UTILITY MAXIMIZATION UNDER RESTRICTED INFORMATION FOR JUMP MARKET MODELS

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Abstract

The contribution of this paper is twofold: we study by a BSDE approach power utility maximization problems in a partially observed financial market with jumps and we solve by the innovation method the arising filtering problem. We consider a Markovian model where the risky asset dynamics S_t follows a pure jump process whose local characteristics are not observable by investors. More precisely, the stock price process dynamics depends on an unobservable stochastic factor X_t described by a jump-diffusion process. We assume that agents' decisions are based on the knowledge of an information flow, $\{\mathcal{G}_t\}_{t\in[0,T]}$, containing the asset price history, $\{\mathcal{F}_t^S\}_{t\in[0,T]}$. Using projection on the filtration $\mathcal{G}_t$, the partially observable investment problem is reduced to a full observable problem. In the case where $\mathcal{G}_t = \mathcal{F}_t^S$ the value process and the optimal investment strategy are represented in terms of solutions to a BSDE driven by the $\mathcal{F}^S$-compensated martingale random measure associated to S_t and the $\mathcal{F}^S$-compensated martingale random measure can be obtained by filtering techniques ([7], [5]). Next, we extend the study to the case $\mathcal{G}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^\eta$, where η_t gives observations of X_t in additional Gaussian noise. This setup can be viewed as an abstract form of "insider information". The value process is now characterized as a solution to a BSDE driven by the $\mathcal{G}$-compensated martingale random measure and the so-called innovation process. Computation of these quantities leads to a filtering problem with mixed type observation and whose solution is discussed via the innovation approach.

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1. Introduction

The purpose of this paper is to study portfolio selection problems under partial information in an incomplete financial market with the risky asset dynamics described by a marked point process. With the advent of intraday information, since real asset prices on a very small time scale are piecewise constant and jump in reaction to trades or to significant new informations, jump models have become more popular in the financial literature (see for instance [37, 17, 7, 5]). Jump-times and jump-sizes in stock price processes are often generated by various external events whose impact on the stock market cannot completely be analyzed, thus it is natural to assume that the local characteristics of risky asset prices depend on an unobservable state variable.

We consider an incomplete financial market with one bond and one risky asset. The risky asset price S_t follows a pure jump process whose local characteristics are not observable by investors. More precisely, we study a Markovian model where the dynamics of the stock price S_t depends on an unobservable stochastic factor X_t described by a jump-diffusion having common jump times with S_t. Presence of common jump times means that the trading activity may affect the law of X_t and could be also related to the possibility of catastrophic events. In such a context, we solve the portfolio optimization problem when agents (with power utility functions) want to maximize the expected utility from terminal wealth assuming that they can observe only an information flow $\{\mathcal{G}_t\}_{t\in[0,T]}$ containing the asset price history $\{\mathcal{F}_t^S\}_{t\in[0,T]}$.

Utility maximization problems in a full information setup have been studied extensively in the literature by using different approaches, such as convex duality methods, stochastic control techniques based on Hamilton-Jacobi-Bellman equation or backward stochastic differential equations (BSDEs) (see for example ([32, 12,

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24, 20, 39, 30, 33, 9, 10] and references therein). Portfolio selection problems with partial information have been studied in [34, 38, 29] in a continuous setting and in [1, 27] for jump-diffusions. In [1] it is assumed that investors are only able to observe the stock price process and not the Markov chain which drives the jump intensity. In [27] a default model is studied where investors only observe the asset prices and the default times, while the Brownian motion which drives the asset price dynamics, the drift process and the default intensities are not directly observable.

Our contribution consists in solving the optimization problem in a general discontinuous setting when both the local characteristics of the stock price and not only the jump intensity are unobservable. By projection results we reduce the partially observable problem to one with full information, involving only observable processes. Two situations are studied, in the first one $\{\mathcal{G}_t\}_{t\in[0,T]} = \{\mathcal{F}_t^S\}_{t\in[0,T]}$ (only stock prices are observed) and in the second one $\{\mathcal{G}_t\}_{t\in[0,T]} \supset \{\mathcal{F}_t^S\}_{t\in[0,T]}$.

Where $\{\mathcal{G}_t\}_{t\in[0,T]} = \{\mathcal{F}_t^S\}_{t\in[0,T]}$, by applying the Bellman optimality principle we directly show that the value function solves a BSDE driven by the $\mathcal{F}_t^S$-compensated martingale random measure associated to S_t. A similar procedure is followed in [30] and in [26] in a full information framework and in [27] under restricted information. In [30] the case where the dynamics of asset prices are described by a continuous semimartingale is studied, in [26] and [27] market models where stocks are exposed to a counterpart risk inducing a finite number of jumps in the prices are discussed. Under further assumptions on the model and assuming compactness of the valued set of admissible strategies, we characterize the value process as the unique solution to the BSDE. Since the BSDE involved and the optimal strategy depend on the $\mathcal{F}^S$-compensated martingale random measure we have to solve a filtering problem, which in the case where $\{\mathcal{G}_t\}_{t\in[0,T]} = \{\mathcal{F}_t^S\}_{t\in[0,T]}$ has been studied in [7, 5].

Next, we examine the case $\mathcal{G}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^\eta$, where η_t gives observations of X_t in additional Gaussian noise, that is $\eta_t = \int_0^t \gamma(X_s)ds + W_t^1$, with $\gamma(x)$ a bounded measurable function and W_t^1 a $(P, \mathcal{F}_t)$-standard Brownian motion independent of the Poisson random measure driving the asset price dynamics. In this situation, which can be viewed as an abstract form of "insider information", agents observe stock prices and receive in addition noisy signals on the unobservable stochastic factor X_t. The value process is now characterized in terms of solutions to a BSDE driven by the $\mathcal{G}_t$-compensated martingale random measure and the $\mathcal{G}_t$-innovation process, quantities which depend on the filter. We recall that the filter is the cadlag version of the conditional law of X_t given $\mathcal{G}_t$. By the innovation approach we derive the Kushner-Stratonovich equation that the filter solves and we characterize the filter as the unique weak solution of this equation, via the Filtered Martingale Problem ([25]). The proofs of these Theorems are postponed in Appendix.

Filtering problems with mixed type observations (marked point processes and diffusions) have been studied in [18, 19] in a framework of credit derivatives. In [18] the marked point process Y_t is the default indicator process and in [19] is the loss-state of the portfolio. In both models the intensities of default times depend on a factor X_t and the additional Gaussian noise is assumed independent of X_t and Y_t. Whereas in this paper we examine a more general setup which allows correlation between the additional Gaussian noise and the stochastic factor. The above mentioned assumption is crucial in [18] to apply a reference probability approach (in a setup where common jump times between Y_t and X_t are allowed) in order to reduce the filtering problem to the case where the information flow consists only of the default history. As in [19] we apply an alternative route based on the innovation method, but in [19] the state process X_t is modeled as a finite-state Markov chain without common jump times with Y_t whereas we describe X_t as a jump-diffusion process having common jumps times with the marked point process.

The paper is organized as follows. In Section 2, we describe the model. In Section 3, we formulate the optimization problem. In Section 4, we derive the BSDE representation of the value function in the case $\{\mathcal{G}_t\}_{t\in[0,T]} = \{\mathcal{F}_t^S\}_{t\in[0,T]}$. Section 5 is devoted to the case where agents observe the stock prices and receive in addition noisy signals on the stochastic factor, that is $\mathcal{G}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^\eta \supset \mathcal{F}_t^S$. We show a suitable martingale representation property and we derive the BSDE that the value process solves. In Section 6 we study the related filtering problem by characterizing the filter as the unique weak solution to the Kushner-Stratonovich equation. In Section 7 a particular case where the risky asset dynamics is described by a geometric pure jump process driven by two independent point processes, describing upward and downward jumps and whose intensities are not directly observable by investors, is proposed. For this model an explicit representation of the optimal investment strategy is provided. This result extends to the partial information case some results obtained in [9] in a full information setting.

2. The market model

We consider a finite time horizon investment model on $[0,T]$ with one riskless money market account and a risky asset.

2.1. Preliminaries

On some underlying filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, satisfying the usual hypotheses, we assume that the price B_t of the cash account solves

$$dB_t = rB_t dt, \quad B_0 \in I\!R^+ \tag{2.1}$$

with constant risk-free interest rate, $r \geq 0$, and the price S_t of the risky asset follows a geometric marked point process

$$S_t = S_0 e^{Y_t}. \tag{2.2}$$

The logreturn process Y_t depends on an external stochastic factor X_t and satisfies

$$Y_t = \int_0^t \int_Z \log\, (1 + K(s, X_{s^-}, Y_{s^-}; \zeta))\mathcal{N}(ds, d\zeta). \tag{2.3}$$

Here $\mathcal{N}(dt, d\zeta)$ denotes a $(P, \mathcal{F}_t)$-standard Poisson random measure on $I\!R^+ \times Z$ with mean measure $dt\ \nu(d\zeta)$, with $\nu(d\zeta)$ a σ-finite measure on a measurable space $(Z, \mathcal{Z})$, and $K(t, x, y; \zeta)$ is an $I\!R$-valued measurable function, such that $K(t, x, y; \zeta) + 1 > 0$.

We describe the unobservable hidden state process X_t as a jump-diffusion having common jump-times with S_t. This means that the trading activity may affect the law of X_t and could be also related to the possibility of catastrophic events. A natural way to describe this kind of behavior is to suppose that the pair (X, Y) is a global solution to the following stochastic differential equations (see [7])

$$X_t = x_0 + \int_0^t b(X_s)\ ds + \int_0^t \sigma(X_s)\ dW_s + \int_0^t \int_Z K_0(s, X_{s^-}; \zeta)\ \mathcal{N}(ds, d\zeta) \tag{2.4}$$

$$Y_t = \int_0^t \int_Z K_1(s, X_{s^-}, Y_{s^-}; \zeta)\ \mathcal{N}(ds, d\zeta) \tag{2.5}$$

with

$$K_1(s, x, y; \zeta) = \log(1 + K(s, x, y; \zeta)). \tag{2.6}$$

Here $x_0 \in I\!R$, W_t is a $(P, \mathcal{F}_t)$-standard Brownian motion independent of $\mathcal{N}(dt, d\zeta)$ and the $I\!R$-valued functions $b(x)$, $\sigma(x)$, $K_0(t, x; \zeta)$ are measurable functions of their arguments.

We assume existence and uniqueness (at least in a weak sense) to the system (2.4)-(2.5). Three different classes of sufficient conditions can be found in [7]. This assumption implies that the pair (X_t, Y_t) is a $(P, \mathcal{F}_t)$-Markov process.

Writing Ito's formula we find that the price S_t of the risky asset satisfies

$$dS_t = S_{t^-} \int_Z K(t, X_{t^-}, Y_{t^-}; \zeta)\ \mathcal{N}(dt, d\zeta), \qquad S_0 \in I\!R^+. \tag{2.7}$$

From now on we will write $K(s;\zeta), K_1(s;\zeta), K_0(s;\zeta)$ for $K(s, X_{s^-}, Y_{s^-}; \zeta)$, $K_1(s, X_{s^-}, Y_{s^-}; \zeta)$ and $K_0(s, X_{s^-}; \zeta)$ respectively, unless it is necessary to underline the dependence on the processes involved.

The sequence $\{T_n\}_{n\geq 0}$ of jump times of S_t coincides with that of Y_t and is defined by

$$T_0 = 0, \quad T_1 = inf\{t > 0 : \int_0^t \int_Z K_1(s;\zeta)\ \mathcal{N}(ds, d\zeta) \neq 0\}$$

$$T_{n+1} = inf\{t > T_n : \int_{T_n}^t \int_Z K_1(s;\zeta)\ \mathcal{N}(ds, d\zeta) \neq 0\}.$$

We introduce the sequence of the marks associated to the marked point process Y_t

$$Z_n = Y_{T_n} - Y_{T_{n-1}} = \int_Z K_1(T_n, ; \zeta) \, \mathcal{N}(\{T_n\}, d\zeta), \quad n \geq 1$$

and we denote by $N_t = \sum_{n \geq 1} \mathbb{I}_{\{T_n \leq t\}}$ the point process which counts the total number of jumps of Y_t (i.e. of S_t). The process Y_t is completely described by assigning the sequence $\{T_n, Z_n\}_{n \geq 1}$ or by giving the following discrete random measure ([4],[22])

$$m(dt, dx) = \sum_{n \geq 1} \delta_{\{T_n, Z_n\}}(dt, dx) \, \mathbb{I}_{\{T_n < \infty\}}. \tag{2.8}$$

In [21] it is proved existence and uniqueness of a positive random measure $m^p(dt, dx)$, called the $\mathcal{F}_t$-predictable projection of $m(dt, dx)$, such that $\forall A \in \mathcal{B}(\mathbb{R})$ (where $\mathcal{B}(\mathbb{R})$ denotes the family of Borel sets of $\mathbb{R}$) the process $m^p((0,t], A)$ is predictable and

$$m((0,t], A) - m^p((0,t], A) \tag{2.9}$$

is a $(P, \mathcal{F}_t)$-local martingale or equivalently for each $H(t,x)$ nonnegative and predictable process

$$\mathbb{E}\Big(\int_0^T \int_{\mathbb{R}} H(t,x) m(dt, dx) \Big) = \mathbb{E}\Big(\int_0^T \int_{\mathbb{R}} H(t,x) m^p(dt, dx) \Big).$$

When the predictable projection is of the form $m^p(dt, dx) = \lambda_t \Phi_t(dx) dt$, where λ_t is a nonnegative $\mathcal{F}_t$-predictable process and $\Phi_t(dx)$ is a probability transition kernel, the pair $(\lambda_t, \Phi_t(dx))$ is called the $(P, \mathcal{F}_t)$-local characteristics of Y_t ([4]).

Define

$$\begin{gathered} D^0(t,x) = \{\zeta \in Z : K_0(t,x;\zeta) \neq 0\}, \\ D^A(t,x,y) = \{\zeta \in Z : K_1(t,x,y;\zeta) \in A \setminus \{0\}\} \subseteq D(t,x,y) = \{\zeta \in Z : K_1(t,x,y;\zeta) \neq 0\}, \\ D^1(t,x,y) = D^0(t,x) \cap D(t,x,y). \end{gathered} \tag{2.10}$$

From now and on we will write D_t^0, D_t^A, D_t and D_t^1 for $D^0(t, X_{t^-})$, $D^A(t, X_{t^-}, Y_{t^-})$, $D(t, X_{t^-}, Y_{t^-})$ and $D^1(t, X_{t^-}, Y_{t^-})$ respectively, unless it is necessary to underline the dependence on the processes involved.

In [5] (Proposition 2.2) it is proved the following result.

Proposition 2.1 *Under the assumption*

$$\mathbb{E} \int_0^T \nu(D_s) \, ds < +\infty \quad P - a.s. \tag{2.11}$$

the $(P, \mathcal{F}_t)$-predictable projection of $m(dt, dx)$ is given by

$$m^p(dt, dx) = \lambda_t \Phi_t(dx) dt \tag{2.12}$$

where $\forall A \in \mathcal{B}(\mathbb{R})$

$$m^p(dt, A) = \lambda_t \Phi_t(A) dt = \nu(D_t^A) dt = \nu(D^A(t, X_{t^-}, Y_{t^-})) dt. \tag{2.13}$$

In particular $\lambda_t = \nu(D_t) = \nu(D(t, X_{t^-}, Y_{t^-}))$ provides the $(P, \mathcal{F}_t)$-predictable intensity of the point process N_t.

Remark 2.2 *Since $E \int_0^T \nu(D_s) \, ds < +\infty$ the process given in (2.9) is a $(P, \mathcal{F}_t)$-martingale, $\forall A \in \mathcal{B}(\mathbb{R})$.*

The time-dependency of $(\lambda_t, \Phi_t(dx))$ is introduced in order to incorporate seasonality effects, which are typical for high frequency data. In particular λ_t, the $(P, \mathcal{F}_t)$-intensity of the counting process N_t, corresponds to the rate at which new economic information is absorbed by the market.

Let us observe that the following representation of S_t in terms of the integer-valued random measure $m(dt, dx)$ holds

$$S_t = S_0 + \int_0^t \int_{I\!R} S_{r^-}(e^x - 1) m(dr, dx). \tag{2.14}$$

Moreover, by (2.13) for each $\mathcal{F}_t$-adapted process, $H(t,x)$, jointly measurable on (t,x) we have that

$$\int_Z H(t, K_1(t;\zeta)) 1\!\!I_{D_t}(\zeta)\nu(d\zeta) = \int_{I\!R} H(t,x)\lambda_t \Phi_t(dx) \tag{2.15}$$

and for $H(t,x) = (e^x - 1)^2$

$$\int_Z K(t;\zeta)^2 \nu(d\zeta) = \int_{I\!R} (e^x - 1)^2 \lambda_t \Phi_t(dx).$$

In the next proposition, proved in [11] (Proposition 2.2) we will give the semimartingale structure for the risky asset S_t.

Proposition 2.3 *Under (2.11) and the following condition*

$$\forall t \in [0,T] \qquad \int_Z K(t;\zeta)^2 \nu(d\zeta) < +\infty \quad P - a.s. \tag{2.16}$$

S_t *is a* $(P, \mathcal{F}_t)$*-special semimartingale ([22]) with the decomposition*

$$S_t = S_0 + M_t^S + A_t^S \tag{2.17}$$

where

$$A_t^S = \int_0^t \int_Z S_{r^-} K(r;\zeta)\nu(d\zeta) dr = \int_0^t \int_{I\!R} S_{r^-}(e^x - 1)\lambda_r \Phi_r(dx) dr$$

is a predictable process with bounded variation paths,

$$M_t^S = \int_0^t \int_Z S_{r^-} K(r;\zeta)(\mathcal{N}(dr, d\zeta) - \nu(d\zeta)dr) = \int_0^t \int_{I\!R} S_{r^-}(e^x - 1)(m(dr, dx) - \lambda_r \Phi_r(dx) dr)$$

is a locally square-integrable local martingale whose angle process is given by

$$< M^S >_t = \int_0^t \int_Z S_{r^-}^2 K(r;\zeta)^2 \nu(d\zeta) dr = \int_0^t \int_{I\!R} S_{r^-}^2 (e^x - 1)^2 \lambda_r \Phi_r(dx) dr. \tag{2.18}$$

2.2. Partial Information

We suppose that investors acting in the market have only limited access to the information flow. The flow of observable events, $\{\mathcal{G}_t\}_{t\in[0,T]}$, contains all information on the underlying asset price, that is

$$\mathcal{F}_t^S = \sigma\{S_s; s \le t\} \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t. \tag{2.19}$$

Note that

$$\mathcal{F}_t^S = \mathcal{F}_t^Y = \mathcal{F}_t^m = \sigma\{m((0,s], A); s \le t, A \in \mathcal{B}(I\!R)\}.$$

Moreover we assume that $\{\mathcal{G}_t\}_{t\in[0,T]}$ is right continuous and that $\mathcal{G}_0$ contains all the P-null sets of $\mathcal{F}_T$. This situation is referred as partial information in contrast to the case of full information.

We will reduce our model to a full information model by introducing the $(P, \mathcal{G}_t)$-predictable projection, $\nu^p(dt, dx)$, of the integer-valued measure $m(dt, dx)$. This leads to a filtering problem where X_t represents the signal or state process, $\mathcal{G}_t$ provides all the available information about X_t and the conditional distribution of X_t given $\mathcal{G}_t$ is the most detailed description of our knowledge of X_t. It is known ([25]) that there exists a probability measure-valued optional process $\pi_t(dx)$ such that, for any bounded measurable function $f(t,x)$

$$\pi_t(f) = \mathbb{E}[f(t,X_t) \mid \mathcal{G}_t], \tag{2.20}$$

and since X_t is a cadlag process π_t has a version with cadlag paths.

From now on we will denote by $\widehat{V}$ the $(P,\mathcal{G}_t)$-optional projection of a generic process V defined as the unique optional process (in a P-indistinguishable sense) such that for each $\mathcal{G}_t$-stopping time τ, $\widehat{V}_\tau = \mathbb{E}[V_\tau|\mathcal{G}_\tau]$, $P-a.s.$. Hence

$$\widehat{V}_t = \mathbb{E}[V_t|\mathcal{G}_t] \quad P-a.s.$$

In particular $\widehat{f(X_t)} = \pi_t(f)$, $P-a.s.$ and this implies that $\widehat{f(X_t)}$ has cadlag trajectories. From now on we will use both the notations $\widehat{f(X_t)}$ and $\pi_t(f)$ to denote the $(P,\mathcal{G}_t)$-optional projection of a process $f(X_t)$.

Proposition 2.4 *The $(P,\mathcal{G}_t)$-predictable projection of the integer-valued measure $m(dt,dx)$ is given by*

$$\nu^p(dt,dx) = \widehat{\lambda_t\Phi_t}(dx)|_{t=t^-}dt = \pi_{t^-}(\lambda_t\Phi_t(dx)), \tag{2.21}$$

that is, for any $A \in \mathcal{B}(\mathbb{R})$

$$\nu^p((0,t],A) = \int_0^t \pi_{s^-}(\lambda_s\Phi_s(A))ds = \int_0^t \pi_{s^-}\big(\nu(D^A(.,.,Y_{s^-}))\big)ds. \tag{2.22}$$

where π_{t^-} denotes the left version of the process π_t.

Proof.
By definition of $(P,\mathcal{F}_t)$-predictable projection of the integer-valued measure $m(dt,dx)$ we have that, for each $H(t,x)$ $\{P,\mathcal{F}_t\}$-predictable process jointly measurable w.r.t. $(t,x) \in [0,T] \times \mathbb{R}$, verifying the condition $\mathbb{E}\int_0^T\int_{\mathbb{R}}|H(r,x)|\lambda_r\Phi_r(dx)dr < +\infty$, the process

$$m_t = \int_0^t\int_{\mathbb{R}} H(r,x)(m(dr,dx) - \lambda_r\Phi_r(dx)dr) \tag{2.23}$$

is a $\{P,\mathcal{F}_t\}$-martingale.
We will use two well-known facts: for every $(P,\mathcal{F}_t)$-martingale m_t, the projection $\widehat{m}_t$ is a $(P,\mathcal{G}_t)$-martingale and that for any progressively measurable process Ψ_t with $\mathbb{E}\int_0^T \Psi_t dt < +\infty$

$$\widehat{\int_0^T \Psi_t dt} - \int_0^T \widehat{\Psi}_t dt$$

is a $(P,\mathcal{G}_t)$-martingale. Note that this implies that $E\int_0^T \Psi_t dt = \mathbb{E}\int_0^T \widehat{\Psi}_t dt$.

Let us now consider in (2.23) a process $H(t,x)$ which is $(P,\mathcal{G}_t)$-predictable. By projection on $\mathcal{G}_t$ we get that

$$\int_0^t\int_{\mathbb{R}} H(r,x)m(dr,dx) - \mathbb{E}[\int_0^t\int_{\mathbb{R}} H(r,x)\lambda_r\Phi_r(dx)dr|\mathcal{G}_t]$$

is a $(P,\mathcal{G}_t)$-martingale, and

$$\int_0^t\int_{\mathbb{R}} H(r,x)m(dr,dx) - \int_0^t\int_{\mathbb{R}} H(r,x)\widehat{\lambda_r\Phi_r}(dx)dr$$

is a $(P,\mathcal{G}_t)$-martingale.
Finally, since $\mathbb{E}\int_0^T\int_{\mathbb{R}}|H(r,x)|\widehat{\lambda_r\Phi_r}dr = \mathbb{E}\int_0^T\int_{\mathbb{R}}|H(r,x)|\lambda_r\Phi_r(dx)dr$, we get that, for any $(P,\mathcal{G}_t)$-predictable process $H(t,x)$ verifying the condition $\mathbb{E}\int_0^T\int_{\mathbb{R}}|H(r,x)|\widehat{\lambda_r\Phi_r}(dx)dr < +\infty$, the process

$$\int_0^t\int_{\mathbb{R}} H(r,x)m(dr,dx) - \int_0^t\int_{\mathbb{R}} H(r,x)\widehat{\lambda_r\Phi_r}(dx)|_{r=r^-}dr$$

is a $(P, \mathcal{G}_t)$-martingale and this concludes the proof.
$\square$

Let us introduce the $(P, \mathcal{G}_t)$-compensated martingale random measure

$$m^\pi(dt, dx) = m(dt, dx) - \nu^p(dt, dx) = m(dt, dx) - \pi_{t^-}(\lambda_t \Phi_t(dx))dt. \tag{2.24}$$

From now on we will consider the following $(P, \mathcal{G}_t)$-semimartingale representation of S_t

$$S_t = \int_0^t \int_{I\!R} S_{r^-}(e^x - 1)m^\pi(dr, dx) + \int_0^t \int_{I\!R} S_{r^-}(e^x - 1)\pi_{r^-}(\lambda_r \Phi_r(dx))dr. \tag{2.25}$$

This model may now be treated as a full information model with respect to the observable flow $\{\mathcal{G}_t\}_{t\in[0,T]}$.

Remark 2.5 *Recalling that the $(P, \mathcal{F}_t)$-semimartingale representation of S_t can be written also as*

$$S_t = \int_0^t \int_Z S_{r^-} K(r;\zeta)(\mathcal{N}(dr, d\zeta) - \nu(d\zeta)dr) + \int_0^t \int_Z S_{r^-} K(r;\zeta)\nu(d\zeta)dr,$$

by projection on $\mathcal{G}_t$ we obtain that $S_t - \int_0^t \int_Z S_{r^-} \widehat{K}(r;\zeta)\nu(d\zeta)dr$ is a $(P, \mathcal{G}_t)$-martingale, which in turn implies

$$\int_0^t \int_Z S_{r^-} \widehat{K}(r;\zeta)\nu(d\zeta)dr = \int_0^t \int_{I\!R} S_{r^-}(e^x - 1)\pi_{r^-}(\lambda_r \Phi_r(dx)).$$

In a more general framework, for each $\mathcal{F}_t$-adapted process, $H(t, x)$, by projecting on $\mathcal{G}_t$ equation (2.15) we get

$$\int_Z I\!E[H(t, K_1(t;\zeta)) 1\!I_{D_t}(\zeta) \mid \mathcal{G}_t]\nu(d\zeta) = I\!E[\int_{I\!R} H(t, x)\lambda_t \Phi_t(dx) \mid \mathcal{G}_t].$$

In particular, for $H(t, x) = (e^x - 1)^2$, we find that

$$\int_Z \widehat{K(t;\zeta)^2}\nu(d\zeta) = \int_{I\!R} (e^x - 1)^2 \widehat{\lambda_t \Phi_t}(dx).$$

From now on we will assume (2.11), (2.16) and

$$\forall t \in [0, T] \quad \int_Z \widehat{K(t;\zeta)^2}\nu(d\zeta) < +\infty \quad P - a.s. \tag{2.26}$$

This condition is not a consequence of (2.16) but it is for example fulfilled if

$$I\!E[\int_Z K(t;\zeta)^2 \nu(d\zeta)] < +\infty.$$

Following the same lines of Proposition 2.3, hypothesis (2.26) implies that S_t is a $(P, \mathcal{G}_t)$-locally square integrable special semimartingale.

3. Utility maximization problem

We are interested in solving an optimal portfolio problem for a small investor who has access only to the observable flow $\{\mathcal{G}_t\}_{t\in[0,T]}$. In this section we discuss the problem without specifying the structure of $\{\mathcal{G}_t\}_{t\in[0,T]}$. We assume that the agent does not affect prices and that continuous trading with perfect liquidity is allowed. The agent with initial capital $z_0 > 0$, invests at any time $t \in [0, T]$ the part θ_t of the wealth Z_t, in stock S_t and his remaining wealth in the bond B_t. The dynamics of B_t and S_t are given in (2.1) and (2.7), respectively. The amount of money invested in stock S_t at time t is $\theta_t Z_{t^-}$. Since the agent's information is described by the filtration $\{\mathcal{G}_t\}_{t\in[0,T]}$ the decision θ_t must be adapted to $\mathcal{G}_t$. By considering $\mathcal{G}_t$-predictable, self-financing strategies, taking values in a set $A^\theta \subset I\!R$, the dynamics of the wealth process controlled by the investment process θ_t can be written as

$$dZ_t = Z_{t^-}\Big(\theta_t \frac{dS_t}{S_{t^-}} + (1-\theta_t) r dt\Big) \tag{3.1}$$

and, taking into account (2.25), as

$$dZ_t = Z_{t^-}\Big(\theta_t \int_{I\!R} (e^x - 1) m^\pi(dt, dx) + \theta_t \int_{I\!R} (e^x - 1)\widehat{\lambda_t \Phi}_t(dx) dt + (1-\theta_t) r dt\Big). \tag{3.2}$$

Let us observe that (3.2) makes sense only when the following inequalities hold

$$\int_0^T \int_{I\!R} |\theta_t(e^x - 1)| \widehat{\lambda_t \Phi}_t(dx) dt < +\infty, \qquad \int_0^T |\theta_t| dt < +\infty \quad P-a.s.. \tag{3.3}$$

For a given strategy $\{\theta_t\}_{t\in[0,T]}$ the solution Z_t to (3.1) will of course depend on θ. To be precise, we should denote the process Z_t by Z_t^θ, but sometimes we will suppress θ.

For an agent with power utility

$$U(x) = \frac{x^\alpha}{\alpha} \quad \alpha < 1, \alpha \neq 0$$

the objective is to maximize the expected utility of his terminal wealth

$$I\!E\Big[U(Z_T)\Big] = I\!E\Big[\frac{Z_T^\alpha}{\alpha}\Big]$$

for a suitable class Θ of admissible strategies. This class consists of all A^θ-valued, $(P, \mathcal{G}_t)$-predictable processes, $\{\theta_t\}_{t\in[0,T]}$, such that equation (3.2) has a unique solution $Z_t^\theta > 0$, $P-a.s.$, $\forall t \in [0,T]$. We shall assume furthermore that

$$\sup_{\theta\in\Theta} I\!E|U(Z_T^\theta)| < +\infty. \tag{3.4}$$

Let us observe that $\theta_t = 0$, $\forall t \in [0,T]$, is an admissible strategy, since the associated wealth, $Z_t^0 = z_0 e^{rt}$, is a positive and deterministic process.

Proposition 3.1 *Let $\{\theta_t\}_{t\in[0,T]}$ be an admissible strategy. Then*

$$1 + \int_{I\!R} \theta_s(e^x - 1) m(\{s\}, dx) > 0 \quad P-a.s. \tag{3.5}$$

and the wealth process Z_t^θ can be written as

$$Z_t^\theta = z_0 \; exp\Big[\int_0^t \int_{I\!R} \log(1 + \theta_s(e^x - 1)) m(ds, dx) + \int_0^t (1-\theta_s) r ds\Big]. \tag{3.6}$$

Furthermore, the following inequality is fulfilled

$$\int_{I\!R} |1 + \theta_t(e^x - 1)|^\alpha 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx) < +\infty \quad P-a.s. \tag{3.7}$$

with

$$\mathcal{D}(t,x) = \{\omega \in \Omega : 1 + \theta_t(\omega)(e^x - 1) > 0\}. \tag{3.8}$$

Proof.
Equation (3.2) can be written as $dZ_t = Z_{t^-} dM_t^\theta$, where

$$M_t^\theta = \int_0^t \int_{I\!R} \theta_s(e^x - 1) m^\pi(ds, dx) + \int_0^t \int_{I\!R} \theta_s(e^x - 1)\widehat{\lambda_s \Phi}_s(dx) ds + \int_0^t (1-\theta_s) r ds$$

is a $(P, \mathcal{G}_t)$-local semimartingale. From Doléans-Dade formula we get that

$$Z_t = z_0\, e^{M_t^\theta} \Pi_{s\le t}(1+\Delta M_s^\theta)e^{-\Delta M_s^\theta}$$

and since $Z_t > 0$ then $\forall s \le t$, $1+\Delta M_s^\theta = 1 + \int_{I\!R} \theta_s(e^x-1)m(\{s\},dx) > 0$.

By standard computation we derive expression (3.6). Moreover we have that

$$dZ_t^\alpha = Z_{t^-}^\alpha dM_t^\theta(\alpha) \tag{3.9}$$

where

$$M_t^\theta(\alpha) = \int_0^t \int_{I\!R} [(1+\theta_s(e^x-1))^\alpha - 1]m(ds,dx) + \int_0^t \alpha(1-\theta_s)r ds. \tag{3.10}$$

Finally, if (3.7) does not hold we get that

$$I\!E[\int_0^T \int_{I\!R} Z_{s^-}^\alpha (1+\theta_s(e^x-1))^\alpha m(ds,dx)] = I\!E[\int_0^T \int_{I\!R} Z_{s^-}^\alpha (1+\theta_s(e^x-1))^\alpha 1\!\!I_{\mathcal{D}(s,x)} \widehat{\lambda_s \Phi}_s(dx)ds] = +\infty$$

which in turn implies

$$I\!E[Z_T^\alpha] = z_0^\alpha + I\!E[\int_0^T Z_{s^-}^\alpha dM^\theta(\alpha)] = +\infty$$

and this is in contrast with (3.4).

□

As usual in stochastic control frame we introduce the associated value process ([14]), which gives a dynamic extension of the initial problem to each initial time $t \in [0,T]$

$$V_t(z) = \operatorname*{ess\,sup}_{\theta\in\Theta_t} I\!E\Big[\frac{Z_T^\alpha}{\alpha} \mid \mathcal{G}_t\Big], \tag{3.11}$$

where z denotes the amount of capital at time t and Θ_t the set of the admissible strategies on the interval $[t,T]$. Equation (3.6) implies

$$V_t(z) = \frac{z^\alpha}{\alpha} J_t$$

where for $0<\alpha<1$

$$J_t = \operatorname*{ess\,sup}_{\theta\in\Theta_t} I\!E\Big[\frac{Z_T^\alpha}{Z_t^\alpha} \mid \mathcal{G}_t\Big], \tag{3.12}$$

and

$$J_t = \operatorname*{ess\,sup}_{\theta\in\Theta_t} I\!E\Big[exp\Big\{\alpha \int_t^T \int_{I\!R} \log(1+\theta_s(e^x-1))m(ds,dx) + \alpha\int_t^T (1-\theta_s)r ds\Big\} \mid \mathcal{G}_t\Big]. \tag{3.13}$$

For $\alpha<0$ the $\operatorname{ess\,sup}$ is replaced by the $\operatorname{ess\,inf}$ in (3.12) and (3.13).

By a duality approach in [24] it is proved that the optimal strategy exists in a general incomplete semimartingale market under suitable assumptions on the utility function (verified by the power one).
Moreover, the Bellman optimality principle ([14]) can be stated as in Proposition 6.9 of [26].

Proposition 3.2 *The following properties hold true*
(i) For $0<\alpha<1$, $\{J_t\}_{t\in[0,T]}$ is the smallest cadlag $\mathcal{G}_t$-adapted process such that for each $\theta_t \in \Theta$ the process $\{(Z_t^\theta)^\alpha J_t\}_{t\in[0,T]}$ is a $(P,\mathcal{G}_t)$-supermartingale with $J_T = 1$.

(i') For $\alpha<0$, $\{J_t\}_{t\in[0,T]}$ is the greatest cadlag $\mathcal{G}_t$-adapted process such that for each $\theta_t\in\Theta$ the process $\{(Z_t^\theta)^\alpha J_t\}_{t\in[0,T]}$ is a $(P,\mathcal{G}_t)$-submartingale with $J_T=1$.

(ii) $\theta_t^* \in \Theta$ is an optimal strategy if and only if the process $\{(Z_t^\theta)^\alpha J_t\}_{t\in[0,T]}$ is a $(P,\mathcal{G}_t)$-martingale.

We give now some further properties of the process J_t.

Proposition 3.3 *We have that*
(i) For $0 < \alpha < 1$, *for any* $t \in [0,T]$, $J_t \geq 1$, $P - a.s.$ *and* $\sup_{t\in[0,T]} \mathbb{E}[J_t] \leq J_0 < +\infty$.

(ii) For $\alpha < 0$, *for any* $t \in [0,T]$, $0 < J_t \leq 1$, $P - a.s.$.

Proof.
Since $\theta_t = 0$ is an admissible strategy, by (3.12) we get that

for $0 < \alpha < 1$, $J_t \geq e^{\alpha r(T-t)} \geq 1$ and for $\alpha < 0$, $J_t \leq e^{\alpha r(T-t)} \leq 1$.

By (i) of Proposition 3.2, for $0 < \alpha < 1$, $z_0^\alpha e^{rt} J_t$ is a $(P, \mathcal{G}_t)$-supermartingale, hence $\mathbb{E}(J_t) \leq e^{-rt} J_0$, where $J_0 = \sup_{\theta\in\Theta_t} \mathbb{E}[Z_T^\alpha] < +\infty$.

Finally for $\alpha < 0$, since $\frac{Z_T^\alpha}{Z_t^\alpha} > 0$, we have that $J_t \geq 0$, $P - a.s.$ and by the existence of an optimal strategy that $J_t > 0$.
□

In the next sections we will use the following notations

- $\mathcal{S}^p, 1 \leq p \leq +\infty$, denotes the space of $\mathbb{R}$-valued $\mathcal{G}_t$-adapted stochastic processes $\{H_t\}_{t\in[0.T]}$ with $\|H\|_{\mathcal{S}^p} = \| \sup_{t\in[0.T]} \mid H_t \mid \|_{L^p} < +\infty$.
- $\mathcal{L}^2_{\nu^p}$ ($\mathcal{L}^2_{\nu^p,loc}$) denotes the space of $\mathbb{R}$-valued $\mathcal{G}_t$-predictable processes $\{U(t,x)\}_{t\in[0.T]}$ indexed by x with

$$\mathbb{E}\Big(\int_0^T \int_{\mathbb{R}} \mid U(t,x) \mid^2 \widehat{\lambda_t \Phi_t}(dx) dt \Big)^{\frac{1}{2}} < +\infty \quad \Big(\text{resp.} \quad \int_0^T \int_{\mathbb{R}} \mid U(t,x) \mid^2 \widehat{\lambda_t \Phi_t}(dx) dt \Big)^{\frac{1}{2}} < +\infty \quad P - a.s.\Big).$$

- $\mathcal{L}^2$ ($\mathcal{L}^2_{loc}$) denotes the space of $\mathbb{R}$-valued $\mathcal{G}_t$-adapted processes $\{R_t\}_{t\in[0.T]}$ with

$$\mathbb{E}\Big(\int_0^T \mid R_t) \mid^2 dt \Big)^{\frac{1}{2}} < +\infty \quad \Big(\text{resp.} \quad \int_0^T \mid R_t \mid^2 dt \Big)^{\frac{1}{2}} < +\infty \quad P - a.s.\Big).$$

- If $U(t,x) \in \mathcal{L}^2_{\nu^p}$ we define

$$\|U\|_{\mathcal{L}^2_{\nu^p}} = \mathbb{E}\Big(\int_0^T \int_{\mathbb{R}} \mid U(t,x) \mid^2 \widehat{\lambda_t \Phi_t}(dx) dt \Big)^{\frac{1}{2}}.$$

4. The case $\mathcal{G}_t = \mathcal{F}_t^S$

In this section we deal with the case where investors can only observe stock prices, that is we assume $\mathcal{G}_t = \mathcal{F}_t^S$. The filtering problem in this frame has been already examined in [7] and [5]. The filter, π_t, is characterized as the unique (in a weak sense) solution to the so called Kushner-Stratonovich equation and an explicit representation via the Feynman-Kac formula is provided. For details we refer to [7] if Y is a discrete valued process and to [5] if Y is a real-valued process.

By using Proposition 3.2 and existence of an optimal strategy [24] we show that the process J_t is a solution to a BSDE, driven by the $\mathcal{F}_t^S$- compensated martingale random measure $m^\pi(dt,dx) = m(dt,dx) - \pi_{t^-}(\lambda_t \Phi_t(dx))dt$.

Since $\theta_t = 0$ is an admissible strategy, Proposition 3.2 implies that $\{e^{\alpha rt} J_t\}_{t\in[0.T]}$, for $0 < \alpha < 1$ ($\alpha < 0$) is a $(P, \mathcal{F}_t^S)$-supermartingale (submartingale) hence it admits a unique Doob-Meyer decomposition

$$e^{\alpha rt} J_t = m_t^J - A_t \tag{4.1}$$

with m_t^J a $(P, \mathcal{F}_t^S)$-local martingale and A_t a nondecreasing (nonincreasing) $(P, \mathcal{F}_t^S)$-predictable process with $A_0 = 0$. By a classical martingale representation Theorem (see for example, Theorem III 4.37 in [22]) there exists a $(P, \mathcal{F}_t^S)$-predictable process $\Gamma(t,x) \in \mathcal{L}^2_{\nu^p,loc}$ such that

$$m_t^J = \int_0^t \int_{\mathbb{R}} \Gamma(s,x) m^\pi(ds,dx). \tag{4.2}$$

Theorem 4.1 *For $0 < \alpha < 1$ ($\alpha < 0$) the process $\{J_t, \Gamma(t,x)\}_{t\in[0.T]}$ solves the following BSDE*

$$J_t = 1 - \int_t^T \int_{I\!R} e^{-\alpha r s}\Gamma(s,x) m^\pi(ds,dx) + \int_t^T \operatorname*{ess\,sup}_{\theta\in\Theta} f(s,J,\Gamma,\theta) ds \tag{4.3}$$

($\operatorname{ess\,inf}$ *in the case $\alpha < 0$), where*

$$f(s,J,\Gamma,\theta) = \int_{I\!R} \left(J_s + \Gamma(s,x)e^{-\alpha r s}\right)\left[\{1+\theta_s(e^x-1)\}^\alpha - 1\right] I\!I_{\mathcal{D}(s,x)} \widehat{\lambda_s \Phi}_s(dx) - \alpha J_s r(\theta_s - 1). \tag{4.4}$$

The strategy $\theta_t^* \in \Theta$ is optimal if and only if for $0 < \alpha < 1$ it realizes the $\operatorname{ess\,sup}$ *of $f(s,J,\Gamma,\theta)$ (*$\operatorname{ess\,inf}$ *in the case $\alpha < 0$).*
Moreover, $\{J_t, \Gamma(t,x)\}_{t\in[0.T]}$ is the smallest for $0 < \alpha < 1$ (the greatest for $\alpha < 0$) solution to (4.3) such that M_t^J defined in (4.6) below is a $(P, \mathcal{F}_t^S)$- local martingale.

Proof.
Let us consider the case $0 < \alpha < 1$. The proof for $\alpha < 0$ can be performed in an analogous way.
By applying the product rule to $(Z_t^\theta)^\alpha J_t$, $\forall \theta \in \Theta$

$$(Z_t^\theta)^\alpha J_t = z_0^\alpha J_0 + \int_0^t J_{s^-} d(Z_s^\theta)^\alpha + \int_0^t (Z_{s^-}^\theta)^\alpha dJ_s + \sum_{s\le t} \Delta(Z_s^\theta)^\alpha \Delta J_s,$$

since by (4.1), (4.2)

$$dJ_t = -\alpha r J_t dt + e^{-\alpha r t}(dm_t^J - dA_t), \quad \Delta J_s = e^{-\alpha r s}\int_{I\!R} \Gamma(s,x) m(\{s\},dx) \tag{4.5}$$

and by (3.9)

$$\Delta(Z_s^\theta)^\alpha = (Z_{s^-}^\theta)^\alpha \int_{I\!R} [\{1+\theta_s(e^x-1)\}^\alpha - 1] m(\{s\},dx),$$

we get

$$d\big((Z_t^\theta)^\alpha J_t\big) = (Z_{t^-}^\theta)^\alpha e^{-\alpha r t} dm_t^J + e^{-\alpha r t}\int_{I\!R} \left(J_{t^-}e^{\alpha r t} + \Gamma(t,x)\right)(Z_{t^-}^\theta)^\alpha \left[\{1+\theta_t(e^x-1)\}^\alpha - 1\right] m(dt,dx) +$$

$$-\alpha r (Z_t^\theta)^\alpha J_t \theta_t dt - (Z_{t^-}^\theta)^\alpha e^{-\alpha r t} dA_t.$$

Then

$$d\big((Z_t^\theta)^\alpha J_t\big) = dM_t^J - (Z_{t^-}^\theta)^\alpha e^{-\alpha r t}\Big[dA_t - \int_{I\!R} \left(J_t e^{\alpha r t} + \Gamma(t,x)\right)[\{1+\theta_t(e^x-1)\}^\alpha - 1] \mathbb{I}_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx) dt + \alpha J_t r \theta_t e^{\alpha r t} dt\Big]$$

where

$$M_t^J = M_0^J + \int_0^t \int_{I\!R} (Z_{s^-}^\theta)^\alpha e^{-\alpha r s}\Gamma(s,x)\{1+\theta_s(e^x-1)\}^\alpha \mathbb{I}_{\mathcal{D}(s,x)} m^\pi(ds,dx) + \tag{4.6}$$

$$\int_0^t \int_{I\!R} (Z_{s^-}^\theta)^\alpha J_{s^-}\left[\{1+\theta_s(e^x-1)\}^\alpha - 1\right] \mathbb{I}_{\mathcal{D}(s,x)} m^\pi(ds,dx).$$

Since $(Z_t^\theta)^\alpha J_t$ is a supermartingale $\forall \theta_t \in \Theta$, it follows that (4.6) is a $(P, \mathcal{F}_t^S)$-local martingale and

$$dA_t - \int_{I\!R} \left(J_t e^{\alpha r t} + \Gamma(t,x)\right)\left[\{1+\theta_t(e^x-1)\}^\alpha - 1\right] \mathbb{I}_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx) dt + \alpha J_t r \theta_t e^{\alpha r t} dt \ge 0$$

which in turn implies

$$dA_t \ge \operatorname*{ess\,sup}_{\theta\in\Theta} \Big[\int_{I\!R} \left(J_t e^{\alpha r t} + \Gamma(t,x)\right)\left[\{1+\theta_t(e^x-1)\}^\alpha - 1\right] \mathbb{I}_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx) dt + \alpha J_t r \theta_t e^{\alpha r t} dt\Big].$$

By Theorem 2.2 in [24] there exists an optimal strategy $\theta^*_t \in \Theta$ and by the Bellman optimality principle $J_t(Z^{\theta^*}_t)^\alpha$ is a $(P, \mathcal{F}^S_t)$- martingale. Thus

$$dA_t = \operatorname*{ess\,sup}_{\theta\in\Theta} \Big[\int_{I\!R} \big(J_t e^{\alpha r t} + \Gamma(t,x)\big)[\{1+\theta_t(e^x-1)\}^\alpha - 1] 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx)dt + \alpha J_t r \theta_t e^{\alpha r t} dt \Big]. \tag{4.7}$$

Finally (4.5), (4.2) and (4.7) yield that $(J_t, \Gamma(t,x))$ solves BSDE (4.3).

We now prove that $\{J_t, \Gamma(t,x)\}_{t\in[0.T]}$ is the smallest solution such that M^J_t in (4.6) is a $(P, \mathcal{F}^S_t)$-local martingale. Let $\{\widetilde{J}_t, \widetilde{\Gamma}(t,x)\}_{t\in[0.T]}$ be a solution of BSDE (4.3). The product rule implies that

$$d\Big((Z^\theta_t)^\alpha \widetilde{J}_t\Big) = dM^{\widetilde{J}}_t - (Z^\theta_t)^\alpha \Big[\operatorname*{ess\,sup}_{\theta\in\Theta} f(t, \widetilde{J}, \widetilde{\Gamma}, \theta) - f(t, \widetilde{J}, \widetilde{\Gamma}, \theta)\Big] dt$$

with $M^{\widetilde{J}}_t$ a $(P, \mathcal{F}^S_t)$-local martingale such that $M^{\widetilde{J}}_0 = z^\alpha_0 \widetilde{J}_0$ and

$$dM^{\widetilde{J}}_t = \int_{I\!R} (Z^\theta_{t^-})^\alpha e^{-\alpha r t} \Big\{ \widetilde{\Gamma}(t,x) + \big(\widetilde{J}_{t^-} e^{\alpha r t} + \widetilde{\Gamma}(t,x)\big)[\{1+\theta_t(e^x-1)\}^\alpha - 1] \Big\} 1\!\!I_{\mathcal{D}(t,x)} m^\pi(dt,dx).$$

We have that $M^{\widetilde{J}}_t \geq (Z^\theta_t)^\alpha \widetilde{J}_t \geq 0$, and $M^{\widetilde{J}}_t$ is a supermartingale since every lower bounded local martingale is a supermartingale. This implies that $(Z^\theta_t)^\alpha \widetilde{J}_t$ is a supermartingale for each $\theta \in \Theta$ and the thesis follows by (i) in Proposition 3.2 .
□

Under further assumptions we characterize J_t as the unique solution to the BSDE (4.3).

Proposition 4.2 *Assume the valued set of admissible portfolios, A^θ, to be compact and that there exists a constant $C > 0$ such that $|K(t;\zeta)| \leq C$ and $\lambda_t = \nu(D_t) \leq C$, $P-a.s.$ (see Section 2). Then the process $\{J_t, \Gamma(t,x)\}_{t\in[0.T]}$ is the unique solution in $\mathcal{S}^2 \times \mathcal{L}^2_{\nu^p}$ to the BSDE (4.3). Moreover J_t is a bounded process.*

Proof.
First, let us observe that $|K(t;\zeta)| \leq C$ and $\lambda_t \leq C$ imply

$$\widehat{\lambda_t \Phi}_t(I\!R) = I\!E[\lambda_t | \mathcal{F}^S_t] \leq C, \qquad \int_{I\!R} (e^x-1)^2 \widehat{\lambda_t \Phi}_t(dx) = I\!E[\int_Z K(t;\zeta)^2 \nu(d\zeta) | \mathcal{F}^S_t] \leq C^3. \tag{4.8}$$

As in [2] we consider the space $L(I\!R, \nu^p)$ of measurable functions $u(x)$ with the topology of convergence in measure and define for $u, \widetilde{u} \in L(I\!R, \nu^p)$,

$$\|u - \widetilde{u}\|_t = \Big(\int_{I\!R} |u(x) - \widetilde{u}(x)|^2 \widehat{\lambda_t \Phi}_t(dx) \Big)^{\frac{1}{2}}. \tag{4.9}$$

For $0 < \alpha < 1$, the generator of BSDE (4.3) is given by

$$g(t,y,u) = \operatorname*{ess\,sup}_{\theta\in\Theta} \Big[\int_{I\!R} \big(y + u(x) e^{-\alpha r t}\big) \big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big] 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx) - \alpha y r(\theta_t - 1) \Big], \tag{4.10}$$

(for $\alpha < 0$ the ess sup is replaced by the ess inf).
We show that $g(t,y,u)$ is Lipschitz in (y,u), namely there exists a constant $\widetilde{C} > 0$ such that

$$|g(t,y,u) - g(t,\widetilde{y},\widetilde{u})| \leq \widetilde{C}\big(|y - \widetilde{y}| + \|u - \widetilde{u}\|_t\big) \quad P \times dt - a.e.$$

Let $0 < \alpha < 1$. We have successively

$$g(t,y,u) = \operatorname*{ess\,sup}_{\theta\in\Theta} \Big\{ \int_{I\!R} \big((y-\widetilde{y}) + (u(x) - \widetilde{u}(x)) e^{-\alpha r t}\big) \big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big] 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx) - \alpha(y - \widetilde{y}) r(\theta_t - 1) +$$

$$\int_{I\!R} \big(\widetilde{y} + \widetilde{u}(x) e^{-\alpha r t}\big) \big[\{1+\theta_t(e^x-1)\}^\alpha - 1\big] 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t \Phi}_t(dx) - \alpha \widetilde{y} r(\theta_t - 1) \Big\},$$

$$g(t,y,u)-g(t,\widetilde{y},\widetilde{u}) \leq \operatorname*{ess\,sup}_{\theta\in\Theta} \Big[\int_{I\!R} \big((y-\widetilde{y})+(u(x)-\widetilde{u}(x))e^{-\alpha r t}\big)\left[\{1+\theta_t(e^x-1)\}^\alpha-1\right] 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t\Phi_t}(dx)-\alpha(y-\widetilde{y})r(\theta_t-1)\Big],$$

$$g(t,y,u)-g(t,\widetilde{y},\widetilde{u}) \leq \operatorname*{ess\,sup}_{\theta\in\Theta} \Big\{ \int_{I\!R} \left[\{1+\theta_t(e^x-1)\}^\alpha-1\right] 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t\Phi_t}(dx)+\alpha r(\theta_t-1)\Big\}\ |y-\widetilde{y}|+$$

$$\operatorname*{ess\,sup}_{\theta\in\Theta} \Big(\int_{I\!R} \left[\{1+\theta_t(e^x-1)\}^\alpha-1\right]^2 1\!\!I_{\mathcal{D}(t,x)} \widehat{\lambda_t\Phi_t}(dx)\Big)^{\frac{1}{2}} \|u-\widetilde{u}\|_t.$$

Hence (4.8) and compactness of A^θ imply

$$g(t,y,u)-g(t,\widetilde{y},\widetilde{u}) \leq \widetilde{C}\big(|y-\widetilde{y}|+\|u-\widetilde{u}\|_t\big) \quad P\times dt-a.e.$$

for a suitable constant $\widetilde{C}>0$ and by symmetry the Lipschitz property follows. By classical results we get that there exists unique solution in $\mathcal{S}^2\times\mathcal{L}^2_{\nu^P}$ to the BSDE (4.3) (see for instance [2], Proposition 3.2). In an analogous way the case $\alpha<0$ can be performed.

Finally, to obtain boundedness of J_t, we recall that

$$dZ^\alpha_t = Z^\alpha_{t^-} dM^\theta_t(\alpha)$$

where

$$M^\theta_t(\alpha) = \int_0^t \int_{I\!R} [\{1+\theta_s(e^x-1)\}^\alpha-1] m(ds,dx) + \int_0^t \alpha(1-\theta_s) r ds.$$

Since A^θ is a compact set, there exist two positive constants $C_i>0$, $i=1,2$, such that

$$I\!E\Big[\frac{Z^\alpha_T}{Z^\alpha_t}|\mathcal{G}_t\Big] \leq 1 + I\!E\Big[\int_t^T \int_{I\!R} \frac{Z^\alpha_{s^-}}{Z^\alpha_t}[(1+C_1(e^x-1))^\alpha-1] m(ds,dx)] + C_1\int_t^T \frac{Z^\alpha_s}{Z^\alpha_t} ds|\mathcal{G}_t\Big] \leq$$

$$1 + I\!E\Big[\int_t^T \int_{I\!R} \frac{Z^\alpha_s}{Z^\alpha_t}(1+C_1(e^x-1))^\alpha \lambda_s\Phi_s(dx)ds + C_1\int_t^T \frac{Z^\alpha_s}{Z^\alpha_t} ds|\mathcal{G}_t\Big] =$$

$$1 + I\!E\Big[\int_t^T \int_{Z} \frac{Z^\alpha_s}{Z^\alpha_t}(1+C_1 K(t;\zeta))^\alpha 1\!\!I_{D_t}(\zeta)\nu(d\zeta)ds + C_1\int_t^T \frac{Z^\alpha_s}{Z^\alpha_t} ds|\mathcal{G}_t\Big],$$

where the last equality is a consequence of (2.15). Recalling that $D_t=\{\zeta\in Z : K(t;\zeta)\neq 0\}$, by boundedness of $K(t;\zeta)$ and $\lambda_t=\nu(D_t)$ we have that

$$I\!E\Big[\frac{Z^\alpha_T}{Z^\alpha_t}|\mathcal{G}_t\Big] \leq 1 + C_2\int_t^T I\!E\Big[\frac{Z^\alpha_s}{Z^\alpha_t}|\mathcal{G}_t\Big] ds.$$

Finally, by Gronwall Lemma we find that, for a suitable constant $C_3>0$

$$I\!E\Big[\frac{Z^\alpha_T}{Z^\alpha_t}|\mathcal{G}_t\Big] \leq e^{C_3(T-t)}$$

which implies that $J_t \leq e^{C_3 T}$, $\forall t\in[0,T]$.
□

5. A model with $\mathcal{G}_t \supset \mathcal{F}^S_t$

In this section we assume that investors receive informations on the stock price and in addition noisy signals on the stochastic factor X_t. Thus they have access to the information given by the filtration $\mathcal{G}_t=\mathcal{F}^S_t\vee\mathcal{F}^\eta_t$ generated by the past asset prices and by a process η_t, giving observations of X in additional Gaussian noise, namely

$$\eta_t = \int_0^t \gamma(X_s)ds + W^1_t. \tag{5.1}$$

Here W_t^1 is a $(P, \mathcal{F}_t)$-standard Brownian motion independent of $\mathcal{N}(dt, d\zeta)$ and $\gamma(x)$ is a bounded measurable function. The Brownian motions W_t^1 and W_t may be correlated, with correlation $\rho \in [-1, 1]$.

The $(P, \mathcal{G}_t)$-predictable projection of the integer-valued measure $m(dt, dx)$, $\nu^p(dt, dx)$, can be now computed as in (2.22), where the filter in this framework is defined by

$$\pi_t(f) = I\!E[f(t, X_t) \mid \mathcal{G}_t] = I\!E[f(t, X_t) \mid \mathcal{F}_t^S \vee \mathcal{F}_t^\eta]. \tag{5.2}$$

We introduce the innovation process

$$I_t = \eta_t - \int_0^t \pi_s(\gamma) ds. \tag{5.3}$$

It is well known, in the case $\mathcal{G}_t = \mathcal{F}_t^\eta$ that I_t is a $(P, \mathcal{F}_t^\eta)$-Brownian motion. This result can be easily extended to our setting following the same lines of that used in [28], Theorem 7.12.

Proposition 5.1 *The random process $\{I_t\}_{t \in [0,T]}$ is a $(P, \mathcal{G}_t)$-Wiener process.*

It is known ([25]) that there exists a functional H_t defined on the space of cadlag trajectories $D_{I\!R^2}[0, T]$ with values in the space of probability measures on $I\!R$ such that the filter is given by $\pi_t = H_t(Y_{.\wedge t}, \eta_{.\wedge t})$. Thus, according to the definition of I_t we have that

$$\mathcal{F}_t^S \vee \mathcal{F}_t^I \subseteq \mathcal{G}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^\eta$$

and in general this inclusion can be strict, hence a martingale representation theorem for $(P, \mathcal{G}_t)$-martingales with respect to $m^\pi(dt, dx)$ and I_t cannot be directly derived from usual martingale representation theorems.

Proposition 5.2 *Every $(P, \mathcal{G}_t)$-locally square integrable local-martingale M_t admits the decomposition*

$$M_t = M_0 + \int_0^t \int_{I\!R} \phi(t, x) m^\pi(ds, dx) + \int_0^t \psi_s dI_s \tag{5.4}$$

where $\phi(t, x)$ is a $\mathcal{G}_t$-predictable process and ψ_t is a $\mathcal{G}_t$-adapted process such that

$$\int_0^T \int_{I\!R} \mid \phi(t, x) \mid^2 \pi_t(\lambda_t \Phi_t(dx)) dt < +\infty, \qquad \int_0^T \psi_s^2 ds < +\infty \quad P - a.s.$$

Proof.
Let Q be the probability measure defined, $\forall t \in [0, T]$, as

$$L_t = \frac{dQ}{dP}|_{\mathcal{G}_t} = \mathcal{E}\Big(- \int_0^t \widehat{\gamma}(X_s) dI_s \Big) = exp\Big\{ - \int_0^t \widehat{\gamma}(X_s) dI_s - \frac{1}{2} \int_0^t \widehat{\gamma}(X_s)^2 ds \Big\}$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential. Since $\gamma(x)$ is bounded the exponential local martingale L_t is actually a $(P, \mathcal{G}_t)$-martingale and it is a bounded martingale such that $I\!E\Big[(\sup_{t \in [0,T]} L_t)^2\Big] < +\infty$. From Girsanov Theorem,

$$I_t + \int_0^t \widehat{\gamma}(X_s) ds = \eta_t$$

is a $(Q, \mathcal{G}_t)$-Brownian motion. Let us observe the Q-distribution of the pair (Y_t, η_t) is uniquely determined by the $(Q, \mathcal{G}_t)$-predictable characteristics. This can be proved by conditioning on $\mathcal{F}_t^\eta$ and then averaging over the Wiener measure. In fact, for any bounded functions f and g since $\mathcal{F}_t^\eta \subseteq \mathcal{G}_t$ we have that

$$I\!E^Q[f(\eta_t) g(Y_t)] = I\!E^Q[f(\eta_t) I\!E^Q[g(Y_t) | \mathcal{F}_t^\eta]] =$$

$$I\!E^Q[f(\eta_t) I\!E^Q[g(Y_0) + \int_0^t \int_{I\!R} \{g(Y_{s^-} + x) - g(Y_{s^-}\} \pi_{s^-}(\lambda_s \phi_s(dx)) ds | \mathcal{F}_t^\eta]]$$

and this expectation can be evaluated by integration with respect to the Wiener measure.

Thus, as in [3], we apply Corollary III.4.31 of [22] which asserts that for any $\widetilde{M}_t$, $(Q, \mathcal{G}_t)$-locally square integrable local-martingale, there exist two $\mathcal{G}_t$-adapted processes $\widetilde{\phi}(t,x)$ and $\widetilde{\psi}_t$, with $\widetilde{\phi}(t,x)$ predictable, such that

$$\widetilde{M}_t = \widetilde{M}_0 + \int_0^t \int_{I\!R} \widetilde{\phi}(s,x) m^{\pi}(ds,dx) + \int_0^t \widetilde{\psi}_s d\eta_s \tag{5.5}$$

and

$$\int_0^T \int_{I\!R} \mid \widetilde{\phi}(t,x) \mid^2 \pi_t(\lambda_t \Phi_t(dx)) dt < +\infty, \quad \int_0^T \widetilde{\psi}_t^2 dt < +\infty \quad Q-a.s..$$

Let M_t be a $(P, \mathcal{G}_t)$-local martingale, by Kallianpur-Striebel formula $\widetilde{M}_t = M_t L_t^{-1}$ is a $(Q, \mathcal{G}_t)$-local martingale. We can write $M_t = \widetilde{M}_t L_t$ and by the product rule we get

$$\begin{aligned} dM_t &= \widetilde{M}_{t^-} dL_t + L_{t^-} d\widetilde{M}_t + d\langle \widetilde{M}^c, L^c \rangle_t + d(\sum_{s \le t} \Delta \widetilde{M}_s \Delta L_s) = \\ &= L_t(\widetilde{\psi}_t - \widehat{\gamma}(X_t)\widetilde{M}_t) dI_t + \int_{I\!R} L_{t^-} \widetilde{\phi}(t,x) m^{\pi}(dt,dx) \end{aligned}$$

which gives the martingale representation for M_t with $\psi_t = L_t \widetilde{\psi}_t - \widehat{\gamma}(X_t) M_t$ and $\phi(t,x) = L_{t^-}\widetilde{\phi}(t,x)$.
□

As in Section 4 we want to characterize the value function by a BSDE. In this frame the Doob-Meyer decomposition of $e^{\alpha r t} J_t$ is given again by (4.1) but, by Proposition 5.2, m_t^J has the representation

$$m_t^J = \int_0^t \int_{I\!R} \Gamma(s,x) m^{\pi}(ds,dx) + \int_0^t R_s dI_s, \tag{5.6}$$

with $\Gamma(t,x) \in \mathcal{L}^2_{\nu^p, loc}$ and $R_t \in \mathcal{L}^2_{loc}$.

By using exactly the same arguments as in the proofs of Theorem 4.1 and Proposition 4.2 we get the following results

Theorem 5.3 *For $0 < \alpha < 1$, the process $\{J_t, \Gamma(t,x), R_t\}_{t \in [0.T]}$ is a solution to the BSDE*

$$J_t = 1 - \int_t^T \int_{I\!R} e^{-\alpha r s} \Gamma(s,x) m^{\pi}(ds,dx) - \int_t^T e^{-\alpha r s} R_s dI_s + \int_t^T \operatorname*{ess\,sup}_{\theta \in \Theta} f(s, J, \Gamma, \theta) ds \tag{5.7}$$

($\operatorname{ess\,inf}$ *in the case* $\alpha < 0$*), where*

$$f(s,J,\Gamma,\theta) = \int_{I\!R} \left(J_s + \Gamma(s,x) e^{-\alpha r s}\right) \left[\{1 + \theta_s(e^x - 1)\}^{\alpha} - 1\right] 1\!\!I_{\mathcal{D}(s,x)} \widehat{\lambda_s \Phi_s}(dx) - \alpha J_s r(\theta_s - 1). \tag{5.8}$$

The strategy $\theta_t^* \in \Theta$ is optimal if and only if for $0 < \alpha < 1$ it realizes the $\operatorname{ess\,sup}$ *of $f(s,J,\Gamma,\theta)$ (*$\operatorname{ess\,inf}$ *in the case $\alpha < 0$). Moreover $\{J_t, \Gamma(t,x), R_t\}_{t \in [0.T]}$ is the smallest for $0 < \alpha < 1$ (the greatest for $\alpha < 0$) solution to (5.7) such that*

$$\begin{aligned} M_t^J = M_0^J &+ \int_0^t \int_{I\!R} (Z^{\theta}_{s^-})^{\alpha} e^{-\alpha r s} \Gamma(s,x) \{1 + \theta_s(e^x - 1)\}^{\alpha} m^{\pi}(ds,dx) + \\ &\int_0^t \int_{I\!R} (Z^{\theta}_{s^-})^{\alpha} J_{s^-} \left[\{1 + \theta_s(e^x - 1)\}^{\alpha} - 1\right] m^{\pi}(ds,dx) + \int_0^t (Z^{\theta}_{s^-})^{\alpha} e^{-\alpha r s} R_s dI_s. \end{aligned} \tag{5.9}$$

is a $(P, \mathcal{G}_t)$-local martingale.

Proposition 5.4 *Assume the valued set of admissible portfolios, A^{θ}, to be compact and that there exists a constant $C > 0$ such that $|K(t;\zeta)| \le C$ and $\lambda_t \le C$, $P-a.s.$. The process $\{J_t, \Gamma(t,x), R_t\}_{t \in [0.T]}$ is the unique solution in $\mathcal{S}^2 \times \mathcal{L}^2_{\nu^p} \times \mathcal{L}^2$ to BSDE (5.7). Moreover J_t is a bounded process.*

6. The mixed type filtering problem

The filtering problem consists in computing the conditional distribution of a signal process, which is not directly observable, given observations up to time. The case of diffusion observations has been widely studied in literature and textbook treatments can be found in Kallianpur [23], Lipster Shiryaev [28] and Rogers and Williams [36]. More recently, there are known results also for pure-jump observations (see [4, 7, 5, 16, 17] and references therein). While few results can be found for mixed type information which involves pure-jump processes and diffusions ([18, 19]) and to the author's knowledge it is the first time that mixed type observations filtering is studied for a general jump-diffusion signal allowing both correlation and common jump times with the observations.

Several approaches have been considered in nonlinear filtering literature. Among them we recall the innovation method and the reference probability approach. In the case where W_t^1 in (5.1) is independent of W_t (see (2.4)) we can apply the reference probability approach proposed in [18] in order to reduce the filtering problem with mixed type informations to a filtering problem with pure jump observations. But since we deal with a mixed type observations model allowing correlation between W_t^1 and W_t we choose the innovation method. It consists in deriving the dynamics of the filter, the so called Kushner-Stratonovich equation (KS) and to characterize the filter as the unique solution to this equation. The KS-equation plays an essential role in the study of partially observable control problems by the Hamilton-Jacobi-Bellman approach (see for instance [8, 31, 1]) and in credit risk setups with restricted information for describing the dynamic evolution of a portfolio of credit risk securities ([19]).

Before writing down the filtering equation we give a result proved in [7], Corollary 2.2.

Lemma 6.1 *Under the assumptions*

$$I\!\!E \int_0^T \nu(D_t^0)dt < +\infty, \quad I\!\!E \int_0^T \sigma^2(X_t)dt < +\infty \tag{6.1}$$

X_t is a $(P, \mathcal{F}_t)$-Markov process with generator

$$L^X f(t,x) = \frac{\partial f}{\partial t}(t,x) + b(x)\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x) + \int_Z \{f(t, x + K_0(t,x;\zeta)) - f(t,x)\}\nu(d\zeta). \tag{6.2}$$

More precisely, for any any function $f(t,x) \in C_b^{1,2}(I\!\!R^+ \times I\!\!R)$ the following semimartingale decomposition holds

$$f(t, X_t) = f(t, x_0) + \int_0^t L^X f(s, X_s)ds + m_t^f \tag{6.3}$$

where m_t^f is a $(P, \mathcal{F}_t)$-martingale.

The next Theorem whose proof is postponed in Appendix is the main result of this section.

Theorem 6.2 *Let us assume the first of (6.1) and*

$$\lambda_t = \nu(D_t) \leq C \quad P-a.s., \quad \sigma(x) \leq C, \tag{6.4}$$

with $C > 0$. The filter (5.2) is a solution of the KS-equation, that, for any function $f(t,x) \in C_b^{1,2}(I\!\!R^+ \times I\!\!R)$ is given by

$$\pi_t(f) = f(x_0, y_0) + \int_0^t \pi_s(L^X f)ds + \int_0^t \int_{I\!\!R} \phi_s^\pi(f,x) m^\pi(ds,dx) + \int_0^t \psi_s^\pi(f) dI_s \tag{6.5}$$

where

$$\phi_s^\pi(f,x) = \frac{d\pi_{s^-}(\lambda_s \Phi_s f)}{d\pi_{s^-}(\lambda_s \Phi_s)}(x) - \pi_{s^-}(f) + \frac{d\pi_{s^-}(\bar{L}_s f)}{d\pi_{s^-}(\lambda_s \Phi_s)}(x) \tag{6.6}$$

$$\psi_s^\pi(f) = \pi_s(\gamma f) - \pi_s(\gamma)\pi_s(f) + \rho\pi_s(\sigma\frac{\partial f}{\partial x}), \tag{6.7}$$

and

$$\frac{d\pi_{s^-}(\lambda_s\Phi_s f)}{d\pi_{s^-}(\lambda_s\Phi_s)}(x), \quad \frac{d\pi_{s^-}(\bar{L}_s f)}{d\pi_{s^-}(\lambda_s\Phi_s)}(x)$$

denote the Radon-Nikodym derivatives of the measures $\pi_{s^-}(\lambda_s\Phi_s f)$*,* $\pi_{s^-}(\bar{L}f)$ *respectively, with respect to* $\pi_{s^-}(\lambda_s\Phi_s)$*, and the operator* $\bar{L}_t f$ *is defined as,* $\forall A \in \mathcal{B}(I\!R)$

$$\bar{L}_t f = \bar{L}f(., Y_{t^-}, dz), \quad \bar{L}f(t,x,y,A) = \int_{D^A(t,x,y)} [f(t, x + K_0(t,x;\zeta)) - f(t.x)]\nu(d\zeta)$$

($D^A(t,x,y)$ *is defined in (2.10)).*

Remark 6.3 *Let us observe that, since*

$$\int_{I\!R} |\phi_s^\pi(f,x)|\pi_{s^-}(\lambda_s\Phi_s(dx)) \leq |\pi_{s^-}(\lambda_s f)| + |\pi_{s^-}(\lambda_s)\pi_{s^-}(f)| + |\pi_{s^-}(\bar{L}f)(I\!R)|,$$

assumption (6.4) and boundedness of γ *imply that for any* $f(t,x) \in C_b^{1,2}(I\!R^+ \times I\!R)$ *it is possible to find a constant* $A_f > 0$ *such that*

$$\psi_s^\pi(f) \leq A_f, \quad \int_{I\!R} |\phi_s^\pi(f,x)|\pi_{s^-}(\lambda_s\Phi_s(dx)) \leq A_f. \tag{6.8}$$

Thus the integrals in (6.6) with respect to the compensated martingale random measure $m^\pi(dt,dx)$ *(defined in (2.24)) and to the innovation process* I_t *(defined in (5.3)) are* $(P, \mathcal{G}_t)$*-martingales.*
We could weaken conditions (6.1) and (6.4) in order to get just $(P,\mathcal{G}_t)$*-local martingales, but we assume them to avoid further technicalities in the proof of Theorem 6.2.*

In order to characterize the filter we introduce the notion of weak solution to the filtering equation. Let $\Pi(I\!R)$ be the space of probability measure on $I\!R$.

Definition 6.4 *As weak solution to KS-equation (6.5) we mean a process* $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ *defined on a probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}_t, \widetilde{P})$ *taking values in* $\Pi(I\!R) \times I\!R^2$ *such that*

- μ_t *is* $\mathcal{F}_t^{\widetilde{Y}} \vee \mathcal{F}_t^{\widetilde{\eta}}$*-adapted with cadlag sample paths*

- $\widetilde{Y}_t$ *is marked point process whose associated random measure* $\widetilde{m}(dt,dx)$ *has* $\mathcal{F}_t^{\widetilde{Y}} \vee \mathcal{F}_t^{\widetilde{\eta}}$*-predictable projection given by* $\mu_{t^-}(\lambda_t\phi_t(dx))dt$

- $\widetilde{\eta}_t$ *is a* $\mathcal{F}_t^{\widetilde{Y}} \vee \mathcal{F}_t^{\widetilde{\eta}}$*-Brownian motion.*

- the triple $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ *solves the KS-equation (6.5) with* $m^\pi(dt,dx)$*,* I_t*,* $\phi_t^\pi(f,x)$ *and* ψ_t^π *replaced by* $m^\mu(dt,dx) = \widetilde{m}(dt,dx) - \mu_{t^-}(\lambda_t\phi_t(dx))dt$*,* $I_t^\mu = \widetilde{\eta}_t - \int_0^t \mu_s(\gamma_s)ds$*,* $\phi_t^\mu(f,x)$ *and* ψ_t^μ*, respectively .*

Let us observe that, by performing a measure change that turns η_t into a Brownian motion, from Theorem 6.2 the triple (π_t, Y_t, η_t) provides a weak solution to equation (6.5). More precisely, the new probability measure $\widetilde{P}$ equivalent to P is defined as

$$\frac{d\widetilde{P}}{dP}\Big|_{\mathcal{F}_t} = \mathcal{E}\Big(-\int_0^t \gamma(X_s)dW_s^1\Big) = exp\Big\{-\int_0^t \gamma(X_s)dW_s^1 - \frac{1}{2}\int_0^t \gamma(X_s)^2 ds\Big\}.$$

From Girsanov Theorem $\eta_t = \int_0^t \gamma(X_s)ds + W_s^1$ is a $(\widetilde{P}, \mathcal{F}_t)$-Brownian motion, which in turn implies that η_t is a $(\widetilde{P}, \mathcal{F}_t^Y \vee \mathcal{F}_t^\eta)$-Brownian motion.

In the next Theorem the filter is characterized as the unique weak solution to the KS-equation (6.2). The proof is given in Appendix.

Theorem 6.5 *Let us assume the first of (6.1) and (6.4). Under one of the following conditions*

(i) $b(x)$, $\sigma(x)$, $\gamma(x) \in C_b^2(I\!R)$, $\nu(D_t^0) \leq C$, *with* C *a positive constant,*

(ii) $b(x)$, $\sigma(x)$ *and* $\gamma(x)$ *continuous functions in* $x \in I\!R$, $K_0(t, x; \zeta)$, $K_1(t, x, y; \zeta)$ *jointly continuous in* (t, x) *and* (t, x, y) *respectively,*

all weak solutions $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ *of the KS-equation have the same law. In particular* μ_t *has the same law of the filter* π_t.

To conclude this section, let us observe that the KS-equation can be written also as

$$\pi_t(f) = f(x_0, y_0) + \int_0^t \{\pi_s(L_0^X f) + \pi_s(f)\pi_s(\lambda_s) - \pi_s(f\lambda_s)\}ds + \int_0^t \int_{I\!R} \phi_s^\pi(f, x)m(ds, dx) + \int_0^t \psi_s^\pi(f)dI_s \quad (6.9)$$

where

$$L_0^X f(t, x, y) = L^X f(t, x) - \bar{L}f(t, x, y, I\!R) =$$

$$\frac{\partial f}{\partial t}(t, x) + b(x)\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(x) + \int_{D^c(t,x,y)} \{f(t, x + K_0(t, x; \zeta)) - f(t, x)\}\nu(d\zeta)$$

$(D^c(t, x, y) = Z - D(t, x, y) = \{\zeta \in Z : K_1(t, x, y; \zeta) = 0\})$ and it has a natural recursive structure. This can be seen if we write the equation at the jump times and between two consecutive jump times. In fact, at a jump time T_n

$$\pi_{T_n}(f) = \frac{d\pi_{T_n^-}(\lambda_{T_n}\Phi_{T_n} f)}{d\pi_{T_n^-}(\lambda_{T_n}\Phi_{T_n})}(Z_n) + \frac{d\pi_{T_n^-}(\bar{L}_{T_n} f)}{d\pi_{T_n^-}(\lambda_{T_n}\Phi_{T_n})}(Z_n), \quad Z_n = Y_{T_n} - Y_{T_{n-1}}.$$

Hence $\pi_{T_n}(f)$ is completely determined by the observed data (T_n, Z_n) and by the knowledge of $\pi_t(f)$ in the interval $[T_{n-1}, T_n)$, since $\pi_{T_n^-}(f) = \lim_{t \to T_n^-} \pi_t(f)$.

For $t \in [T_n, T_{n+1})$

$$\pi_t(f) = \pi_{T_n}(f) + \int_{T_n}^t \{\pi_s(L_0^X f) + \pi_s(f)\pi_s(\lambda_s) - \pi_s(f\lambda_s)\}ds + \int_{T_n}^t \psi_s^\pi(f)dI_s.$$

7. A particular case

As example we deal with the particular case when the risky asset follows a geometric pure jump process driven by two independent point processes. In [6, 9, 10] similar models driven by two independent Poisson processes have been considered in full information frameworks. We now examine in the frame of restricted information the utility maximization problem in the case where the intensities of the point processes driving the stock price dynamics are not directly observable by investors.

This particular model is presented since it allows us to obtain explicit expressions for the quantities of interest.

Let us define

$$D^+(t, X_{t^-}, Y_{t^-}) = \{\zeta \in Z : K(t, X_{t^-}, Y_{t^-}; \zeta) > 0\}, \quad D^-(t, X_{t^-}, Y_{t^-}) = \{\zeta \in Z : K(t, X_{t^-}, Y_{t^-}; \zeta) < 0\}.$$

From now on we will write $K(t; \zeta)$, D_t^+, D_t^- for $K(t, X_{t^-}, Y_{t^-}; \zeta)$, $D^+(t, X_{t^-}, Y_{t^-})$ and $D^-(t, X_{t^-}, Y_{t^-})$ respectively, unless it is necessary to underline the dependence on the processes involved.
Assume

$$K(t; \zeta) = K^+(t, S_{t^-})1\!\!1_{D_t^+}(\zeta) - K^-(t, S_{t^-})1\!\!1_{D_t^-}(\zeta) \quad (7.1)$$

with $K^+(t, x), K^-(t, x)$ two positive jointly measurable functions (as in Section 2 we assume $K^-(t, x) < 1$).

In this particular case equation (2.7) can be written as

$$dS_t = S_{t^-}(K^+(t,S_{t^-})dN_t^1 - K^-(t,S_{t^-})dN_t^2) \tag{7.2}$$

where $N_t^1 = \mathcal{N}((0,t),D_t^+)$ and $N_t^2 = \mathcal{N}((0,t),D_t^-)$ are two independent counting processes with $(P,\mathcal{F}_t)$-predictable intensities given by $\lambda_1(t) = \nu(D_t^+)$ and $\lambda_2(t) = \nu(D_t^-)$. The counting processes N_t^1 and N_t^2 describe upwards and downwards jumps of S_t and $N_t^1 + N_t^2 = N_t$ (with N_t the point process which counts the total number of jumps of S_t). In this model the agent can observe $K^+(t,S_{t^-})$ and $K^-(t,S_{t^-})$ but not the intensities $\lambda_i(t), i = 1,2$, since they depend on the unobservable stochastic factor X_t.

In [6] and [9], the full information optimal investment problem for CRRA preferences has been solved by using the Hamilton-Jacobi-Bellman approach (in [9] by considering K^+ and K^- Markovian in the stochastic factor X_{t^-} instead of S_{t^-}). In [10] the exponential utility case has been studied by a BSDE approach, for K^+ and K^- general predictable processes. In all these papers the intensities, $\lambda_i(t), i = 1,2$, have been assumed deterministic functions on time while in this note they are stochastic processes which depend on the unobservable endogenous process X_t.

From now on we denote by K_t^+ and K_t^- the processes $K^+(t,S_{t^-})$ and $K^-(t,S_{t^-})$, respectively and

$$H_1(t) = \log(1+K_t^+) > 0, \quad H_2(t) = \log(1-K_t^-) < 0. \tag{7.3}$$

The integer-valued random measure $m(dt,dx)$ defined in (4.2) becomes

$$m(dt,dx) = \delta_{H_1(t)}(dx)dN_t^1 + \delta_{H_2(t)}(dx)dN_t^2, \tag{7.4}$$

and its $(P,\mathcal{F}_t)$-predictable dual projection is given by

$$m^p(dt,dx) = \Big(\delta_{H_1(t)}(dx)\lambda_1(t) + \delta_{H_2(t)}(dx)\lambda_2(t)\Big)dt. \tag{7.5}$$

By introducing the filter, $\pi_t(f) = I\!E[f(t,X_t)|\mathcal{G}_t]$, the $(P,\mathcal{G}_t)$-predictable dual projection of $m(dt,dx)$ can be written as

$$\nu^p(dt,dx) = \Big(\delta_{H_1(t)}(dx)\pi_{t^-}(\lambda_1) + \delta_{H_2(t)}(dx)\pi_{t^-}(\lambda_2)\Big)dt,$$

with $\pi_{t^-}(\lambda_1) = \pi_{t^-}(\nu(D^+(.,.,Y_{t^-}))$ and $\pi_{t^-}(\lambda_2) = \pi_{t^-}(\nu(D^-(.,.,Y_{t^-}))$ the $(P,\mathcal{G}_t)$- predictable intensities of N_t^1 and N_t^2.

By applying Theorem 6.2 and Theorem 6.5 the filter is characterized as the unique weak solution to the KS-equation, that in this special case can be written in the simplified form

$$\pi_t(f) = f(x_0,y_0) + \sum_{i=1,2}\int_0^t \pi_s(\lambda_i)^+\{\pi_{s^-}(\lambda_i f) - \pi_{s^-}(\lambda_i)\pi_{s^-}(f) + \pi_{s^-}(R^i f)\}(dN_s^i - \pi_{s^-}(\lambda_i)ds) + \tag{7.6}$$

$$\int_0^t \pi_s(L^X f)ds + \int_0^t \psi_s^\pi(f)dI_s$$

where $a^+ = \frac{1}{a}1\!\!I_{\{a>0\}}$, $\psi_s^\pi(f)$ is given in (6.7) and

$$R^i f(t,x) = \int_Z [f(t,x+K_0(t,x;\zeta)) - f(t.x)]\nu(d\zeta)1\!\!I_{\{H_i(t)\neq 0\}} \quad i = 1,2.$$

In the last part of this section we examine in this special case the utility maximization problem under restricted information by using the BSDE approach. We will assume the riskless interest rate $r = 0$. First, note that the Doob-Meyer decomposition of J_t given in (4.1) and (4.2) can be now written as

$$J_t = m_t^J - A_t, \quad m_t^J = \int_0^t \Gamma_1(s)\big(dN_s^1 - \pi_{s^-}(\lambda_1)ds\big) + \int_0^t \Gamma_2(s)\big(dN_s^2 - \pi_{s^-}(\lambda_2)ds\big) + \int_0^t R_s dI_s \tag{7.7}$$

with $\Gamma_1(s)=\Gamma(s,H_1(s))$ and $\Gamma_2(s)=\Gamma(s,H_2(s))$ and $R_t\in\mathcal{L}^2_{loc}$.

We shall denote by $\mathcal{L}^2_i$ ($\mathcal{L}^2_{i,loc}$), $i=1,2$, the space of $I\!R$-valued $\mathcal{G}_t$-predictable processes $\{U(t)\}_{t\in[0.T]}$ with

$$I\!E\Big(\int_0^T \mid U(t)\mid^2 \pi_{t^-}(\lambda_i)dt\Big)^{\frac{1}{2}}<+\infty \quad \Big(\text{resp.} \quad \int_0^T \mid U(t)\mid^2 \pi_{t^-}(\lambda_i)dt\Big)^{\frac{1}{2}}<+\infty \quad P-a.s.\Big).$$

Proposition 7.1 *The process $\{J_t,\Gamma_1(t),\Gamma_2(t),R_t\}_{t\in[0.T]}$ solves the BSDE*

$$J_t=1-\int_t^T\Gamma_1(s)(dN^1_s-\pi_{s^-}(\lambda_1)ds)-\int_t^T\Gamma_2(s)(dN^2_s-\pi_{s^-}(\lambda_2)ds)+\int_t^T g(s,J_s,\Gamma_1(s),\Gamma_2(s))ds-\int_t^T R_s dI_s \tag{7.8}$$

where

$$g(t,y,z_1,z_2)=(y+z_1)\big[\{1+\theta^*(t,y,z_1,z_2)K^+_t\}^\alpha-1\big]\pi_t(\lambda_1)+(y+z_2)\big[\{1-\theta^*(t,y,z_1,z_2)K^-_t\}^\alpha-1\big]\pi_t(\lambda_2)$$

with

$$\theta^*(t,y,z_1,z_2)=\frac{1-(\frac{y+z_1}{y+z_2}\Big)^{\frac{1}{\alpha-1}}\Big(G_t\Big)^{\frac{1}{\alpha-1}}}{K^-_t+\Big(\frac{y+z_1}{y+z_2}\Big)^{\frac{1}{\alpha-1}}\Big(G_t\Big)^{\frac{1}{\alpha-1}}K^+_t},\quad G_t=\frac{K^+_t\pi_{t^-}(\lambda_1)}{K^-_t\pi_{t^-}(\lambda_2)}. \tag{7.9}$$

and the optimal investment strategy is given by $\theta^*_t=\theta^*(t,J_{t^-},\Gamma_1(t),\Gamma_2(t))$.

Moreover, $\{J_t,\Gamma_1(t),\Gamma_2(t),R_t\}_{t\in[0.T]}$ is the smallest for $0<\alpha<1$ (the greatest for $\alpha<0$) solution to (7.8) such that M^J_t defined (5.9) is a $(P,\mathcal{G}_t)$-local martingale. Let us note that in this frame M^J_t has the representation

$$M^J_t=M^J_0+\int_0^t(Z^\theta_{s^-})^\alpha R_s dI_s+\int_0^t(Z^\theta_{s^-})^\alpha(J_{s^-}+\Gamma_1(s))\{1+\theta_s(K^+_s\}^\alpha(dN^1_s-\pi_{s^-}(\lambda_1)ds)+$$
$$\int_0^t(Z^\theta_{s^-})^\alpha(J_{s^-}+\Gamma_2(s))\{1-\theta_s K^-_s\}^\alpha(dN^2_s-\pi_{s^-}(\lambda_2)ds)-\int_0^t(Z^\theta_{s^-})^\alpha J_{s^-}(dN_s-\pi_{s^-}(\lambda_s)ds).$$

Proof.
The claim is a direct consequence of Theorem 4.1. In this particular case, for any fixed (t,ω,y,z_1,z_2) such that $y+z_i>0$, $i=1,2$ (observe that $J_t=J_{t^-}+\Gamma_i(t)>0$ if $\Delta N^i_t\neq 0$) we are able to compute explicitly $\theta^*(t,y,z_1,z_2)$ which maximizes for $0<\alpha<1$ (minimize for $\alpha<0$) the function

$$H(\theta)=(y+z_1)\big[\{1+\theta K^+_t\}^\alpha-1\big]\pi_t(\lambda_1)+(y+z_2)\big[\{1-\theta K^-_t\}^\alpha-1\big]\pi_t(\lambda_2)$$

over $\theta\in\Big(-\frac{1}{K^+_t},\frac{1}{K^-_t}\Big)$. By a direct computation we get that the maximum for $0<\alpha<1$ (the minimum for $\alpha<0$) is achieved in $\theta^*(t,y,z_1,z_2)$ given in (7.9).
□

The analogous of Proposition 4.2 can be stated avoiding compactness of the valued set of admissible strategies, A^θ, assuming (7.10) below.

Proposition 7.2 *Assume the existence of positive constants A_1, A_2 such that $\forall t\in[0,T]$*

$$A_1\le K^+_t\le A_2\quad A_1\le K^-_t\le A_2\quad A_1\le\lambda_i(t)\le A_2,\quad i=1,2\quad P-a.s.. \tag{7.10}$$

The process $\{J_t,\Gamma_1(t),\Gamma_2(t),R_t\}_{t\in[0,T]}$ is the unique solution in $\mathcal{S}^2\times\mathcal{L}^2_1\times\mathcal{L}^2_2\times\mathcal{L}^2$ to the BSDE (7.8). Moreover J_t is a bounded process.

Proof.
Since any admissible strategy θ_t necessarily satisfies $\theta_t \in \Big(-\frac{1}{K_t^+}, \frac{1}{K_t^-}\Big)$, $P-a.s.$, we get that

$$(1+\theta_t K_t^+)^\alpha \le C, \quad (1-\theta_t K_t^-)^\alpha \le C, \quad [(1+\theta_t K_t^+)^\alpha - 1]^2 \le C, \quad [(1-\theta_t K_t^-)^\alpha - 1]^2 \le C \quad P-a.s.$$

with C positive constant. By proceeding similarly to the proof of Proposition 4.2 we can prove that

$$g(t,y,z_1,z_2) = \operatorname*{ess\,sup}_{\theta\in\Theta} [(y+z_1)\big[\{1+\theta_t K_t^+\}^\alpha - 1\big]\pi_t(\lambda_1) + (y+z_2)\big[\{1-\theta_t K_t^-\}^\alpha - 1\big]\pi_t(\lambda_2)$$

is Lipschitz in (y,z_1,z_2), that is there exists a constant $\widetilde{C}>0$ such that

$$|g(t,y,z_1,z_2) - g(t,\widetilde{y},\widetilde{z}_1,\widetilde{z}_2)| \le \widetilde{C}\big(|y-\widetilde{y}| + |z_1-\widetilde{z}_1| + |z_2-\widetilde{z}_2|\big) \quad P\times dt - a.e.$$

□

Remark 7.3 *In the case where agents have access only to the flow generated by asset prices, that is $\mathcal{G}_t = \mathcal{F}_t^S$, the KS-equation and the BSDE involved are given by equations (7.6) and (7.8), respectively, without the part driven by the innovation process, I_t.*
Let us observe that the optimal investment strategies in the case $\mathcal{G}_t = \mathcal{F}_t^S$ and $\mathcal{G}_t = \mathcal{F}_t^S \vee \mathcal{F}_t^\eta$ do not coincide since both depend on the solution $(J_t, \Gamma_1(t), \Gamma_2(t))$ of different BSDEs and $\theta^*(\omega,t,y,z_1,z_2)$ given in (7.9) depends on the filter which has different dynamics in the two situations.

Remark 7.4 *In the case where K_t^+, K_t^- and $\lambda_i(t)$, $i=1,2$, are deterministic functions on time we are in a full information setup with $\mathcal{G}_t = \mathcal{F}_t^{N^1} \vee \mathcal{F}_t^{N^2}$. Equation (7.8) becomes*

$$J_t = 1 - \int_t^T \Gamma_1(s)(dN_s^1 - \pi_{s^-}(\lambda_1)ds) - \int_t^T \Gamma_2(s)(dN_s^2 - \pi_{s^-}(\lambda_2)ds) + \int_t^T g(s,J_s,\Gamma_1(s),\Gamma_2(s))ds. \quad (7.11)$$

Note that the generator $g(t,y,z_1,z_2)$ is a deterministic function hence the unique solution to equation (7.11) is given by $(J_t,0,0)$, with J_t and the optimal strategy, θ_t^*, deterministic processes given by

$$J_t = e^{\int_t^t (\Lambda(s)-\lambda(s))ds}, \quad \Lambda(t) = \{1+\theta_t^* K_t^+\}^\alpha \lambda_1(t) + \{1-\theta_t^* K_t^-\}^\alpha \lambda_2(t),$$

$$\theta_t^* = \frac{(G_t)^{\frac{1}{\alpha-1}} - 1}{K_t^- + (G_t)^{\frac{1}{\alpha-1}} K_t^+}, \quad G_t = \frac{K_t^+\lambda_1(t)}{K_t^-\lambda_2(t)},$$

respectively. This result has been obtained in Theorem 11 of [6] by using Verification results for the Hamilton-Jacobi-Bellman equation.

8. Appendix

8.1. Proof of Theorem 6.2

In order to deduce equation (6.2) let us consider the semimartingale given in (6.3)

$$f_t = f(t,X_t) = f(0,x_0) + \int_0^t L^X f(s,X_s)ds + m_t^f$$

with

$$m_t^f = \int_0^t \frac{\partial f}{\partial x}(s,X_s)\sigma(X_s)dW_s + \int_0^t \int_Z \{f(s,X_{s^-} + K_0(s,X_{s^-};\zeta)) - f(s,X_{s^-})\}\big(\mathcal{N}(ds,d\zeta) - \nu(d\zeta)ds\big).$$

Next we perform a projection on $\mathcal{G}_t$. Taking into account that $\widehat{m}_t$ is a $\mathcal{G}_t$-martingale and that for any progressively measurable process h_t such that $I\!E\int_0^T \mid h_s \mid ds < +\infty$, $\widehat{\int_0^t h_s ds} - \int_0^t \widehat{h}_s ds$ is a $\mathcal{G}_t$-martingale, we get that

$$\widehat{f}_t - \widehat{f}_0 - \int_0^t \widehat{L^X f}(s, X_s) ds$$

is a $\mathcal{G}_t$-martingale. Then by Proposition 5.2 there exist $\phi_s^\pi(f, x)$ and $\psi_s^\pi(f)$ such that

$$\widehat{f}_t = \widehat{f}_0 + \int_0^t \widehat{L^X f}(s, X_s) ds + \int_0^t \int_{I\!R} \phi_s^\pi(f, x) m^\pi(ds, dx) + \int_0^t \psi_s^\pi(f) dI_s.$$

We assume that $\phi_s^\pi(f, x)$ and $\psi_s^\pi(f)$ satisfy (6.8) thus we can find $(P, \mathcal{F}_t)$-martingales, M_t^i, $i = 1, 2$, and $(P, \mathcal{G}_t)$-martingales, m_t^i, $i = 1, 2, 3, 4$, such the following relations hold true.

First we derive (6.7) by applying the product rule to the processes f_t and η_t

$$d(f_t \eta_t) = f_{t^-} d\eta_t + \eta_{t^-} df_t + d\big\langle f^c, \eta \big\rangle_t = \big\{ \eta_t L^X f(t, X_t) + f_t \gamma(X_t) + \frac{\partial f}{\partial x}(t, X_t) \sigma(X_t) \rho \big\} dt + dM_t^1$$

and considering the projection on $\mathcal{G}_t$ we get

$$d(\widehat{f_t \eta_t}) = \big\{ \eta_t \widehat{L^X f}(t, X_t) + \widehat{f_t \gamma}(X_t) + \rho \widehat{\frac{\partial f}{\partial x} \sigma(t, X_t)} \big\} dt \big\} dt + dm_t^1. \tag{8.1}$$

On the other hand let us observe that $\widehat{f_t \eta_t} = \widehat{f}_t \eta_t$, and again by the product rule

$$d(\widehat{f}_t \eta_t) = \widehat{f_{t^-}} d\eta_t + \eta_{t^-} d\widehat{f}_t + \big\langle \eta, \widehat{f} \big\rangle_t = \big\{ \eta_t \widehat{L^X f}(t, X_t) + \psi_s^\pi(f) - \widehat{f}_t \widehat{\gamma(X_t)} \big\} dt + dm_t^2. \tag{8.2}$$

Since the finite variation parts in (8.1) and (8.2) have to coincide we obtain (6.7).

Next to derive (6.6) we apply the product rule to the processes f_t and U_t, where $U_t = \int_0^t \int_{I\!R} \Gamma(t, x) m(dt, dx)$, for $\Gamma(s, x)$ any bounded $\mathcal{G}_t$-predictable process. Since

$$[f, U]_t = \int_0^t \int_Z 1\!\!I_{D_s^1}(\zeta) \{ f(s, X_{s^-} + K_0(s, X_{s^-}; \zeta)) - f(s, X_{s^-}) \} \Gamma(s, K_1(s, X_{s^-}, Y_{s^-}; \zeta)) \mathcal{N}(ds, d\zeta),$$

we get that

$$d(f_t U_t) = f_{t^-} dU_t + U_{t^-} df_t + d[f, U]_t = \big\{ U_t L^X f(t, X_t) + \int_R f_{t^-} \Gamma(t, x) \lambda_t \phi_t(dx) + V_t \big\} dt + dM_t^2$$

where

$$V_t = \int_Z 1\!\!I_{D_t^1}(\zeta) \{ f(t, X_{t^-} + K_0(t, X_{t^-}; \zeta)) - f(t, X_{t^-}) \} \Gamma(t, K_1(t, X_{t^-}, Y_{t^-}; \zeta)) \nu(d\zeta) \tag{8.3}$$

and again by performing a $\mathcal{G}_t$-projection

$$d(\widehat{f_t U_t}) = \big\{ U_t \widehat{L^X f}(t, X_t) + \int_R \Gamma(t, x) \widehat{f_{t^-} \lambda_t} \Phi_t(dx) + \widehat{V}_t \big\} dt + dm_t^3. \tag{8.4}$$

Again let us observe that U_t is $\mathcal{G}_t$-measurable thus

$$d(\widehat{f}_t U_t) = \widehat{f}_{t^-} dU_t + U_{t^-} d\widehat{f}_t + d[\widehat{f}, U]_t = \big\{ U_t \widehat{L^X f}(t, X_t) + \int_R (\phi_s^\pi(f) + \widehat{f}_t) \Gamma(t, x) \widehat{\lambda_t \Phi_t}(dx) \big\} dt + dm_t^4. \tag{8.5}$$

Again we claim that the finite variation parts in (8.4) and (8.5) have to coincide

$$\int_R \phi_t^\pi(f) \Gamma(t, x) \widehat{\lambda_t \Phi_t}(dx) = - \int_R \widehat{f}_{t^-} \Gamma(t, x) \widehat{\lambda_t \Phi_t}(dx) + \int_R \Gamma(t, x) \widehat{f_{t^-} \lambda_t} \phi_t(dx) + \widehat{V}_t.$$

Setting $\phi_t^\pi(f) = -\widehat{f}_{t^-} + \xi_2(t, x) + \xi_3(t, x)$ with $\xi_i(t, x)$, $i = 2, 3$, such that

$$\int_R \xi_2(t, x) \Gamma(t, x) \widehat{\lambda_t \Phi_t}(dx) = \int_R \Gamma(t, x) \widehat{f_{t^-} \lambda_t} \Phi_t(dx) \tag{8.6}$$

$$\int_R \xi_3(t, x) \Gamma(t, x) \widehat{\lambda_t \Phi_t}(dx) = \widehat{V}_t, \tag{8.7}$$

and choosing $\Gamma(t,x) = C_t 1\!\!1_A(x)$, $A \in \mathcal{B}(I\!\!R)$, $C_t > 0$, $(P, \mathcal{G}_t)$-predictable and bounded, we get that

$$V_t = \int_Z C_t 1\!\!1_{D_t^A}(\zeta)\{f(t, X_{t^-} + K_0(t, X_{t^-}; \zeta)) - f(t, X_{t^-})\}\nu(d\zeta) = C_t \int_A \bar{L} f(X_{t^-}, Y_{t^-}, dx).$$

Finally equations (8.6) and (8.7) reduce to

$$\forall A \in \mathcal{B}(I\!\!R), \quad \int_A \xi_2(t,x)\widehat{\lambda_t \Phi_t}(dx) = \int_A f_{t^-}\widehat{\lambda_t}\Phi_t(dx), \quad \int_A \xi_3(t,x)\widehat{\lambda_t\Phi_t}(dx) = \int_A \widehat{\bar{L}f}(X_{s^-}, Y_{s^-}, dx)$$

respectively, which imply (6.6).

□

8.2. Proof of Theorem 6.5

First, we need some preliminaries

Lemma 8.1 *Under the assumption (6.1) and*

$$I\!\!E \int_0^T \nu(D_t)dt < +\infty \tag{8.8}$$

(X_t, Y_t, η_t) is a $(P, \mathcal{F}_t)$-Markov process with generator

$$Lf(t,x,y,z) = \frac{\partial f}{\partial t} + b(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2} + \gamma(x)\frac{\partial f}{\partial z} + \rho\sigma(x)\frac{\partial^2 f}{\partial x \partial z} + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} + \tag{8.9}$$

$$+ \int_Z \{f(t, x + K_0(t,x;\zeta), y + K_1(t,x,y;\zeta), z) - f(t,x,y,z)\}\nu(d\zeta).$$

More precisely, for any any bounded function $f(t,x,y,z) \in C_b^{1,2,0,2}(I\!\!R^+ \times I\!\!R^3)$ the semimartingale decomposition holds

$$f(t, X_t, Y_t, \eta_t) = f(t, X_0, Y_0, \eta_0) + \int_0^t Lf(s, X_s, Y_s, \eta_s)ds + M_t^f \tag{8.10}$$

where M_t^f is a $(P, \mathcal{F}_t)$-martingale.

Proof.

From Ito formula we get (8.10) with

$$M_t^f = \int_0^t \frac{\partial f}{\partial x}(s, X_s, Y_s, \eta_s)\sigma(X_s)dW_s + \int_0^t \frac{\partial f}{\partial z}(s, X_s, Y_s, \eta_s)dW_s^1 +$$

$$\int_0^t \int_Z \{f(t, X_{s^-} + K_0(t, X_{s^-}; \zeta), y + K_1(t, X_{s^-}, Y_{s^-}; \zeta), \eta_s) - f(t, X_{s^-}, Y_{s^-}, \eta_s)\}(\mathcal{N}(ds, d\zeta) - \nu(d\zeta)ds)$$

and by assumptions (6.1) and (8.8) M_t^f is $(P, \mathcal{F}_t)$-martingale.

□

By projecting on $\mathcal{G}_t$ we get that (8.10) implies that

$$\pi_t\big(f(., Y_t, \eta_t)\big) - \int_0^t \pi_s\big(Lf(., Y_s, \eta_s)\big)ds$$

is a $(P, \mathcal{G}_t)$-martingale. This martingale property will allows us to apply the idea proposed in [25] to characterize the distribution of (π_t, Y_t, η_t) by introducing the notion of Filtered Martingale Problem (FMP).

Definition 8.2 *Let $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ be a process taking values in $\Pi(I\!R) \times I\!R^2$ with cadlag sample paths. This process is a solution of the Filtered Martingale Problem for L, given in (8.9), with initial condition $(x_0, y_0, 0)$ $(FMP(L,(x_0,y_0,0)))$ if μ_t is $\mathcal{F}_t^{\widetilde{Y}} \vee \mathcal{F}_t^{\widetilde{\eta}}$-adapted and for any $F \in C_b^{2,0,2}(I\!R^3)$*

$$\mu_t\big(F(.,\widetilde{Y}_t,\widetilde{\eta}_t)\big) - \int_0^t \mu_s\big(LF(.,\widetilde{Y}_s,\widetilde{\eta}_s)\big)ds \tag{8.11}$$

is an $\mathcal{F}_t^{\widetilde{Y}} \vee \mathcal{F}_t^{\widetilde{\eta}}$-martingale and $I\!E[\mu_0(F(.,\widetilde{Y}_0,\widetilde{\eta}_0)] = F(x_0,y_0,0)$.

Finally we consider the proof of Theorem 6.5. We begin by proving that any weak solution to the KS-equation solves the $FMP(L,(x_0,y_0,0))$. It is sufficient to prove (8.11) for functions of the form $F(x,y,z) = f(x)g(y,z)$. Let $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ a weak solution to equation (6.5). By Ito formula we find that

$$g(\widetilde{Y}_t,\widetilde{\eta}_t) = g(y_0,\eta_0) + \int_0^t \frac{\partial g}{\partial z}(\widetilde{Y}_s,\widetilde{\eta}_s)d\widetilde{\eta}_s + \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial z^2}(\widetilde{Y}_s,\widetilde{\eta}_s)ds + \int_0^t\int_{I\!R}[g(\widetilde{Y}_{s^-}+x,\widetilde{\eta}_s) - g(\widetilde{Y}_{s^-},\widetilde{\eta}_s)]\widetilde{m}(ds,dx).$$

By the product rule we get that

$$d\{\mu_t(f)g(\widetilde{Y}_t,\widetilde{\eta}_t)\} = \mu_t(f)\Big\{\frac{\partial g}{\partial z}\mu_t(\gamma) + \frac{1}{2}\frac{\partial^2 g}{\partial z^2}\Big\}dt + g(\widetilde{Y}_t,\widetilde{\eta}_t)\mu_t(L^X f)dt + \psi_t^\mu(f)\frac{\partial g}{\partial z}dt + \tag{8.12}$$

$$\int_{I\!R}\{\mu_t(f) + \phi_t^\mu(f,x)\}[g(\widetilde{Y}_{t^-}+x,\widetilde{\eta}_t) - g(\widetilde{Y}_{t^-},\widetilde{\eta}_t)]\mu_{t^-}(\lambda_t\Phi_t(dx))dt + dM_t^{f,g}$$

where $M_t^{f,g}$ is given by

$$M_t^{f,g} = \int_0^t\{\mu_s(f)\frac{\partial g}{\partial z} + \psi_s^\mu(f)g(\widetilde{Y}_s,\widetilde{\eta}_s)\}dI_s^\mu + \int_0^t g(\widetilde{Y}_{s^-},\widetilde{\eta}_s)\int_{I\!R}\phi_s^\mu(f,x)m^\mu(ds,dx) +$$

$$\int_0^t\int_{I\!R}\{\mu_{s^-}(f) + \phi_s^\mu(f,x)\}\{g(\widetilde{Y}_{s^-}+x,\widetilde{\eta}_s) - g(\widetilde{Y}_{s^-},\widetilde{\eta}_s)\}m^\mu(ds,dx).$$

Defining $\widetilde{\mathcal{G}}_t = \mathcal{F}_t^{\widetilde{Y}} \vee \mathcal{F}_t^{\widetilde{\eta}}$, let $\widetilde{Q}$ be the probability measure defined by, $\forall t \in [0,T]$, as

$$\widetilde{L}_t = \frac{d\widetilde{Q}}{d\widetilde{P}}|_{\widetilde{\mathcal{G}}_t} = \mathcal{E}\Big(\int_0^t \mu_s(\gamma_s)d\widetilde{\eta}_s\Big) = exp\Big\{\int_0^t \mu_s(\gamma_s)d\widetilde{\eta}_s - \frac{1}{2}\int_0^t \mu_s(\gamma_s)^2 ds\Big\}.$$

From Girsanov Theorem, $I_t^\mu = \widetilde{\eta}_t - \int_0^t \mu_s(\gamma_s)ds$ is a $(\widetilde{Q},\widetilde{\mathcal{G}}_t)$-Brownian motion and by (6.4) $M_t^{f,g}$ is a $(\widetilde{Q},\widetilde{\mathcal{G}}_t)$-martingale.

Finally taking into account the expressions of $\phi_t^\mu(f,x)$, $\psi_t^\mu(f)$ and L^X, (8.12) can be written as

$$d\{\mu_t(f)g(\widetilde{Y}_t,\widetilde{\eta}_t)\} = \mu_t(f)\frac{1}{2}\frac{\partial^2 g}{\partial z^2}dt + g(\widetilde{Y}_t,\widetilde{\eta}_t)\mu_t\Big(b\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big)dt + \mu_t(\gamma f)\frac{\partial g}{\partial z} + \rho\mu_t(\sigma\frac{\partial f}{\partial x})\frac{\partial g}{\partial z} + \tag{8.13}$$

$$\mu_t\Big(\int_{D_1(t,x,\widetilde{Y}_{t^-})}\Big\{g\big(\widetilde{Y}_{t^-} + K_1(t,x,\widetilde{Y}_{t^-};\zeta),\widetilde{\eta}_t\big)f(x+K_0(t,x;\zeta)) - g(\widetilde{Y}_{t^-},\widetilde{\eta}_t)f(x)\Big\}\nu(d\zeta)\Big) + dM_t^{f,g} =$$

$$\mu_t(Lf(x)g(\widetilde{Y}_{t^-},\widetilde{\eta}_t))dt + dM_t^{f,g}$$

which proves that $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ solves the FMP.

Next we observe that if uniqueness holds for the $FMP(L,(x_0,y_0,0))$ all weak solutions $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ to the KS-equation have the same law. The last claim can be achieved, under (i) by applying [25], Theorem 3.2, while under (ii) by [25], Theorem 3.3.

Furthermore, since μ_t is $\mathcal{F}_t^{\widetilde{Y}} \vee \mathcal{F}_t^{\widetilde{\eta}}$-adapted, for each t there exists a measurable function h_t from $D_{I\!R^2}[0,T]$ (space of cadlag trajectories from $[0,T]$ into $I\!R^2$) to $\Pi(I\!R)$ such that $\mu_t = h_t(\widetilde{Y}(.\wedge t),\widetilde{\eta}(.\wedge t))$, *a.s.*. Thus

uniqueness for solutions of the $FMP(L,(x_0,y_0,0))$ implies that $(\mu_t, \widetilde{Y}_t, \widetilde{\eta}_t)$ has the same distribution as (π_t, Y_t, η_t) and hence $\pi_t = h_t(Y(.\wedge t), \eta(.\wedge t))$, *a.s.*. Consequently μ_t has the same law of the filter π_t.
$\square$

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