Finding maxmin allocations in cooperative and competitive fair division

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Abstract

We consider upper and lower bounds for maxmin allocations of a completely divisible good in both competitive and cooperative strategic contexts. We then derive a subgradient algorithm to compute the exact value up to any fixed degree of precision.

1 Introduction

The notion of what is fair in the allocation of completely divisible goods to a finite number of agents with subjective preferences has long been debated. Predictably, no agreement has been reached on the subject. The situation is often exemplified with children (players) at a birthday party who are around a table waiting for their slice of the cake to be served, with the help of some parent (an impartial referee). If we think about a special class of resolute children who are able to specify their preferences in terms of utility set functions, the parent in charge of the division could ease his task by using a social welfare function to summarize the children's utility values. Among the many proposals, the maxmin – or Rawlsian – division was extensively studied in the seminal work of Dubins and Spanier [10], who showed the existence of maxmin optimal partitions of the cake for any completely divisible cake and its main properties. They also showed that when a condition of mutual appreciation holds (assumption *(MAC)* below) the optimal partition is also *equitable*, i.e. it assigns the same level of utility for each child.

The study of the maximin optimal partition and its properties has continued in more recent years. In particular, its relationship with other important notions such as efficiency (or Pareto optimality) and, above all, envy-freeness has been investigated with alternating success: each maxmin partition is efficient, but while for the two children case Brams and Taylor [6] showed that it also envy-free, the same may not hold when three or more children are to be served, as shown in Dall'Aglio and Hill [9].

It is worth pointing out the relationship with n player bargaining solutions. If we think about the division as deriving from a bargaining procedure among children, it is straightforward to show that the egalitarian bargaining solution proposed by Kalai [12] coincides with the equitable maximin division. Therefore if the conditions proposed by Dubins and Spanier hold, the two solutions actually coincide.

Little attention has been devoted, however, to finding optimal maximin partitions with few notable exceptions: the case with two players with additive and linear utility over several goods has been considered by Brams and Taylor [6], with the popular Adjusted Winner procedure. As a bonus in this case, the resulting partition is also envy-free.

For the more general case of general preferences (expressed as set functions) and arbitrary number of players, Legut and Wilczinski [17] gave a characterization of the optimal maxmin allocation in terms of weighted density functions. Moreover Elton et al. [11] and Legut [14] provided lower bounds on the maxmin value. The optimization problem was later analysed by Dall'Aglio [7]. The general problem was reformulated as the minimization of a convex function with a particular attention to the case where the maxmin allocation is not equitable and the allocation of the cake occurs in stages to subsets of players. No detail was given on how to proceed with the minimization.

In most of the fair division literature, little is assumed about the strategic behavior of the children. Brams and Taylor [6] discuss the issue of the manipulability of the preferences: in most cases children may benefit from declaring false preferences. A different approach takes into account the possibility for the children to form coalitions after (Legut [15] and Legut Potters Tijs [16]) or before, Dall'Aglio et al. [8], the division of the cake. In both cases coalitional games are defined and analysed. In the case of early cooperation among children, the game is based on a maxmin allocation problem among coalitions, each one having a joint utility function and a weight. The computation of the maxmin problem becomes essential to compute the coalitional game values and its indices.

The coalitional maxmin problem is indeed a generalization of the classical maxmin problem introduced by Dubins and Spanier. Therefore, we consider a common approach to set up an algorithm which, at each step, will compute an approximating allocation, together with lower and upper bounds for the maxmin value. The algorithm is based on a subgradient method proposed by Shor [19] and it yields an approximation of the optimal allocation with any fixed degree of precision.

2 The model and the maximin fair division problem with coalitions

We represent our completely divisible good with the set C, usually $C \subset \mathbb{R}$ or $C \subset \mathbb{R}^2$, and let $\mathcal{B}(C)$ be the Borel $\sigma-$algebra of subsets of C. Let $N = \{1, \ldots, n\}$ be the set of players, whose preferences on the good are $\mu_1, \ldots, \mu_n$, where μ_i is a probability measures on C. We will consider the following assumptions:

a) *complete divisibility of the cake (CD).* Each μ_i $(i \in N)$ does not contain atoms: If $\mu_i(A) > 0$, $A \in \mathcal{B}(C)$, then there exists a measurable $B \subset A$ such that $\mu_i(A \cap B) > 0$ and $\mu_i(A \cap B^c) > 0$;

b) *mutual absolutely continuity (MAC).* If $\mu_i(A) > 0$ for some i, then $\mu_j(A) > 0$ for every other $j \neq i$.

   As consequence, and by the Radon-Nikodym theorem,

$$\mu_i(A) = \int_A f_i(x)dx \quad \forall\, i \in N, \forall A \in \mathcal{B}(C),$$

   where $f_1, \ldots, f_n$ are density functions and f_i is the density of μ_i, $\mathcal{B}(C)$ are the Borel sets in C;

Throughout the rest of the work we will assume that (CD) always holds, while (MAC) is a useful, though restrictive assumption that we will employ only when strictly needed.

Let Π_n be the set of all possible $n-$allocations of C to agents in N. How do players behave in the division procedure? In the simplest case, each player compete with the others to get a part of the cake with no strategic interaction with other players. Therefore, individual players seek an allocation with values as high as possible. A fair compromise between the conflicting interests is given by the maximin allocation $(A_1^*, \ldots, A_n^*) \in \Pi_n$ that achieves

$$v_m = \max_{(A_1,\ldots,A_n)\in\Pi_n} \left\{ \min_{i\in N} \mu_i(A_i) \right\}. \tag{1}$$

With a completely divisible cake, the allocation $(A_1^*, \ldots, A_n^*)$ is fair, i.e. $\mu_i(A_i^*) \geq \frac{1}{n}$ for all $i \in N$, while if also (MAC) holds, it is also egalitarian, i.e. $\mu_i(A_i^*) = \mu_j(A_j^*)$ for all $i, j \in N$ (See [10]). Therefore, under this assumption, an optimal allocation is also the bargaining solution proposed by Kalai and Smorodinski [13] for the two players case and by Kalai [12] for any numbers of players.

Dall'Aglio, Branzei and Tijs ([8]) proposed a strategic model of interaction among players, where players, before the division takes place, gather into mutually disjoint coalitions. Within each coalition, players pursue an efficient allocation of their collective share of the cake.

Let $\mathcal{G}$ be the family of all partitions of N and, for each $\Gamma \in \mathcal{G}$, let be $|\Gamma| = m$, $m \leq n$, and let be the coalitions indexes set $M = \{1, \ldots, m\}$. Thus, players cluster into coalitions specified by the structure $\Gamma = \{S_1, \ldots, S_m\}$. In any coalition S_j, $j \in M$, players state their joint preferences as follows

$$\mu_{S_j}(B) = \max_{\{D_i\}_{i \in S_j} \text{ partition of } B} \mu_i(D_i) = \int_B f_{S_j}(x)dx \tag{2}$$

with $f_{S_j}(x) = \max_{i \in S_j} f_i(x)$.

Once the global coalition structure is known, a fair allocation of the cake among the competing coalitions is sought. Fairness here must take into account the different importance that coalitions may assume and this is taken into account by a weight function $w : \mathcal{P}(N) \to \mathbb{R}_+$.

In this framework, coalitions take up the role that in (1) was assumed by single players and the will agree on a division of the cake which achieves the following value

$$v(\Gamma, w) = \max_{(B_1, \ldots, B_m) \in \Pi_m} \left\{ \min_{j \in M} \frac{\mu_{S_j}(B_j)}{w(S_j)} \right\}, \tag{3}$$

Single coalitions can evaluate their performance in the division by considering the following coalitional game

$$\eta(S, w) = w(S)v(\Gamma_S, w) \qquad S \subset N \tag{4}$$

where $\Gamma_S = \{S, \{j\}_{j \notin S}\}$. $\eta(S, w)$ can be interpreted as the minimal utility that coalition S is going to receive in the division when the system of weight w is enforced, independently of the behavior of the other players.

A crucial question lies in the definition of the weight system. We list three proposals:

- $w_{\text{card}} = |S|$, $S \subset N$. This is certainly the most intuitive setting. Although very natural, this proposal suffers from a serious drawback, since players participating in the game $\eta(\cdot, w)$ may be better off waiting to seek for cooperation well after the cake has been divided (see [8]);

- $w_{\text{post}} = \mu_S\left(\cup_{i\in S} A_i^*\right)$, $S \subset N$, where $\{A_1^*, A_2^*, \ldots, A_n^*\}$ is the partition maximizing (1). By seeking early agreements among them, players will be better off than postponing such agreements until the cake is cut. The above mentioned problem is overcome at the cost of a less intuitive (and more computationally challenging) formulation. (see [8]). It is interesting to note that to find these weights we need to solve (1);
- $w_{\text{barg}} = \mu_S(C)$, $S \subset N$. Here (3) returns the Kalai-Smorodinski bargaining solution in the coaltional context.

It is easy to verify that the fully competitive division (1) is a special case of (3), whenever $\Gamma = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and $w = w_{\text{card}}$. We thus turn our attention to the latter problem.

The optimization problem (3) can be seen as an infinite dimensional assignment problem. In principle we could attribute any point of the cake C to any of the participating players (provided certain measurability assumptions are met). For very special instances this becomes a linear program: when the preferences have piecewise constant densities, or when the cake is made of a finite number of parts (or indivisible pieces).

2.1 A geometrical setting

We now describe a geometrical setting already employed in [3], [7] and [2] to explore fair division problems. In what follows we consider the weighted preferences and densities, μ_j^w and f_j^w , given respectively by

$$\mu_j^w = \frac{\mu_{S_j}}{w_j} \qquad f_j^w = \frac{f_{S_j}}{w_j}.$$

Let Δ_{m-1} denote the $(m-1)-$simplex. The partition range is defined as

$$\mathcal{P} := \{(\mu_1^w(B_1), \ldots, \mu_h^w(B_m)) : (B_1, \ldots, B_m) \in \Pi_m\} \subset \mathbb{R}_+^m.$$

Let us consider some of its features. Each point $p \in \mathcal{P}$ is the image, under μ, of a m-partition of C. Moreover, $\mathcal{P}$ is a set contained in $[0,1]^m$, which includes Δ_{m-1}. Then, for any $\{B_1, \ldots, B_m\} \in \Pi_m$,

$$1 \le \sum_{j=1}^{m} \mu_j^w(B_j) \le m. \tag{5}$$

Finally, $\mathcal{P}$ is compact and, if (CD) holds, $\mathcal{P}$ is also convex ([18]). Therefore $v(\Gamma, w) = \max\{x > 0 : (x, x, \ldots, x) \cap \mathcal{P} \neq \emptyset\}$. So, the point $v(\Gamma, w)$ is the intersection between the Pareto frontier of $\mathcal{P}$ and the egalitarian line

$$\ell = \{x^w \in \mathbb{R}^m : x_1^w = x_2^w = \ldots = x_m^w\},$$

where $x_j^w = x_j/w_j$, for all $j \in M$.

3 Upper and lower bounds for the maximin value

We turn our attention to a simpler optimization problem, which in general may return an unfair solution, but it also gives upper and lower easy-to-compute bounds for the original problem. This bounds depend on a weighted maxsum partition, which we can derive through a straightforward extension of a result by Dubins and Spanier ([10]).

Proposition 3.1. *([7]) Let $\alpha \in \Delta_{m-1}$. A $m-$partition of C, $\boldsymbol{B}^\alpha$ is such that for all $k, \ell = 1, 2, \ldots, m$, if*

$$\alpha_k f_k^w(x) \geq \alpha_\ell f_l^w(x) \quad \textit{for all } x \in B_k^\alpha, \tag{6}$$

then

$$(B_1^\alpha, \ldots, B_m^\alpha) \in \underset{(B_1,\ldots,B_m)\in\Pi_m}{\operatorname{argmax}} \sum_{j=1}^m \alpha_j \mu_j^w(B_j). \tag{7}$$

The value of this maxsum problem is itself an upper bound for problem (3). For any choice of $\alpha \in \Delta_{m-1}$ we have a maxsum partition $\boldsymbol{B}^\alpha$ corresponding to α.

Definition 3.2. *The **partition value vector (PVV)** $u^\alpha = (u_1^\alpha, \ldots, u_m^\alpha)$, is defined by*

$$u_j^\alpha = \mu_j^w(B_j^\alpha), \quad j = 1, \ldots, m.$$

The PVV u^α is a point where the hyperplane $\sum_{j=1}^m \alpha_j x_j = k$ touches the partition range $\mathcal{P}$, so u^α lies on the Pareto border of $\mathcal{P}$. Moreover, for any $\alpha \in \Delta_{m-1}$ there exists at least one PVV ([2]). We are ready to state the first approximation result.

Proposition 3.3. *Let be $g : \Delta_{m-1} \to \mathbb{R}^+$ as it follows:*

$$g(\alpha) := \int_C \max_{j \in M} \{\alpha_j f_j^{w_j}(x)\} dx.$$

Then,

$$v(\Gamma, w) \leq g(\alpha).$$

Moreover,

(i) *$g(\alpha)$ is convex;*

(ii) *$v(\Gamma, w) = \min_{\alpha \in \Delta_{m-1}} g(\alpha)$.*

Proof. Let us consider the intersection point between the hyperplane $\sum_{j=1}^m \alpha_j x_j^w = k$ and the egalitarian line,

$$\begin{cases} \alpha_1 x_1^w + \alpha_2 x_2^w + \ldots + \alpha_{m-1} x_{m-1}^w + \alpha_m x_m^w = k \\ x_1^w - x_2^w = 0 \\ \vdots \\ x_{m-1}^w - x_m^w = 0 \end{cases}$$

which is $x_1^w = \ldots = x_m^w = k$. Then, the hyperplane touching $\mathcal{P}$, i.e. $\sum_{j=1}^m \alpha_j u_j^\alpha = g(\alpha)$, intersects the egalitarian line in the point which has all coordinates equal to $g(\alpha)$. Since $\mathcal{P}$ is convex and the hyperplane $\sum_{j=1}^m \alpha_j u_j^\alpha = g(\alpha)$ is a supporting plane at u^α, by separability properties of convex sets the intersection point is external to $\mathcal{P}$ for all $\alpha \in \Delta_{m-1}$. So, $g(\alpha) \geq x_j^{\mathcal{P}}$, where $x^{\mathcal{P}} \in \{x \in \mathbb{R}^m : x \in \ell \cap \mathcal{P}\}$.

(i) The domain of g is convex, so we have to show that for all $\hat{\alpha}, \tilde{\alpha} \in \Delta_{m-1}$ and $t \in [0,1]$ the following holds:

$$g(t\hat{\alpha} + (1-t)\tilde{\alpha}) \leq t g(\hat{\alpha}) + (1-t) g(\tilde{\alpha}). \tag{8}$$

Take $x \in C$. For an index $j^* \in M$ we have

$$\max_{j \in M}(t\hat{\alpha}_j f_j(x) + (1-t)\tilde{\alpha}_j f_j(x)) = t\hat{\alpha}_{j^*} f_{j^*}(x) + (1-t)\tilde{\alpha}_{j^*} f_{j^*}(x),$$

hence, $t\hat{\alpha}_{j^*} f_{j^*}(x) \leq t \max_{j \in M} \hat{\alpha}_j f_j(x)$ and $(1-t)\tilde{\alpha}_{j^*} f_{j^*}(x) \leq (1-t) \max_{j \in M} \tilde{\alpha}_j f_j(x)$. Thus

$$t\hat{\alpha}_{j^*} f_{j^*}(x) + (1-t)\tilde{\alpha}_{j^*} f_{j^*}(x) \leq t \max_{j \in M} \hat{\alpha}_j f_j(x) + (1-t) \max_{j \in M} \tilde{\alpha}_j f_j(x).$$

Taking the integral over C, we obtain (8).

(ii) We get the minimum of $g(\alpha)$ when the touching point u^α lies also on the egalitarian line. The only point belongs both egalitarian line and Pareto boundary of $\mathcal{P}$ is just $v(\Gamma, w)$. □

We now turn our attention to a lower bound for $v(\Gamma, w)$. Although we will see later only one PVV is enough to assure such a bound, we give a general result for the case where several PVVs have already been computed. We derive the second approximation result through a convex combination of these easily computable points in $\mathcal{P}$, which lies close to $v(\Gamma, w)$.

Proposition 3.4. *Suppose that there are m PVV's $u^1, \ldots, u^m$, with $u^\ell = (u_{1\ell}, \ldots, u_{m\ell})$ and*

$$\max_{i=1,\ldots,m} u_{i\ell} = u_{\ell\ell} \quad \ell = 1, \ldots, m. \tag{9}$$

Then, for any $\Gamma \in \mathcal{G}$,

$$v(\Gamma, w) \geq \underline{v}(U) := \frac{1}{\sum_{\ell=1}^{m} \sum_{j=1}^{m} \left[U^{-1}\right]_{j\ell}}, \tag{10}$$

where U^{-1} is the inverse matrix of $U = (u^1, \ldots, u^m)$.

Proof. Consider the hyperplane through the m PVVs. We can describe it as the convex combination among the m PVVs,

$$\mathcal{H} := t_1 u^1 + t_2 u^2 + \ldots + t_m u^m = k,$$

where $t_1, \ldots, t_m \in \Delta_{m-1}$. Now, let us obtain its interection with the egalitarian line, the point $(x_w, \ldots, x_w)$ as it follows:

$$\begin{cases} t_1 u_{11} + t_2 u_{12} + \ldots + t_m u_{1m} = x_w \\ t_1 u_{21} + t_2 u_{22} + \ldots + t_h u_{2m} = x_w \\ \vdots \\ t_1 u_{m1} + t_2 u_{m2} + \ldots + t_m u_{mm} = x_w \\ t_1 + t_2 + \ldots + t_m = 1 \end{cases}$$

We are dealing with a linear system with $m + 1$ unknown quantities, $t_1, t_2, \ldots, t_m, x_w$.

Then, we can get x_w,

$$
x_w = \frac{det \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1m} & 0 \\ u_{21} & u_{22} & \dots & u_{2m} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mm} & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}}{det \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1m} & -1 \\ u_{21} & u_{22} & \dots & u_{2m} & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mm} & -1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}}
$$

$$
= \frac{det(U)}{det \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & u_{11} & u_{12} & \dots & u_{1m} \\ -1 & u_{21} & u_{22} & \dots & u_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & u_{m1} & u_{m2} & \dots & u_{mm} \end{pmatrix}}
$$

$$
= \frac{det(U)}{\sum_{\ell=1}^{m} \sum_{j=1}^{m} (-1)^{\ell+j} det(U_{j\ell})} = \frac{1}{\sum_{\ell=1}^{m} \sum_{j=1}^{m} \left[U^{-1}\right]_{j\ell}},
$$

where $U = (u^1, \dots, u^m)$, $det(U_{j\ell})$ is the (j, ℓ)-th minor of U, and the second equality derives by adeguate exchanges of rows and columns in the denominator matrix. In fact, we get the second after an even number of exchanges on the first. □

An illustration of the position of the bounds with respect to the partition range in the case of two coalitions is shown in Figure 1

As stated before, the result can be used even if we have fewer vectors satisfying (9). We replace any missing PVV u^q by the following vector

$$
e^q = (0, \dots, 0, \mu_q^w(C), 0, \dots, 0), \tag{11}
$$

where $\mu_q^w(C)$ is the weighted joint utility of the whole cake by coalition S_q. Actually, even a single PVV is enough to establish a lower bound.

Corollary 3.5. *Let u^ℓ be such that* (9) *holds. Then, for any $\Gamma \in \mathcal{G}$,*

$$
v(\Gamma, w) \geq \frac{u_{\ell\ell}}{1 + \sum_{j \neq \ell} \frac{u_{\ell\ell} - u_{j\ell}}{\mu_j^w(C)}}, \tag{12}
$$

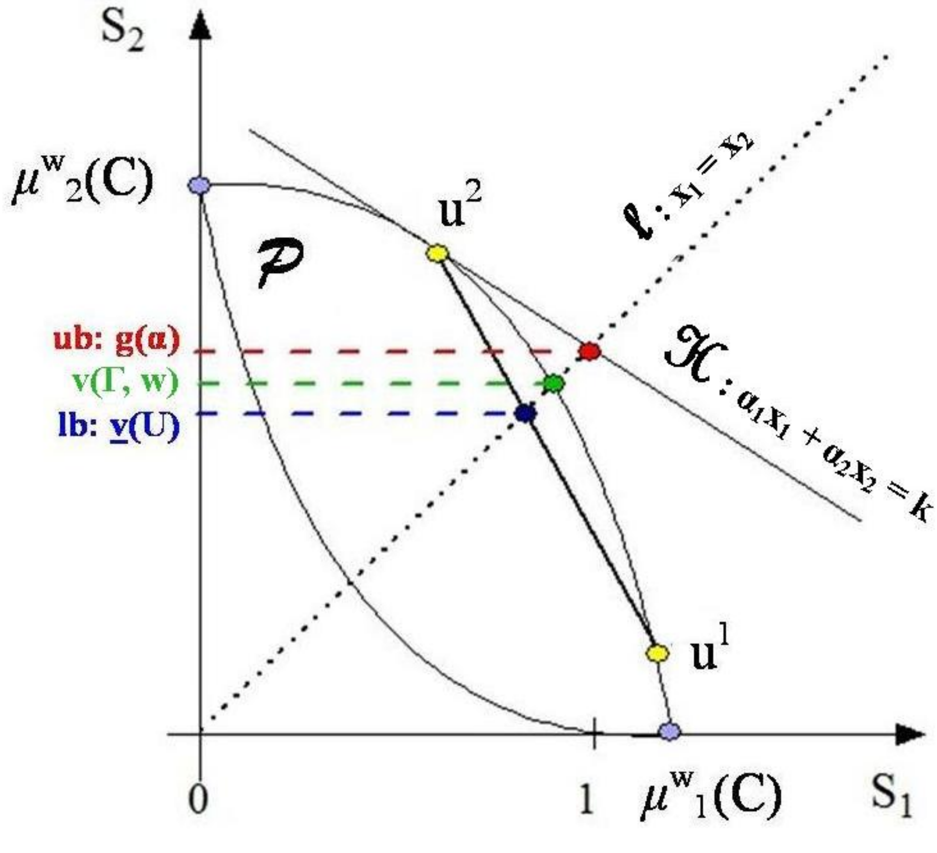

Figure 1: Upper and lower bounds for the two-coalition case

Proof. For any $q \neq \ell$, replace any other PVV with the corresponding vector (11). Without loss of generalization, let us suppose $\ell = 1$. Then

$$
\begin{aligned}
v(\Gamma, w) &\geq \frac{det \begin{pmatrix} u_{11} & 0 & \dots & 0 & 0 \\ u_{21} & \mu_2^w(C) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m1} & 0 & \dots & \mu_m^w(C) & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}}{det \begin{pmatrix} u_{11} & 0 & \dots & 0 & -1 \\ u_{21} & \mu_2^w(C) & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m1} & 0 & \dots & \mu_m^w(C) & -1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}} \\
&= \frac{u_{11} \prod_{j \neq 1} \mu_j^w(C)}{det \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & u_{11} & 0 & \dots & 0 \\ -1 & u_{21} & \mu_2^w(C) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & u_{m1} & 0 & \dots & \mu_m^w(C) \end{pmatrix}} \\
&= \frac{u_{11} \prod_{j \neq 1} \mu_j^w(C)}{\prod_{j \neq 1} \mu_j^w(C) + \sum_{j \neq 1} \prod_{i \neq j} \mu_j^w(C)(u_{11} - u_{j1})} \\
&= \frac{u_{11} \prod_{j \neq 1} \mu_j^w(C)}{\prod_{j \neq 1} \mu_j^w(C) \left[1 + \sum_{j \neq 1} \frac{(u_{11} - u_{j1})}{\mu_j^w(C)}\right]} \\
&= \frac{u_{11}}{1 + \sum_{j \neq 1} \frac{(u_{11} - u_{j1})}{\mu_j^w(C)}}.
\end{aligned}
$$

□

In the case of complete competition, we recognize the results of Elton et al. ([11]) and Legut ([14]).

Corollary 3.6. *Suppose $\ell \in N$ such that u^ℓ satisfies (9). Then,*

$$
v(\Gamma, w) \geq \frac{u_{\ell\ell}}{n u_{\ell\ell} + 1 - K} \geq \frac{1}{n + 1 - K} \tag{13}
$$

where $K = \sum_{i \in N} u_{i\ell}$.

Proof. When each agent plays on his own, $\Gamma = \{1, 2, \ldots, n\}$ (therefore, $M = N$),

and $w_i = 1$, for any $i \in N$. Note that $\mu_i^w(C) = 1$, for all $i \in N$. Hence,

$$v(\Gamma, w) \geq \frac{u_{\ell\ell}}{1 + \sum_{i \neq \ell} u_{\ell\ell} - u_{i\ell}} = \frac{u_{\ell\ell}}{n u_{\ell\ell} + 1 - K} \geq \frac{1}{n + 1 - K},$$

with $K = \sum_{i \in N} u_{i\ell}$.

□

The lower bound in Prop. 3.4 employs the same geometric setting by Legut [14]. The same framework also shows us how the lower bound $\underline{v}(U)$ can be improved by an accurate replacement of the vectors in U. Let $p(u^\alpha)$ be the projection of u^α on Δ_{m-1}, and $CH(p(u^1), \ldots, p(u^m))$ be the convex hull of the PVV's projections on Δ_{m-1}. Pick an $\alpha^* \in \Delta_{m-1}$ and compute u^{α^*}. If $p(u^{\alpha^*}) \in CH(p(u^1), \ldots, p(u^m))$, and $u_{\ell\ell}^{\alpha^*} = \max_{i \in M} u_{i\ell}^{\alpha^*}$, then replace u^ℓ in U with u^{α^*}. It is easy to show that the lower bound has not decreased with this substitution.

4 The subgradient method

In the previous section we have seen that for each choice of the coefficients α we can derive upper and lower bounds for $v(\Gamma, w)$. In principle, if we were extremely lucky, a particular choice of the α, corresponding to the $\mathcal{P}$ supporting hyperplane, had an egalitarian PVV, we would guess the optimal value and the optimal partition achieved. More realistically, we describe a way of improving the coefficients α so that eventually the bounds shrink to the desired value.

Since in general $g(\alpha)$ is a nondifferentiable convex function, we can rely on a simple minimizing algorithm developed by Shor [19], the subgradient method. In particular, since g domain is constrained, we should use an its extension, the projected subgradient method, which solves constrained convex optimization problems. Let us start by describing the method through some basic definitions and the essential convergence result.

Definition 4.1. *Let f be a convex function with domain D and let x_0 an interior point of D. A vector $g(x_0)$ is called a subgradient or a generalized gradient of f at x_0 if it satisfies*[1]

$$f(x) - f(x_0) \leq (g(x_0), x - x_0) \quad \textit{for all } x \in D. \tag{14}$$

[1] Here $(\cdot, \cdot)$ indicates the inner product.

Let us named $\partial_x f(x)$ the set of subgradients of a convex function f at any interior point x of the f domain.

Definition 4.2. *Let D be a closed convex set and let $||\cdot||$ be the Euclidean norm. The projection of $x \in \mathbb{R}^n$ on D is denoted by $p(x)$ and it is defined as*

$$p(x) = \operatorname*{argmin}_{z \in D} ||z - x||. \tag{15}$$

Proposition 4.3. *([19], [5]) Let f be a convex function defined on $D \subseteq \mathbb{R}^m$, which has a bounded set of minimum points D^* and let $g(x) \in \partial_x f(x)$ be a bounded subgradient, i.e. there exists $G > 0$ such that $||g(x)|| \leq G$ for all $x \in D$. Moreover, let $\{s_t\}_{t=1}^{+\infty}$ a sequence of positive numbers satisfying the conditions:*

- $\lim_{t \to +\infty} s_t = 0$,
- $\sum_{t=0}^{+\infty} s_t = +\infty$.

Then for any $x^0 \in D$ the sequence $\{x^t\}_{t=0}^{+\infty}$ generated according to the formula

$$x^{t+1} = p[x^t - s_t g(x^t)] \tag{16}$$

has the following property: either an index t^* exists such that $x^{t^*} \in D^*$, or $\lim_{t \to +\infty} f_{best}^t - f^* = 0$, *where*

- $f_{best}^t = \min_{i=1,\dots,t} f(x^t)$,
- $f^* = \min_{x \in D} f(x)$.

Let us check that $g(\alpha)$ can be minimized through the projected subgradient method. First of all, $g(\alpha)$ is convex with $\min_{\alpha \in \Delta_{m-1}} = v(\Gamma, w)$, and we can easily show that u^α is a bounded subgradient of $g(\alpha)$:

$$\begin{aligned}\partial_\alpha g(\alpha) &= \partial_\alpha \int_C \max_{j \in M} \{\alpha_j f_j^{w_j}(x)\} dx = \int_C \partial_\alpha \max_{j \in M} \{\alpha_j f_j^{w_j}(x)\} dx \\ &= \int_C \partial_\alpha \alpha_j f_j^{*w_j}(x) dx = \int_C f_j^{*w_j}(x) dx = u^\alpha.\end{aligned}$$

Let us choose a sequence $\{s_t\}_{t=1}^{\infty}$ such that $\lim_{t \to +\infty} s_t = 0$ and $\sum_{t=0}^{+\infty} s_t = +\infty$. For a given set $\alpha^{(t)} \in \Delta_{m-1}$ of coefficients compute the subgradient u^t. The new set of coefficients is

$$\alpha^{t+1} = p[\alpha^t - s_t u^t] = (\alpha^t - s_t u^t + \lambda)_+$$

where λ is the normalizing constant such that

$$\sum_{i=1}^{m}(\alpha_i^t - s_t u_i^t + \lambda)_+ = 1.$$

We state that there exists a sequence $s_t^{'}$ such that $s_t^{'} \leq s_t$, $\lim_{t\to+\infty} s_t^{'} = 0$, $\sum_{t=0}^{+\infty} s_t^{'} = +\infty$ and $\alpha_i^t - s_t^{'} u_i^t + \lambda > 0$ for all $t \in \mathbb{N}$ and for all $i = 1, \ldots, m$. If this is the case, then

$$\sum_{i=1}^{m}(\alpha_i^t - s_t^{'} u_i^t + \lambda) = 1,$$

i. e.

$$\sum_{i=1}^{m}\alpha_i^t - s_t^{'}\sum_{i=1}^{m} u_i^t + m\lambda = 1,$$

hence

$$\lambda = s_t^{'}\bar{u}^t,$$

where $\bar{u}^t = \frac{\sum_{i=1}^{m} u_i^t}{m}$ is the average of the subgradient *vector components*.

Therefore our claim becomes

Proposition 4.4. *There exists a sequence $s_t^{'} \leq s_t$ which satisfies*

(1) $\lim_{t\to+\infty} s_t^{'} = 0$,

(2) $\sum_{t=0}^{+\infty} s_t^{'} = +\infty$,

(3) $\alpha_i^t - s_t^{'}(u_t^t - \bar{u}^t) > 0$ *for all* $t \in \mathbb{N}$ *and for all* $i = 1, \ldots, m$.

Proof. Firstly, notice that constraint (3) involves only those indexes $i \in \{1, \ldots, m\}$ for which $u_i^t > \bar{u}^t$. Let be $\mathcal{I}_t$ the set of those indexes in the step t. For each of them we would get

$$s_t^{'} < \frac{\alpha_i^t}{u_i^t - \bar{u}^t} = \frac{m\alpha_i^t}{(m-1)u_i^t - \sum_{j\neq i} u_j^t}.$$

Now, let us name

$$\tau_t = \min_{i\in\mathcal{I}_t}\left\{\frac{m\alpha_i^t}{(m-1)u_i^t - \sum_{j\neq i} u_j^t}\right\} - \varepsilon,$$

with $\varepsilon > 0$. Hence , let us define

$$s_t^{'} = \min\{s_t, \tau_t\}.$$

Thus, $s_t^{'}$ satisfies (3) and (1), as trivially, $\lim_{t\to+\infty} s_t^{'} = 0$. To show (2), let us suppose $\sum_{t=0}^{+\infty} s_t^{'} < +\infty$. This implies for some $i^* \in \{1, \ldots, m\}$ and some sequence $\{t_r\} \subset \mathbb{N}$

$$\lim_{t_r\to+\infty} \frac{m\alpha_{i^*}^{t_r}}{(m-1)u_{i^*}^{t_r} - \sum_{j\neq i^*} u_j^{t_r}} = 0.$$

Since $(m-1)u_{i^*}^{t_r} - \sum_{j\neq i^*} u_j{}^{t_r} > 0$, taking a further subsequence $\{t_p\} \subset \{t_r\}$ we have

$$\begin{aligned}
&\alpha_{i^*}^{t_p} \to \tilde{\alpha}_{i^*} = 0, \text{ so } \sum_{j\neq i^*} \alpha_j^{t_p} \to \sum_{j\neq i^*} \tilde{\alpha}_j = 1, \text{ and} \\
&u_j^{t_p} \to \tilde{u}_j \text{ for all } j \in \{1, \ldots, m\}, \text{ with } (m-1)\tilde{u}_{i^*} > \sum_{j\neq i^*} \tilde{u}_j \quad (*).
\end{aligned}$$

By the continuity, $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ lies on the upper surface of $\mathcal{P}$ and it is supported by the hyperplane $\sum_{j=1}^{m} \tilde{\alpha}_j x_j = k$.

First of all, we show that $\tilde{u}_{i^*} \geq \frac{1}{m} > 0$. In fact, $\sum_{j\in M} \tilde{u}_j \geq 1$ by (5). Therefore, (*) becomes

$$\begin{aligned}
(m-1)\tilde{u}_{i^*} \geq \sum_{j\neq i^*} \tilde{u}_j = \sum_{j\in M} \tilde{u}_j - \tilde{u}_{i^*} \geq 1 - \tilde{u}_{i^*}, \text{ so} \\
\tilde{u}_{i^*} \geq \frac{1}{m} > 0.
\end{aligned}$$

Now, the coexistence of $\tilde{u}_{i^*} > 0$ and $\tilde{\alpha}_{i^*} = 0$ clashes with the hypothesis (MAC). In fact, $(\tilde{u}_1, \ldots, \tilde{u}_m) \in \operatorname{argmax}_{x\in\mathcal{P}} \sum_{j\in M} \tilde{\alpha}_j x_j$, $\sum_{j\in M} \tilde{\alpha}_j \tilde{u}_j = \sum_{j\neq i^*} \tilde{\alpha}_j \tilde{u}_j = k$ and there is not $(x_1, \ldots, x_m) \in \mathcal{P}$ for which $\sum_{j\in M} \tilde{\alpha} x_j > k$. Since $\tilde{u}_{i^*} \geq \frac{1}{m} > 0$, there exists $\tilde{A}_{i^*}$ such that $\tilde{u}_{i^*} = \mu_{i^*}(\tilde{A}_{i^*}) \geq \frac{1}{m}$. By (MAC) we can derive a partition from $\tilde{A}_{i^*}$ of $(m-1)$ subsets $\{B_j\}_{j\neq i^*}$, with $\cup_{j\neq i^*} B_j = \tilde{A}_{i^*}$ and $B_j \cap B_l = \emptyset$ if $j \neq l$, such that $\mu_j(B_j) \geq \varepsilon > 0$ for all $j \neq i^*$.

If we consider the partition $\tilde{\tilde{A}}$ defined as $\tilde{\tilde{A}}_{i^*} = \emptyset$, $\tilde{\tilde{A}}_j = \tilde{A}_j \cup B_j$, we get

$$\sum_{j\in M} \tilde{\alpha}_j \mu_j(\tilde{\tilde{A}}_j) = \sum_{j\neq i^*} \tilde{\alpha}_j (\mu_j(\tilde{A}_{i^*}) + \mu_j(B_j)) = k + (m-1)\varepsilon > k,$$

which is a contradiction. □

While α converges to α^*, $g(\alpha)$ converges to $v(\Gamma, w)$, which is the desidered PVV u^*. Since the lower bound $\underline{v}(U)$ is a continuos function, for u^α converging to u^* we have that $\underline{v}(U)$ converge to $\underline{v}(U^*)$, where U^* is a matrix containing u^*. It is easy to show that $\underline{v}(U^*) = u^*$. Without loss of generality, let us suppose that u^* is the first PVV in U^* and let us remember that u^* has egalitarian component, that we indicate with u_{11}. Then,

$$\begin{aligned}
\underline{v}(U^*) &= \frac{det(U^*)}{\sum_{\ell=1}^{m}\sum_{j=1}^{m}(-1)^{\ell+1}det(U_{j\ell})} \\
&= \frac{u_{11}\sum_{j=1}^{m}(-1)^{j+1}det(U^*_{j1})}{\sum_{j=1}^{m}(-1)^{j+1}det(U^*_{j1}) + \sum_{\ell=2}^{m}\sum_{j=1}^{m}(-1)^{j+\ell}det(U^*_{j\ell})} \\
&= u_{11},
\end{aligned}$$

where the last equality holds because the second sum in the denominator is equal to zero.

4.1 The algorithm

Now, we present an algorithm where the two bounds converge to the efficient and fair value $v(\Gamma, w)$. The algorithm elements are:

- α_i, $i = 1, \ldots, m$ are the supporting hyperplane coefficients;
- u is the PVV vector associated to α;
- $V = \{v^1, \ldots, v^m\}$ is the set of PVVs, such that $\max_i v_i^\ell = v_\ell^\ell$, for $\ell = 1, \ldots, m$;
- $g(\alpha) = \int_C \max_{j\in M}\alpha_j f_j^w(x)\,dx$;
- $\underline{v}(V) = \frac{det(V)}{\sum_{\ell=1}^{m}\sum_{j=1}^{m}(-1)^{\ell+j}det(V_{j\ell})}$;
- ub is the upper bound;
- lb is the lower bound;
- $\{s_t^{'}\}$ is a sequence such that
  - **(1)** $\lim_{t\to+\infty} s_t^{'} = 0$,
  - **(2)** $\sum_{t=0}^{+\infty} s_t^{'} = +\infty$,
  - **(3)** $\alpha_i^t - s_t^{'}(u_t^t - \bar{u}^t) > 0$ for all $t \in \mathbb{N}$ and for all $i = 1, \ldots, m$.

Let us initialize the elements:

- $\alpha^0 = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$;
- u^0 is the PVV associated to α^0;
- $V^0 = \{e^1, \ldots, e^m\}$, with $e^\ell = \{0, \ldots, \mu^w_\ell(C), \ldots, 0\}$;
- $ub = g(\alpha^0)$;
- $lb = \underline{v}(V^0)$;

A generic step t follows:

1. update $\alpha : \alpha^t = \alpha^{t-1} - s_t^{'}(u^{t-1} - \bar{u}^{t-1})$;
2. compute $u^t = PVV(\alpha^t)$;
3. update ub. Compute $g(\alpha^t)$. If $g(\alpha^t) < ub$, then $ub = g(\alpha^t)$;
4. update lb. Replace one vector in V^{t-1} : if $\max_i u_i^t = u_\ell^t$, then replace v^ℓ with u^t, and save the new matrix in V^t. Compute $\underline{v}(V^t)$. If $\underline{v}(V^t) > lb$, then $lb = \underline{v}(V^t)$.
5. If $ub - lb < \varepsilon$, then STOP. Else, go back to 1.

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