ShLin}^\omega$, together with the sequential and parallel unification operators, we show the optimality results and give an alternative, algebraic characterization of the unification operator. Finally, in Section 5 and 6 we exploits our results to devise the optimal operators for $\mathtt{ShLin}^2$ and $\mathtt{Sharing} \times \mathtt{Lin}$. We conclude with some open questions for future work.

2 Notation

Given a set A, let $\wp(A)$ be the powerset of A and $\wp_f(A)$ be the set of finite subsets of A. Given two posets $(A, \leq_A)$ and $(B, \leq_B)$, we denote by $A \to B$ the poset of monotonic functions from A to B ordered pointwise. We use $\leq_{A\to B}$ to denote the order relation over $A \to B$. When an order for A or B is not specified, we assume the least informative order ($x \leq y \iff x = y$). We also use $A \uplus B$ to denote disjoint union and $|A|$ for the cardinality of the set A.

2.1 Terms and substitutions

In the following, we fix a first order signature and a denumerable set of variables $\mathcal{V}$. Given a term or other syntactic object o, we denote by $\mathit{vars}(o)$ the set of variables occurring in o and by $\mathit{occ}(v, o)$ the number of occurrences of v in o. When it does not cause ambiguities, we abuse the notation and prefer to use o itself in the place of $\mathit{vars}(o)$. For example, if t is a term and $x \in \mathcal{V}$, then $x \in t$ should be read as $x \in \mathit{vars}(t)$.

We denote with ϵ the empty substitution, by $\{x_1/t_1, \ldots, x_n/t_n\}$ a substitution θ with $\theta(x_i) = t_i \neq x_i$ and by $\mathrm{dom}(\theta)$ and $\mathrm{rng}(\theta)$ the domain and range of θ. Let $\mathit{vars}(\theta)$ be the set $\mathrm{dom}(\theta) \cup \mathrm{rng}(\theta)$ and, given $U \in \wp_f(\mathcal{V})$, let $\theta_{|U}$ be the projection of θ over U, i.e., the only substitution such that $\theta_{|U}(x) = \theta(x)$ if $x \in U$ and $\theta_{|U}(x) = x$ otherwise. Given θ_1 and θ_2 two substitutions with disjoint domains, we denote by $\theta_1 \uplus \theta_2$ the substitution θ such that $\mathrm{dom}(\theta) = \mathrm{dom}(\theta_1) \cup \mathrm{dom}(\theta_2)$ and $\theta(x) = \theta_i(x)$ if $x \in \mathrm{dom}(\theta_i)$, for each $i \in \{1, 2\}$. The application of a substitution θ to a term t is written as $t\theta$ or $\theta(t)$. Given two substitutions θ and δ, their composition, denoted by $\theta \circ \delta$, is given by $(\theta \circ \delta)(x) = \theta(\delta(x))$. A substitution ρ is called renaming if it is a bijection from $\mathcal{V}$ to $\mathcal{V}$ (this is equivalent to say that there exists a substitution ρ^{-1} such that $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho = \epsilon$). Instantiation induces a preorder on substitutions: θ is more general than δ, denoted by $\delta \leq \theta$, if there exists σ such that $\sigma \circ \theta = \delta$. If $\approx$ is the equivalence relation induced by $\leq$, we say that σ and θ are equal up to renaming when $\sigma \approx \theta$. The set of substitutions, idempotent substitutions and renamings are denoted by *Subst*, *ISubst* and *Ren* respectively. Given a set of equations E, we write $\sigma = \mathrm{mgu}(E)$ to denote that σ is a most general unifier of E. Any idempotent substitution σ is a most general unifier of the corresponding set of equations $\mathrm{Eq}(\sigma) = \{x = \sigma(x) \mid x \in \mathrm{dom}(\sigma)\}$. In the following, we will abuse the notation and denote by $\mathrm{mgu}(\sigma_1, \ldots, \sigma_n)$, when it exists, the substitution $\mathrm{mgu}(\mathrm{Eq}(\sigma_1) \cup \ldots \cup \mathrm{Eq}(\sigma_n))$.

Let $\mathbb{N}^+$ be the set of natural numbers without zero. A *position* is a sequence of natural numbers without zero. If we denote with Ξ the set of positions, we have $\Xi = (\mathbb{N}^+)^*$. Given a term t and a position ξ, we define $t(\xi)$ inductively as follows:

$$t(\epsilon) = t \qquad \text{(where } \epsilon \text{ denotes the empty sequence)}$$
$$t(i \cdot \xi') = \begin{cases} t_i(\xi') & \text{if } t \equiv f(t_1, \ldots, t_n) \text{ and } i \leq n; \\ \text{undefined} & \text{otherwise} \end{cases}$$

For any variable x, an *occurrence* of x in t is a position ξ such that $t(\xi) = x$.

In the rest of the paper, we use: U, V, W to denote finite sets of variables; h, k, u, v, w, x, y, z for variables; c, s, t for term symbols or terms; a, b for constants; $\eta, \theta, \sigma, \delta$ for substitutions; ρ for renamings. The same holds for derivatives of these symbols, obtained by adding subscripts, superscripts or both.

2.2 Multisets

A *multiset* is a map $X : \mathcal{X} \to \mathbb{N}^+$ where $\mathcal{X}$ is a set, called the *support* of X, denoted also by $\llbracket X \rrbracket$. With $\{\!\{v_1, \ldots, v_n\}\!\}$, where $v_1, \ldots, v_n$ is a sequence with (possible) repetitions, we denote a multiset X with support $\{v_1, \ldots, v_n\}$ and $X(v) = |\{i \mid v_i = v\}|$. We also use the polynomial notation $v_1^{i_1} \ldots v_n^{i_n}$, where $v_1, \ldots, v_n$ is a sequence without repetitions, to denote a multiset X with support $\{v_1, \ldots, v_n\}$ such that $X(v_k) = i_k$ for each $k \in \{1, \ldots, n\}$. We use $\{\!\{\}\!\}$ for the multiset whose support is the empty set. Finally, $|X|$ is the cardinality of the set $\{\langle x, i\rangle \mid x \in \llbracket X \rrbracket \wedge i \in \{1, \ldots, X(x)\}\}$. When $\llbracket X \rrbracket$ is finite, $|X|$ boils down to $\sum_{x \in \llbracket X \rrbracket} X(x)$. Note that, formally, a map $X : \mathcal{X} \to \mathbb{N}$ is not a multiset since it may be the case that $X(v) = 0$ for some $v \in \mathcal{X}$. However, to ease notation, we regard this map as the multiset $X|_{\{v \in \mathcal{X} \mid X(v) \neq 0\}}$. In the same way, given the multiset $X : \mathcal{X} \to \mathbb{N}^+$, we assume $X(v) = 0$ for any $v \notin \mathcal{X}$.

The collection of all the multisets is not a set in standard set theories. However, we denote with $\wp_m(\mathcal{X})$ the set of all the multisets whose support is *any finite subset* of $\mathcal{X}$. For example, both a^2c^4 and $a^1b^2c^3$ are elements of $\wp_m(\{a, b, c\})$.

The standard operations on sets, namely *union* and *intersection*, may be lifted to multisets. The same holds for the *subset* relation. We obtain:

$$A \cup B = \lambda v \in \llbracket A \rrbracket \cup \llbracket B \rrbracket.\, \max(A(v), B(v)) \ ,$$
$$A \cap B = \lambda v \in \llbracket A \rrbracket \cap \llbracket B \rrbracket.\, \min(A(v), B(v)) \ ,$$
$$A \subseteq B \iff \llbracket A \rrbracket \subseteq \llbracket B \rrbracket \wedge \forall v \in \llbracket A \rrbracket.\ A(v) \leq B(v) \ .$$

In particular, as for sets, it holds that $A \subseteq B \iff A \cap B = A \iff A \cup B = B$. The new fundamental operation for multisets is the *sum*, defined as

$$A \uplus B = \lambda v \in \llbracket A \rrbracket \cup \llbracket B \rrbracket . A(v) + B(v) \ .$$

Multiset sum is associative, commutative and $\{\!\{\}\!\}$ is the neutral element. Note that we also use $\uplus$ to denote disjoint union for standard sets. The context will allow us to identify the proper semantics of $\uplus$.

For the sake of notation, given any multiset A and natural number n, we denote by nA the multiset $\lambda v \in \llbracket A \rrbracket .\ nA(v)$ obtained by taking n copies of each element in A. Given $X : \mathcal{X} \to \mathbb{N}^+$ and $\mathcal{Y} \subseteq \mathcal{X}$, the *restriction* of X over $\mathcal{Y}$, denoted by $X_{|\mathcal{Y}}$, is the only multiset Y such that $\llbracket Y \rrbracket = \mathcal{Y}$ and $X(v) = Y(v)$ for each $v \in \mathcal{Y}$. Finally, if $A \in \wp_m(\mathcal{X})$ and $E[x]$ is a numeric expression when $x \in \mathcal{X}$, we define

$$\sum_{x \in A} E[x] = \sum_{x \in \llbracket A \rrbracket} A(x) \cdot E[X] \ .$$

According to this definition, we have $|A| = \sum_{x \in A} 1$.

2.3 Abstract Interpretation

Given two sets C and A of concrete and abstract objects respectively, an *abstract interpretation* [11] is given by an approximation relation $\rhd \subseteq A \times C$. When $a \rhd c$ holds, this means that a is a correct abstraction of c. In particular, we are interested in the case when $(A, \leq_A)$ is a poset and $a \leq_A a'$ means that a is more precise than a'. In this case we require that, if $a \rhd c$ and $a \leq_A a'$, then $a' \rhd c$ too. More in detail, we require what [11] calls the *existence of a best abstract approximation assumption*, i.e., the existence of a map $\alpha : C \to A$ such that for all $a \in A, c \in C$, it holds that $a \rhd c \iff \alpha(c) \leq_A a$. The map α is called the *abstraction function* and maps each c to its best approximation in A.

Given a (possibly partial) function $f : C \to C$, we say that $\tilde{f} : A \to A$ is a correct abstraction of f, and we write $\tilde{f} \rhd f$, whenever

$$a \rhd c \Rightarrow \tilde{f}(a) \rhd f(c) \ ,$$

assuming that $\tilde{f}(a) \rhd f(c)$ is true whenever $f(c)$ is not defined. We say that $\tilde{f} : A \to A$ is the *optimal* abstraction of f when it is the best correct approximation of f, i.e., when $\tilde{f} \rhd f$ and

$$\forall f' : A \to A.\ f' \rhd f \Rightarrow \tilde{f} \leq_{A \to A} f' \ .$$

In some cases, we prefer to deal with a stronger framework, in which the domain C is also endowed with a partial order $\leq_C$ and $\alpha : C \to A$ is a left adjoint to $\gamma : A \to C$, i.e.,

$$\forall c \in C. \forall a \in A. \alpha(c) \leq_A a \iff c \leq_C \gamma(a) \ .$$

The pair $\langle \alpha, \gamma \rangle$ is called a *Galois connection*. In particular, we will only consider the case of *Galois insertions*, which are Galois connections such that $\alpha \circ \gamma$ is the identity map. If $\langle \alpha, \gamma \rangle$ is a Galois insertion and $f : C \to C$ is a monotone map, the optimal abstraction $\tilde{f}$ always exists and it is definable as $\tilde{f} = \alpha \circ f \circ \gamma$.

3 The domain of existential substitutions

The choice of the concrete domain depends on the observable properties we are interested in analyzing. Most of the semantics suited for program analysis are based on computed answer substitutions, and most of the domains are expressed as abstractions of sets of substitutions. In general, we are not really interested in the substitutions, but in their quotient-set w.r.t. an appropriate equivalence relation. Let us consider a one-clause program $\mathtt{p(x,x)}$, the goal $p(x, y)$, and the following answer substitutions: $\theta_1 = \{y/x\}$, $\theta_2 = \{x/y\}$, $\theta_3 = \{x/u, y/u\}$ and $\theta_4 = \{x/v, y/v\}$. Although θ_1 and θ_2 are equal up to renaming, the same does not hold for θ_3 and θ_4. Nonetheless, they essentially represent the same answer, since u and v are just two different variables we chose when renaming apart the clause $\mathtt{p(x,x)}$ from the goal $p(x, y)$, and therefore are not relevant to the user. On the other side, if θ_3 and θ_4 are answer substitutions for the goal $q(x, y, u)$, then they correspond to computed answers $q(u, u, u)$ and $q(v, v, u)$ and therefore are fundamentally different. As a consequence, the equivalence relation we need to consider must be coarser then renaming, and must take into account the set of variables of interest, i.e., the set of variables which appear in the goal. For these reasons, we think that the best solution is to use a domain of equivalence classes of substitutions. Among the various domains proposed in the literature (e.g. [16, 22, 21]), we will adopt the domain of existential substitutions [2], since it is explicitly defined as quotient set of substitutions, w.r.t. a suitable equivalence relation. Moreover, the domain is equipped with all the necessary operators for defining a denotational semantics, namely projection, renaming and unification. We briefly recall the basic definitions of the domain and the unification operator.

Given $\theta_1, \theta_2 \in \mathit{Subst}$ and $U \in \wp_f(\mathcal{V})$, the preorder $\preceq_U$ is defined as follows:

$$\theta_1 \preceq_U \theta_2 \iff \exists \delta \in \mathit{Subst}. \forall v \in U.\ \theta_1(v) = \delta(\theta_2(v)) \ . \tag{1}$$

Intuitively, $\theta_1 \preceq_U \theta_2$, when θ_1 is an instance of θ_2, provided we are interested in the variables in U only. The equivalence relation induced by the preorder $\preceq_U$ is given by:

$$\theta_1 \sim_U \theta_2 \iff \exists \rho \in \mathit{Ren}. \forall v \in U.\ \theta_1(v) = \rho(\theta_2(v)) \ . \tag{2}$$

This relation precisely captures the extended notion of renaming which is needed to work with computed answers substitutions.

Example 1. It is easy to check that $\{x/v, y/u\} \sim_{\{x,y\}} \epsilon$. It is enough to choose $\rho = \{x/v, v/x, y/u, u/y\}$. Note that $\sim_U$ is coarser than the standard equivalence relation $\approx$: there is no renaming ρ such that $\epsilon = \rho \circ \{x/v, y/u\}$. As it happens for $\preceq$, if we enlarge the set of variables of interest, not all equivalences between substitutions are preserved: for instance, $\{x/v, y/u\} \not\sim_{\{x,y,v\}} \epsilon$.

Let $\mathit{ISubst}_{\sim_U}$ be the quotient set of ISubst w.r.t. $\sim_U$. The domain $\mathit{ISubst}_\sim$ of *existential substitutions* is defined as the disjoint union of all the $\mathit{ISubst}_{\sim_U}$ for $U \in \wp_f(\mathcal{V})$, in formulas:

$$\mathit{ISubst}_\sim = \biguplus_{U \in \wp_f(\mathcal{V})} \mathit{ISubst}_{\sim_U} \ . \tag{3}$$

In the following we write $[\theta]_U$ for the equivalence class of θ w.r.t. $\sim_U$. The partial order $\preceq$ over $\mathit{ISubst}_\sim$ is given by:

$$[\theta]_U \preceq [\theta']_V \iff U \supseteq V \wedge \theta \preceq_V \theta' \ . \tag{4}$$

Intuitively, $[\theta]_U \preceq [\theta']_V$ means that θ is an instance of θ' w.r.t. the variables in V, provided that they are all variables of interest of θ.

To ease notation, we often omit braces from the sets of variables of interest when they are given extensionally. So we write $[\theta]_{x,y}$ instead of $[\theta]_{\{x,y\}}$ and $\sim_{x,y,z}$ instead of $\sim_{\{x,y,z\}}$. When the set of variables of interest is clear from the context or it is not relevant, it will be omitted. Finally, we omit the braces which enclose the bindings of a substitution when it occurs inside an equivalence class, i.e., we write $[x/y]_U$ instead of $[\{x/y\}]_U$.

Unification Given $U, V \in \wp_f(\mathcal{V})$, $[\theta_1]_U, [\theta_2]_V \in \mathit{ISubst}_\sim$, the most general unifier between these two classes is defined as the mgu of suitably chosen representatives, where variables not of interest are renamed apart. In formulas:

$$\mathrm{mgu}([\theta_1]_U, [\theta_2]_V) = [\mathrm{mgu}(\theta'_1, \theta'_2)]_{U \cup V} \tag{5}$$

where $\theta_1 \sim_U \theta'_1 \in \mathit{ISubst}$, $\theta_2 \sim_V \theta'_2 \in \mathit{ISubst}$ and $(U \cup \mathit{vars}(\theta'_1)) \cap (V \cup \mathit{vars}(\theta'_2)) \subseteq U \cap V$. The last condition is needed to avoid variables clashes between the chosen representatives θ'_1 and θ'_2. Moreover, mgu is the greatest lower bound of $\mathit{ISubst}_\sim$ ordered by $\preceq$.

Example 2. Let $\theta_1 = \{x/a, y/t(v_1, v_1, v_2)\}$ and $\theta_2 = \{y/t(a, v_2, v_1), z/b\}$. Then

$$\mathrm{mgu}([\theta_1]_{x,y}, [\theta_2]_{y,z}) = [\{x/a, y/t(a, a, v), z/b\}]_{x,y,z}$$

by choosing $\theta'_1 = \theta_1$ and $\theta'_2 = \{y/t(a, w, v), z/b\}$. In this case we have

$$\begin{aligned}&\{x/a, y/t(a, a, v), z/b\} \sim_{x,y,z} \\ &\qquad\qquad \mathrm{mgu}(\theta'_1, \theta'_2) = \{x/a, y/t(a, a, v), z/b, v_1/a, w/a, v_2/v\} \ .\end{aligned}$$

A different version of unification is obtained when one of the two arguments is an existential substitution, and the other one is a standard substitution. In this case, the latter argument may be viewed as an existential substitution where all the variables are of interest:

$$\mathrm{mgu}([\theta]_U, \delta) = \mathrm{mgu}([\theta]_U, [\delta]_{\mathit{vars}(\delta)}) \ . \tag{6}$$

Note that deriving the general unification in (5) from the special case in (6) is not possible. This is because there are elements in $\mathit{ISubst}_\sim$ which cannot be obtained as $[\delta]_{\mathit{vars}(\delta)}$ for any $\delta \in \mathit{ISubst}$ (see [2]).

This is the kind of unification which is more useful for giving the semantics of logic programs [2]. Therefore, the rest of the paper will be concerned with the problem of devising optimal abstract operators corresponding to (6), for three different abstract domains. We remember that unification is not the only operator needed to give semantics to logic programs: we also need projection, renaming and union. However, providing optimal abstract counterparts for the latter is generally a trivial task, and will not be considered here.

We want to conclude the section with a small remark about our choice of the concrete domain. By adopting existential substitutions and the corresponding notion of unification, the semantics definitions which are heavily based on renaming apart objects to avoid variable clashes are greatly simplified. This is because all the details concerning renamings are moved to the inner level of the semantic domain, where they are more easily manageable [16, 2].

4 The abstract domain $\mathtt{ShLin}^\omega$

In this section we define a new abstract domain $\mathtt{ShLin}^\omega$ which can be used to approximate $\mathit{ISubst}_\sim$. Since $\mathtt{ShLin}^\omega$ is infinite, in most of the cases it cannot be directly used for the analysis. It should be thought of as a general framework from which other domains can be easily derived by abstraction.

In this sense, $\mathtt{ShLin}^\omega$ closes the gap between the concrete domain of substitutions and the abstractions like $\mathtt{Sharing} \times \mathtt{Lin}$ or $\mathtt{ASub}$ which focus on the properties of aliasing and linearity of variables. Developing optimal operators for such abstract domains has been a difficult (and unsuccessful until now) task. In our opinion, this is because the gap between $\mathit{ISubst}_\sim$ and $\mathtt{Sharing} \times \mathtt{Lin}$ (or $\mathtt{ASub}$) is too wide and the effect of aliasing and linearity is difficult to grasp on the concrete side. On the opposite, $\mathtt{ShLin}^\omega$ is a domain for aliasing and linearity properties with a structure which has made it possible to develop clean and optimal abstract operators. From these, optimal operators for the simpler domains are easy to obtain, just by simple case analysis.

The idea underlying $\mathtt{ShLin}^\omega$ is to count the exact number of occurrences of the same variable in a term. In this way, it extends the standard domain $\mathtt{Sharing}$ by recording, for each $v \in \mathcal{V}$ and $\theta \in \mathit{ISubst}$, not only the set $\{w \in \mathcal{V} \mid v \in \theta(w)\}$ but the multiset $\lambda w \in \mathcal{V}.\mathit{occ}(v, \theta(w))$.

Definition 1 (ω-Sharing Group). *We call* ω-sharing group *a multiset of variables, i.e., an element of $\wp_m(\mathcal{V})$.*

Given a substitution θ and a variable $v \in \mathcal{V}$, we denote by $\theta^{-1}(v)$ the ω-sharing group $\lambda w \in \mathcal{V}.\mathit{occ}(v, \theta(w))$, which maps each variable w to the number of occurrences of v in $\theta(w)$.

Definition 2 (Correct Approximation). *Given $S \subseteq \wp_m(V)$, we say that $[S]_V$ correctly approximates a substitution $[\theta]_U$ when $V = U$ and for each $v \in \mathcal{V}$, $\theta^{-1}(v)|_U \in S$. We write $[S]_V \rhd [\theta]_U$.*

In other words, $[S]_U$ correctly approximates $[\theta]_U$ when S contains at least all the ω-sharing groups which may arise in θ, restricted to the variables U.

Theorem 1. *The relation $\rhd$ is well defined.*

Proof. It is enough to prove that $\{{\theta_1}^{-1}(v)|_U \mid v \in \mathcal{V}\} = \{{\theta_2}^{-1}(v)|_U \mid v \in \mathcal{V}\}$ when $\theta_1 \sim_U \theta_2$. Assume $\theta_1 \sim_U \theta_2$, then by definition of $\sim_U$ there exists a renaming ρ such that $\rho(\theta_1(u)) = \theta_2(u)$ for each $u \in U$. Given $S = \theta_1^{-1}(v)|_U$, if $w = \rho(v)$ we have $\theta_2^{-1}(w)|_U = \theta_1^{-1}(v)|_U = S$. This concludes the proof.

We may build a domain $\mathtt{ShLin}^\omega$ for ω-sharing groups, defined as

$$\mathtt{ShLin}^\omega = \{[S]_U \mid U \in \wp_f(\mathcal{V}), S \subseteq \wp_m(U), S \neq \emptyset \Rightarrow \{\!\!\{\}\!\!\} \in S\} \ , \tag{7}$$

and ordered by $[S_1]_{U_1} \leq_\omega [S_2]_{U_2}$ iff $U_1 = U_2$ and $S_1 \subseteq S_2$. The order relation corresponds to the approximation ordering, since bigger (w.r.t $\leq_\omega$) elements correctly approximates a larger number of substitutions than smaller elements. In order to simplify notation, in the following we often write an object $[\{B_1, \ldots, B_n\}]_U \in \mathtt{ShLin}^\omega$ as $[B_1, \ldots, B_n]_U$ by omitting the braces. Moreover, if $X \in \mathtt{ShLin}^\omega$, we write $B \in X$ in place of $X = [S]_U \wedge B \in S$.

We also define the abstraction for a substitution $[\theta]_U$ as

$$\alpha_\omega([\theta]_U) = [\{\theta^{-1}(v)|_U \mid v \in \mathcal{V}\}]_U \ . \tag{8}$$

This is the least element of $\mathtt{ShLin}^\omega$ which correctly approximates $[\theta]_U$. Note that by the proof of Theorem 1 immediately follows that α_ω is well defined, i.e., it does not depend from the choice of the representative for $[\theta]_U$.

Example 3. Given $\theta = \{x/t(y, u, u), z/y, v/u\}$ and $U = \{w, x, y, z\}$, we have $\theta^{-1}(u) = x^2vu$, $\theta^{-1}(y) = xyz$, $\theta^{-1}(z) = \theta^{-1}(v) = \theta^{-1}(x) = \{\!\!\{\}\!\!\}$ and $\theta^{-1}(s) = s$ for all the other variables (included w). Projecting over U we obtain $\alpha_\omega([\theta]_U) = [\{x^2, xyz, w, \emptyset\}]_U$.

Example 4. As promised, we show an element of $ISubst_\sim$, namely the existential substitution $[x/t(v, v), y/t(v, v)]_{x,y}$, which cannot be obtained as $[\delta]_{vars(\delta)}$ for any substitution δ. Note that if $B = \delta^{-1}(v)|_{vars(\delta)}$, then either $v \notin \mathrm{rng}(\delta)$ and $B = \{\!\!\{\}\!\!\}$ or $v \in \mathrm{rng}(\delta)$ and $B(v) = 1$. Therefore if $\{\!\!\{\}\!\!\} \neq B \in \alpha_\omega([\delta]_{vars(\delta)})$, there exists a variable v such that $B(v) = 1$. However, $\alpha([x/t(v, v), y/t(v, v)]_{x,y}) = [x^2y^2]_{x,y}$ and x^2y^2 does not satisfy the above property.

4.1 Multigraphs

In order to define an abstract unification operator over $\mathtt{ShLin}^\omega$, we need to introduce the concept of multigraph and the operation of flattening. We call (directed) *multigraph* a graph where multiple distinguished edges are allowed between nodes. We use the definition of multigraph which is customary in category theory [19].

Definition 3 (Multigraph). *A* multigraph *G is a tuple $\langle N_G, E_G, \mathrm{src}_G, \mathrm{tgt}_G\rangle$ where $N_G \neq \emptyset$ and E_G are the sets of* nodes *and* edges *respectively,* $\mathrm{src}_G : E_G \to N_G$ *is the* source *function which maps each edge to its starting node, and* $\mathrm{tgt}_G : E_G \to N_G$ *is the* target *function which maps each edge to its ending node.*

A labeled *multigraph G is a multigraph equipped with a* labelling *function $l_G : N_G \to L_G$ which maps each node to its* label *in the given set L_G.*

We write $e : n_1 \to n_2 \in G$ to denote the edge $e \in E_G$ such that $\mathrm{src}_G(e) = n_1$ and $\mathrm{tgt}_G(e) = n_2$. We also write $n_1 \to n_2 \in G$ to denote any edge $e \in E_G$ such that $\mathrm{src}_G(e) = n_1$ and $\mathrm{tgt}_G(e) = n_2$. Moreover, with $|n_1 \to n_2 \in G|$ we denote the cardinality of the set $\{e \in E_G \mid \mathrm{src}_G(e) = n_1 \wedge \mathrm{tgt}_G(e) = n_2\}$. In the notation above, we omit "$\in G$" whenever the multigraph G is clear from the context.

We call *in-degree* (respectively *out-degree*) of a node n the cardinality of the set $\{e \in E_G \mid \mathrm{tgt}(e) = n\}$ (respectively $\{e \in E_G \mid \mathrm{src}(e) = n\}$).

Given a multigraph G, a *path* π is a *non-empty* sequence of nodes $n_1 \dots n_k$ such that, for each $i \in \{1, \dots, k-1\}$, there is either and edge $n_i \to n_{i+1} \in G$ or an edge $n_{i+1} \to n_i \in G$. Nodes n_1 and n_k are the *endpoints* of π, and we say that π *connects* n_1 and n_k. A multigraph is *connected* when all pairs of nodes are connected by at least a path.

Definition 4 (Flattening of Multigraphs). *Given a family $\mathcal{G} = \{G_i\}_{i \in I}$ of multigraphs over the same set of nodes N and such that the sets E_{G_i} are pairwise disjoint, we define the* flattening *of $\mathcal{G}$ as the multigraph $\underline{\mathcal{G}} = \langle N, E, \mathrm{src}_{\mathcal{G}}, \mathrm{tgt}_{\mathcal{G}} \rangle$ where $E = \cup_{i \in I} E_i$ and $\mathrm{src}_{\mathcal{G}} : E \to N$ (respectively $\mathrm{tgt}_{\mathcal{G}}$) is the only map such that $\mathrm{src}_{\mathcal{G}}|_{E_i} = \mathrm{src}_{G_i}$ (respectively $\mathrm{tgt}_{\mathcal{G}}|_{E_i} = \mathrm{tgt}_{G_i}$).*

4.2 Abstract Unification

We need to find the abstract counterpart of mgu over $\mathtt{ShLin}^\omega$, i.e., an operation mgu_ω such that, if $[S]_U \rhd [\theta]_U$, then

$$\mathrm{mgu}_\omega([S]_U, \delta) \rhd \mathrm{mgu}([\theta]_U, \delta) \tag{9}$$

for each $\delta \in \mathit{ISubst}$. Note that we are looking for an abstract counterpart to the mixed unification in (6), where one of the two arguments is a plain substitution. This is the form which is better suited for analysis of logic programs, where existential substitutions are the denotations of programs while standard substitutions are the result of unification between goals and heads of clauses.

In particular, we would like to find mgu_ω to be the minimum element which satisfy the condition in (9), i.e., the *optimal* abstract counterpart of mgu. Observe that, fixed U, the subset of $\mathtt{ShLin}^\omega$ made of all the elements of the kind $[S]_U$ is a complete lattice w.r.t. $\leq_\omega$ with top element given by $[\wp_m(U)]_U$ and

$$\textstyle\bigwedge_\omega \{[S_i] \mid i \in I\} = \left[\bigcap_{i \in I} S_i\right]_U \ . \tag{10}$$

Moreover, the relation $\rhd$ is meet-preserving on the left, since if $[S_i]_U \rhd [\theta]_U$ for each $i \in I$, then $\bigwedge_\omega \{[S_i]_U \mid i \in I\} \rhd [\theta]_U$. Therefore, we may define the abstract mgu as follows

$$\mathrm{mgu}_\omega([S]_U, \delta) = \bigwedge\nolimits_\omega \left\{[S']_{U \cup V} \mid \forall [\theta]_U . [S]_U \rhd [\theta]_U \Rightarrow [S']_{U \cup V} \rhd \mathrm{mgu}([\theta]_U, \delta)\right\} \ , \tag{11}$$

and it will enjoy (9).

This definition is completely non-constructive. The rest of this section is devoted to provide an algorithmic method for computing $\mathrm{mgu}_\omega([S]_U, \delta)$. We begin to characterize the operation of abstract unification by means of graph theoretic notions. We first need to define the *multiplicity* of an ω-sharing group B in a term t as follows:

$$\chi(B,t) = \sum_{v \in \llbracket B \rrbracket} B(v) \cdot occ(v,t) = \sum_{v \in \mathcal{V}} B(v) \cdot occ(v,t) \ , \tag{12}$$

where in the last equality we are using the assumption that $B(v) = 0$ when $v \notin \llbracket B \rrbracket$. For instance, $\chi(x^3yz^4, t(x,y,f(x,y,z))) = 3 \cdot 2 + 1 \cdot 2 + 4 \cdot 1 = 12$. The meaning of the map χ is made clear by the following Proposition.

Proposition 1. *Given an ω-sharing group B, a substitution θ, a variable v and a term t such that $B = \theta^{-1}(v)$, then $\chi(B,t) = occ(v, \theta(t))$. The same holds if, given a set U of variables, $B = \theta^{-1}(v)|_U$ and $vars(t) \subseteq U$.*

Proof. The proof is by induction on the structure of the term t. If $t \equiv a$ is a constant, then $occ(v, \theta(a)) = occ(v, a) = 0$ which is equal to $\chi(B,a)$ since $occ(w,a) = 0$ for each $w \in \mathcal{V}$. If $t \equiv w$ is a variable, then $occ(v, \theta(w)) = \theta^{-1}(v)(w) = B(w)$. In the same time, $\chi(B,t) = B(w)$ since $occ(w,w) = 1$ and $occ(y,w) = 0$ for $y \not\equiv w$. For the inductive case, if $t \equiv f(t_1, \ldots, t_n)$, we have $occ(v,t) = \sum_{i=1}^{n} occ(v,t_i) = \sum_{i=1}^{n} \chi(B,t_i)$ by inductive hypothesis. Moreover $\chi(B,t) = \sum_{v \in \llbracket B \rrbracket}(B(v) \cdot \sum_{i=1}^{n} occ(v,t_i)) = \sum_{i=1}^{n} \sum_{v \in \llbracket B \rrbracket} B(v) \cdot occ(v,t_i) = \sum_{i=1}^{n} \chi(B,t_i)$. The special case when we fix a set U of variables of interest is trivial.

Example 5. Assume $B = xy^2z^3$ and $\theta = \{y/r(x,x), z/r(x,x,x)\}$, in such a way that $\theta^{-1}(x) = \{xy^2z^3\}$. Given $t \equiv s(x,z)$ we have

$$occ(x, \theta(t)) = occ(x, s(x, r(x,x,x))) = 4 \ ,$$

and

$$\chi(B,t) = B(x) occ(x,t) + B(z) occ(z,t) = 1 \cdot 1 + 3 \cdot 1 = 4 \ .$$

If $[S]_U \rhd [\theta]_U$ and we unify $[\theta]_U$ with δ, some of the ω-sharing groups in S may be glued together to obtain a bigger resultant group.

It happens that the gluing of the sharing groups during the unification process of $[\theta]_U$ with a single binding substitution $\{x/t\}$ may be represented by the edges of a multigraph labelled with elements in S. Not all these graphs are relevant for a given $[S]_U$ and a binding x/t. Those which actually represents possible ways in which the sharing groups in S are merged will be called *sharing graphs*. They are defined using the multiplicity of ω-sharing groups in terms as introduced before.

Given any labelled multigraph G, in the rest of the paper we assume that the codomain of the labelling function l_G is $\wp_m(\mathcal{V})$, the set of ω-sharing groups.

Definition 5 (Sharing Graph). *A* sharing graph *for the binding x/t and a set of ω-sharing groups S is a labelled multigraph G such that*

1. *G is connected;*
2. *for each node $s \in N_G$, $l_G(s) \in S$;*
3. *for each node $s \in N_G$, the out-degree of s is equal to $\chi(l_G(s), x)$ and the in-degree of s is equal to $\chi(l_G(s), t)$.*

Given a labelled multigraph G, we define the *resultant ω-sharing group* of G as

$$res(G) = \biguplus_{s \in N_G} l_G(s) \ . \tag{13}$$

The set of resultant ω-sharing groups for the set S of ω-sharing groups and the binding x/t is given by

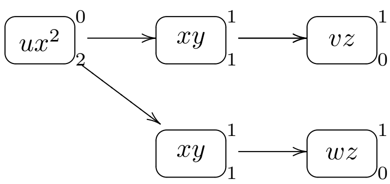
$$\mathrm{mgu}_\omega(S, \{x/t\}) = \{res(G) \mid G \text{ is a sharing graph for } S \text{ and } x/t\} \ . \tag{14}$$

It is possible to lift mgu_ω to an operation over $\mathtt{ShLin}^\omega$. What we obtain is a particular case of the abstract unification operator we are looking for. If $[S]_U \in \mathtt{ShLin}^\omega$ and $vars(\{x/t\}) \subseteq U$, we define

$$\mathrm{mgu}_\omega([S]_U, \{x/t\}) = [\mathrm{mgu}(S, \{x/t\})]_U \ . \tag{15}$$

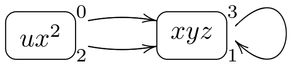
We give here an intuition of the way sharing graphs work. Assume given $[S]_U \rhd [\theta]_U$ and a binding x/t with $vars(\{x/t\}) \subseteq U$. We want to compute $\alpha_\omega(\mathrm{mgu}([\theta]_U, \{x/t\}))$. For each $j \in \{1, 2\}$, let $B_j \in S$, i.e., there exist $v_j \in \mathcal{V}$ such that $B_j = \theta^{-1}(v_j)|_U$. When unifying θ with the binding x/t, we know that $\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \{x = t\}) = \mathrm{mgu}(\{\theta(x) = \theta(t)\}) \circ \theta$ and that $\theta(x)$ (respectively $\theta(t)$) contains $\chi(B_j, x)$ (respectively $\chi(B_j, t)$) instances of v_j by Prop. 1. Assume that $\theta(x)$ and $\theta(t)$ only differs for the variables occurring in them. Then, an arrow from the sharing group B_1 to B_2 represents the fact that, in $\mathrm{mgu}(\{\theta(x) = \theta(t)\})$, one of the copies of v_1 is aliased with one of the copies of v_2, i.e., that there are corresponding positions in $\theta(x)$ and $\theta(t)$ where the two terms contain the variables v_1 and v_2 respectively. The third condition for sharing graphs implies that each occurrence of v_j is aliased to some other variable. Although here we are only considering the case when $\theta(x)$ and $\theta(t)$ differs for the variables occurring in them, we will show that it is enough to reach correctness and optimality.

Example 6. Let $S = \{ux^2, xy, vz, wz, xyz\}$ and $U = \{u, v, w, x, y, z\}$. The following is a sharing graph for $x = r(y, z)$ and S:



where pedices and apices on a sharing group B are respectively the value of $\chi(B, x)$ and $\chi(B, r(y, z))$. Therefore $uvwx^4y^2z^2 \in \mathrm{mgu}_\omega(S, \{x/r(y, z)\})$.

Example 7. Let $S = \{ux^2, xy, vz, wz, xyz\}$ and $U = \{u, v, w, x, y, z\}$. The following is a sharing graph for $x = r(y, y, z)$ and S:



where pedices and apices on a sharing group B are respectively the value of $\chi(B, x)$ and $\chi(B, r(y, y, z))$. Therefore $ux^3yz \in \mathrm{mgu}_\omega(S, \{x/r(y, y, z)\})$. Note that this sharing group can actually be generated by the substitution $\theta = \{x/r(v_1, v_1, v_2), y/v_2, z/v_2, u/v_1, v/a, w/a\}$ where a is a ground term. It is the case that $[S]_U \rhd [\theta]_U$ and $\mathrm{mgu}([\theta]_U, \{x/r(y, y, z)\})$ performs exactly the variable alasings depicted by the sharing graph. Actually $\mathrm{mgu}([\theta]_U, \{x/r(y, y, z)\}) = [x/r(v_1, v_1, v_1), y/v_1, u/v_1, v/a, w/a]_U = [\eta]_U$ and $\eta^{-1}(v_1)|_U = ux^3yz$.

We define $\mathrm{mgu}_\omega([S]_U, \delta)$ with $\delta \in \mathit{ISubst}$ and $\mathit{vars}(\delta) \subseteq U$ by induction on the number of bindings:

$$\begin{aligned} \mathrm{mgu}_\omega([S]_U, \epsilon) &= [S]_U \\ \mathrm{mgu}_\omega([S]_U, \{x/t\} \uplus \delta) &= \mathrm{mgu}_\omega(\mathrm{mgu}_\omega([S]_U, \{x/t\}), \delta) \ . \end{aligned} \tag{16}$$

For the general case, when $\mathit{vars}(\delta) \not\subseteq U$, we extend the abstract object $[S]_U$ by adjoining informations regarding the variables in $\mathit{vars}(\delta) \setminus U$. We actually use the identity $\mathrm{mgu}([\theta]_U, \delta) = \mathrm{mgu}(\mathrm{mgu}([\theta]_U, [\epsilon]_{\mathit{vars}(\delta)}), \delta)$. When computing $\mathrm{mgu}([\theta]_U, [\epsilon]_{\mathit{vars}(\delta)})$ all the variables in $\mathit{vars}(\delta) \setminus U$ occurring in θ are renamed apart from δ itself. Therefore each $v \in \mathit{vars}(\delta) \setminus U$ is free (hence linear) in $\mathrm{mgu}([\theta]_U, [\epsilon]_{\mathit{vars}(\delta)})$, i.e., $\alpha_\omega(\mathrm{mgu}([\theta]_U, [\epsilon]_{\mathit{vars}(\delta)})) = [S \cup \{\{\!\{v\}\!\} \mid v \in \mathit{vars}(\delta) \setminus U\}]_{U \cup \mathit{vars}(\delta)}$. Finally, we obtain

$$\mathrm{mgu}_\omega([S]_U, \delta) = \mathrm{mgu}_\omega([S \cup \{\{\!\{v\}\!\} \mid v \in \mathit{vars}(\delta) \setminus U\}]_{U \cup \mathit{vars}(\delta)}, \delta) \ . \tag{17}$$

Note that, for a generic abstract domain, the method of extending the abstract object to include all the variables in the concrete substitution δ may result in a non-optimal abstract unification. For example, this is what happens in the case of the domain `Sharing`, as shown in [1]. However, we will prove that, in the case of $\mathtt{ShLin}^\omega$, the abstract mgu in (17) is optimal.

4.3 Correctness of Abstract Unification

If θ is a substitution, we extend the definition of θ^{-1} to the case when it is applied to a sharing group B. In formulas:

$$\theta^{-1}(B) = \lambda v \in \mathcal{V}. \chi(B, \theta(v)) \ . \tag{18}$$

In order to prove the correctness of abstract unification, we need the following property:

Proposition 2. *Given substitutions θ, η and a sharing group B, it is the case that*

$$(\eta \circ \theta)^{-1}(B) = \theta^{-1}(\eta^{-1}(B)) \ .$$

Proof. Using the definitions and simple algebraic manipulations, we have

$$
\begin{aligned}
&\theta^{-1}(\eta^{-1}(B)) \\
&= \lambda w.\chi\big(\lambda v.\chi(B,\eta(v)),\theta(w)\big) \\
&= \lambda w.\sum_y \chi(B,\eta(y))\cdot \mathit{occ}(y,\theta(w)) \\
&= \lambda w.\sum_y \left(\sum_x B(x)\cdot \mathit{occ}(x,\eta(y))\right)\cdot \mathit{occ}(y,\theta(w)) \\
&= \lambda w.\sum_x B(x)\cdot \sum_y \mathit{occ}(x,\eta(y))\cdot \mathit{occ}(y,\theta(w)) \\
&= \lambda w.\sum_x B(x)\cdot \sum_y \eta^{-1}(x)(y)\cdot \mathit{occ}(y,\theta(w)) \\
&= \lambda w.\sum_x B(x)\cdot \chi(\eta^{-1}(x),\theta(w))
\end{aligned}
$$

By Prop. 1, we have that $\chi(\eta^{-1}(x),\theta(w)) = \mathit{occ}(x,\eta(\theta(w))$ and therefore

$$\theta^{-1}(\eta^{-1}(B)) = (\eta\circ\theta)^{-1}(B) \enspace .$$

Theorem 2. *The operation* mgu_ω *is correct w.r.t.* mgu.

Proof. Assume $[S]_U \rhd [\theta]_U$ and $\delta \in \mathit{ISubst}$. We need to prove that $\mathrm{mgu}_\omega([S]_U,\delta) \rhd \mathrm{mgu}([\theta]_U,\delta)$ or the equivalent property $\alpha_\omega(\mathrm{mgu}([\theta]_U,\delta)) \leq_\omega \mathrm{mgu}_\omega([S]_U,\delta)$. Note that $\mathrm{mgu}([\theta]_U,\delta) = \mathrm{mgu}(\mathrm{mgu}([\theta]_U,[\epsilon]_{\mathit{vars}(\delta)}),\delta)$ and

$$\alpha_\omega(\mathrm{mgu}([\theta]_U,[\epsilon]_{\mathit{vars}(\delta)})) = [S \cup \{\{\!\{v\}\!\} \mid v \in \mathit{vars}(\delta)\setminus U\}]_{U\cup \mathit{vars}(\delta)} \enspace .$$

Therefore, since mgu_ω is defined inductively on the number of bindindings in δ, it is enough to prove $\mathrm{mgu}_\omega([S]_U,\delta) \rhd \mathrm{mgu}([\theta]_U,\delta)$ when $\delta = \{x/t\}$ and $\mathit{vars}(\delta) \subseteq U$. Let B be a sharing group in $\alpha_\omega(\mathrm{mgu}([\theta]_U,\{x/t\}))$, we prove that $B \in \mathrm{mgu}_\omega([S]_U,\{x/t\})$.

Without loss of generality, we assume $\mathrm{dom}(\theta) = U$. Let $\theta' = \mathrm{mgu}(\theta,\{x/t\}) = \eta\circ\theta$ with $\eta = \mathrm{mgu}(\{\theta(x) = \theta(t)\})$ and we have $[\theta']_U = \mathrm{mgu}([\theta]_U,[x/t]_U)$. Due to the assumption that $\mathrm{dom}(\theta) = U$, we have that $\mathit{vars}(\eta)\cap U = \emptyset$. Consider η' obtained from η by replacing each occurrence of a variable in $\mathrm{rng}(\eta)$ with a different fresh variable. This means there exists a $\rho \in \mathit{Subst}$ mapping variables to variables such that $\rho(\eta'(x)) = \eta(x)$ for each $x \in \mathrm{dom}(\eta)$. In formulas, we have

$$\rho = \{v_1/v_2 \mid \exists x \in \mathrm{dom}(\eta), \xi \in \Xi \text{s.t.} \eta'(x)(\xi) = v_1 \wedge \eta(x)(\xi) = v_2\} \enspace .$$

Note that $\beta = \eta'\circ\theta$ has the property that $[S]_U \rhd [\beta]_U$. Actually, assuming $C \in \alpha([\beta]_U)$, we may distinguish three cases:

- $C = \{\!\{\}\!\}$. In this case $C \in S$ by definition of $\mathtt{ShLin}^\omega$;
- $C = \beta^{-1}(w)|_U$ for $w \in \mathrm{rng}(\theta)\setminus\mathrm{dom}(\eta)$. In this case $\mathit{occ}(w,(\eta'\circ\theta)(v)) = \mathit{occ}(w,\theta(v))$ for each $v \in \mathcal{V}$, therefore $\beta^{-1}(w)|_U = \theta^{-1}(w)|_U \in S$;

– $C = \beta^{-1}(w)|_U$ for $w \in \mathrm{rng}(\eta')$. Hence there exists $v \in \mathrm{rng}(\theta)$ such that $\mathit{occ}(w, \eta'(v)) = 1$ and $\mathit{occ}(w, \eta'(v')) = 0$ for each $v' \notin \{v', w\}$. Hence, for each $u \in U$, $\mathit{occ}(w, \eta'(\theta(u))) = n$ iff $\mathit{occ}(v, \theta(u)) = n$ and this implies $C = \theta^{-1}(v)|_U \in S$.

Moreover $\rho(\beta(u)) = \theta'(u)$ for each $u \in U$, therefore $\theta' \sim_U \rho \circ \beta$.

Consider the labelled multigraph G such that $N_G = \{v \mid v \in \mathit{vars}(\beta(U))\}$, $l_G(v) = \beta^{-1}(v)|_U \in S$ and $E_G = \{\xi \mid \beta(x)(\xi) \in \mathcal{V}\}$. Note that if $\beta(x)(\xi) \in \mathcal{V}$, then $\beta(t)(\xi) \in \mathcal{V}$, too. Each position ξ in E_G is an arrow such that $\mathrm{src}_G(\xi) = \beta(x)(\xi)$ and $\mathrm{tgt}_G(\xi) = \beta(t)(\xi)$. Observe that the second condition in the definition of sharing graph for S and x/t is satisfied, since $[S]_U \rhd [\beta]_U$.

Let us check the third condition. For each node $v \in N_G$, if $\chi(\beta^{-1}(v)|_U, x) = n$ by Prop. 1 we have $\mathit{occ}(v, \beta(x)) = n$, i.e., there are n positions in $\beta(x)$ corresponding to v. Therefore the outdegree of v is n. In the same way, we have that $\chi(\beta^{-1}(v)|_U, t)$ is the in-degree of v.

Given $B = \theta'^{-1}(u)_{|U}$, by Prop. 2 we have $B = \beta^{-1}(\rho^{-1}(u))_{|U}$. Since $\theta' \preceq_U \beta \preceq_U \theta$, then $[\theta']_U = \mathrm{mgu}([\beta]_U, \{x/t\}) = [\mathrm{mgu}(\beta, \{x/t\})]_U$. Therefore $\rho \circ \beta' \sim_U \theta' = \mathrm{mgu}(\theta, \{x/t\}) \sim_U \mathrm{mgu}(\beta, \{x/t\}) = \mathrm{mgu}(\beta(x) = \beta(t)) \circ \beta$. We call δ the result of $\mathrm{mgu}(\beta(x) = \beta(t))$, and note that $\beta(x) = \beta(t)$ is equivalent to the set of equations $X = \{v_1 = v_2 \mid \text{there is a position } \xi \text{ such that } \beta(x)(\xi) = v_1 \wedge \beta(t)(\xi) = v_2\}$. The relation $\rho \circ \beta \sim_U \delta \circ \beta$ means that, if $w_1, w_2 \in \beta(U)$ and $\rho(w_1) = \rho(w_2)$ then $\delta(w_1) = \delta(w_2)$. The latter implies that there are in X equations of the kind $x_1 = x_2$, $x_2 = x_3$, $\ldots$, $x_{n-1} = x_n$ with $x_1 = w_1$ and $x_n = w_2$, i.e., that w_1 and w_2 are connected in the graph G.

Therefore, let $Y = \{w \mid \rho(w) = u\} = \llbracket \rho^{-1}(u) \rrbracket$. If ξ is an edge such that $\mathrm{src}_G(\xi) \in Y$, then $\mathrm{tgt}_G(\xi) \in Y$, since $\beta(x)(\xi) = \beta(t)(\xi) \in X$. The converse also holds. Hence, if we restrict the graph G to the set of nodes Y, we obtain a sharing graph whose resultant ω-sharing group is $\biguplus_{w \in Y} \beta^{-1}(w)|_U = \beta^{-1}(\rho^{-1}(u))_{|U} = B$.

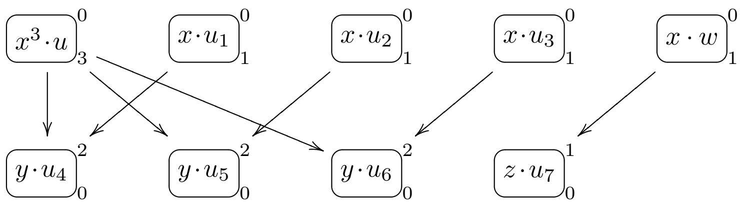
Example 8. Let us consider $\theta = \{x/t(s(u,u),v,w), y/v', z/w'\}$, $U = \{x,y,z\}$ and $\delta = \{x/t(y,z,z)\}$. Therefore $\alpha_\omega([\theta]_U) = [x^2, x, y, z]_U$. If we proceed with the concrete unification of $[\theta]_U$ with δ, we have $\mathrm{mgu}([\theta]_U, \delta) = [\theta']_U$ with $\theta' = \mathrm{mgu}(\theta, \delta) = \eta \circ \theta$ and $\eta = \mathrm{mgu}(\theta(x) = \theta(t(y,y,z)))$. This gives the following results:

$$\eta = \{v'/s(u,u,u), v/s(u,u,u), w'/w\} \ ,$$
$$\theta' = \{x/t(s(u,u,u), s(u,u,u), w), y/s(u,u,u), z/w, v'/s(u,u,u), w'/w\} \ ,$$

with $[\theta']_U = [\theta]_U$. Now, let η' obtained from η by replacing each occurrence of a variable in $\mathrm{rng}(\eta)$ with a different fresh variable, $\beta = \eta' \circ \theta$ and ρ a substitution mapping variables to variables such that $\rho(\beta(x)) = \theta'(x))$ for each $x \in U$. We have:

$$\eta = \{v/s(u_1,u_2,u_3), v'/s(u_4,u_5,u_6), w'/u_7\}$$
$$\beta = \{x/t(s(u,u,u), s(u_1,u_2,u_3), w), y/s(u_4,u_5,u_6), z/u_7, v'/s(u_4,u_5,u_6), w'/u_7\}$$
$$\rho = \{u_1/u, u_2/u, u_3/u, u_4/u, u_5/u, u_6/u, u_7/w\} \ .$$

Following the proof, we build a multigraph G as follows:



Note that we have chosen to annotate every node of the multigraph with the corresponding variable in $vars(\beta(U))$. This is not a sharing graph since it is not connected, but if we take $Y = \llbracket \rho^{-1}(u) \rrbracket = \{u, u_1, u_2, u_3, u_4, u_5, u_6\}$, the restriction of G to the nodes annotated with a variable in Y is a sharing graph whose resultant ω-sharing group is $x^6 y^3$.

4.4 Parallel Abstract Unification

We would like to prove that mgu_ω is not only correct, but also optimal w.r.t. the concrete mgu, i.e., it is the least correct abstraction. This means proving that, given $[S]_U \in \mathtt{ShLin}^\omega$, $\delta \in \mathit{ISubst}$, and $B \in \mathrm{mgu}_\omega([S]_U, \delta)$, there exists $[\theta]_U$ such that $[S]_U \rhd [\theta]_U$ and $B \in \alpha_\omega(\mathrm{mgu}([\theta]_U, \delta))$.

It is quite easy to prove optimality of mgu_ω in the case when the second argument is a single binding substitution. However, proving optimality in the general case is much more difficult. We pursue this goal in three steps:

1. we define a new operator which computes the abstract unification with a multi-binding substitution in one step. This is based on a generalization of the concept of sharing graph with multiple layers. For this reason, we speak of *parallel sharing graph* and *parallel abstract unification*;
2. we prove that parallel abstract unification is actually the same as mgu_ω (which we may call *sequential abstract unification*);
3. we prove that the parallel abstract unification is optimal w.r.t. standard unification.

Definition 6 (Parallel sharing graph). *A* parallel *sharing graph for a set of ω-sharing groups S and the idempotent substitution $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$ is a family $\mathcal{G} = \{G_i\}_{i \in [1,n]}$ of multigraphs over the same set of nodes $N_\mathcal{G}$, equipped with a labelling function $l_\mathcal{G} : N_\mathcal{G} \to S$ such that*

- *the set of edges E_{G_i} are all pairwise disjoint;*
- *for each node $s \in N_\mathcal{G}$ and each $i \in [1, n]$, the out-degree of s in G_i is equal to $\chi(l_\mathcal{G}(s), x_i)$ and the in-degree of s in G_i is equal to $\chi(l_\mathcal{G}(s), t_i)$;*
- *$\underline{\mathcal{G}}$ (the flattening of $\mathcal{G}$) is connected.*

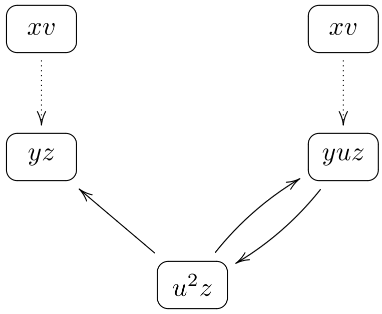
Definition 7 (Parallel abstract mgu). *Given a set of ω-sharing groups S and an idempotent substitution θ, the* abstract parallel unification *of S and θ is given by*

$$\mathrm{mgu}_\mathrm{P}(S, \theta) = \{res(\underline{\mathcal{G}}) \mid \mathcal{G} \text{ is a parallel sharing graph for } S \text{ and } \theta\} \ . \qquad (19)$$

This is lifted to the domain $\mathtt{ShLin}^{\omega}$*:*

$$\mathrm{mgu}_{\mathrm{P}}([S]_U, \theta) = [\mathrm{mgu}_{\mathrm{P}}(S \cup \{\{\!\!\{v\}\!\!\} \mid v \in \mathit{vars}(\theta) \setminus U, \theta)]_{U \cup \mathit{vars}(\theta)} \ . \tag{20}$$

Example 9. Let $S = \{yz, u^2z, xv, yuz\}$ and $\theta = \{x/y, u/t(z)\}$. The following is a parallel sharing graph for S and θ:

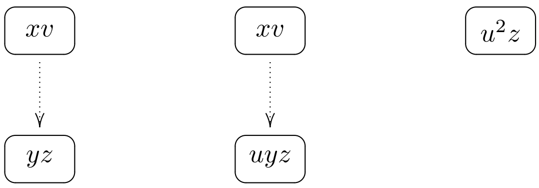


The dotted lines identify the edges relative to the binding x/y, while the single continuous lines refer to the binding u/z. The resultant sharing group is $x^2y^2z^3u^3v^2$.

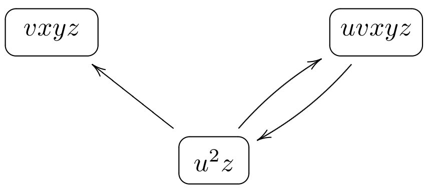
4.5 Coincidence of Parallel and Sequential Abstract Unification

This section is devoted to prove that sequential and parallel abstract unification do coincide. The idea is to show that parallel abstract unification may be executed sequentially without affecting the result.

Example 10. In the Example 9 we have shown a parallel sharing graph for $S = \{yz, u^2z, xv, yuz\}$ and $\theta = \{x/y, u/t(z)\}$. The same sharing group may be obtained by first computing $S' = \mathrm{mgu}_{\omega}(S, \{x/y\})$ and later $\mathrm{mgu}_{\omega}(S', \{u/t(z)\})$. The connected components given by the dotted arrows, i.e



are three sharing graphs for S and x/y. Therefore, $vxyz$, $uvxyz$ and u^2z are elements of S'. Now, in the graph given in the Example 9, we collapse these collected components and obtain



which is a sharing graph for S' and $y/t(z)$. We obtain what we were expecting, namely that $x^2y^2z^3u^3v^2 \in \mathrm{mgu}_\omega(S', y/t(z))$.

Lemma 1. $\mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\} \uplus \theta) = \mathrm{mgu}_\mathrm{P}(\mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\}), \theta)$

Proof. If $\theta = \epsilon$ the result follows easily since $\mathrm{mgu}_\mathrm{P}(S, \epsilon) = S$. In the case $\theta \neq \epsilon$, we prove the two sides of the equality separately.

Assume that $\theta = \{x_2/t_2, \ldots, x_l/t_l\}$ and let $B \in \mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\} \uplus \theta)$. We want to prove that $B \in \mathrm{mgu}_\mathrm{P}(\mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\}), \theta)$. By definition, there exists a parallel sharing graph $\mathcal{G} = \{G^i\}_{i \in [1,l]}$ such that $B = \mathit{res}(\underline{\mathcal{G}})$. We decompose G^1 in its connected components $G^1_1, \ldots, G^1_k$. Note that each G^1_j, labelled with the obvious restriction of $l_\mathcal{G}$, is a sharing graph for S and $x_1 = t_1$, therefore $\mathit{res}(G^1_j) \in \mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\})$.

We now show a parallel sharing graph $\mathcal{G}'$ for θ and $\mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\})$ such that $\mathit{res}(\underline{\mathcal{G}'}) = B$. This will prove that $B \in \mathrm{mgu}_\mathrm{P}(\mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\}), \theta)$. For any $i \in [2, l]$, we consider the multigraph G_i obtained from G^i by collapsing each of the connected components $G^1_1, \ldots G^1_k$ to a single node. Formally:

- $N_{G_i} = \{1, \ldots, k\}$;
- $E_{G_i} = E_{G^i}$;
- $\mathrm{src}_{G_i}(e) = j$ iff $\mathrm{src}_{G^i}(e) \in G^1_j$.
- symmetrically for tgt_{G_i};

We want to prove that $\mathcal{G}' = \{G_i\}_{i \in [2,l]}$ is a parallel sharing graph for θ and $\mathrm{mgu}_\mathrm{P}(S, \{x_1/t_1\})$, when endowed with the labelling function $l_{\mathcal{G}'}(j) = \mathit{res}(G^1_j)$.

It is immediate to check that the sets of edges E_{G_i} are pairwise disjoint. Given any node $j \in [1, k]$ we have that the out-degree of j in G_i is

$$\begin{aligned}
|\{e \in E_{G_i} \mid \mathrm{src}_{G_i}(e) = j\}| &= |\{e \in E_{G^i} \mid \mathrm{src}_{G^i}(e) \in G^1_j\}| \\
&= \sum_{n \in N_{G^1_j}} |\{e \in E_{G^i} \mid \mathrm{src}_{G^i}(e) = n\}| = \sum_{n \in N_{G^1_j}} \chi(l_\mathcal{G}(n), x_i) \\
&= \sum_{n \in N_{G^1_j}} \sum_{v \in \mathcal{V}} l_\mathcal{G}(n)(v) \cdot \mathit{occ}(v, x_i) = \sum_{v \in \mathcal{V}} \sum_{n \in N_{G^1_j}} l_\mathcal{G}(n)(v) \cdot \mathit{occ}(v, x_i) \\
&= \sum_{v \in \mathcal{V}} \mathit{res}(G^1_j)(v) \cdot \mathit{occ}(v, x_i) = \sum_{v \in \mathcal{V}} l_{\mathcal{G}'}(j)(v) \mathit{occ}(v, x_i) \\
&= \chi(l_{\mathcal{G}'}(j), x_i)
\end{aligned}$$

Simmetrically, we have that the in-degree of j in G_i is $\chi(l_{\mathcal{G}'}(j), t_i)$. The only thing we still need to prove is that $\underline{\mathcal{G}'}$ is connected.

Assume we want to find a path from i to j. Since $\underline{\mathcal{G}}$ is connected, there is a path π from some $n_1 \in N_{G^1_i}$ to some $n_2 \in N_{G^1_j}$. A path from i to j may be obtained in two steps:

1. replacing each node n in π with $\bar{n}$ where $\bar{n}$ is the only $j \in [1, k]$ such that $n \in N_{G^1_j}$;

2. replacing each subsequence $\bar{n}\bar{n}$ with a single node $\bar{n}$. Such a situation may appear when π contains the subsequence nm with $n \to m \in G_p^1$ for some p. The corresponding edge $p \to p$ may not exists in $\mathcal{G}'$, but being a self-loop it may be deleted.

Finally, we need to show that $res(\underline{\mathcal{G}'}) = B$. It is easy to check that $res(\underline{\mathcal{G}'}) = \biguplus_{i\in[1,k]} l_{\mathcal{G}'}(i) = \biguplus_{i\in[1,k]} res(G_i^1) = \biguplus_{i\in[1,k]} \biguplus_{n\in N_{G_i^1}} l_{\mathcal{G}}(n) = \biguplus_{n\in N_{\mathcal{G}}} l_{\mathcal{G}}(n) = res(\underline{\mathcal{G}})$.

On the other way around, let $S' = \mathrm{mgu}_{\mathrm{P}}(S, \{x_1/t_1\})$ and $B \in \mathrm{mgu}_{\mathrm{P}}(S', \theta)$ where $\theta = \{x_2/t_2, \ldots, x_l/t_l\}$. By definition, there is a parallel sharing graph $\mathcal{G} = \{G^i\}_{i\in[2,l]}$ such that $res(\underline{\mathcal{G}}) = B$. Since $S' = \mathrm{mgu}_{\mathrm{P}}(S, \{x_1/t_1\})$, for each node $k \in N_{\mathcal{G}}$ we have a sharing graph G_k such that $res(G_k) = l_{\mathcal{G}}(k)$. Without loss of generality, we may choose these graphs in such a way that the sets N_{G_k} are pairwise disjoint and disjoint from $N_{\mathcal{G}}$.

For any multigraph G^i, we build a new multigraph $\bar{G}^i$ obtained by replacing each node k in G^i with the set of nodes of the generating graph G_k. Formally:

- $N_{\bar{G}^i} = \bigcup_{k\in N_{\mathcal{G}}} N_{G_k}$;
- $E_{\bar{G}^i} = E_{G^i}$;
- $\mathrm{src}_{\bar{G}^i}$ is chosen in such a way that $\mathrm{src}_{\bar{G}^i}(e)$ is a node of N_{G_k} iff $\mathrm{src}_{G^i}(e) = k$;
- symmetrically for $\mathrm{tgt}_{\bar{G}^i}$.

Consider the labelling function $l : N_{\bar{G}^i} \to \wp_m(\mathcal{V})$ which is the disjoint union of all the l_{G_k}. In formulas, $l(n) = l_{G_k}(n)$ iff $n \in N_{G_k}$.

Note we may choose $\mathrm{src}_{\bar{G}^i}$ and $\mathrm{tgt}_{\bar{G}^i}$ in such a way that the out-degree and the in-degree of each node n in $\bar{G}^i$ are $\chi(l(n), x_i)$ and $\chi(l(n), t_i)$. This is always possible since $l_{\mathcal{G}}(k) = res(G_k) = \uplus_{n\in N_{G_k}} l_{G_k}(n)$ and therefore $\chi(l_{\mathcal{G}}(k), x_i)) = \sum_{n\in N_{G_k}} \chi(l_{G_k}(n), x_i)$ (the same holds for t_i).

Finally, we define $\bar{G}^1$ as the union of the graphs G_k. We now want to prove that $\mathcal{G}' = \{\bar{G}^i\}_{i\in[1,l]}$ with the labelling function l is a parallel sharing graph for $\{x_1/t_1, \ldots, x_l/t_l\}$ and S. The only thing we need to prove is the fact that $\underline{\mathcal{G}'}$ is connected.

Assume there is an edge $i \to j$ in G^k, and consider nodes $n_i \in N_{G_i}$ and $n_j \in N_{G_j}$. We may prove there is a path in $\underline{\mathcal{G}'}$ from n_i to n_j. Actually, we have in $\bar{G}^k$ at least an edge $m_i \to m_j$ from a node $m_i \in N_{G_i}$ to $m_j \in N_{G_j}$. Since G_i and G_j are connected, there are in $\bar{G}^1$ two paths $\pi : n_i \to m_i$ and $\pi' : m_j \to n_j$. Therefore $\pi\pi'$ is a path in $\underline{\mathcal{G}'}$ from n_i to n_j.

Now, given two generic nodes n_i, n_j where $n_i \in N_{G_i}$ and $n_j \in N_{G_j}$, we know there is a path π in $\underline{\mathcal{G}}$ from i to j. Applying the result of the previous paragraph to each edge in π, we immediately get that n_i and n_j are connected.

Finally it is easy to check that $res(\underline{\mathcal{G}'}) = B$ and this concludes the proof of the theorem.

Theorem 3. *The abstract operators* mgu_ω *and* $\mathrm{mgu}_{\mathrm{P}}$ *do coincide.*

Proof. We now show that mgu_ω and $\mathrm{mgu}_{\mathrm{P}}$ do coincide. First note that, by construction, mgu_ω and $\mathrm{mgu}_{\mathrm{P}}$ coincide for a single binding, that is to say that

$\mathrm{mgu}_\omega(S, \{x/t\}) = \mathrm{mgu}_\mathrm{P}(S, \{x/t\})$. In the general case, the proof is by induction on the number of bindings in θ. Clearly $\mathrm{mgu}_\omega(S, \epsilon) = S = \mathrm{mgu}_\mathrm{P}(S, \epsilon)$. Assume that $\mathrm{mgu}_\omega(S, \theta) = \mathrm{mgu}_\mathrm{P}(S, \theta)$ for each S. It follows that

$$\begin{aligned}
&\mathrm{mgu}_\omega(S, \{x/t\} \uplus \theta) \\
&= \mathrm{mgu}_\omega(\mathrm{mgu}_\omega(S, \{x/t\}), \theta) && \text{by definition of } \mathrm{mgu}_\omega \\
&= \mathrm{mgu}_\mathrm{P}(\mathrm{mgu}_\omega(S, \{x/t\}), \theta) && \text{by induction hypothesis} \\
&= \mathrm{mgu}_\mathrm{P}(\mathrm{mgu}_\mathrm{P}(S, \{x/t\}), \theta) && \mathrm{mgu}_\omega \text{ and } \mathrm{mgu}_\mathrm{P} \text{ coincide for a single binding} \\
&= \mathrm{mgu}_\mathrm{P}(S, \{x/t\} \uplus \theta) && \text{by Lemma 1}
\end{aligned}$$

and this proves the theorem.

4.6 Optimality of Abstract Unification

We now want to prove that parallel unification is optimal w.r.t. the concrete mgu. We already now it is correct, therefore we only need to prove that it is the best correct approximation. First of all, we prove optimality in the special case when, given $\mathrm{mgu}_\mathrm{P}([S]_U, \theta)$, it is the case that $\mathit{vars}(\theta) \subseteq U$. Next, we extend this result to the general case.

Example 11. We show how to find a substitution δ for the Example 9 such that $x^2y^2z^3u^3v^2 \in \alpha_\omega(\mathrm{mgu}(\delta, \theta))$. Let $U = \{u, v, x, y, z\}$. For each node n of the sharing graph, we consider a different fresh variable w_n. In our case, we assume that the node labelled with xv in the upper-left corner is node 1, and we proceed clockwise to number the other nodes. The two nodes labelled with xv yields to different variables w_1 and w_2

We define $\delta(y) = s(w_3, w_5)$ where w_3 and w_5 corresponds to the nodes containing y and s is an arbitrary term symbol. The same holds for all the other variables in $U \setminus \mathrm{dom}(\theta)$, and therefore $\delta(z) = s(w_3, w_4, w_5)$, $\delta(v) = s(w_1, w_2)$.

For the variables in $\mathrm{dom}(\theta)$, we define δ in a different way. In particular, we define $\delta(x) = s(w_1, w_2)$. It is obtained by replacing in $\theta(x)$ the variable y with a term similar to $\delta(y)$, with the difference that w_3 and w_5 are replaced with the (variables corresponding to the) nodes w_1 and w_2. The choice of w_1 and w_2 is obvious by looking at the sharing graph, since the first and second node are the sources of the two edges targeted at nodes three and five respectively. The same holds for all the other variables in $\mathrm{dom}(\theta)$, therefore we obtain $\delta(u) = t(s(w_4, w_3, w_4))$.

Summing up, we have

$$\delta = \{u/t(s(w_4, w_3, w_4)), v/s(w_1, w_2), x/s(w_1, w_2), \\ y/s(w_3, w_5), z/s(w_3, w_4, w_5)\} \enspace .$$

It is easy to check that $[S]_U \rhd [\delta]_U$ and $\eta = \mathrm{mgu}(\delta, \theta)$ is

$$\eta = \{u/t(s(w_1, w_1, w_1), v/s(w_1, w_1), x/s(w_1, w_1), y/s(w_1, w_1), z/s(w_1, w_1, w_1) \\ w_2/w_1, w_3/w_1, w_4/w_1, w_5/w_1\} \enspace ,$$

hence $\alpha_\omega([\eta]_U) = [\{\!\{\}\!\}, x^2y^2z^3u^3v^2]_U$.

The main ideas shown in this example are:

1. to associate a new fresh variable to each node in the sharing graph;
2. to define an appropriate substitution such that these fresh variables are unified according to the arrows in the sharing graph.

This approach is pursued further in the following:

Theorem 4. *The parallel unification* $\mathrm{mgu_P}([S]_U, \theta)$ *is optimal w.r.t.* mgu*, under the assumption that* $vars(\theta) \subseteq U$*.*

Proof. Let $\theta = \{x_1/t_1, \ldots, x_n/t_n\}$ and $X \in \mathrm{mgu_P}(S, \theta)$. By definition of $\mathrm{mgu_P}$, there exists a parallel sharing graph $\mathcal{G} = \{G^i\}_{i \in [1,n]}$ such that $X \in res(\underline{\mathcal{G}})$. Let $N_{\mathcal{G}} = \{n_1, \ldots, n_k\}$.

In the rest of the proof, we define a substitution δ such that $[S]_U \rhd [\delta]_U$ and $X \in \alpha_\omega(\mathrm{mgu}([\delta]_U, \theta))$. If $X = \{\!\!\{\}\!\!\}$ this is trivial, hence we assume $X \neq \{\!\!\{\}\!\!\}$. For each node $n \in N_{\mathcal{G}}$ we consider a fresh variable w_n and we denote by W the set of all these new variables.

For any $y \in U \setminus \mathrm{dom}(\theta)$ we define a term t_y of arity $\sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(y)$ as follows:

$$t_y = t(\underbrace{w_{n_1}, \ldots, w_{n_1}}_{l_{\mathcal{G}}(n_1)(y) \text{ times}}, \underbrace{w_{n_2}, \ldots, w_{n_2}}_{l_{\mathcal{G}}(n_2)(y) \text{ times}}, \ldots, \underbrace{w_{n_k}, \ldots, w_{n_k}}_{l_{\mathcal{G}}(n_k)(y) \text{ times}})$$

Given any binding $x_i/t_i \in \theta$, there exists a map $f^i : E_{G^i} \to \mathcal{V}$ such that, for each variable y and node n, the set of edges targeted at n and labelled with y by f^i is exactly $l_{\mathcal{G}}(n)(y) \cdot occ(y, t_i)$. In formulas, we require

$$|\{e \in E_{G^i} \mid f^i(e) = y \wedge \mathrm{tgt}_{G^i}(e) = n\}| = l_{\mathcal{G}}(n)(y) \cdot occ(y, t_i) \ .$$

The idea is that each edge targeted at the node n is actually targeted at one of the specific variables in $l_{\mathcal{G}}(n)$. In particular, each variable $y \in \llbracket l_{\mathcal{G}}(n) \rrbracket$ should have exactly $l_{\mathcal{G}}(n)(y) \cdot occ(y, t_i)$ edges targeted at it, so that the total number of edges pointing n is $\sum_{y \in U} l_{\mathcal{G}}(n)(y) \cdot occ(y, t_i) = \chi(l_{\mathcal{G}}(n), t_i)$, i.e., the in-degree of n. The map f^i chooses, for each edge targeted at n, a variable in $l_{\mathcal{G}}(n)$ according to the previous idea.

Now, for each node n and variable $y \in U$, we denote by $M^i_{n,y}$ the set of edges pointing at y in n, i.e., $M^i_{n,y} = \{e \in E_{G^i} \mid \mathrm{tgt}_{G^i}(e) = n \wedge f^i(e) = y\}$. Each $M^i_{n,y}$ may be partitioned in $occ(y, t_i)$ sets of $l_{\mathcal{G}}(n)(y)$ elements, denoted by $M^i_{n,y,\xi}$ such that $\cup\{M^i_{n,y,\xi} \mid t_i(\xi) = y\} = M^i_{n,y}$.

We may define some variations of the terms t_y by replacing the variables occurring in them with those in the set $M^i_{n,y,l}$. In particular, for $y \in U \setminus \mathrm{dom}(\theta)$ and any occurrence ξ of a variable y in t_i, we define the term $t^{i,y}_\xi$ of arity $\sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(y)$ as

$$t^{i,y}_\xi = t(w(M^i_{n_1,y,\xi}), w(M^i_{n_2,y,\xi}), \ldots, w(M^i_{n_k,y,\xi})) \ ,$$

where, if $M = \{e_1, \ldots, e_q\}$, we define $w(M)$ as the sequence $w_{n'_1}, \ldots, w_{n'_q}$ where $n'_j = \mathrm{src}_{E_{G^i}}(e_j)$.

Note that t_y and $t_\xi^{i,y}$ have, in corresponding positions, variables related to nodes which are connected through edges. We are now ready to define the substitution δ in the following way:

- for $y \in U \setminus \mathrm{dom}(\theta)$ then $\delta(y) = t_y$;
- for $x_i \in \mathrm{dom}(\theta)$ then $\delta(x_i)$ is the same as $\theta(x_i)$ with the difference that each occurrence ξ of a variable $y \in \theta(x_i)$ is replaced by the term $t_\xi^{i,y}$;
- in all the other cases, i.e $v \notin U$, $\delta(v) = v$.

Now we show that $[S]_U \rhd [\delta]_U$. We need to consider all the variables $v \in \mathcal{V}$ and check that $\delta^{-1}(v)|_U \in S$. We distinguish several cases:

- if we choose the variable w_n for some $n \in N$, by construction $occ(w_n, t_y) = l_\mathcal{G}(n)(y)$. Moreover, since $\mathcal{G}$ is a parallel sharing graph, for any $x_i \in \mathrm{dom}(\theta)$ there are $l_\mathcal{G}(n)(x_i)$ edges in E^i departing from n and targeted to nodes m such that $\chi(l_\mathcal{G}(m), t_i) \neq 0$. Therefore

$$\sum_{y \in vars(t_i), m \in N_\mathcal{G}} |\{e \in M_{m,y}^i | \mathrm{src}_{E_{G^i}}(e) = n\}| = l_\mathcal{G}(n)(x_i)$$

  and $occ(\delta(x_i), w_n) = l_\mathcal{G}(n)(x_i)$. Since for each $v \in U$ we have $occ(\delta(v), w_n) = l_\mathcal{G}(n)(v)$, we obtain the required result which is $\delta^{-1}(w_n)|_U = l_\mathcal{G}(n) \in S$.
- if we choose a variable $v \in U$ then $v \in \mathrm{dom}(\delta)$ and $\delta^{-1}(v) = \{\!\{\}\!\} \in S$;
- finally, if $v \notin U \cup W$, then $\delta^{-1}(v) = \{\!\{v\}\!\}$ and $\delta^{-1}(v)_{|U} = \{\!\{\}\!\} \in S$.

We now show that $X \in \alpha_\omega(\mathrm{mgu}([\delta]_U, \theta))$. By definition of mgu over $ISubst_\sim$, we have that $\mathrm{mgu}([\delta]_U, \theta) = [\mathrm{mgu}(\delta, \theta)]_U$. We obtain:

$$\begin{aligned}\eta =&\mathrm{mgu}(\delta, \theta) = \\ &\theta \circ \mathrm{mgu}\big(\{y = t_y \mid y \in U \setminus \mathrm{dom}(\theta)\} \cup \\ &\quad \{y = t_\xi^{i,y} \mid x_i \in \mathrm{dom}(\theta), \theta(x_i)(\xi) = y\}\big) = \\ &\theta \circ \{y/t_y \mid y \in U \setminus \mathrm{dom}(\theta)\} \circ \\ &\quad \mathrm{mgu}\{t_y = t_\xi^{i,y} \mid x_i \in \mathrm{dom}(\theta), \theta(x_i)(\xi) = y\} \ .\end{aligned} \tag{21}$$

Let F be the set of equations $\{t_y = t_j^{i,y} \mid x_i \in \mathrm{dom}(\theta), \theta(x_i)(j) = y\}$. We show that, for any edge $n \to m \in E_{G^i}$, it follows from F that $w_n = w_m$. Since $n \to m \in E_{G^i}$, then for some $y \in vars(t_i)$ it holds that $f^i(n \to m) = y$. This implies that $n \to m \in M_{m,y}^i$ and therefore there exists a position ξ such that $n \to m \in M_{m,y,\xi}^i$. By definition of $t_\xi^{i,y}$, it means that $w_n \in vars(t_\xi^{i,y})$, hence $w_n = w_m$ follows from $t_y = t_\xi^{i,y} \in F$.

Since this holds for any edge in E_{G^i} and for any $i \in [1, n]$, it follows that for any edge $n \to m \in E_{\underline{\mathcal{G}}}$ the equation $w_m = w_n$ is entailed by F. We know that $\underline{\mathcal{G}}$ is connected, hence for any $n, m \in N_\mathcal{G}$, the set of equations in F implies $w_n = w_m$. We choose a particular node $\bar{n} \in N_\mathcal{G}$ and, for what we said before, we have $\mathrm{mgu}(F) = \{w_n/w_{\bar{n}} \mid n \in N_{\mathcal{G}}\}$. We show that $\eta^{-1}(w_{\bar{n}})_{|U} = X$.

$$
\begin{aligned}
&\eta^{-1}(w_{\bar{n}})_{|U}\\
&= \theta^{-1}(\{y/t_y \mid y \in U \setminus \mathrm{dom}(\theta)\}^{-1}(\{\!\{w_{n_1}, \ldots, w_{n_k}\}\!\}))_{|U} =\\
&= \theta^{-1}(\{\!\{w_{n_1}, \ldots, w_{n_k}\}\!\} \uplus \lambda y \in U \setminus \mathrm{dom}(\theta). \sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(y))_{|U} =\\
&= \lambda y \in U \setminus \mathrm{dom}(\theta). \sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(y) \uplus\\
&\qquad\qquad \lambda x \in \mathrm{dom}(\theta). \sum_{y \in \mathcal{V}} \mathit{occ}(y, \theta(x)) \cdot \sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(y)\\
&= \lambda y \in U \setminus \mathrm{dom}(\theta). \sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(y) \uplus \lambda x \in \mathrm{dom}(\theta). \sum_{n \in N_{\mathcal{G}}} \chi(l_{\mathcal{G}}(n), \theta(x)) \enspace .
\end{aligned}
$$

Since $\mathcal{G}$ is a parallel sharing graph, the total out-degree for the graph G^i, i.e., $\sum_{n \in N_{\mathcal{G}}} \chi(l_{\mathcal{G}}(n), \theta(x_i))$, is equal to the total in-degree $\sum_{n \in N_{\mathcal{G}}} \chi(l_{\mathcal{G}}(n), x_i)$. Hence

$$
\begin{aligned}
&\eta^{-1}(w_{\bar{n}})_{|U}\\
&= \lambda y \in U \setminus \mathrm{dom}(\theta). \sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(y) \uplus \lambda x \in \mathrm{dom}(\theta). \sum_{n \in N_{\mathcal{G}}} \chi(l_{\mathcal{G}}(n), x)\\
&= \lambda x \in U. \sum_{n \in N_{\mathcal{G}}} l_{\mathcal{G}}(n)(x)\\
&= \mathit{res}(\underline{\mathcal{G}}) \enspace .
\end{aligned}
$$

This concludes the proof.

The previous proof only works when $\mathit{vars}(\theta) \subseteq U$. However, nothing changes when this condition is not satisfied.

Example 12. Let $U = \{x, y\}, S = \{x^2, x^2y\}, \theta = \{x/t(y, z)\}$ and assume we want to compute $\mathrm{mgu}_\omega([S]_U, \theta)$. By extending the domain of variable of interests, we obtain $[S']_V = [x^2, x^2y, z]_{x,y,z}$. One of the sharing graph for θ and $[S']_V$ is

$$
\begin{tikzcd}
\boxed{x^2} \arrow[r] \arrow[d] & \boxed{x^2y} \arrow[d] \arrow[dr] & \\
\boxed{z} & \boxed{z} & \boxed{z}
\end{tikzcd}
$$

Following the proof of the previous theorem, we obtain the substitution

$$\delta' = \{x/t(t(w_1), t(w_1, w_2, w_2)), y/t(w_2), z/t(w_3, w_4, w_5)\}$$

such that $[S']_V \rhd [\delta']_V$ and $x^4yz^3 \in \alpha_\omega(\mathrm{mgu}_\omega([\delta']_V, \theta))$. However, we are looking for a substitution δ such that $[S]_U \rhd [\delta]_U$ and $x^4yz^3 \in \alpha_\omega(\mathrm{mgu}_\omega([\delta]_U, \theta))$. Nonetheless, we may choose $\delta' = \delta$ (or, if we prefer $\delta = \delta'|_{x,y}$) to get the required substitution.

This is not a fortuitous coincidence. We may show that it consistently happens every time we apply Theorem 4 to an abstract unification where $\mathit{vars}(\theta) \nsubseteq U$. Therefore, we may prove the main result of the paper:

Theorem 5. *The abstract parallel unification* $\mathrm{mgu_P}$ *is optimal w.r.t.* mgu.

Proof. Let us define $S' = S \cup \{\{\!\{v\}\!\} \mid v \in \mathit{vars}(\theta) \setminus U\}$, $V = U \cup \mathit{vars}(\theta)$, $\theta = \{x_1/t_1, \dots, x_n/t_n\}$ and $X \in \mathrm{mgu_P}(S', \theta)$. We want to find $[\delta]_U$ such that $[S]_U \rhd [\delta]_U$ and $X \in \alpha_\omega(\mathrm{mgu}([\delta]_U, \theta))$.

First of all, following the previous theorem, we find a substitution δ such that $X \in \alpha_\omega(\mathrm{mgu}([\delta]_V, \theta))$ and $[S']_V \rhd [\delta]_V$. We want to prove that $[S]_U \rhd [\delta]_U$ and $\mathrm{mgu}([\delta]_V, \theta) \leq_\omega \mathrm{mgu}([\delta]_U, \theta)$, so that $[\delta]_U$ is the existential substitutions we are looking for.

Note that, with the same considerations which led to (4.6), we have

$$\begin{aligned}\mathrm{mgu}(\theta, \delta) &= \mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \mathrm{Eq}(\delta|_{V \setminus U})) = \\ &= \mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \{y = t_y \mid y \in \mathrm{rng}(\theta) \setminus U\} \cup \\ &\qquad \{t_y = t^{i,y}_\xi \mid x_i \in \mathrm{dom}(\theta) \setminus U, \theta(x_i)(\xi) = y\})\end{aligned}$$

If $x_i \in \mathrm{dom}(\theta) \setminus U$, then x_i appears in S' only in the multiset $\{\!\{x_i\}\!\}$. Moreover, if n is a node labelled by $\{\!\{x_i\}\!\}$, there is only one edge which departs from n and there are no edges which arrive in n. This means that

- w_n does not appear in any t_y for $y \in V \setminus \mathrm{dom}(\theta)$ and in any $t^{j,y}_\xi$ with $j \neq i$,
- $\delta(x_i)$ is linear since given edges $e \neq e'$, we have that $\mathrm{src}_{E_{G^i}}(e) \neq \mathrm{src}_{E_{G^i}}(e')$.

As a result, $\delta(x_i)$ is linear and does not share variables with θ or the other bindings in δ. The last formula may be rewritten as

$$\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \{y = t_y \mid y \in \mathrm{rng}(\theta) \setminus U\}) \uplus \beta$$

where β is a substitution such that $\mathrm{dom}(\beta) = \mathrm{rng}(\delta_{|\mathrm{dom}(\theta) \setminus U})$. It is obvious that

$$\begin{aligned}&\alpha_\omega([\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \{y = t_y \mid y \in \mathrm{rng}(\theta) \setminus U\})]_V) \\ = &\alpha_\omega([\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \{y = t_y \mid y \in \mathrm{rng}(\theta) \setminus U\}) \uplus \beta]_V)\ .\end{aligned}$$

since $\mathrm{dom}(\beta) \cap V = \emptyset$.

Now, we split the set of variables $\mathrm{rng}(\theta) \setminus U$ in two parts: $U_1 = (\mathrm{rng}(\theta) \setminus U) \cap \mathit{vars}(\theta(U))$ and $U_2 = (\mathrm{rng}(\theta) \setminus U) \setminus \mathit{vars}(\theta(U))$.

If $y \in U_1$ there exist $x_{i_y} \in U \cap \mathrm{dom}(\theta)$ and a position ξ_y such that $\theta(x_{i_y})(\xi_y) = y$ and $\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \{y = t_y\}$ is equivalent to $\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \{t^{i_y,y}_{\xi_y} = t_y\}$. Note that, since $y \notin U$, then t_y (which is actually $\delta(y)$) is linear and independent from θ and the other bindings in δ. Therefore

$$\begin{aligned}&\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \mathrm{Eq}(\delta|_{U_1}) \cup \mathrm{Eq}(\delta|_{U_2})) \\ =&\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \{t^{i_y,y}_{\xi_y} = t_y \mid y \in U_1\} \cup \mathrm{Eq}(\delta|_{U_2})) \\ =&\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \mathrm{Eq}(\delta|_{U_2})) \uplus \beta'\end{aligned}$$

where $\beta' = \mathrm{mgu}(\{t^{i_y,y}_{\xi_y} = t_y \mid y \in U_1\})$ and $\mathrm{dom}(\beta') = \mathit{vars}(\{t_y \mid y \in U_1\})$. As before, it is obvious that

$$\begin{aligned}
&\alpha_\omega([\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \mathrm{Eq}(\delta|_{U_2})) \uplus \beta']_V) \\
=&\alpha_\omega([\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \mathrm{Eq}(\delta|_{U_2}))]_V) \ .
\end{aligned}$$

Let $\eta = \mathrm{mgu}(\mathrm{Eq}(\theta|_U) \cup \mathrm{Eq}(\delta|_U))$, then

$$\begin{aligned}
&\mathrm{mgu}(\mathrm{Eq}(\theta) \cup \mathrm{Eq}(\delta|_U) \cup \mathrm{Eq}(\delta|_{U_2})) \\
=&\mathrm{mgu}(\mathrm{Eq}(\theta|_U) \cup \mathrm{Eq}(\theta|_{V\setminus U}) \cup \mathrm{Eq}(\delta|_U) \cup \mathrm{Eq}(\delta|_{U_2})) \\
=&\eta \circ \mathrm{mgu}(\eta(\mathrm{Eq}(\theta|_{V\setminus U}) \cup \mathrm{Eq}(\delta|_{U_2}))) \\
=&\eta \circ \mathrm{mgu}(\{x = \eta(\theta(x)) \mid x \in \mathrm{dom}(\theta) \setminus U\} \cup \mathrm{Eq}(\delta|_{U_2})) \\
&[\text{since } \mathit{vars}(\delta|_{U_2}) \cap \mathit{vars}(\eta) = \emptyset \text{ and } \mathit{vars}(\eta) \cap (\mathrm{dom}(\theta) \setminus U) = \emptyset] \\
=&\eta \circ (\theta \circ \eta)|_{\mathrm{dom}(\theta)\setminus U} \circ \delta|_{U_2} \\
&[\text{since } \mathrm{dom}(\theta) \setminus U \text{ is disjoint from } \mathit{vars}(\delta|_{U_2})]
\end{aligned}$$

where $\eta \circ (\theta \circ \eta)|_{\mathrm{dom}(\theta)\setminus U}$ is the mgu of θ and $\delta|_U$. We call $\gamma = \eta \circ (\theta \circ \eta)|_{\mathrm{dom}(\theta)\setminus U}$ and we prove that $\alpha_\omega([\gamma]_V) \geq_\omega \alpha_\omega([\gamma \circ \delta|_{U_2}]_V)$.

Consider a variable $v \in \mathcal{V}$. If $v \notin \mathit{vars}(\delta|_{U_2})$ there is nothing to prove. If $v \in \mathrm{rng}(\delta_{|U_2})$ we know that v does not occur anywhere else in $\delta|_{U_2}$ and γ. Then $(\gamma \circ \delta|_{U_2})^{-1}(v) = \gamma^{-1}(\{\!\{y, v\}\!\}) = \gamma^{-1}(y) \uplus \{\!\{v\}\!\}$ for the only y such that $v \in \mathit{vars}(\delta|_{U_2}(y))$. Therefore, since $v \notin V$, the sharing group over V we obtain in $\gamma \circ \delta|_{U_2}$ from v may be obtained in γ from the variable y. If $v \in \mathrm{dom}(\delta_{|U_2})$ then $(\gamma \circ \delta|_{U_2})^{-1}(v) = \{\!\{\}\!\}$ which occurs in every element of $\mathtt{ShLin}^\omega$.

Note that the operation mgu_ω is designed by first extending the domain in order to include all the variables in V and then performing the operation, and that this construction yields to an optimal abstraction of the concrete unification. This is not the case for other abstract domains, e.g. `Sharing`, as shown in [1].

4.7 A Characterization for Resultant Sharing Groups

The concept of resultant ω-sharing group, while suggestive and very intuitive, does not help in practice in the implementation of the operations. Although $\mathtt{ShLin}^\omega$ has not been designed to be directly implemented, some of its abstractions could. Providing a simpler definition for the set of resultant ω-sharing groups could help in developing the abstract operators for its abstractions. We show that given a set S of ω-sharing groups and a binding x/t, the set of resultant ω-sharing groups has an elegant algebraic characterization.

Theorem 6. *Let S be a set of ω-sharing groups, $x \in \mathcal{V}$ and t a term. Then $B \in \mathrm{mgu}_\omega(S, \{x/t\})$ iff there exists $\{B_i\}_{i \in I} \in \wp(S)$ which satisfies the following conditions:*

1. *I is a finite set,*
2. *$B = \uplus_{i \in I} B_i$,*

3. $\sum_{i\in I}\chi(B_i,x)=\sum_{i\in I}\chi(B_i,t)\geq |I|-1$,
4. *either* I *is a singleton or* $\forall i\in I.\ \chi(B_i,x)+\chi(B_i,t)>0$.

Proof. We first prove the conditions are necessary. Assume B is a resultant sharing group for S and x/t, obtained by the sharing graph G. Take $I=N_G$ and $B_i=l_G(i)$ for each $i\in I$, so that $B=\uplus_{i\in I}B_i$. Note that, since then in-degree of each node is $\chi(B_i,x)$, the sum of the in-degres of all the nodes is $\sum_{i\in I}\chi(B_i,x)$ and the sum of the out-degree is $\sum_{i\in I}\chi(B_i,t)$. Both of them must be equal to the number of edges in E_G. Moreover, each connected graph with $|I|$ nodes has at least $|I|-1$ edges. Finally, if a connected graph has more than one node, then every node i has an adjacent edge. Therefore, either $\chi(B_i,x)$ or $\chi(B_i,t)$ is not null.

Now we prove that the condition is sufficient. Let $n=|I|$. If $n=1$ and $\chi(B_i,x)+\chi(B_i,t)=0$ for the only $i\in I$, simply consider a sharing graph with a single node labelled with B_i and no edges. Otherwise, we partition the set I in three parts:

- $N_x=\{i\in I\mid \chi(B_i,x)=0\}$;
- $N_t=\{i\in I\mid \chi(B_i,t)=0\}$;
- $N=\{i\in I\mid \chi(B_i,x)\neq 0,\chi(B_i,t)\neq 0\}$;

Note that this is a partition of I since, by hypothesis, $\forall i\in I.\ \chi(B_i,x)+\chi(B_i,t)>0$. Now we define a connected labelled multigraph G whose sets of nodes is I and whose labelling function is $\lambda i\in I.B_i$. In order to define the edges, we distinguish two cases.

$N\neq\emptyset$: Let $N=\{b_1,\dots,b_m\}$ with $m\geq 1$ and consider the set of edges:

$$\{a\to b_1\mid a\in N_t\}\cup\{b_1\to c\mid c\in N_x\}\cup\{b_i\to b_{i+1}\mid i\in\{1,\dots,m-1\}\}\ .$$

$N=\emptyset$: If $N_t=\emptyset$, then also $N_x=\emptyset$ and there is nothing to prove. We assume $N_t\neq\emptyset$, and thus $N_x\neq\emptyset$. Let $\bar a\in N_t$, $\bar c\in N_x$ and consider the set of edges:

$$\{\bar a\to c\mid c\in N_x\}\cup\{a\to\bar c\mid a\in N_t\setminus\{\bar a\}\}\ .$$

Note that, in both cases, we obtain a multigraph with the following properties:

1. it is connected;
2. it has exactly $n-1$ edges, i.e. it is a tree (if we do not consider the direction of edges);
3. there is no edge targeted at a node i with $\chi(i,t)=0$ and no edge whose source is a node i with $\chi(i,x)=0$.

In the rest of the proof, we call *pre-sharing graph* a multigraph which satisfies the above properties.

If $indeg(i)$ is the in-degree of a node and $outdeg(i)$ the outdegree, we call *unbalancement factor* of the graph the value:

$$\sum\{outdeg(i)-\chi(B_i,x)\mid i\in I, outdeg(i)>\chi(B_i,x)\}+$$

$$+\sum\{\mathit{indeg}(i)-\chi(B_i,t) \mid i\in I, \mathit{indeg}(i)>\chi(B_i,t)\} \ .$$

We prove that given a pre-sharing graph with unbalancement factor k, we can build another pre-sharing graph with unbalancement factor strictly less then k. As a result, there is a pre-sharing graph with unbalancement factor equals to zero.

Assume the graph has unbalancement factor k. There is at least an unbalanced node. Assume without loss of generality that the unbalanced node is j and that $\mathit{outdeg}(j) > \chi(B_j, x)$. Since $\sum_{i\in I}\chi(B_i,x) \geq n-1$, there exists a node l such that $\mathit{outdeg}(l) < \chi(B_l, x)$. Let e be the unique edge with source j such that, if we remove e from the graph, l becomes disconnected from j. Since no edge starts from a node i with $\chi(B_i, x) = 0$, then $\chi(B_j, x) > 0$. This means that $\mathit{outdeg}(j) > 1$ and there is at least another edge starting from j. Assume it is $e' : j \to j'$. Remove this edge and replace it with an edge $l \to j'$. It is obvious that the result is a pre-sharing graph with a smaller unbalancement factor than the original one. The case for $\mathit{indeg}(j) > \chi(B_j, t)$ is symmetric.

Once the unbalancement factor is zero, since $\sum_{i\in I}\chi(B_i,x) = \sum_{i\in I}\chi(B_i,t)$ we can freely add other edges in such a way to complete the graph w.r.t. the condition on the degree of nodes. We obtain a sharing graph G such that $\mathit{res}(G) = B$.

Following the above theorem, we can give an algebraic characterization of the abstract unification operator as follows. If we denote by $\mathit{rel}(S, x, t)$ the set

$$\begin{aligned}\mathit{rel}(S,x,t) &= \{B\in S.\ \chi(B,x)+\chi(B,t)>0\}\\ &= \{B\in S.\ \llbracket B\rrbracket \cap \mathit{vars}(\{x/t\})\neq\emptyset\}\ ,\end{aligned} \tag{22}$$

we have the following characterization for the abstract mgu:

$$\begin{aligned}&\mathrm{mgu}_\omega(S,\,\{x/t\}) = (S\setminus \mathit{rel}(S,x,t))\cup\\ &\qquad\left\{\uplus\mathcal{S} \mid \mathcal{S}\in\wp_m(\mathit{rel}(S,x,t)), \sum_{B\in\mathcal{S}}\chi(B,x)=\sum_{B\in\mathcal{S}}\chi(B,t)\geq|\mathcal{S}|-1\right\}.\end{aligned} \tag{23}$$

Example 13. Consider $S = \{xa, xb, z^2, zc\}$ and the equation $x = z$. Then if we choose $X = \{\!\{xa, xb, z^2\}\!\}$, we have $\chi(X, x) = 2 = \chi(X, z) \geq |X| - 1$. Therefore $x^2z^2ab \in \mathrm{mgu}_\omega(S, \{x/z\})$. If we take $X = \{\!\{xa, xb, zc, zc\}\!\}$, although $\chi(X, x) = 2 = \chi(X, z)$, we have $|X| - 1 = 3$. This only proves that $z^2c^2x^2ab$ cannot be obtained by the multiset X. If we check for every possible multiset over S, we have that $z^2c^2x^2ab \notin \mathrm{mgu}_\omega(S, \{x/z\})$.

5 An Application: King's Domain for Linearity and Aliasing

In the rest of the paper we show that two domains for sharing analysis with linearity information, namely the domain proposed by King in [17] and the classic reduced product $\texttt{Sharing} \times \texttt{Lin}$, may be obtained as abstraction of $\texttt{ShLin}^\omega$. This allows us to design optimal abstract operators for both domains, by exploiting the results for $\texttt{ShLin}^\omega$.

We first consider the domain for combined analysis of sharing and linearity in [17]. We call *2-sharing group* a map $o : \mathcal{V} \rightarrow \{0, 1, \infty\}$ such that its support $\llbracket o \rrbracket = \{v \in \mathcal{V} \mid o(v) \neq 0\}$ is finite. We write $o_m(x)$ to denote $\min\{o(x), 2\}$ (where $n < \infty$ for each $n \in \mathbb{N}$). Intuitively, a 2-sharing group o represents the sets $\gamma_2(o)$ of ω-sharing group given by:

$$\gamma_2(o) = \{B \in \wp_m(\mathcal{V}) \mid \llbracket o \rrbracket = \llbracket B \rrbracket \wedge \forall x \in \llbracket o \rrbracket . o_m(x) \leq B(x) \leq o(x)\} \ .$$

We denote by $Sg^2(V)$ the set of 2-sharing groups whose support is a subset of V. We use a polynomial notation for 2-sharing groups as for ω-sharing groups: a group o such that $\llbracket o \rrbracket = \{x, y, z\}$, $o(x) = o(z) = 1$ and $o(y) = \infty$ will be denoted by $xy^\infty z$. We also use $\emptyset$ for the 2-sharing group with empty support.

The idea is to use 2-sharing groups to keep track of linearity: If $o(x) = \infty$, it means that the variable x is not linear in the sharing group o. In the rest of this subsection, we use the term "sharing group" as a short form of 2-sharing group[1].

We first need to define an order relation over sharing groups as follows:

$$o \leq o' \iff \llbracket o \rrbracket = \llbracket o' \rrbracket \wedge \forall x \in \llbracket o \rrbracket .\ o(x) \leq o'(x) \ . \tag{24}$$

The domain we are interested in is the following:

$$\texttt{ShLin}^2 = \left\{[S]_U \mid S \in \wp_\downarrow(Sg^2(U)), U \in \wp_f(\mathcal{V}), S \neq \emptyset \Rightarrow \emptyset \in S\right\} \ . \tag{25}$$

where $\wp_\downarrow(Sg^2(U))$ is the powerset of downward closed 2-sharing groups according to $\leq$ and $[S_1]_{U_1} \leq_2 [S_2]_{U_2}$ iff $U_1 = U_2$ and $S_1 \subseteq S_2$. Since we only consider downward closed sets, we are not able to state that some variable is definitively non-linear. We define an adjunction with $\texttt{ShLin}^\omega$ via the following concretization map $\gamma_2 : \texttt{ShLin}^2 \rightarrow \texttt{ShLin}^\omega$.

$$\gamma_2([S]_U) = \left[\bigcup \{\gamma_2(o) \mid o \in S\}\right]_U \ . \tag{26}$$

The adjoint map $\alpha_2 : \texttt{ShLin}^\omega \rightarrow \texttt{ShLin}^2$ is:

$$\alpha_2([S]_U) = [\downarrow \{\alpha_2(B) \mid B \in S\}]_U \ . \tag{27}$$

where

$$\alpha_2(B) = \lambda v \in \llbracket B \rrbracket . \begin{cases} 1 \text{ if } B(x) = 1 \\ \infty \text{ otherwise} \ . \end{cases}$$

With an abuse of notation, we also apply γ_2 and α_2 to subsets of ω-sharing groups and 2-sharing groups respectively, by ignoring the set of variables of interest.

[1] In [17] the number 2 is used as an exponent instead of ∞, but we prefer this notation to be coherent with ω-sharing groups.

Theorem 7. $\langle \alpha_2, \gamma_2 \rangle : \mathtt{ShLin}^2 \leftrightarrows \mathtt{ShLin}^\omega$ *is a Galois insertion.*

Proof. It is obvious that α_2 and γ_2 are monotone functions and that they are both join-morphisms. Extensionality of $\gamma_2 \circ \alpha_2$ follows from the fact that, given an ω-sharing group B, we have $B \in \gamma_2(\alpha_2(B))$. Finally, given a 2-sharing group o, we have $\alpha_2(\gamma_2(o)) = \{o\}$. This implies that $\alpha_2 \circ \gamma_2$ is the identity.

Since $\langle \alpha_2, \gamma_2 \rangle$ is a Galois insertion, we may obtain the abstract optimal mgu for $\mathtt{ShLin}^2$ with $\mathrm{mgu}_2([S]_U, \theta) = \alpha_2(\mathrm{mgu}_\omega(\gamma_2([S]_U), \theta))$. In the case where $\theta = \{x/t\}$ and $vars(\theta) \subseteq U$, by using additivity of α_2 we get

$$\begin{aligned}\mathrm{mgu}_2([S]_U, \{x/t\}) = \Big[&\alpha_2(\gamma_2(S) \setminus rel(\gamma_2(S), x, t)) \cup \\ &\alpha_2\Big(\{\uplus \mathcal{S} \mid \mathcal{S} \in \wp_m(rel(\gamma_2(S), x, t)), \\ &\qquad \sum_{B \in \mathcal{S}} \chi(B, x) = \sum_{B \in \mathcal{S}} \chi(B, t) \geq |\mathcal{S}| - 1\}\Big)\Big]_U \ . \qquad (28)\end{aligned}$$

Now we want to simplify (28). In particular we would like to get rid of the abstraction and concretization functions and to express the result using only objects in $\mathtt{ShLin}^2$. We need to define operations in $\mathtt{ShLin}^2$ which correspond to $\uplus$ and χ in $\mathtt{ShLin}^\omega$.

The operation on 2-sharing groups, which corresponds to multiset union on ω-sharing groups, is given by

$$o \uplus o' = \lambda v \in \mathcal{V}. o(v) \oplus o'(v) \qquad (29)$$

where $0 \oplus x = x \oplus 0 = x$ and $\infty \oplus x = x \oplus \infty = 1 \oplus 1 = \infty$. We will use $\biguplus \{\!\{ o_1, \ldots, o_n \}\!\}$ for $o_1 \uplus \cdots \uplus o_n$. Given a sharing group o, we also define the *delinearization* operator $o^2 = o \uplus o$. Note that $o^2 = \lambda x \in \llbracket o \rrbracket . \infty$. The operator is extended pointwise to sets and multisets.

A fundamental role is also played by the way we define the notion of multiplicity of a sharing group in a term. While the multiplicity of an ω-sharing group in a term is a single natural number, every object in $\mathtt{ShLin}^2$ represents a set of ω-sharing groups, hence its multiplicity should be a set of natural number. Actually, it is enough to consider intervals. We define the minimum χ_m and maximum χ_M multiplicity of o in t as follows:

$$\chi_m(o, t) = \sum_{v \in \llbracket o \rrbracket} o_m(v) \cdot occ(v, t) \qquad \chi_M(o, t) = \sum_{v \in \llbracket o \rrbracket} o(v) \cdot occ(v, t) \ . \qquad (30)$$

Sum and product on integers are lifted in the obvious way, namely the sum is ∞ if and only if at least one of the addenda is ∞ and $n \cdot \infty = \infty \cdot n = \infty$ for any $n \in \mathbb{N}^+$. Note that, if B is an ω-sharing group represented by o, i.e., $B \in \gamma_2(o)$, then $\chi_m(o, t) \leq \chi(B, t) \leq \chi_M(o, t)$. Actually, not all the values between $\chi_m(o, t)$ and $\chi_M(o, t)$ may be assumed by $\chi(B, t)$, but this will not affect the precision of the abstract operators. Moreover, the maximum multiplicity $\chi_M(o, t)$ either is equal to the minimum multiplicity $\chi_m(o, t)$ or it is infinite.

Example 14. Let $o = x^\infty$ and $t = t(x, x)$. According, to our definition, $\chi(o, t) = [4, \infty)$. However, it is obvious that if $B \in \gamma_2(o)$, then $\chi(B, t)$ is an even number, therefore $\chi(o, t)$ is a superset of $\{\chi(B, t) \mid B \in \gamma_2(o)\}$.

According to the above definitions, we can define the multiplicity of a multiset of sharing groups as

$$\chi(Y, t) = \left\{ n \in \mathbb{N} \mid \sum_{o \in Y} \chi_m(o, t) \leq n \leq \sum_{o \in Y} \chi_M(o, t) \right\} . \tag{31}$$

As we said before, this is a superset of all the possible values which can be obtained by combining the multiplicities of all the sharing groups in Y. But, as we will show later, this definition is sufficiently accurate to allows us to design the optimal abstract unification operator.

We extend in the obvious way the definition of *rel* to 2-sharing groups, as $\mathit{rel}(S, x, t) = \{o \in S \mid \llbracket o \rrbracket \cap \mathit{vars}(\{x/t\}) \neq \emptyset\}$, and we prove:

Proposition 3. *The following properties hold:*

1. $\alpha_2(\biguplus \mathcal{S}) = \biguplus \alpha_2(\mathcal{S})$.
2. $\mathit{rel}(\gamma_2(S), x, t)) = \gamma_2(\mathit{rel}(S, x, t))$.

Proof. We begin by proving the first property.

$$\begin{aligned}
&\alpha_2(\uplus\{\!\{B_1, \ldots, B_n\}\!\}) \\
=&\alpha_2\Big(\lambda v \in \bigcup_{1 \leq i \leq n} \llbracket B_i \rrbracket . \sum_{1 \leq i \leq n} B_i(v)\Big) \\
=&\lambda v \in \bigcup_{1 \leq i \leq n} \llbracket B_i \rrbracket . \begin{cases} 1 \text{ if } \sum_{1 \leq i \leq n} B_i(v) = 1 \\ \infty \text{ otherwise} \end{cases} \\
=&\uplus \{\!\{o_1, \ldots, o_n\}\!\} \text{ where } o_i = \lambda v \in \llbracket B_i \rrbracket . \begin{cases} 1 \text{ if } B_i(v) = 1 \\ \infty \text{ otherwise} \end{cases} \\
=&\uplus \{\!\{\alpha_2(B_1), \ldots, \alpha_2(B_n)\}\!\} \ .
\end{aligned}$$

Now we proceed with the proof of the second property.

$$\begin{aligned}
&\mathit{rel}(\gamma_2(S), x, t)) \\
=&\bigcup\{\gamma_2(o) \mid o \in S, \llbracket \gamma_2(o) \rrbracket \cap \mathit{vars}(x = t) \neq \emptyset\} \\
=&\bigcup\{\gamma_2(o) \mid o \in S, \llbracket o \rrbracket \cap \mathit{vars}(x = t) \neq \emptyset\} \qquad (\text{since } \llbracket o \rrbracket = \llbracket \gamma_2(o) \rrbracket) \\
=&\gamma_2(\mathit{rel}(S, x, t))) \ .
\end{aligned}$$

The next step is to design the abstract unification in $\texttt{ShLin}^2$ by using the abstract operators defined above.

Theorem 8. *If $vars(\{x/t\}) \subseteq U$, then*

$$\mathrm{mgu}_2([S]_U, \{x/t\}) = [(S \setminus S') \cup \\ \downarrow\{\biguplus Y \mid Y \in \wp_m(S'), n \in \chi(Y, x) \cap \chi(Y, t).\ n \geq |Y| - 1\}]_U \tag{32}$$

where $S' = rel(S, x, t)$.

Proof. By using Proposition 3(2), and since $o \neq o' \Rightarrow \gamma_2(o) \cap \gamma_2(o') = \emptyset$, we get:

$$\begin{aligned} &\alpha_2(\gamma_2(S) \setminus rel(\gamma_2(S), x, t)) \\ =&\alpha_2(\gamma_2(S) \setminus \gamma_2(rel(S, x, t))) \\ =&\alpha_2(\gamma_2(S \setminus rel(S, x, t))) \\ =&S \setminus rel(S, x, t) \ . \end{aligned}$$

Therefore, we obtain the equality

$$\mathrm{mgu}_2([S]_U, \{x/t\}) = \Big[S \setminus rel(S, x, t) \cup \\ \alpha_2\Big(\{\uplus \mathcal{S} \mid \mathcal{S} \in \wp_m(rel(\gamma_2(S), x, t)), \sum_{B \in \mathcal{S}} \chi(B, x) = \sum_{B \in \mathcal{S}} \chi(B, t) \geq |\mathcal{S}| - 1\}\Big)\Big]_U \tag{33}$$

Now, with simple algebraic manipulations, we obtain:

$$\begin{aligned} &\alpha_2(\{\uplus \mathcal{S} \mid \mathcal{S} \in \wp_m(rel(\gamma_2(S), x, t)), \sum_{B \in \mathcal{S}} \chi(B, x) = \sum_{B \in \mathcal{S}} \chi(B, t) \geq |\mathcal{S}| - 1\}) \\ =&\alpha_2(\{\uplus \mathcal{S} \mid \mathcal{S} \in \wp_m(\gamma_2(rel(S, x, t))), \sum_{B \in \mathcal{S}} \chi(B, x) = \sum_{B \in \mathcal{S}} \chi(B, t) \geq |\mathcal{S}| - 1\}) \\ =&\alpha_2(\{\uplus \{\!\{B_1, \ldots, B_k\}\!\} \mid k \in \mathbb{N}, \\ &\qquad \forall i. B_i \in \gamma_2(rel(S, x, t)), \sum_{1 \leq i \leq k} \chi(B_i, x) = \sum_{1 \leq i \leq k} \chi(B_i, t) \geq k - 1\}) \\ =&\alpha_2(\{\uplus \{\!\{B_1, \ldots, B_k\}\!\} \mid k \in \mathbb{N}, \{\!\{o_1, \ldots, o_k\}\!\} \in \wp_m(rel(S, x, t)), \\ &\qquad \forall i. B_i \in \gamma_2(o_i), \sum_{1 \leq i \leq k} \chi(B_i, x) = \sum_{1 \leq i \leq k} \chi(B_i, t) \geq k - 1\}) \\ =&\alpha_2(\{\uplus \{\!\{B_1, \ldots, B_k\}\!\} \mid k \in \mathbb{N}, \{\!\{o_1, \ldots, o_k\}\!\} \in \wp_m(rel(S, x, t)), \\ &\qquad \forall i. \alpha_2(B_i) = o_i, \sum_{1 \leq i \leq k} \chi(B_i, x) = \sum_{1 \leq i \leq k} \chi(B_i, t) \geq k - 1\}) \\ &\qquad\qquad \text{(such } o_i\text{'s do always exist since } rel(S, x, t) \text{ is downworld closed)} \\ =&\downarrow\{\alpha_2(\uplus \{\!\{B_1, \ldots, B_k\}\!\}) \mid k \in \mathbb{N}, \{\!\{o_1, \ldots, o_k\}\!\} \in \wp_m(rel(S, x, t)), \\ &\qquad \forall i. \alpha_2(B_i) = o_i, \sum_{1 \leq i \leq k} \chi(B_i, x) = \sum_{1 \leq i \leq k} \chi(B_i, t) \geq k - 1\} \\ =&\downarrow\{\uplus \{\!\{\alpha_2(B_1), \ldots, \alpha_2(B_k)\}\!\} \mid k \in \mathbb{N}, \{\!\{o_1, \ldots, o_k\}\!\} \in \wp_m(rel(S, x, t)), \\ &\qquad \forall i. \alpha_2(B_i) = o_i, \sum_{1 \leq i \leq k} \chi(B_i, x) = \sum_{1 \leq i \leq k} \chi(B_i, t) \geq k - 1\} \end{aligned}$$

$$
\begin{aligned}
&= \downarrow\{\uplus\{\!\{o_1,\dots,o_k\}\!\} \mid k\in\mathbb{N}, \{\!\{o_1,\dots,o_k\}\!\}\in\wp_m(rel(S,x,t)),\\
&\qquad \forall i.\alpha_2(B_i)=o_i, \sum_{1\le i\le k}\chi(B_i,x)=\sum_{1\le i\le k}\chi(B_i,t)\ge k-1\}\}\\
&= \downarrow\{\uplus\{\!\{o_1,\dots,o_k\}\!\} \mid k\in\mathbb{N}, \{\!\{o_1,\dots,o_k\}\!\}\in\wp_m(rel(S,x,t)),\\
&\qquad \forall i.\alpha_2(B_i)=o_i, \forall i.\alpha_2(B'_i)=o_i, \sum_{1\le i\le k}\chi(B_i,x)=\sum_{1\le i\le k}\chi(B'_i,t)\ge k-1\}\\
&\qquad\qquad \text{(we discuss later why this is faithful)}\\
&= \downarrow\{\uplus\{\!\{o_1,\dots,o_k\}\!\} \mid k\in\mathbb{N}, \{\!\{o_1,\dots,o_k\}\!\}\in\wp_m(rel(S,x,t)), n\ge k-1,\\
&\qquad n\in\{\sum_{1\le i\le k}B_i(x)\mid \forall i.\alpha_2(B_i)=o_i\}\cap\{\sum_{1\le i\le k}\chi(B'_i,t)\mid \forall i.\alpha_2(B'_i)=o_i\}\}\\
&= \downarrow\{\uplus\{\!\{o_1,\dots,o_k\}\!\} \mid k\in\mathbb{N}, \{\!\{o_1,\dots,o_k\}\!\}\in\wp_m(rel(S,x,t)), n\ge k-1\\
&\qquad n\in[\sum_{1\le i\le k}o_{i\,m}(x), \sum_{1\le i\le k}o_i(x)]\cap\{\sum_{1\le i\le k}\chi(B'_i,t)\mid \forall i.\alpha_2(B'_i)=o_i\}\}
\end{aligned}
$$

The move from a single family $\{B_i\}_{1\le i\le k}$ to different families $\{B_i\}_{1\le i\le k}$ and $\{B'_i\}_{1\le i\le k}$ is possible since, whether

$$
\forall i.\alpha_2(B_i)=o_i, \forall i.\alpha_2(B'_i)=o_i, \sum_{1\le i\le k}\chi(B_i,x)=\sum_{1\le i\le k}\chi(B'_i,t)\ge k-1\ ,
$$

we may define a family $\{C_i\}_{1\le i\le k}$ such that $C_i(x)=B_i(x)$ and $C_i(v)=B'(v)$ for each $v\neq x$. It is immediate to check that the C_i's satisfy the condition

$$
\forall i.\alpha_2(C_i)=o_i, \sum_{1\le i\le k}\chi(C_i,x)=\sum_{1\le i\le k}\chi(C_i,t)\ge k-1\ .
$$

If we denote with $c(\{\!\{o_1,\dots,o_k\}\!\},t)$ the set $\big\{\sum_{1\le i\le k}\chi(B_i,t)\mid \forall i.\alpha_2(B_i)=o_i\big\}$, what remains to prove is that

$$
\begin{aligned}
&\downarrow\{\biguplus X\mid X\in\wp_m(rel(S,x,t)), n\in\chi(X,x)\cap c(X,t).n\ge|X|-1\} \qquad (34)\\
&=\downarrow\{\biguplus X\mid X\in\wp_m(rel(S,x,t)), n\in\chi(X,x)\cap\chi(X,t), n\ge|X|-1\}\ ,
\end{aligned}
$$

where the only difference is that we replaced $c(X,t)$ with $\chi(X,t)$.

We begin by examining the relationship between $c(X,t)$ and $\chi(X,t)$. First of all, it is obvious that $c(X,t)\subseteq\chi(X,t)$, therefore we only need to prove half of the equality.

If there exists $o\in X$ such that $\chi_M(o,t)=\infty$, then $c(X,t)$ is an infinite set. We call n its least element. Under the same conditions, $\chi(X,t)$ is the interval $[n,\infty]$. If there is no $o\in X$ such that $\chi_M(o,t)=\infty$, then $c(X,t)=\chi(X,t)$ and they are both singletons.

In the same way, if there exists some $o\in X$ such that $o(x)=\infty$ then $\chi(X,x)$ is an interval of the kind $[n,\infty)$. However, if there is no such o, then $\chi(X,x)$ is a singleton, whose only element is $|\{o\in X\mid o(x)=1\}|$.

Assume we have $X\in\wp_m(rel(S,x,t))$ such that there exists $n\in\chi(X,x)\cap\chi(X,t)$ with $n\ge|X|-1$. We want to prove that we may find a multiset $Y\in$

$\wp_m(rel(S,x,t))$ such that there exists $m \geq |Y|-1$ with $m \in \chi(Y,x) \cap c(Y,t)$ and $\biguplus X \leq \biguplus Y$. This is enough to complete the proof of the theorem.

We distinguish several cases.

- $\chi(X,x)$ and $\chi(X,t)$ are both infinite. In this case, $c(X,t)$ is infinite. Moroever, since $\chi(X,x)$ is an interval, there are infinite natural numbers in $\chi(X,x) \cap c(X,t)$. We may take $Y = X$.
- $\chi(X,t)$ is infinite and $\chi(X,x)$ is a singleton $\{v\}$, then $v = |\{o \in X \mid o(x) = 1\}| \leq k$. Since it must be $v \geq k-1$, there are only two choices: either $v = k$ or $v = k-1$. We distinguish the two subcases.
  - $v = k-1$. In this case, there exists $o \in X$ such that $\chi_m(o,t) = 0$ and $o(x) = 1$, otherwise it is not possible that $v \geq \chi_m(X,t)$. Since $\chi(X,t)$ is infinite, the same holds for $c(X,t)$, hence we may find an $n \in c(X,t)$ such that $n \geq v$. Consider $Y = X \uplus (n-v) * \{\!\{o\}\!\}$. We have $\chi(Y,x) = \{v+(n-v)\} = n$, $c(Y,t) = c(X,t)$ and $|Y| = |X|+n-v = n+1$. Therefore $n \in c(Y,x) \cap c(Y,t)$ and $n \geq |Y|-1$. $\biguplus Y$ is a valid result, and $\biguplus X \leq \biguplus Y$.
  - $v = k$. If there is an $o \in X$ such that $\chi_m(o,t) = 0$, the proof proceeds as in the previous case. Otherwise, $\chi_m(X,t) \geq k$ and since it should be $v = k \geq \chi_m(X,t)$, we have $\chi_m(X,t) = k$. Therefore $k \in c(X,t)$ too, since $\min c(X,t) = \min \chi(X,t)$, and we may take $Y = X$.
- if $\chi(X,t)$ is finite, then $\chi(X,t) = c(X,t)$ and we take $Y = X$.

Example 15. Let $S = \downarrow\{\emptyset, ux^\infty, vx^\infty, x^\infty y, z^\infty\}$ and $Y = \{\!\{ux^\infty, vx^\infty, xy, z^\infty\}\!\}$. We have $\chi(Y,x) = \{n \mid n \geq 5\}$ and $\chi(Y,t(z,z)) = \{n \mid n \geq 4\}$. Since $t(z,z)$ contains two occurrences of z, the "actual" multiplicity of the sharing group z^∞ in $t(z,z)$ should be a multiple of 2. But we do not need to check this condition and can safely approximate this set with $\{n \mid n \geq 4\}$. Intuitively, this works because we can always choose a number which is contained in both $\chi(Y,x)$ and $\chi(Y,t)$ and which is an "actual" multiplicity. For instance, we can take $n = 6 \in \chi(Y,x) \cap \chi(Y,t(z,z))$ and since we have $6 \geq 3 = |Y|-1$, we get that the sharing group $\biguplus Y = uvx^\infty yz^\infty$ belongs to $\mathrm{mgu}_2([S]_U, \{x/t(z,z)\})$. This sharing group can be generated by the substitution $\{x/t(t(u,u,y), t(v,v,y)), z/t(w,w,w)\}$ when the variables of interest are $\{u,v,x,y,z\}$.

Note 1. It would be interesting to ask ourself whether it is possible to modify the definition of $\chi_m(o,t)$ for a 2-sharing group o in such a way that $\chi_m(o,t) = \chi_m(l(o),t)$. The answer is no. For example, consider the substitution $\theta = \{x/y\}$ and $S = \{\emptyset, ux, ux^\infty, y\}$. The sharing group $o = ux^\infty y$ cannot be generated by $\mathrm{mgu}_2(S,\theta)$: the only way we may obtain o is from the multiset $\{\!\{ux^\infty, y\}\!\}$. However, $\chi(X,x) = [2,\infty)$ and $\chi(X,y) = \{1\}$ and their intersection is empty. If we modify the definition of χ as stated above, we have $\chi(X,x) = [1,\infty)$ and $\chi(X,y) = \{1\}$. Since $|X| = 2$, we obtain o as the valid result.

By Theorem 8 we have a characterization of the abstract unification over $\mathtt{ShLin}^2$. However, this is not amenable of a direct implementation, since it requires to check a certain condition for each element of $\wp_m(S \setminus rel(S,x,t))$, which is an infinite set. However, this is the starting point to prove correctness and completeness of our abstract unification algorithm, which we are going to introduce.

5.1 An algorithm for abstract unification in $\mathtt{ShLin}^2$

We first need to give some definitions which will be used in the abstract unification algorithm. Given $X \subseteq \mathit{rel}(S, x, t)$, we partition X in three subsets $X_x = \{o \in X \mid \chi_M(o, t) = 0\}$, $X_t = \{o \in X \mid \chi_M(o, x) = 0\}$ and $X_{xt} = X \setminus (X_x \cup X_t)$.

Definition 8. *Let $X \subseteq \mathit{rel}(S, x, t)$. We say that X is:*

- linear *for the term t when for all $o \in X$ it holds that $\chi_M(o, t) \leq 1$;*
- non-linear *for the term t when there exists $o \in X$ such that $\chi_M(o, t) > 1$;*
- strongly non-linear *for the term t when there exists a 2-sharing group $o \in X$ such that $\chi_M(o, t) = \infty$ or there exists $o \in X_{xt}$ such that $\chi_M(o, t) > 1$.*

Note that, if t is a variable, the non-linear and strongly non-linear cases do coincide. We now present our algorithm for computing the abstract unification in $\mathtt{ShLin}^2$.

Definition 9. *We define:*

$$\mathrm{mgu}_2'([S]_U, \{x/t\}) = [(S \setminus S') \cup \downarrow \bigcup_{X \subseteq S'} \mathit{res}(X, x, t)]_U \tag{35}$$

where $S' = \mathit{rel}(S, x, t)$ and $\mathit{res}(X, x, t)$ is defined as follows:

- *if X is non-linear for x and t, then $\mathit{res}(X, x, t) = \{\biguplus X^2\}$;*
- *if X is non-linear for x and linear for t, $|X_x| \leq 1$ and $|X_t| \geq 1$, then we have $\mathit{res}(X, x, t) = \{(\biguplus X_x) \uplus (\biguplus X_{xt}^2) \uplus (\biguplus X_t^2)\}$;*
- *if X is linear for x and strongly non-linear for t, $|X_x| \geq 1$ and $|X_t| \leq 1$, then we have $\mathit{res}(X, x, t) = \{(\biguplus X_x^2) \uplus (\biguplus X_{xt}^2) \uplus (\biguplus X_t)\}$;*
- *if X is linear for x and not strongly non-linear for t, $|X_t| \leq 1$, then we have $\mathit{res}(X, x, t) = \{(\biguplus Z) \uplus (\biguplus X_{xt}^2) \uplus (\biguplus X_t) \mid Z \in \wp_m(X_x), |Z| = \chi_M(X_t, t) = \chi_m(X_t, t), \llbracket Z \rrbracket = X_x\}$;*
- *otherwise $\mathit{res}(X, x, t) = \emptyset$.*

The biggest difference of mgu_2' w.r.t. the characterization given in Theorem 8 is that here X is a subset of S', while in (33), $\mathcal{S}$ is a multiset over S'. Since the number of subsets of S' is finite, mgu_2' is computable. Obviously, a direct implementation would be very slow, so that appropriate data structures and procedures should be developed for a real implementation. Although this is out of the scope of the paper, a property which may be of help in an actual implementation appears in Proposition 4 in the next section.

Theorem 9 (Unification Algorithm in $\mathtt{ShLin}^2$). *If $\mathit{vars}(\{x/t\}) \subseteq U$, then*

$$\mathrm{mgu}_2([S]_U, \{x/t\}) = \mathrm{mgu}_2'([S]_U, \{x/t\}) \ .$$

Proof. By Theorem 8, we only need to show that:

$$\downarrow\{\biguplus Y \mid Y \in \wp_m(S'), n \in \chi(Y,x) \cap \chi(Y,t).\ n \geq |Y|-1\} = \downarrow \bigcup_{X \subseteq S'} \mathit{res}(X,x,t) \tag{36}$$

where $S' = \mathit{rel}(S,x,t)$. We prove the two different inclusions separately.

Left to Right Inclusion. Let $\bar{o} \in \mathit{res}(X,x,t)$ for some $X \subseteq \mathit{rel}(S,x,t)$. We want to prove that there exist $Y \in \wp_m(S')$ and $n \in \chi(Y,x) \cap \chi(Y,t)$ such that $n \geq |Y|-1$ and $\uplus Y = \bar{o}$. We distinguish several cases:

- if X is non-linear for x and t, it is $\uplus X^2 = \bar{o}$. We distinguish two subcases:
  - if $\chi_M(X,t) = \infty$, it is enough to take $Y = X \uplus X$.
  - if $\chi_M(X,t)$ is finite, since X is non-linear for t, there exists $o' \in X$ such that $\chi_m(o',t) > 1$. Since S' is downward closed, consider $o \in S$ such that $o(x) = \min(o'(x),1)$ and $o(v) = o'(v)$ if $v \neq x$. We show that there exists a natural number n such that, for $Y = X \uplus X \uplus n\{\!\{o\}\!\}$, we have $\chi_m(Y,t) \geq \chi_m(Y,x)$ and $\chi_m(Y,t) \geq |Y|-1$. Since $\chi_m(Y,x) \leq 2\chi_m(X,x) + n$, we need to solve the inequalities $2\chi_m(X,t) + n\chi_m(o,t) \geq 2\chi_m(X,x) + n$ and $2\chi_m(X,t) + n\chi_m(o,t) \geq 2|X| + n$. Since $\chi_m(o,t) \geq 2$, there always exists a solution for n. Since $\chi_M(X,x) = \infty$, we have that $\biguplus Y = \bar{o}$ is in the left hand side of (36).
- if X is non-linear for x and linear for t. We need to find m such that, if we take $Y = X_x \uplus 2X_{xt} \uplus 2mX_t$, we have $\chi_m(Y,t) \geq \chi_m(Y,x)$. In other words, we need to solve the disequation $2\chi_m(X_{xt},t) + 2m\chi_m(X_t,t) \geq \chi_m(X_x,x) + 2\chi_m(X_{xt},x)$, which is always possible, since $|X_t| \geq 1$. Since $|Y| \leq 1+2|X_{xt}| + 2m|X_t|$ we have $\chi_m(X,t) \geq |Y|-1$.
- if X is linear for x and strongly non-linear for t, we distinguish two subcases:
  - $\chi_M(X,t) = \infty$. Let $n = 2\chi_m(X_{xt},t) + \chi_m(X_t,t)$ and consider any number m such that $2m|X_x| + 2|X_{xt}| \geq n$ (such an m always exists since $|X_x| \geq 1$). Then, consider the multiset $Y = 2mX_x \uplus 2X_{xt} \uplus X_t$, and we have that $\chi_m(Y,x) = \chi_M(Y,x) = 2m|X_x| + 2|X_{xt}| \geq \chi_m(Y,t)$ by construction. Moreover $\chi_M(Y,t) = \infty$ and $|Y| \leq 2m|X_x| + 2|X_{xt}| + 1$. Then $\biguplus Y \in \mathit{res}(X,x,t)$ is a valid resultant sharing group.
  - $\chi_M(X,t)$ is finite. Let $o \in X_{xt}$ be a sharing group such that $\chi_M(o,t) > 1$ and o' be a generic sharing group in X_x. We need to find two natural numbers n and m such that, if we take $Y = 2X_x \uplus 2X_{xt} \uplus X_t \uplus m\{\!\{o\}\!\} \uplus n\{\!\{o'\}\!\}$, we obtain $\chi_m(Y,x) = \chi_m(Y,t)$ (from this immediately follows $\chi_M(Y,x) = \chi_M(Y,t)$) and $\chi_m(Y,x) \geq |Y|-1$. This means we need to solve the equations:

    $$2|X_x| + 2|X_{xt}| + m + n = 2\chi_m(X_{xt},t) + \chi_m(X_t,t) + m\chi_m(o,t)$$

    $$2|X_x| + 2|X_{xt}| + m + n \geq 2|X_x| + 2|X_{xt}| + |X_t| + m + n - 1$$

    Since $|X_t| \leq 1$, the second equation is always satisfied. A solution for the first equation always exists, since the greatest common divisor of $\chi_m(o,t) - 1$ and 1 is 1.

– if X is linear for x and X is not strongly non-linear for t, consider the multiset $Y = Z \uplus X_{xt} \uplus X_{xt} \uplus X_t$. Then $\chi_m(Y,x) = \chi_M(Y,x) = |Z| + 2|X_{xt}|$ and $\chi_m(Y,t) = \chi_M(Y,t) = 2|X_{xt}| + \chi_m(X_t,t)$. Since $|Z| = \chi_m(X_t,t)$, we have that $\chi_m(Y,x) = \chi_m(Y,t)$. Moreover, $|Y| = |Z| + 2|X_{xt}| + |X_t| \leq \chi_m(X_t,t) + 2|X_{xt}| + 1 = \chi_m(Y,t) + 1$.

Right to left inclusion. Let $o = \biguplus X$ where $X \in \wp_m(S')$ and there exists $n \geq |X| - 1$ such that $n \in \chi(X,x) \cap \chi(X,t)$. We show that there exists $Y \subseteq S'$ and $o' \in \mathit{res}(Y,x,t)$ such that $o' \geq_2 o$. Let $k = |X|$. We partition X in three multisets $X_x = X|_{\{o \mid \chi_M(o,t)=0\}}$, $X_t = X|_{\{o \mid \chi_M(o,x)=0\}}$ and $X|_{xt} = X|_{\{o \mid \chi_M(o,t)>0 \wedge \chi_M(o,t)>0\}}$. Note that X_x, X_t and X_{xt} here are multisets and not ordinary set as in the definition of mgu_2'. We distinguish several cases:

– if $\llbracket X \rrbracket$ is linear for x and strongly non-linear for t, it is the case that $\chi_m(X,x) = \chi_M(X,x) = |X_x| + |X_{xt}| \leq k$. Since $\chi_m(X,x) \geq k-1$, there are two cases: either $|X_x| + |X_{xt}| = k-1$ or $|X_x| + |X_{xt}| = k$, which implies that $|X_t| \leq 1$.
  Since $\llbracket X \rrbracket$ is strongly non-linear for t, there exists $o'' \in X_t \uplus X_{xt}$ such that $\chi_M(o'',t) \geq 2$, and thus $\chi_m(X,t) \geq 2$. Therefore $\chi_m(X,t) > |X_{xt}|$. Since $\chi_m(X,x) = \chi_M(X,x) \geq \chi_m(X,t)$, we have that $|X_x| \geq 1$. It follows that $o = \biguplus(X_x \uplus X_{xt} \uplus X_t) \leq_2 (\biguplus\llbracket X_x \rrbracket)^2 \uplus (\biguplus\llbracket X_{xt} \rrbracket)^2 \uplus (\biguplus\llbracket X_t \rrbracket) \in \mathit{res}(\llbracket X \rrbracket, x, t)$.
– if $\llbracket X \rrbracket$ is linear for x and not strongly non-linear for t, then, as in the previous case we have $|X_t| \leq 1$. Since X is not strongly non-linear for t, $\chi_M(X,t) = \chi_m(X,t) = |X_{xt}| + \chi_M(X_t,t)$. Moreover, $\chi_M(X,x) = \chi_m(X,x) = |X_x| + |X_{xt}|$. By the condition $n \in \chi(X,x) \cap \chi(X,t)$, we get $\chi_M(X_t,t) = |X_x|$. Therefore $o \leq_2 \biguplus\llbracket X_x \rrbracket \uplus (\biguplus\llbracket X_{xt} \rrbracket)^2 \uplus (\biguplus X_t) \in \mathit{res}(\llbracket X \rrbracket, x, t)$.
– if $\llbracket X \rrbracket$ is non-linear for x and t, then $o \leq_2 (\biguplus\llbracket X \rrbracket)^2 \in \mathit{res}(\llbracket X \rrbracket, x, t)$.
– if $\llbracket X \rrbracket$ is non-linear for x and linear t, the proof is symmetric to the one of the first case.

5.2 `ShLin`^{2} and multi-binding substitutions

We have developed an abstract unification operator for single-binding substitutions in the domain `ShLin`^{2}. It is possible to extend this operator to multi-binding substitutions in the obvious way,

$$\mathrm{mgu}_2'([S]_U, \{x/t\} \uplus \theta) = \mathrm{mgu}_2'(\mathrm{mgu}_2'([S]_U, \{x/t\}), \theta) \ . \tag{37}$$

However, defined in such a way, mgu_2' is not equal to the optimal abstract unification mgu_2. Consider, for example $S = \{\emptyset, xz, yw\}$, $\theta = \{x/t(y,y), z/w\}$ and $U = \{x,y,z,w\}$. We have $\mathrm{mgu}_2'([S]_U, \{x/t(y,y)\}) = [\downarrow \{\emptyset, x^\infty z^\infty yw\}]_U$. Since $x^\infty zyw \leq_2 x^\infty z^\infty yw$, by applying the third case of mgu_2' to $Y = \{x^\infty zyw\}$ we get

$$\mathrm{mgu}_2'([\downarrow \{\emptyset, z^\infty x^\infty yw\}]_U, \{z/w\}) = [\downarrow \{\emptyset, x^\infty y^\infty z^\infty w^\infty\}]_U \ .$$

However,

$$\begin{aligned}
&\alpha_2(\mathrm{mgu}_\omega(\gamma_2([\{\emptyset, xz, yw\}]_U, \theta)))\\
&= \alpha_2(\mathrm{mgu}_\omega([\{\{\!\!\{\}\!\!\}, xz, yw\}]_U, \theta))\\
&= \alpha_2(\mathrm{mgu}_\omega([\{\{\!\!\{\}\!\!\}, wx^2yz^2\}]_U, \{z/w\}))\\
&= \alpha_2([\{\!\!\{\}\!\!\}]_U) = [\emptyset]_U \ ,
\end{aligned} \tag{38}$$

which shows that mgu_2 is not optimal. The problem is that, to be able to conclude that the unification of S with θ is ground, we need to keep track of the fact that, after the first binding, w is linear and z is definitively non-linear. Since $\texttt{ShLin}^2$ is downward closed, we are not able to state this property. Note that, in the case we have presented here, by changing the order of the binding we get an optimal result in $\texttt{ShLin}^2$, but this happens just by accident.

Now, consider the substitution $\theta = \{x/t(y, ..., y), z/s(y, ..., y), u/v\}$ with $S = \{\emptyset, xu, zv, y\}$ and $U = \{u, v, x, y, z\}$. Assume $t(y, ..., y)$ is an n-ary term, $s(y, ..., y)$ is an m-ary term with $n \neq m$ and $n, m \geq 2$. We have that:

$$\begin{aligned}
&\mathrm{mgu}'_2([S]_U, \{x/t(y, ..., y)\}) = [\downarrow\{\emptyset, x^\infty u^\infty y, zv\}]_U \ ,\\
&\mathrm{mgu}'_2([\downarrow\{\emptyset, x^\infty u^\infty y, zv\}]_U, \{z/s(y, ..., y)\}) = [\downarrow\{\emptyset, x^\infty u^\infty z^\infty v^\infty y\}]_U \ ,\\
&\mathrm{mgu}'_2([\downarrow\{\emptyset, x^\infty u^\infty z^\infty v^\infty y\}]_U, \{u/v\}) = [\downarrow\{\emptyset, x^\infty u^\infty z^\infty v^\infty y\}]_U \ .
\end{aligned} \tag{39}$$

On the other side, we have that:

$$\begin{aligned}
&\alpha_2(\mathrm{mgu}_\omega(\gamma_2([\{\emptyset, xu, zv, y\}]_U, \theta)))\\
&= \alpha_2(\mathrm{mgu}_\omega([\{\{\!\!\{\}\!\!\}, xu, zv, y\}]_U, \theta))\\
&= \alpha_2(\mathrm{mgu}_\omega([\{\{\!\!\{\}\!\!\}, x^n u^n y, zv\}]_U, \{z/s(y, ..., y), u/v\}))\\
&= \alpha_2(\mathrm{mgu}_\omega([\{\{\!\!\{\}\!\!\}, x^n u^n y z^m v^m\}]_U, \{u/v\}))\\
&= \alpha_2([\{\!\!\{\}\!\!\}]_U) = [\emptyset]_U \ .
\end{aligned} \tag{40}$$

However, if $n = m$, we have:

$$\begin{aligned}
&\alpha_2(\mathrm{mgu}_\omega(\gamma_2([\{\emptyset, xu, zv, y\}]_U, \theta)))\\
=&\alpha_2([\{\{\!\!\{\}\!\!\}\} \cup \{x^{kn} u^{kn} y^k z^{kn} v^{kn} \mid k \in \mathbb{N}\}]_U)\\
=&[\downarrow\{\emptyset, x^\infty u^\infty z^\infty v^\infty y\}]_U \ .
\end{aligned} \tag{41}$$

In this case, keeping track of the variables which are definitively non-linear does not help. It seems that, in order to compute abstract unification one binding at a time, we need to work in a domain which is able to keep track of the exact multiplicity of variables in a sharing group. Actually, this is how $\texttt{ShLin}^\omega$ works. Obviously, we could try to develop a different algorithm for unification in $\texttt{ShLin}^2$ which directly works with multi-binding substitutions. However, since the algorithm for single-binding substitutions is already quite complex, we think this is not worth the effort.

6 An Application: the Domain $\mathtt{Sharing} \times \mathtt{Lin}$

In this section we deal with the reduced product $\mathtt{ShLin} = \mathtt{Sharing} \times \mathtt{Lin}$. This domain has been used from the beginning of the studies in the analysis of aliasing properties, since it was recognized that the precision of these analysis could be greatly improved by keeping track of the linear variables. Among the papers which treat the domain ShLin, we refer to [12] and [14]. Actually, these papers also deal with freeness properties to further improve precision, which we do not consider here.

Although the domain ShLin has been used for so many years, an optimal abstract unification was not known, not event for a single binding substitution. We provide here a new abstract unification operator for ShLin, designed from the abstract unification for $\mathtt{ShLin}^2$, and we prove that it is optimal.

We briefly recall the definition of the abstract domain and show the abstraction function from King's domain $\mathtt{ShLin}^2$ to ShLin. The domain is

$$\begin{aligned}\mathtt{ShLin} = \{[S, L, U] \mid S \subseteq \wp(U), (S \neq \emptyset \Rightarrow \emptyset \in S),\\ L \supseteq U \setminus \mathit{vars}(S), U \in \wp_f(\mathcal{V})\}\ ,\end{aligned}$$

with the approximation relation $\leq_{sl}$ defined as $[S, L, U] \leq_{sl} [S', L', U']$ iff $U = U'$, $S \subseteq S'$, $L \supseteq L'$. The abstraction map from $\mathtt{ShLin}^2$ to ShLin is defined in the obvious way:

$$\alpha_{sl}([S]_U) = [\{\llbracket o \rrbracket \mid o \in S\}, \{x \in U \mid \forall o \in S.\ o(x) \leq 1\}, U]\ . \tag{42}$$

The composition of α_ω, α_2 and α_{sl} gives the standard abstraction map from substitutions to ShLin.

We call *sharing group* an element of $\wp_f(\mathcal{V})$. As for the previous domains, we use the polynomial notation to represent sharing groups, but now all the exponents are fixed to one. Note that the last component U in $[S, L, U]$ is redundant since it can be computed as $L \cup \mathit{vars}(S)$. This is because the set L keeps track of both linear and ground variables.

6.1 Abstract unification

We want to design an abstract operator over ShLin which is optimal for the unification of a single binding. To obtain a correct and optimal abstract unification over ShLin, the trivial way is to compute $\alpha_{sl}(\mathrm{mgu}_2(\gamma_{sl}([S, L, U]), \{x/t\}))$. However, we prefer to give an unification operator which is similar (to the extent it is possible) to the other unification operators for ShLin known in the literature [15, 4, 14]. Given a set L of linear variables, we define the maximum multiplicity of a sharing group o in a term t as follows:

$$\chi_M^L(o, t) = \begin{cases} \sum_{v \in o} \mathit{occ}(v, t) & \text{if } o \cap \mathit{vars}(t) \subseteq L \\ \infty & \text{otherwise} \end{cases} \tag{43}$$

According to the similar definition for 2-sharing groups, given $[S, L, U] \in \mathtt{ShLin}$, we say that (S, L) is *linear* for a term t when for all $o \in S$ it holds that $\chi^L_M(o, t) \leq 1$. Note that, when t is a variable, the definition boils down to check whether $t \in L$.

Given $X \subseteq \mathit{rel}(S, x, t)$ and a set L, we partition X in three subsets $X^L_x = \{o \in X \mid \chi^L_M(o, t) = 0\}$, $X^L_t = \{o \in X \mid \chi^L_M(o, x) = 0\}$ and $X^L_{xt} = X \setminus (X_x \cup X_t)$. In the following, we will abuse the notation and write X_x, X_t and X_{xt} when the set of linear variables L is clear from the context. The same applies to the following definitions:

$$\begin{aligned}
X^{=\infty}_t &= \{B \in X_t \mid \chi^L_M(B, t) = \infty\} \\
X^{\in \mathbb{N}}_t &= \{B \in X_t \mid \chi^L_M(B, t) \in \mathbb{N}\} \\
X^{=1}_t &= \{B \in X_t \mid \chi^L_M(B, t) = 1\} \\
X^{>1}_t &= \{B \in X_t \mid \chi^L_M(B, t) > 1\} \\
X^{>1}_{xt} &= \{B \in X_{xt} \mid \chi^L_M(B, t) > 1\} \\
X^{=1}_{xt} &= \{B \in X_{xt} \mid \chi^L_M(B, t) = 1\} \ .
\end{aligned} \tag{44}$$

We also need to denote the set:

$$X^U_{xt} = \{B \in X_{xt} \mid \chi^U_M(B, t) = 1\} \tag{45}$$

which corresponds to the *linearizable* sharing groups. Since we deal with downword closed domains, we need to take into account also the sharing groups which can be build by linearizing variables: with X^U_{xt} we consider all the variables as linear by using the set U instead of L when computing the multiplicity.

Moreover, given sets $A_1, \ldots, A_n$ with $n \geq 2$ we denote by $\mathrm{bin}(A_1, \ldots, A_n)$ the set $\{\{a_1, \ldots, a_n\} \mid a_1 \in A_1, \ldots, a_n \in A_n\}$, by A^* the set $\{\bigcup B \mid B \subseteq A\}$ and by A^+ the set $\{\cup B \mid B \subseteq A, B \neq \emptyset\}$[2]. Now we can define the abstract unification mgu_{sl} as follows:

$$\mathrm{mgu}_{sl}([S, L, U], \{x/t\}) = [(S \setminus X) \cup K, U' \cup L', U] \tag{46}$$

where $X = \mathit{rel}(S, x, t) = \{B \in S \mid B \cap \mathit{vars}(\{x/t\}) \neq \emptyset\}$ and $U' = U \setminus \mathit{vars}((S \setminus X) \cup K)$. Here, K is the set of new sharing groups created by the unification process and U' is the set of variables which do not appear in any sharing group of the result, i.e. the set of ground variables. Finally, L' is the set of linear variables in the result which are not ground. The set K of new sharing groups may be defined as follows:

[2] This notation slightly deviates from most of other literature on `Sharing`, where A^* does not include the empty set. We prefer to adopt a double notation, namely A^* and A^+, which is more standard in the rest of the research community.

– if $x \in L$ then

$$\begin{aligned} K = &\operatorname{bin}(X_t^{=\infty}, X_x^+, X_{xt}^*) \cup \\ &\operatorname{bin}(X_t \cup \{\emptyset\}, X_{xt}^{>1}, X_x^+, X_{xt}^*) \cup \\ &\operatorname{bin}(\{\{o\} \cup (\cup Z) \mid o \in X_t^{\in\mathbb{N}}, Z \subseteq X_x, 1 \leq |Z| \leq \chi_M^L(o,t)\}, (X_{xt}^{=1})^*) \cup \\ &(X_{xt}^U)^+ \end{aligned} \tag{47}$$

– if $x \notin L$ then

$$\begin{aligned} K = &\operatorname{bin}(X_t^{>1} \cup X_{xt}^{>1}, X_x \cup X_{xt}, X^*) \cup \\ &\operatorname{bin}((X_t^{=1})^+, X_x \cup X_{xt}^{=1}, (X_{xt}^{=1})^*) \cup \\ &(X_{xt}^{=1})^+ \end{aligned} \tag{48}$$

while the set L' of linear variables which are not ground is

$$L' = \begin{cases} L \setminus (\mathit{vars}(X_x \cup X_{xt}) \cap \mathit{vars}(X_t \cup X_{xt})) & \text{if } (S,L) \text{ is linear for } x \text{ and } t \\ L \setminus \mathit{vars}(X_x \cup X_{xt}) & \text{otherwise, if } (S,L) \text{ is linear for } x \\ L \setminus \mathit{vars}(X_t \cup X_{xt}) & \text{otherwise, if } (S,L) \text{ is linear for } t \\ L \setminus \mathit{vars}(X) & \text{otherwise} \end{cases} \tag{49}$$

In the rest of this section we will prove that mgu_{sl} is indeed the optimal abstract unification for `ShLin`.

6.2 Proof of optimality

First of all, we show that the definition of $\mathrm{mgu}_2([S]_U, \{x/t\})$ may be modified to only consider *maximal* subsets of S. This is a property which is interesting in itself, since it can help in the actual implementation of mgu_2, and it will be useful in the proof of optimality of `ShLin`.

Given $[A]_U \in \mathtt{ShLin}^2$, let $\max A$ be the set of maximal elements of A, in formula $\{a \in A \mid \nexists b \in A. b >_2 a\}$. Given a 2-sharing group o, we define the *linearized* version of o, denote by $l(o)$, as

$$l(o)(v) = \begin{cases} 1 & \text{if } v \in \llbracket o \rrbracket \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

The linearization operator l is extended pointwise to sets of 2-sharing groups. We show that instead of choosing X as a subset of S' in the definition of mgu_2', we may only consider those X which are subsets of $\max S'$, with only a slight change in Definition 9.

Proposition 4.

$$\mathrm{mgu}_2([S]_U, \{x/t\}) = [(S \setminus S') \cup \downarrow \bigcup_{X \subseteq \max S'} (\mathit{res}(X,x,t) \cup \mathit{res}'(X,x,t))]_U \tag{51}$$

where $S' = rel(S,x,t)$ *and*

$$res'(X,x,t) = \begin{cases} \{\biguplus X^2\} & \text{if } X = X_{xt} \text{ and } l(X) \text{ is linear for } t \\ \emptyset & \text{otherwise} \end{cases}$$

Proof. It clearly holds that:

$$\mathrm{mgu}_2([S]_U, x/t) \supseteq [(S \setminus S') \cup \downarrow \bigcup_{X \subseteq \max S'} (res(X,x,t) \cup res'(X,x,t))]_U \tag{52}$$

since, for each $X \subseteq \max S'$, if $res'(X,x,t)$ is non-empty then $\biguplus X^2$ may be generated by Definition 9. It is enough to take $X' = \{l(o) \mid o \in X\}$, hence $\biguplus X' = \biguplus X^2 \in res(X',x,t)$ according to the last case of Definition 9.

We prove the opposite inclusion. Let $X \subseteq S'$ and assume that $X \nsubseteq \max S'$. There exists $X' \subseteq \max S'$ obtained by replacing each $a \in X$ with $b \in \max S'$ such that $a \leq_2 b$. We have that $|X'| \leq |X|$ since two different elements in X may be replaced with the same maximal element in X'. We want to prove that either $res(X,x,t) = \emptyset$, or $res(X,x,t) \subseteq \downarrow res(X',x,t)$ or $res(X,x,t) \subseteq \downarrow res'(X',x,t)$. Therefore, we assume $res(X,x,t) \neq \emptyset$ and compare the linearity properties (linear, non linear, strongly non-linear) of X' w.r.t. those of X.

If they coincide, then it follows that $res(X,x,t) \subseteq \downarrow res(X',x,t)$. This happens because both $res(X,x,t)$ and $res(X',x,t)$ are obtained by the same case of Definition 9. However, note that X' may have less elements than X and therefore some variable which is non-linear in $res(X,x,t)$ could be linear in $res(X',x,t)$. Actually, this never happens since the elements in X' which are not explicitly delinearized are either elements of the multiset Z in the third case of Definition 9 (and therefore may appear multiple times) or elements of X_t (X_x) subject to the condition $|X_t| \leq 1$ ($|X_x| \leq 1$).

Assume that the linearity properties of X and X' do not coincide. The only interesting case is when X is linear for x and not strongly non-linear for t. In all the other cases, it is immediate from the definition that $res(X,x,t) \subseteq \downarrow res(X',x,t)$.

If X' is not linear for x and for t, then it holds $res(X,x,t) \subseteq \downarrow res(X',x,t)$ by definition.

If X' is linear for x and strongly non-linear for t, then it is immediate from the definition that $res(X,x,t) \subseteq \downarrow res(X',x,t)$, provided that $|X_x| \geq 1$. Otherwise, it must be $|X_t| = 0$ and therefore, in order to be $res(X,x,t) \neq \emptyset$, we have $X = X_{xt}$ and $\chi_M(X,t) = 1$, which means $l(X') = X$ is linear for t. It follows that $res(X,x,t) = \{\biguplus X^2\} = res'(X',x,t)$.

If X' is not linear for x and linear for t, we show that $|X_x| \leq 1$. Assume, by contradiction, that $|X_x| > 1$. Since X' is linear for t and $|X_t| \leq 1$, then $\chi_M(X_t,t) = \chi_m(X_t,t) \leq 1$, while $|\llbracket Z \rrbracket| = |X_x| > 1$, which is a contradiction. Thus it must be $|X_x| \leq 1$. If $|X_x| = 0$ then $|X_t| = 0$, hence $res(X,x,t) = \{\biguplus X^2\}$ and $res(X,x,t) = res'(X,x,t)$. If $|X_x| = 1$, since X' is linear for t, it follows that $|Z| = 1$. Thus $res(X,x,t) \subseteq \downarrow res(X',x,t)$.

Theorem 10. mgu_{sl} *is correct and optimal w.r.t.* mgu.

Proof. It is enough to prove that mgu_{sl} is correct and optimal w.r.t. mgu_2, that is to say that:

$$\mathrm{mgu}_{sl}([S,L,U]),\{x/t\}) = \alpha_{sl}(\mathrm{mgu}_2(\gamma_{sl}([S,L,U]),\{x/t\})) \enspace .$$

Let $\gamma_{sl}([S,L,U]) = [T]_U$. By Proposition 4, it holds that:

$$\begin{aligned}
&\alpha_{sl}(\mathrm{mgu}_2(\gamma_{sl}([S,L,U]),\{x/t\})\\
=&\alpha_{sl}([(T \setminus T') \cup \downarrow \bigcup_{Y \subseteq \max T'} (\mathit{res}(Y,x,t) \cup \mathit{res}'(Y,x,t)\})]_U\\
=&\alpha_{sl}\big([T \setminus T']_U \sqcup_2 \bigsqcup_{Y \subseteq \max T'} ([\downarrow \mathit{res}(Y,x,t)]_U \sqcup_2 [\downarrow \mathit{res}'(Y,x,t)\}]_U)\big)
\end{aligned}$$

where $T' = \mathit{rel}(T,x,t)$ and $\sqcup_2$ is the lowest upper bound in $\mathtt{ShLin}^2$. By additivity of α_{sl}, this is equivalent to

$$\alpha_{sl}([T \setminus T']_U) \sqcup_{sl} \bigsqcup_{Y \subseteq \max T'} (\alpha_{sl}([\mathit{res}(Y,x,t)]_U) \sqcup_{sl} \alpha_{sl}([\mathit{res}'(Y,x,t)]_U)) \enspace . \tag{53}$$

Let X, L', U' and K as in (46), we have that $\mathrm{mgu}_{sl}([S,L,U],\{x/t\})$ is equivalent to

$$[(S \setminus X) \cup K, U' \cup L', U] \enspace . \tag{54}$$

We need to prove that equations (53) and (54) do coincide. In the rest of the paper, we assume the result of (53) is $[S'', L'', U]$.

Sharing. We first prove that the $\mathtt{Sharing}$ components of the two equations are equal, i.e. that $S'' = (S \setminus X) \cup K$. Given $B \in S''$, there are several cases. If $B = \llbracket o \rrbracket$ with $o \in T \setminus T'$, then $B \in S \setminus X$.

If $B = \llbracket o \rrbracket$, for $o \in \mathit{res}'(Y,x,t)$ with $Y \subseteq \max T'$, then $B = \bigcup\{\llbracket o \rrbracket \mid o \in Y\}$ with $Y = Y_{xt}$ and $l(Y)$ is linear for t. If $x \in L$ then is generated by $(X^U_{xt})^+$, since $l(Y)$ is linear for t. If $x \notin L$ there are two cases: if Y is linear for t then it is generated by $(X^{=1}_{xt})^+$, otherwise by $\mathrm{bin}(X^{>1}_t \cup X^{>1}_{xt}, X_x \cup X_{xt}, X^*)$. Thus $B \in K$.

Now, assume $B = \llbracket o \rrbracket$ with $o \in \mathit{res}(Y,x,t)$ and $\emptyset \neq Y \subseteq \max T'$. Then $B = \bigcup W$ where $W = \{\llbracket o \rrbracket \mid o \in Y\}$. Since Y is made of maximal elements and $[T]_U = \gamma_2([S,L,U])$, we have that Y is linear for x iff $x \in L$. For the same reason, Y is linear for t iff (W,L) is linear for t. As a consequence, if Y is non-linear for t, then (X,L) is non-linear for t.

We proceed by cases:

- Y **non-linear for** x **and** t**.** Then $\mathit{res}(Y,x,t) = \{\biguplus Y^2\}$. Since (X,L) is non-linear for x and t, we have $X^{>1}_t \cup X^{>1}_{xt} \neq \emptyset$ and $X_x \cup X_{xt} \neq \emptyset$. Thus $B \in \mathrm{bin}(X^{>1}_t \cup X^{>1}_{xt}, X_x \cup X_{xt}, X^*) \subseteq K$.
- Y **non-linear for** x **and linear for** t**.** By hypothesis $|Y_x| \leq 1$ and $|Y_t| \geq 1$, hence $o = (\biguplus Y_x) \uplus (\biguplus Y^2_{xt}) \uplus (\biguplus Y^2_t)$ and

  $$B \in \mathrm{bin}((X^{=1}_t)^+, X_x \cup X^{=1}_{xt}, (X^{=1}_{xt})^*) \subseteq K \enspace .$$

  In particular, $B \in \mathrm{bin}((X^{=1}_t)^+, X_x, (X^{=1}_{xt})^*)$ when $|Y_x| = 1$, otherwise $B \in \mathrm{bin}((X^{=1}_t)^+, X^{=1}_{xt}, (X^{=1}_{xt})^*)$.

Y **linear for x and strongly non-linear for t.** In this case $o = (\biguplus Y_x^2) \uplus (\biguplus Y_{xt}^2) \uplus (\biguplus Y_t)$ with $|Y_x| \geq 1$ and $|Y_t| \leq 1$. By definition of strongly non-linearity, we have two cases:

- there exists $o \in Y_{xt}$ such that $\chi_M(o,t) > 1$: in this case
$$B \in \mathrm{bin}(X_t \cup \{\emptyset\}, X_{xt}^{>1}, X_x^+, X_{xt}^*) \subseteq K \ \ ;$$
- there exists $o \in Y_t$ such that $\chi_M(o,t) = \infty$: in this case
$$B \in \mathrm{bin}(X_t^{=\infty}, X_x^+, X_{xt}^*) \subseteq K \ \ .$$

Y **linear for x and non strongly non-linear for t.** In this case
$$o = (\biguplus Z') \uplus (\biguplus Y_{xt}^2) \uplus (\biguplus Y_t)$$
with $|Y_t| = 1$, for some $Z' \in \wp_m(Y_x)$ such that $|Z'| = \chi_m(Y_t, t)$ and $\llbracket Z' \rrbracket = Y_x$. It is obvious that
$$B \in \mathrm{bin}(\{\{o\} \cup (\cup Z) \mid o \in X_t^{\in \mathbb{N}}, Z \subseteq X_x, 1 \leq |Z| \leq \chi_M^L(o,t)\}, (X_{xt}^{=1})^*) \subseteq K$$
by choosing $Z = \{\llbracket o \rrbracket \mid o \in Z'\}$.

This proves that if $B \in S''$, then $B \in (S \setminus X) \cup K$. Now, we need to prove the converse implication. If $B \in S \setminus X$, then $B = \llbracket o \rrbracket$ for some $o \in T$, and it is obvious that $o \in T \setminus T'$, hence $B \in S''$.

Therefore, assume $B \in K$, and consider the case when $x \in L$ and $B \in \mathrm{bin}(X_t^{=\infty}, X_x^+, X_{xt}^*)$. We have that $B = A \cup (\cup A') \cup (\cup A'')$ for some $A \in X_t^{=\infty}$, A' non-empty subset of X_x and $A'' \subseteq X_{xt}$. We may find $o' \in \max T'$, $Y', Y'' \subseteq \max T'$ such that $\llbracket o' \rrbracket = A$, $\llbracket Y' \rrbracket = A'$ and $\llbracket Y'' \rrbracket = A''$. We have that $Y''' = \{o'\} \cup Y' \cup Y''$ is linear for x and strongly non-linear for t (due to the element o'), with $|Y_x'''| \geq 1$ and $|Y_t'''| \leq 1$. Therefore, we may apply the definition of *res* to obtain $\mathit{res}(Y''', x, t) = \{o\}$ with $\llbracket o \rrbracket = B$, hence $B \in S''$.

With similar reasonings, we may prove that for every $B \in K$, we have $B \in S''$. In particular: the second line of (47) corresponds to the case we choose a Y''' which is linear for x and strongly non-linear for t, due to an element $o \in Y_{xt}'''$ which $\chi_M(o,t) > 1$; the third line of (47) corresponds to the case Y''' is linear for X and is not strongly non-linear for t; the first line of (48) corresponds to the case Y''' is non-linear for both x and t; the second line of (48) corresponds to the case Y''' is linear for t and non-linear for x.

Finally, if $x \notin L$ and $B \in (X_{xt}^{=1})^+$, it is possible that B cannot be obtained as $\mathit{res}(Y''', x, t)$ for any $Y''' \subseteq \max T'$. However, B may be obtained as $\mathit{res}'(Y''', x, t)$, choosing Y''' as in the previous cases. The same happens if $x \in L$and $B \in (X_{xt}^U)^+$.

Linearity. We want to prove that $L'' = L' \cup U'$. First of all, let us define $L_g'' = U \setminus \mathit{vars}(\mathrm{mgu}_2([T]_U, \{x/t\}))$ the set of ground variables in $\mathrm{mgu}_2([T]_U, \{x/t\})$, hence $L_g'' \subseteq L''$. We are going to prove that $U' = L_g''$ and $L' \setminus U' = L'' \setminus L_g''$. The first equality trivially follows from the fact that the sharing component of mgu_{sl}

is optimal, hence a variable occurs in a sharing group of $S \setminus S \cup K$ iff it occurs in a 2-sharing group of $\mathrm{mgu}_2([T]_U, \{x/t\})$.

Now, we consider a variable $v \in U \setminus U'$, and prove that $v \in L'$ iff $v \in L''$. There are several cases. If we assume $v \notin L$, by (49) we have $v \notin L'$. Moreover, if $Y \in \max T'$ and $v \in \llbracket Y \rrbracket$, by maximality of Y we have $Y(v) = \infty$. Hence, by Proposition 4, we have $v \notin L''$. If we assume $v \notin X$, by (49) we have $v \in L'$ iff $v \in L$. Since $\mathit{vars}(X) = \mathit{vars}(T)$, we also have $v \in L''$ iff $v \in L$ and therefore $v \in L'$ iff $v \in L''$.

The only case it remains to prove is $v \in \mathit{vars}(X) \cap L$ which, combined with the condition $v \notin U'$, gives $v \in \mathit{vars}(K) \cap L$. First of all, note that if $v \in \mathit{vars}(X_{xt})$ then $v \notin L'$ (by definition of L') and $v \notin L''$ (since X_{xt}) appears delinearized in every 2-sharing group resulting from *res* or *res*$'$. If $v \notin \mathit{vars}(X_{xt})$, we distinguish four subcases:

- $x \in L$ and (S, L) linear for t. Given $Y \subseteq \max T'$, checking the forth case of Definition 9 when $\chi_M(Y_t, t) = 1$, we have that $\mathit{res}(Y, x, t)$ is not linear for v iff $v \in \mathit{vars}(Y_{xt})$ or $v \in \mathit{vars}(Y_x) \cap \mathit{vars}(Y_t)$. Note that there exists $Y \subseteq \max T'$ s.t. $v \in \mathit{vars}(Y_{xt}) \cup (\mathit{vars}(Y_x) \cap \mathit{vars}(Y_t))$ iff $v \in \mathit{vars}(T'_{xt}) \cup (\mathit{vars}(T'_x) \cap \mathit{vars}(T'_t))$. Finally $v \in L''$ iff $v \in \mathit{vars}(T'_{xt}) \cup (\mathit{vars}(T'_x) \cap \mathit{vars}(T'_t))$ iff $v \in (X_{xt} \cup (X_x \cap X_t))$ iff $v \in L'$.
- $x \in L$ and (S, L) not linear for t. Given $Y \subseteq \max T'$, checking the third and forth cases (when $\chi_M(Y, t) > 1$) of Definition 9, we have that $\mathit{res}(Y, x, t)$ non-linear for v implies $v \in \mathit{vars}(Y_{xt})$ or $v \in \mathit{vars}(Y_x)$, which is equivalent to $v \in X_{xt} \cup X_x$, i.e. $v \notin L'$. On the other side, if $v \in X_x$, we distinguish the cases:
  - (S, L) strongly non-linear for t. There exists $o \in T'$ such that $\chi_M(o, t) = \infty$ or $o \in T'_{xt}$ such that $\chi_M(o, t) > 1$. Moreover, there exists $o' \in T'_x$ such that $v \in \llbracket o' \rrbracket$. If we take $Y = \{o, o'\}$, we have that $\mathit{res}(Y, x, t)$ is not linear for v, hence $v \notin L''$.
  - (S, L) is not strongly non-linear for t. There exists $o \in T'_t$ such that $1 < \chi_M(o, t) < \infty$. Moreover, there exists $o' \in T'_x$ such that $v \in \llbracket o' \rrbracket$. If we take $Y' = \{o, o'\}$, by the fourth case in the definition of *res*, we have $\mathit{res}(Y, x, t)$ is not linear for v, i.e. $v \notin L''$.
- $x \notin L$ and (S, L) linear for t. If $v \notin L''$ then $v \in \mathit{vars}(Y_{xt})$ or $v \in \mathit{vars}(Y_t)$. This implies $v \in X_{xt} \cup X_t$, i.e. $v \notin L'$. On the other side, if $v \in X_t$, there exist $o \in T'_x$ such that $\chi_M(o, x) = \infty$ and $o' \in T'_t$ such that $v \in \llbracket o' \rrbracket$. By definition of *res*, we have that $\mathit{res}(\{o, o'\}, x, y)$ is not linear for v, hence $v \notin L''$.
- $x \notin L$ and (S, L) non-linear for t. Since $L' = L \setminus X$, it is obvious that $v \notin L'$. Moreover, there exist $o \in T'$ such that $\chi_M(o, x) = \infty$, $o' \in T'$ such that $\chi_M(o, t) > 1$ and $o'' \in T'$ such that $v \in \llbracket o'' \rrbracket$. Note that it is possible that $o = o' = o''$. By definition, we have $\mathit{res}(\{o, o', o''\}, x, t)$ is not linear for v, hence $v \notin L''$.

Example 16. Let $S = \{xv, xy, zw\}, L = \{x, y, v, w\}$ and consider the binding $x = t(y, z)$. Then $\chi^L_M(xv, t) = 0$, since $xv \cap \mathit{vars}(t) = \emptyset$, $\chi^L_M(xy, t) = 1$ and $\chi^L_M(zw, t) = \infty$. As a result (S, L) is non-linear for t. In words, it means that the sharing group zw is not linear in t and that t itself is not linear. Note that S is linear for x since $x \in L$.

Applying Definition 9 as stated above, we obtain $S' = \{xy, xyzvw, xzvw\}$ and $L' = \{w\}$. This is more precise that the standard operators for $\mathtt{Sharing} \times \mathtt{Lin}$ [12]. Actually even with the optimizations proposed in [15, 14] or [4], the standard operator is not able to infer that the sharing group xyv is not a possible result of the concrete unification. Note that it would be possible in a domain for rational trees, where the unification of $\{x/t(t(v, y), c), z/w\}$ with $x/t(y, z)$ succeeds with $\{x/t(t(v, y), c), z/c, w/c, y/t(v, y)\}$. This means that we are able to exploit the occur-check of the unification in finite trees. As a consequence, our abstract unification operator is not correct w.r.t. a concrete domain of rational substitutions [18]. However, our results improve over the abstract unification operators of the domains in the literature even in some cases which do not involve the occur-check. For example, if $S = \{xa, xb, xy, z\}$, $L = \{x, a, b, z\}$ and given the binding $x/t(y, z)$, we are able to state that $xzab$ is not a member of $\mathrm{mgu}_{sl}([S, L, \mathit{vars}(S)], \{x = t(y, z)\})$, but the domains in [15, 4, 14] cannot.

An alternative to mgu_{sl} would be to compute $\alpha_{sl} \circ \mathrm{mgu}_2 \circ \gamma_{sl}$ following Definition 9 with χ_M and $\uplus$ replaced by χ_M^L and $\cup$ respectively (we can obviously ignore the delinearization operator $(\_)^2$ since $B \cup B = B$). However, we do not pursue further this approach.

Although the abstract operator mgu_{sl} is optimal for the unification with a single binding, the optimal operator for a multi-binding substitution cannot be obtained by considering one binding at a time. This is a direct consequence of the fact that the corresponding operator for single-binding unification on $\mathtt{ShLin}^2$ can not be extended to an optimal multi-binding operator, by simply considering one binding at a time. In fact, the counterexamples in Section 5.2 are still counterexamples for mgu_{sl} since $[S]_U = \gamma_{sl}(\alpha_{sl}([S]_U))$.

7 Conclusion and Future Work

We summarize the main results of this paper.

- We propose a new domain $\mathtt{ShLin}^\omega$ as a general framework for investigating sharing and linearity properties. We introduce the notion of *sharing graph* as a generalization of the concept of alternating path [24, 18] used for pair sharing analysis and provide optimal abstract operators for $\mathtt{ShLin}^\omega$.
- We show that $\mathtt{ShLin}^\omega$ is a useful starting point for studying further abstractions. We obtain the optimal operators for single binding abstract unification in $\mathtt{Sharing} \times \mathtt{Lin}$ and King's domain $\mathtt{ShLin}^2$. This is the first paper which shows optimality results for a domain obtained by combining sharing and linearity information. Actually in [7] a variant of $\mathtt{Sharing} \times \mathtt{Lin}$ is proposed, based on *set logic programs*. However, despite the claim in the paper, the proposed operators are not optimal, as shown in [14]. Also the operators in [18] for $\mathtt{ASub}$ are not optimal when working over finite trees.

Several things remain to be explored: first of all, we plan to analyze the domain PSD $\times$ Lin [3] in our framework and, possibly, to devise a variant of $\texttt{ShLin}^2$ which enjoys a similar closure property for redundant sharing groups. This could be of great impact on the efficiency of the analysis. Moreover, we need to study the impact on the precision and performance obtained by adopting the new optimal operators and domains. We plan to implement both of them within the CiaoPP [5] static analyzer.

In the recent years, many efforts has been made to study the behavior of logic programs in the domain of *rational trees* [18, 25], since they formalize the standard implementations of logic languages. We have shown that our operators, which are optimal for finite trees, are not correct for rational trees, since they exploit the occur-check to reduce the sharing groups generated by the abstract unification (see Example 16). It would be interesting to adapt our framework to work with rational trees, in order to obtain optimal operators also in this case.

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