INDIFFERENCE VALUATION VIA BACKWARD SDE'S DRIVEN BY POISSON MARTINGALES

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Abstract

We prove the existence and uniqueness of bounded solutions to backward stochastic equations driven by two independent Poisson martingales in the case of locally Lipschitz generator having a certain monotonicity property. This result allows us to solve utility maximization problems with exponential preferences in an incomplete market where the risky asset dynamics is described by a pure jump process driven by two independent Poisson processes. This includes results on portfolio optimization under an additional European claim. Value processes of the optimal investment problems, optimal hedging strategies and the indifference price are represented in terms of solutions to BSDEs with generators satisfying the upper mentioned assumptions. Via a duality result, the solution to the dual problems are derived. In particular an explicit expression for the density of the minimal martingale measure is provided. The Markovian case is also discussed. This includes either asset dynamics dependent on a pure jump stochastic factor or claims written on a correlated non-tradable asset.

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1. Introduction

In this paper we study utility maximization problems with exponential preferences in an incomplete market where the dynamics of the underlying asset price S is described by a pure jump process driven by two independent Poisson processes, describing upwards and downwards jumps . This includes portfolio optimization in presence of a stochastic factor and under an additional liability. The case where the claim is written on a nontraded asset X, described by a pure jump process correlated to the trading asset S is also covered.

Intraday information on financial asset price quotes and the increasing amount of studies on market microstructure show that prices are piecewise constant and jump at irregularly spaced random times in reaction to trades or to significant new information. This is the reason why many authors believe that pure jump processes may be more suitable for modeling the observed price or quantities related to the price. Several models in which the price process is a marked point process are available in the literature, we only quote Rydberg and Shephard [32], Frey [18], Frey and Runggaldier [19], and the references therein.

Optimal investment problems, hedging and derivative pricing are fundamental problems in Mathematical Finance and they are closely related to each other. Different approaches have been proposed in literature to deal with these problems. By using convex duality the solution to the utility maximization problem can be obtained by solving the dual problem ([3, 4, 33] and the references therein). In a Markovian setting the classical dynamic programming approach leads to characterizing the value function of the utility maximization problem as a solution to the Hamilton-Jacobi-Bellman equation ([29, 30, 9, 11]).

In this note, we choose an alternative approach based on the Bellman principle (without the Markovianity assumption) which studies directly the primal problem and leads to characterizing the value process in terms

of a backward differential equation (BSDE). BSDEs are generally known to be useful for studying problems in mathematical finance (see [15]), but have been mainly used in continuous setting thus far.
Among previous studies of utility maximization we refer to [2, 16, 24, 27, 26, 28]. Becherer ([2]) considers a discontinuous filtration but a continuous price dynamics whereas the others authors consider both continuous filtrations and continuous price dynamics. Our contribution consists in solving the optimization problem in a discontinuous setting by using the tool of BSDEs. This approach allows to cover non-Markovian situations and in the Markovian case to improve same results obtained by classical stochastic control techniques ([9, 11]).

In Section 2, we prove existence and uniqueness of bounded solutions to backward stochastic equations driven by two independent Poisson martingales when the generator is locally Lipschitz and possesses a certain monotonicity property.

By an application of this result, in Section 3, we solve the exponential utility optimization problems with an additional claim, where the price dynamics S is described by a geometric marked point process driven by two independent Poisson processes. We give a representation of the value process and provide an optimal strategy in terms of the bounded solution to a BSDE whose generator satisfies the upper mentioned assumptions.

The solution to the dual problem and an explicit representation of the density of the minimal entropy measure (MEMM) for the model considered is provided in Section 4.

Section 5 deals with indifference valuation. The utility price is proved to be the unique bounded solution to a BSDE under the MEMM. This representation allows us to obtain the asymptotic behavior of the indifference price and hedging strategy for vanishing risk aversion. As in [2] and [27], the limit corresponds to risk minimization under the MEMM.

Finally, Section 6 is devoted to the study of Markovian cases. This includes both price dynamics in presence of a stochastic factor and valuation of claims written on a nontraded asset correlated to S, where the stochastic factor (or the level of the non tradable asset) is described by a marked point process driven by the same independent Poisson processes driving the dynamic of S. A relation between the HJB-equation and BSDE is also discussed.

2. Backward stochastic differential equations driven by Poisson martingales

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ carrying two independent Poisson processes, N_t^i, $i = 1, 2$. Here $\mathcal{F}_t = \sigma\{N_u^i, i = 1, 2,\ u \le t\}$ and the $\{P, \mathcal{F}_t\}$-intensities of N_t^i, $i = 1, 2$, are positive deterministic bounded functions denoted by $\lambda_i(t)$, $i = 1, 2$, respectively. We will denote by $\lambda(t) = \lambda_1(t)+\lambda_2(t)$ the intensity of $N_t = N_t^1 + N_t^2$, and we assume $0 < \underline{\lambda} \le \lambda(t) \le \overline{\lambda}$, for $\underline{\lambda}$, $\overline{\lambda}$ positive constants.

Denote by $m_t^i = N_t^i - \int_0^t \lambda_i(s) ds$, $i = 1, 2$, the $(P, \mathcal{F}_t)$-martingales associated to the two Poisson processes, respectively.

We want to study a backward stochastic differential equation of the form

$$Y_t = B - \int_t^T Z_s^1 \ dN_s^1 + \int_t^T Z_s^2 \ dN_s^2 - \int_t^T f(s, Y_{s^-}, Z_s^1, Z_s^2) \ ds \tag{2.1}$$

where T is a fixed time horizon, B is a $\mathcal{F}_T$-random variable and $f(\omega, t, y, z_1, z_2)$ is locally Lipschitz uniformly in (ω, t), verifying some suitable inequalities given in Theorem 2.5 below. In what follows we will say that (B, f) are the coefficients of equation (2.1). We are interested in finding a triple (Y, Z^1, Z^2) solution to (2.1). Later on we show how the value processes of exponential utility optimization problems in a market where the risky asset is described by a geometric pure-jump process driven by N_t^i,$i = 1, 2$, can be described explicitly in terms of these BSDE solutions.

The equation (2.1) can be also written as

$$Y_t = B - \int_t^T Z_s^1 dm_s^1 + \int_t^T Z_s^2 dm_s^2 - \int_t^T \widetilde{f}(s, Y_{s^-}, Z_s^1, Z_s^2) ds \tag{2.2}$$

with generator

$$\widetilde{f}(s, y, z_1, z_2) = f(s, y, z_1, z_2) + \lambda_1(s) z_1 - \lambda_2(s) z_2. \tag{2.3}$$

Remark 2.1 *By introducing the following integer-valued random measure on $[0,T]\times U$, with $U=\{1,2\}$*

$$\mu(dt,dx) := \sum_{s\in(0,T]} \Big\{ 1\!\!I_{\{\Delta N^1_s\neq 0\}}\delta_{(s,\{1\})}(ds,dx) + 1\!\!I_{\{\Delta N^2_s\neq 0\}}\delta_{(s,\{2\})}(ds,dx) \Big\}$$

where $\delta_{(s,\{i\})}(ds,dx)$ denotes the Dirac measure in $(s,\{i\})$, $i=1,2$, the BSDE (2.1) can be written as

$$Y_t = B - \int_t^T \int_U Z_s(x)\mu(ds,dx) - \int_t^T F(s,Y_{s^-},Z_s)\ ds.$$

Here $Z_t(x) = Z^1_t 1\!\!I_{\{x=1\}} + Z^2_t 1\!\!I_{\{x=2\}}$ and $F(t,Y_{t^-},Z_t(x)) = f\Big(t,Y_{t^-},Z_t(x)1\!\!I_{\{x=1\}},Z_t(x)1\!\!I_{\{x=2\}}\Big)$.
This last equation is a particular case of that studied in [2].

Let us fix some notations:

- Let $\mathcal{P}$ denotes the predictable σ-algebra on $\Omega\times[0,T]$.
- $\mathcal{S}^p, 1\le p\le\infty$, denotes the space of $I\!\!R$-valued $\mathcal{F}_t$-adapted stochastic processes $\{Y_t\}_{t\in[0.T]}$ with $\|Y\|_{\mathcal{S}^p} = \|\sup_{t\in[0.T]} \mid Y_t \mid \|_{L^p} < \infty$.
- $\mathcal{L}^2$ denotes the space of $I\!\!R$-valued predictable processes $\{Z_t\}_{t\in[0.T]}$ with $\|Z\|_{\mathcal{L}^2} = I\!\!E\Big(\int_0^T \mid Z_t\mid^2 dt\Big)^{\frac{1}{2}}$.

When the coefficients (B,f) are standard it is straightforward to generalize a classical fixed point method to the present setting (see [31, 15, 7, 2]).

Theorem 2.2 *Let $B\in L^2(\Omega,\mathcal{F}_T,P)$ and assume that*

$$\widetilde{f}:\Omega\times[0,T]\times I\!\!R^3 \to I\!\!R$$

is a $\mathcal{P}\otimes\mathcal{B}(I\!\!R^3)$-measurable mapping such that $\widetilde{f}(t,0,0,0)\in\mathcal{L}^2$ and $\widetilde{f}$ is uniformly Lipschitz:

$$\exists L>0:\quad \mid \widetilde{f}(\omega,t,y,z_1,z_2) - \widetilde{f}(\omega,t,\widetilde{y},\widetilde{z}_1,\widetilde{z}_2) \mid \le L(\mid y-\widetilde{y}\mid + \mid z_1-\widetilde{z}_1\mid + \mid z_2-\widetilde{z}_2\mid) \quad P\times dt - a.e. \tag{2.4}$$

for all $(y,z_1,z_2),(\widetilde{y},\widetilde{z}_1,\widetilde{z}_2)\in I\!\!R^3$. Then there exists a unique $(Y,Z^1,Z^2)\in\mathcal{S}^2\times\mathcal{L}^2\times\mathcal{L}^2$ which solves the BSDE (2.2) (or equivalently (2.1)).

Taking into account the representation given in Remark 2.1 we can apply Proposition 3.3 in [2], that in our framework is given by

Proposition 2.3 *Let (B,f) and (B',f') be data satisfying the assumptions of Theorem 2.2. Let $(Y_t,Z^1_t,Z^2_t)\in\mathcal{S}^2\times\mathcal{L}^2\times\mathcal{L}^2$ and $(Y'_t,Z'^{,1}_t,Z'^{,2}_t)\in\mathcal{S}^2\times\mathcal{L}^2\times\mathcal{L}^2$ the solutions to BSDE (2.2) with coefficients (B,f) and (B',f'), respectively.*
Let $\delta B = B-B'$, $\delta\widetilde{f} = \widetilde{f}-\widetilde{f}'$ (see (2.3) for the definition of $\widetilde{f}$) $\delta Y_t = Y_t - Y'_t$, $\delta Z^i_t = Z^i_t - Z'^{,i}_t$, $i=1,2$. Then there exists a constant $C>0$ such that

$$I\!\!E[\sup_{t\in[0,T]}|\delta Y_t|^2 + \int_0^T|\delta Z^1_t|^2dt + \int_0^T|\delta Z^2_t|^2dt] \le CI\!\!E[|\delta B|^2 + \int_0^T|\delta\widetilde{f}(Y_{t^-},Z^1_t,Z^2_t)|^2dt]. \tag{2.5}$$

In our later applications of BSDEs the Lipschitz condition on the generator will not be satisfied. In Theorem 2.5 we will prove the existence and uniqueness of bounded solutions to (2.1) in the case of locally Lipschitz generator and bounded terminal data B.

Lemma 2.4 *If (Y,Z^1,Z^2) is a solution to the BSDE (2.1), belonging to $\mathcal{S}^2\times\mathcal{L}^2\times\mathcal{L}^2$, then the pair (Z^1,Z^2) is uniquely determined by the knowledge of Y. In fact,*

$$Z^i_t\ 1\!\!I_{\Delta N^i_t\neq 0} = (Y_t - Y_{t^-})\ 1\!\!I_{\Delta N^i_t\neq 0} \qquad i=1,2 \tag{2.6}$$

and if $(\widetilde{Z}^1,\widetilde{Z}^2)$ is another pair of predictable processes in $\mathcal{L}^2\times\mathcal{L}^2$ verifying (2.6), then $Z^i=\widetilde{Z}^i$ $i=1,2$ P-a.s. and for a.a. t.

Proof.
For $i = 1, 2$

$$0 = I\!E\left[\sum_{r\in(0,T]} \left(Z^i_r - \widetilde{Z}^i_r\right)^2 \, 1\!\!1_{\Delta N^i_r \neq 0}\right] = I\!E\left[\int_{(0,T]} \left(Z^i_r - \widetilde{Z}^i_r\right)^2 \, dN^i_r\right] = I\!E\left[\int_0^T \left(Z^i_r - \widetilde{Z}^i_r\right)^2 \, \lambda_i(r)\, dr\right]$$

and the thesis, since $\lambda_i(r)$ are supposed to be positive, for $i = 1, 2$.
□

Theorem 2.5 *Assume that B is a bounded $\mathcal{F}_T$-random variable and that*

$$f : \Omega \times [0,T] \times I\!R^3 \to I\!R$$

is a $\mathcal{P} \otimes \mathcal{B}(I\!R^3)$-measurable mapping such that $f(t, y, 0, 0) \in \mathcal{S}^\infty$, for any $y \in I\!R$, and f is locally uniformly Lipschitz:

$$\begin{aligned} &\forall C > 0,\ \exists L_C > 0 :\ \forall\ (y, z_1, z_2),\ (\widetilde{y}, \widetilde{z}_1, \widetilde{z}_2) \in I\!R^3 \textit{ with } \|(y, z_1, z_2)\| \leq C,\ \|(\widetilde{y}, \widetilde{z}_1, \widetilde{z}_2)\| \leq C, \\ &\mid f(\omega, t, y, z_1, z_2) - f(\omega, t, \widetilde{y}, \widetilde{z}_1, \widetilde{z}_2) \mid \leq L_C(\mid y - \widetilde{y} \mid + \mid z_1 - \widetilde{z}_1 \mid + \mid z_2 - \widetilde{z}_2 \mid) \quad P \times dt - a.e.. \end{aligned} \tag{2.7}$$

Assume furthermore that, $\forall t \in [0,T]$ $y \in I\!R$,

$$f(t, y, \cdot, z_2) \textit{ non increasing } \forall\ z_2 \in I\!R, \quad \textit{and} \quad f(t, y, z_1, \cdot) \textit{ non decreasing } \forall\ z_1 \in I\!R. \tag{2.8}$$

Then there exists a unique solution $(Y, Z^1, Z^2) \in \mathcal{S}^\infty \times \mathcal{L}^2 \times \mathcal{L}^2$ to the BSDE (2.1) where Y, Z^1 and Z^2 are bounded P-a.s..
Moreover, if the pair (B', f') satisfies the assumption of this theorem, denoting by $(Y'_t, Z'^{,1}_t, Z'^{,2}_t)$ the bounded solution to BSDE (2.2) with coefficients (B', f'), then estimate (2.5) still hold.

Proof.
First, let us observe that, since $f(t, y, 0, 0) \in \mathcal{S}^\infty$ and f is locally uniformly Lipschitz then f is locally bounded:

$$\forall C > 0 \quad \mid f(\omega, t, y, z_1, z_2) \mid \leq \mid f(\omega, t, y, 0, 0) \mid + 2CL_C \leq \sup_{y\in I\!R} \|f(\omega, t, y, 0, 0)\|_{\mathcal{S}^\infty} + 2CL_C \quad P \times dt - a.e. \tag{2.9}$$

for all (y, z_1, z_2) such that $\|(y, z_1, z_2)\| \leq C$.

By hypothesis, there exists C_1 such that $\mid B \mid \leq C_1$ and define

$$b(t) = C_1 + C_2(T - t),$$

with $C_2 \geq \sup_{y\in I\!R} \|f(t, y, 0, 0)\|_{\mathcal{S}^\infty}$. We note that $b(t)$ is a decreasing function such that

$$b(T) = C_1 \leq b(t) \leq C_1 + C_2 T = b(0).$$

We consider the same truncation function as in Theorem 3.5 of [2]. More precisely, let $k(t, y)$ the following truncation function

$$k(t, y) = \begin{cases} -b(t) & \text{for } y \leq -b(t) \\ \quad y & \text{for } -b(t) < y < b(t) \\ \ \ b(t) & \text{for } y \geq b(t) \end{cases}$$

which is bounded and Lipschitz in y uniformly in t.
Setting

$$\widetilde{f}_k(\omega, t, y, z_1, z_2) = f\Big(\omega, t, k(t, y), k(t, y + z_1) - k(t, y), -\big(k(t, y - z_2) - k(t, y)\big)\Big) + \tag{2.10}$$

$$\lambda_1(t)\Big(k(t, y+z_1) - k(t, y)\Big) + \lambda_2(t)\Big(k(t, y-z_2) - k(t, y)\Big) = \widetilde{f}\Big(\omega, t, k(t, y), k(t, y+z_1) - k(t, y), -\big(k(t, y-z_2) - k(t, y)\big)\Big),$$

we consider the following BSDE with generator $\widetilde{f}_k$

$$Y_t = B - \int_t^T Z_s^1 dm_s^1 + \int_t^T Z_s^2 dm_s^2 - \int_t^T \widetilde{f}_k(s, Y_{s^-} Z_s^1, Z_s^2) ds \tag{2.11}$$

Since $| k(t,y) | \leq b(0)$, $\forall (t,y) \in [0,T] \times I\!R$, and $\lambda_i(t)$, $i = 1,2$, are bounded functions, by (2.7) follows that f_k satisfies the hypotheses of Theorem 2.2.

Let $(Y, Z^1, Z^2) \in \mathcal{S}^2 \times \mathcal{L}^2 \times \mathcal{L}^2$ the unique solution to (2.11) and define

$$\widetilde{Y}_t = k(t, Y_t) \quad \widetilde{Z}_t^1 = k(t, Y_{t^-} + Z_t^1) - k(t, Y_{t^-}) \quad \widetilde{Z}_t^2 = -\Big(k(t, Y_{t^-} - Z_t^2) - k(t, Y_{t^-})\Big). \tag{2.12}$$

If we will prove that Y_t and $\widetilde{Y}_t$ are indistinguishable, as a consequence of Lemma 2.4, Z_t^i and $\widetilde{Z}_t^i$, $i = 1,2$, will be indistinguishable, then

$$\widetilde{f}_k(\omega, t, Y_{t^-}, Z_t^1, Z_t^2) = \widetilde{f}(\omega, t, \widetilde{Y}_{t^-}, \widetilde{Z}_t^1, \widetilde{Z}_t^2) = \widetilde{f}(\omega, t, Y_{t^-}, Z_t^1, Z_t^2)$$

and (Y, Z^1, Z^2) solves the BSDE (2.2) or equivalently (2.1).

In order to prove that Y_t and $\widetilde{Y}_t$ are indistinguishable, we will show that $| Y_t | \leq b(t)$ for all $t \in [0,T]$.

First, we consider the upper bound. Fix $t \in [0,T]$ and let

$$\tau = \inf\{s \in [t,T] : Y_s \leq b(s)\} \tag{2.13}$$

Since $| Y_T | = | B | \leq b(T) = C_1$ and Y_t is cadlag, $Y_\tau \leq b(\tau)$ and $Y_s > b(s)$ for $(\omega, s) \in [t, \tau)$.

Equations (2.11) implies that for all $t < \tau$, by the optional sampling theorem,

$$\begin{aligned}
Y_t &= I\!E[Y_\tau - \int_t^\tau \widetilde{f}_k(s, Y_{s^-}, Z_s^1, Z_s^2) ds \mid \mathcal{F}_t] = \\
&= I\!E[Y_\tau - \int_t^\tau \Big[f\Big(\omega, s, k(s, Y_{s^-}), k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-}), -k(s, Y_{s^-} - Z_s^2) + k(s, Y_{s^-})\Big) + \\
&+ \lambda_1(s)\Big(k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-})\Big) + \lambda_2(s)\Big(k(s, Y_{s^-} - Z_s^2) - k(s, Y_{s^-})\Big)\Big] ds \mid \mathcal{F}_t].
\end{aligned}$$

By definition of intensity of point process we get that

$$\begin{aligned}
&I\!E[\int_t^\tau \lambda_1(s)\Big(k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-})\Big) ds \mid \mathcal{F}_t] = I\!E[\int_t^T \big(k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-})\big) dN_s^1 \mid \mathcal{F}_t] = \\
&= I\!E[\sum_{t<s\leq\tau} \big(k(s, Y_s) - k(s, Y_{s^-})\big) 1\!\!1_{\{\Delta N_s^1 \neq 0\}} \mid \mathcal{F}_t] = E[\big(k(\tau, Y_\tau) - k(\tau, Y_{\tau^-})\big) 1\!\!1_{\{\Delta N_\tau^1 \neq 0\}} \mid \mathcal{F}_t] = \\
&= I\!E[(Y_\tau - b(\tau)) 1\!\!1_{\{\Delta N_\tau^1 \neq 0\}} \mid \mathcal{F}_t].
\end{aligned}$$

Similarly we get that

$$I\!E[\int_t^\tau \lambda_2(s)\Big(k(s, Y_{s^-} - Z_s^2) - k(s, Y_{s^-})\Big) ds \mid \mathcal{F}_t] = I\!E[(Y_\tau - b(\tau)) 1\!\!1_{\{\Delta N_\tau^2 \neq 0\}} \mid \mathcal{F}_t].$$

Moreover, noting that the integrand is predictable and bounded (see (2.14)), and that the intensities are strictly positive,

$$\begin{aligned}
&I\!E\Big[\int_t^\tau f\Big(s, k(s, Y_{s^-}), k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-}), -k(s, Y_{s^-} - Z_s^2) + k(s, Y_{s^-})\Big)\ ds \mid \mathcal{F}_t\Big] = \\
&= I\!E\Big[\int_t^\tau \frac{1}{\lambda(s)}\ f\Big(s, k(s, Y_{s^-}), k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-}), -k(s, Y_{s^-} - Z_s^2) + k(s, Y_{s^-})\Big)\ dN_s \mid \mathcal{F}_t\Big] = \\
&= I\!E\Big[\int_t^\tau \frac{1}{\lambda(s)}\ f\Big(s, k(s, Y_{s^-}), k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-}), -k(s, Y_{s^-} - Z_s^2) + k(s, Y_{s^-})\Big)\ dN_s^1 \mid \mathcal{F}_t\Big] + \\
&+ I\!E\Big[\int_t^\tau \frac{1}{\lambda(s)}\ f\Big(s, k(s, Y_{s^-}), k(s, Y_{s^-} + Z_s^1) - k(s, Y_{s^-}), -k(s, Y_{s^-} - Z_s^2) + k(s, Y_{s^-})\Big)\ dN_s^2 \mid \mathcal{F}_t\Big] =
\end{aligned}$$

and, by (2.8), (2.13)

$$\mathbb{E}\Big[\int_t^\tau \frac{1}{\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), k(s,Y_{s^-}+Z^1_s)-k(s,Y_{s^-}), -k(s,Y_{s^-}-Z^2_s)+k(s,Y_{s^-})\Big)\; dN^1_s \mid \mathcal{F}_t\Big] =$$
$$= E\Big[\sum_{t<s<\tau} \frac{1}{\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), 0, -k(s,Y_{s^-}-Z^2_s)+k(s,Y_{s^-})\Big)\; \mathbb{I}_{\{\Delta N^1_s\neq 0\}} \mid \mathcal{F}_t\Big] +$$
$$+E\Big[\frac{1}{\lambda(\tau)}\; f\Big(\tau, b(\tau), Y_\tau - b(\tau), -k(\tau,Y_{\tau^-}-Z^2_\tau)+k(\tau,Y_{\tau^-})\Big)\; \mathbb{I}_{\{\Delta N^1_\tau\neq 0\}} \mid \mathcal{F}_t\Big] \geq$$
$$\geq E\Big[\int_t^\tau \frac{\lambda_1(s)}{\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), 0, -k(s,Y_{s^-}-Z^2_s)+k(s,Y_{s^-})\Big)\; ds \mid \mathcal{F}_t\Big] =$$

(recalling again that the integrand is predictable and bounded),

$$= E\Big[\int_t^\tau \frac{\lambda_1(s)}{\lambda_2(s)\;\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), 0, -k(s,Y_{s^-}-Z^2_s)+k(s,Y_{s^-})\Big)\; dN^2_s \mid \mathcal{F}_t\Big] =$$
$$= E\Big[\sum_{t<s<\tau} \frac{\lambda_1(s)}{\lambda_2(s)\;\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), 0, 0\Big)\; \mathbb{I}_{\{\Delta N^2_s\neq 0\}} \mid \mathcal{F}_t\Big] +$$
$$+E\Big[\frac{\lambda_1(\tau)}{\lambda_2(\tau)\;\lambda(\tau)}\; f\Big(\tau, k(\tau,Y_{\tau^-}), 0, -Y_\tau + b(\tau)\Big)\; \mathbb{I}_{\{\Delta N^2_\tau\neq 0\}} \mid \mathcal{F}_t\Big] \geq$$
$$\geq E\Big[\int_t^\tau \frac{\lambda_1(s)}{\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), 0, 0\Big)\; ds \mid \mathcal{F}_t\Big].$$

Analogously

$$\mathbb{E}\Big[\int_t^\tau \frac{1}{\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), k(s,Y_{s^-}+Z^1_s)-k(s,Y_{s^-}), -k(s,Y_{s^-}-Z^2_s)+k(s,Y_{s^-})\Big)\; dN^2_s \mid \mathcal{F}_t\Big] \geq$$
$$= E\Big[\int_t^\tau \frac{\lambda_2(s)}{\lambda(s)}\; f\Big(s, k(s,Y_{s^-}), 0, 0\Big)\; ds \mid \mathcal{F}_t\Big].$$

Thus

$$\mathbb{E}\Big[\int_t^\tau f\Big(s, k(s,Y_{s^-}), k(s,Y_{s^-}+Z^1_s)-k(s,Y_{s^-}), -k(s,Y_{s^-}-Z^2_s)+k(s,Y_{s^-})\Big)\; ds \mid \mathcal{F}_t] \geq$$
$$= E\Big[\int_t^\tau f\Big(s, k(s,Y_{s^-}), 0, 0\Big)\; ds \mid \mathcal{F}_t\Big] \geq -C_2(\tau - t).$$

Since

$$Y_\tau - (Y_\tau - b(\tau))\mathbb{I}_{\{\Delta N^1_\tau\neq 0\}} - (Y_\tau - b(\tau))\mathbb{I}_{\{\Delta N^2_\tau\neq 0\}} = b(\tau),$$

finally, as a conclusion,

$$\begin{aligned} Y_t &= \mathbb{E}[b(\tau) - \int_t^\tau f\Big(s, k(s,Y_{s^-}), k(s,Y_{s^-}+Z^1_s)-k(s,Y_{s^-}), -k(s,Y_{s^-}-Z^2_s)+k(s,Y_{s^-})\Big)\; ds \mid \mathcal{F}_t] \leq \\ &\leq \mathbb{E}[b(\tau) + C_2(\tau - t) \mid \mathcal{F}_t] = \mathbb{E}[C_1 + C_2(T-\tau) + C_2(\tau - t) \mid \mathcal{F}_t] = b(t). \end{aligned}$$

Similarly we can prove the lower bound $Y_t \geq -b(t)\ \forall t \in [0.T]$, by introducing $\widetilde{\tau} = \inf\{s \in [t,T] : Y_s \geq -b(s)\}$. Moreover we get, by (2.12), that

$$\mid Y_t \mid \leq b(0) \quad \text{and} \quad \mid Z^i_t \mid \leq 2b(0), \quad i = 1,2. \tag{2.14}$$

To show uniqueness we proceed as in [2]. Let $(\widehat{Y}, \widehat{Z}^1, \widehat{Z}^2)$ be another solution to the BSDE with $\widehat{Y}$ bounded. Taking $b(0) \geq 2\|\widehat{Y}\|_{\mathcal{S}^\infty}$ we get that $(\widehat{Y}, \widehat{Z}^1, \widehat{Z}^2)$ solves the BSDE also with generator $\widetilde{f}_k$, and by Theorem 2.2 the two solutions must coincide.

The validity of estimate (2.5) follow from the observation that the BSDE solutions to (B, f) and (B', f') also solve the BSDEs with the corresponding truncated generators, which satisfy the Lipschitz condition (2.4). $\square$

3. The model and the hedging problem

We consider a finite time horizon investment model on $[0, T]$ with one riskless money market account and one risky asset. The price of the risk-free asset is taken equal to 1 (that is we suppose the riskless interest rate to be equal zero). The price S of the stock is modeled as a pure jump process verifying, on $[0, T]$

$$dS_t = S_{t^-}\Big(K^1_t\ dN^1_t - K^2_t\ dN^2_t\Big), \tag{3.1}$$

with $S_0 = s_0 \in I\!R^+$.

The $I\!R$-valued stochastic processes K^i_t, $i = 1, 2$ are supposed to be positive and $\{P, \mathcal{F}_t\}$-predictable, $K^2_t < 1$ and, setting $K_t = K^1_t + K^2_t$, we assume $0 < \overline{K} \leq K_t \leq \underline{K}$, for $\overline{K}$, $\underline{K}$ positive constants.

By the Doléans-Dade exponential formula we get that

$$S_t = S_0 e^{Y_t},$$

where the logreturn process Y is given by

$$Y_t = \int_0^t \log\ (1 + K^1_r) dN^1_r + \int_0^t \log\ \left(1 - K^2_r\right) dN^2_r.$$

Let us consider a European contingent claim with maturity T, whose payoff is given by B, $\mathcal{F}_T$-measurable bounded random variable such that, as in the previous section, $|B| \leq C_1$.
The hedging problem consists in finding an investment strategy to trade in the available assets in the time window $[0, T]$ in order to reduce (or avoid) potential losses arising from having to honor the contract B.

The goal of this section is to study and solving an hedging problem. Since in the model we are studying the market is incomplete, perfect replication is not possible. Thus, we have to use an hedging criterion under incompleteness. Many methods are possible. In particular a stochastic control approach can be chosen. Among others, we quote [9, 11, 22] and the references therein. In this frame we will use a method which is along the lines of that proposed by [24, 2, 26].

For a predictable, S-integrable, self-financing strategy $\frac{\pi_t}{S_{t^-}}$ and initial capital $x_0 \geq 0$, the associated wealth process is defined as

$$X_t = x_0 + \int_0^t \frac{\pi_r}{S_{r^-}} dS_r = x_0 + \int_0^t \pi_r\ \Big(K^1_r\ dN^1_r - K^2_r\ dN^2_r\Big). \tag{3.2}$$

For an agent with exponential preferences and risk aversion parameter $\alpha \in I\!R^+$, the objective is to maximize the expected utility of his terminal wealth, which is given by

$$I\!E\Big[-\exp\big\{-\alpha(X_T - B)\big\}\Big] = I\!E\Big[-\exp\big\{-\alpha(x_0 + \int_0^T \frac{\pi_r}{S_{r^-}} dS_r\ - B)\big\}\Big] \tag{3.3}$$

for a suitable class Π of admissible strategies which we characterize later on.

In the frame of the stochastic control approach, we introduce the associated value process

$$V^B_t(x) = \operatorname{ess}\sup_{\pi \in \Pi_t} I\!E\Big[-\exp\big\{-\alpha(x + \int_t^T \frac{\pi_r}{S_{r^-}} dS_r\ - B)\big\} \mid \mathcal{F}_t\Big] = -e^{-\alpha x}\ W^B_t \tag{3.4}$$

where x denotes the amount of capital at time t, and

$$W^B_t = \operatorname{ess}\inf_{\pi \in \Pi_t} I\!E\Big[\exp\big\{-\alpha(\int_t^T \frac{\pi_r}{S_{r^-}} dS_r\ - B)\big\}\Big) \mid \mathcal{F}_t\Big]. \tag{3.5}$$

Here Π_t denotes the set of the admissible strategies on the interval $[t, T]$.

Let us observe that setting, for any $\pi \in \Pi$

$$J_t(\pi) = \exp\big\{ -\alpha \int_0^t \frac{\pi_r}{S_{r^-}} dS_r \big\} \tag{3.6}$$

we get

$$W_t^B = \operatorname{ess}\inf_{\pi\in\Pi} I\!E\Big[e^{\alpha B}\ \frac{J_T(\pi)}{J_t(\pi)} \mid \mathcal{F}_t\Big]. \tag{3.7}$$

Proposition 3.1 *Assume the existence of a family of $\{\mathcal{F}_t\}$-adapted stochastic processes $\{R_t^\pi\}_{\pi\in\Pi}$ such that*

$$\begin{array}{ll} (i) & \forall \pi \in \Pi,\ R_t^\pi \ \textit{is a}\ \{P, \mathcal{F}_t\} - \textit{submartingale},\ R_T^\pi = J_T(\pi)\ e^{\alpha B}. \\ (ii) & \exists \pi^B \in \Pi \ \textit{such that}\ R_t^{\pi^B}\ \textit{is a}\ \{P, \mathcal{F}_t\} - \textit{martingale}. \end{array}$$

Then the process

$$H_t = \frac{R_t^\pi}{J_t(\pi)}$$

verifies

$$H_t = W_t^B \qquad \textit{and} \qquad H_T = e^{\alpha B}.$$

Proof.
The submartingale property of the process R_t^π implies that, $\forall \pi \in \Pi$

$$H_t = \frac{R_t^\pi}{J_t(\pi)} \le \frac{I\!E[R_T^\pi|\mathcal{F}_t]}{J_t(\pi)} = I\!E\Big[\frac{J_T(\pi)\ e^{\alpha B}}{J_t(\pi)} \mid \mathcal{F}_t\Big]$$

which in turn implies that $H_t \le W_t^B$. On the other hand, for $\pi = \pi^B$ for the martingale property we have

$$H_t = \frac{R_t^{\pi^B}}{J_t(\pi^B)} = \frac{I\!E[R_T^{\pi^B}|\mathcal{F}_t]}{J_t(\pi^B)} = I\!E\Big[\frac{J_T(\pi^B)\ e^{\alpha B}}{J_t(\pi^B)} \mid \mathcal{F}_t\Big]$$

and then $H_t \ge W_t^B$.
□

Remark 3.2 *Thus we obtain that the strategy $\frac{\pi_t^B}{S_{t^-}}$, with π^B mentioned in the previous proposition is an optimal control for the problem (3.7), and, as a consequence, for the problem (3.3).*

In what follows we make the following assumption on the set of admissible strategies.

Definition 3.3 *Let us denote by Π_0 the set of processes π_t such that $\frac{\pi_t}{S_{t^-}}$ is predictable, self-financing and S-integrable.*

Hypothesis 3.4 *The set Π consists of processes $\pi_t \in \Pi_0$, taking value in a compact set. This means that there exists a positive real number $\overline{\pi}$ and any $\pi \in \Pi$ verifies $|\pi_t| \le \overline{\pi}$, P-a.s. for all $t \in [0, T]$.*

Let us remark that the class Π will not be modified for an absolutely continuous change of probability measure.

Proposition 3.5 *The following inequalities hold true*

$$e^{-\alpha C_1}\ \exp\big\{\big(e^{-\alpha\overline{\pi}\overline{K})} - 1\big)\ \overline{\lambda}(T-t)\big\} \le W_t^B \le e^{\alpha C_1}\ \exp\big\{\big(e^{\alpha\overline{\pi}\overline{K})} - 1\big)\ \overline{\lambda}(T-t)\big\}. \tag{3.8}$$

Proof.
Recalling (3.1) and (3.5) the inequalities (3.8) are easily obtained by a direct computation taking into account the assumption 3.4.

$$W_t^B \leq e^{\alpha C_1}\ I\!E\left[e^{\alpha\overline{\pi}\overline{K}\ (N_T-N_t)}|\mathcal{F}_t\right] = e^{\alpha C_1}\ \exp\left\{\int_t^T \left(e^{\alpha\overline{\pi}\overline{K}}-1\right)\lambda(s)\ ds\right\} \leq e^{\alpha C_1}\ \exp\left\{\left(e^{\alpha\overline{\pi}\overline{K}}-1\right)\overline{\lambda}\,(T-t)\right\}$$

$$W_t^B \geq e^{-\alpha C_1}\ I\!E\left[e^{-\alpha\overline{\pi}\overline{K}\ (N_T-N_t)}|\mathcal{F}_t\right] = e^{-\alpha C_1}\ \exp\left\{\int_t^T \left(e^{-\alpha\overline{\pi}\overline{K}}-1\right)\lambda(s)\ ds\right\} \geq e^{-\alpha C_1}\ \exp\left\{\left(e^{-\alpha\overline{\pi}\overline{K}}-1\right)\overline{\lambda}\,(T-t)\right\}.$$

□

The construction of a family $\{R_t^\pi\}_{\pi\in\Pi}$ with the properties required by Proposition 3.1 is strictly related with the existence of bounded solution to a suitable BSDE.

Theorem 3.6 *Let $(Y^B, Z^{B,1}, Z^{B,2})$ be a bounded solution to the BSDE*

$$Y_t^B = B - \int_t^T Z_s^{B,1}\ dN_s^1 + \int_t^T Z_s^{B,2}\ dN_s^2 - \int_t^T f_\alpha(s, Z_s^{B,1}, Z_s^{B,2})\ ds \tag{3.9}$$

where the function $f_\alpha(t, z_1, z_2)$ is defined as

$$f_\alpha(t, z_1, z_2) = \frac{\lambda(t)}{\alpha} - \frac{\lambda_1(t)}{\alpha}\ \frac{K_t}{K_t^2}\left(\frac{\lambda_1(t)K_t^1}{\lambda_2(t)K_t^2}\right)^{-K_t^1/K_t}\ \exp\left\{-\alpha\ \frac{K_t^1 z_2 - K_t^2 z_1}{K_t}\right\}. \tag{3.10}$$

An optimal control for the problem (3.7), and, as a consequence, for the problem (3.3) is given by

$$\pi_t^B = \frac{1}{K_t}\left\{\frac{1}{\alpha}\ \log\left(\frac{\lambda_1(t)K_t^1}{\lambda_2(t)K_t^2}\right) + Z_t^{B,1} + Z_t^{B,2}\right\}, \tag{3.11}$$

and the value process is

$$V_t^B(x) = -e^{\alpha(Y_t^B - x)}. \tag{3.12}$$

Proof.
First we observe that the equation (3.9) with the generator defined with (3.10) verifies the assumptions of Theorem 2.5. Thus there exists a bounded solution $(Y^B, Z^{B,1}, Z^{B,2})$ to (3.9).

Let us set, for any $\pi \in \Pi$

$$R_t^\pi = J_t(\pi)\ e^{\alpha Y_t^B} \tag{3.13}$$

with $J_t(\pi)$ defined in (3.6). In order to prove that the family $\{R_t^\pi\}_{\pi\in\Pi}$ verifies the properties required by Proposition 3.1, we introduce the following processes.

Then we claim that the process

$$M_t^\pi = e^{\alpha Y_0^B}\ \exp\left\{\sum_{i=1}^2\left(\int_0^t \log(1+U_s^i(\pi))\ dN_s^i - \int_0^t \lambda_i(s)\ U_s^i(\pi)\ ds\right)\right\} \tag{3.14}$$

with, for $\pi \in \Pi$.

$$U_s^1(\pi) = \exp\Big\{-\alpha\big(\pi_s K_s^1 - Z_s^{B,1}\big)\Big\} - 1 \qquad\qquad U_s^2(\pi) = \exp\Big\{\alpha\big(\pi_s K_s^2 - Z_s^{B,2}\big)\Big\} - 1$$

is a positive $\{P, \mathcal{F}_t\}$-martingale, as a consequence of the assumption made on $\lambda_i(t)$ and K_t^i, $i = 1, 2$. More, we set

$$A_t^\pi = \exp\left\{\int_0^t v_\alpha(s, \pi_s, Z_s^{B,1}, Z_s^{B,2})\ ds\right\}$$

with

$$v_\alpha(s, \pi_s, Z_s^{B,1}, Z_s^{B,2}) = \alpha f_\alpha(s, Z_s^{B,1}, Z_s^{B,2}) + \lambda_1(s)U_s^1(\pi) + \lambda_2(s)U_s^2(\pi) \tag{3.15}$$

and we get that

$$\forall \pi \in \Pi,\ v_\alpha(s,\pi_s,Z_s^{B,1},Z_s^{B,2}) \geq 0 \quad \text{and} \quad \text{for } \pi=\pi^B,\ v_\alpha(s,\pi_s^B,Z_s^{B,1},Z_s^{B,2})=0 \tag{3.16}$$

being π^B given in (3.11). The assertion in (3.16) can be seen by noting that $v_\alpha(s,\pi_s^B,Z_s^{B,1},Z_s^{B,2})=0$ and that this is its minimum value.

Next we compute

$$M_t^\pi A_t^\pi = e^{\alpha Y_0^B}\ \exp\left\{-\alpha\int_0^t \pi_s\big(K_s^1\, dN_s^1 - K_s^2\, dN_s^2\big) + \alpha\int_0^t \big(Z_s^{B,1}\, dN_s^1 - Z_s^{B,2}\, dN_s^2\big) + \alpha\int_0^t f_\alpha(s,Z_s^{B,1},Z_s^{B,2})\, ds\right\}.$$

On the other hand, by (3.9) we can write

$$Y_t^B = Y_0^B + \int_0^t Z_s^{B,1}\, dN_s^1 - \int_0^t Z_s^{B,2}\, dN_s^2 + \int_0^t f_\alpha(s,Z_s^{B,1},Z_s^{B,2})\, ds$$

and recalling (3.6) we get $M_t^\pi A_t^\pi = R_t^\pi$, for any $\pi \in \Pi$.

As a consequence, we obtain that R_t^π can be written as the product of a nondecreasing process and a positive martingale, which implies that it is a $\{P,\mathcal{F}_t\}$-submartingale that turns to be a martingale if the optimal control is chosen.

Finally by Proposition 3.1 and Remark 3.2, we have that $W_t^B = e^{\alpha Y_t^B}$ and the thesis.
□

Remark 3.7 *It turns out to be that, if $(Y_t^B,Z_t^{B,1},Z_t^{B,2})$ is any bounded solution to (3.9) then*

$$Y_t^B = \frac{1}{\alpha}\ \log W_t^B.$$

Thus, recalling Lemma 2.4, we get that uniqueness holds for the solutions to the equation (3.9). Moreover, recalling inequalities (3.8), we have

$$|Y_t^B| \leq C_1 + \frac{\overline{\lambda}}{\alpha}\ D(T-t)$$

with $D = max\big\{1-e^{-\alpha\overline{\pi}\overline{K}}, e^{\alpha\overline{\pi}\overline{K}}-1\big\}$.
Moreover let us observe that our generator does not satisfy the condition required in Theorem 3.5 in [2].

Proposition 3.8 *With the same assumption of Theorem 3.6, if $(Y_t^B,Z_t^{B,1},Z_t^{B,2})$ is the bounded solution to (3.9) (with coefficient (B,f_α)) then*

$$Y_t^B = \frac{\widehat{Y}_t^B}{\alpha},\quad Z_t^{B,i} = \frac{\widehat{Z}_t^{B,i}}{\alpha},\quad i=1,2 \quad P\times dt - a.e. \tag{3.17}$$

where $(\widehat{Y}_t^B \widehat{Z}_t^{B,1},\widehat{Z}_t^{B,2})$ is the unique bounded solution to (3.9) with coefficient $(\alpha B, f_1)$, that is

$$\widehat{Y}_t^B = \alpha B - \int_t^T \widehat{Z}_s^{B,1} dN_s^1 + \int_t^T \widehat{Z}_s^{B,2} dN_s^2 - \int_t^T f_1(s,\widehat{Z}_s^{B,1},\widehat{Z}_s^{B,2}) ds. \tag{3.18}$$

Proof.
Let $(\widehat{Y}_t^B \widehat{Z}_t^{B,1},\widehat{Z}_t^{B,2})$ be the unique bounded solution to (3.18), dividing by α, since $f_1(t,z_1,z_2) = \alpha f_\alpha(t,\frac{z_1}{\alpha},\frac{z_2}{\alpha})$ we get that $(\frac{\widehat{Y}_t^B}{\alpha},\frac{\widehat{Z}_t^{B,1}}{\alpha},\frac{\widehat{Z}_t^{B,2}}{\alpha})$ solves (3.9).
□

In what follows, it will be usefull to consider the case of an investor who seeks to maximize the expected utility of his terminal wealth without taking into account the claim, that is to maximize

$$I\!\!E\Big[-\exp\big\{-\alpha\ X_T\big\}\Big] = I\!\!E\Big[-\exp\big\{-\alpha(x_0 + \int_0^T \frac{\pi_r}{S_{r^-}} dS_r)\big\}\Big]. \tag{3.19}$$

In this case the associated value process is

$$V_t^0(x) = \text{ess}\sup_{\pi\in\Pi_t} I\!E\Big[-\exp\big\{-\alpha(x+\int_t^T \frac{\pi_r}{S_{r^-}} dS_r)\big\} \mid \mathcal{F}_t\Big] = -e^{-\alpha x}\ W_t^0 \tag{3.20}$$

where

$$W_t^0 = \text{ess}\inf_{\pi\in\Pi_t} I\!E\Big[\exp\big\{-\alpha\Big(\int_t^T \frac{\pi_r}{S_{r^-}} dS_r\ - B\big\}\Big) \mid \mathcal{F}_t\Big]. \tag{3.21}$$

It is a particular case of the problem discussed in this section, but we want to emphasize the final result.

Theorem 3.9 *Let $(Y^0, Z^{0,1}, Z^{0,2})$ the bounded solution to the BSDE*

$$Y_t^0 = -\int_t^T Z_s^{0,1}\ dN_s^1 + \int_t^T Z_s^{0,2}\ dN_s^2 - \int_t^T f_\alpha(s, Z_s^{0,1}, Z_s^{0,2})\ ds \tag{3.22}$$

where the function $f_\alpha(t,y,z_1,z_2)$ is defined in (3.10). An optimal control for the problem (3.19) is given by

$$\pi_t^0 = \frac{1}{K_t}\left\{\frac{1}{\alpha}\ \log\left(\frac{\lambda_1(t)K_t^1}{\lambda_2(t)K_t^2}\right) + Z_t^{0,1} + Z_t^{0,2}\right\}, \tag{3.23}$$

and the value process is

$$V_t^0(x) = -e^{\alpha(Y_t^0 - x)}. \tag{3.24}$$

4. Dual problems

The dual problem related to the primal utility maximization problem discussed in Section 3 consists in finding, in the class $\mathcal{M}_f$ of martingale measures equivalent to P with finite entropy, a measure Q_B solution to the problem

$$\max_{Q\in\mathcal{M}_f}\Big(I\!E^Q[\alpha B] - H(Q|P)\Big),$$

where $H(Q|P)$, the relative entropy of a probability measure Q w.r.t. P is defined by

$$H(Q|P) = \begin{cases} I\!E^P\Big[\frac{dQ}{dP}\ log\Big(\frac{dQ}{dP}\Big)\Big] & Q \ll P \\ +\infty & otherwise \end{cases} \tag{4.1}$$

The link between the utility maximization problem and the dual problem is provided by the duality principle discussed, among others, in [12]. We will prove that a duality relation for the model studied in this note can be written as

$$\inf_{\Pi} I\!E\left[\exp\left\{\alpha B - \alpha\ \int_0^T \frac{\pi_t}{S_{t^-}}\ dS_t\right\}\right] = \exp\left\{\sup_{Q\in\mathcal{M}_f}\Big(I\!E^Q[\alpha B] - H(Q|P)\Big)\right\}. \tag{4.2}$$

Thus, for $\pi = \pi^B$, with π^B defined in (3.11), by Theorem 3.6 the l.h.s. of (4.2) takes the value $e^{\alpha Y_0^B}$.

Lemma 4.1 *The probability measure $\widetilde{P}$ defined by the density*

$$\frac{d\widetilde{P}}{dP} = \exp\left\{\sum_{i=1}^2\left(\int_0^t \log(1+\widetilde{U}_s^i)\ dN_s^i - \int_0^t \lambda_i(s)\widetilde{U}_s^i\ ds\right)\right\}$$

with, $\widetilde{U}_s^1 = \exp\Big\{-\widetilde{\theta}_s K_s^1\Big\} - 1$, $\widetilde{U}_s^2 = \exp\Big\{\widetilde{\theta}_s K_s^2\Big\} - 1$, and $\widetilde{\theta}_s = \frac{1}{K_t}\ \log\frac{\lambda_1(s)K_s^1}{\lambda_2(s)K_s^2}$, is a risk-neutral measure equivalent to P with finite entropy w.r.t. P.

Proof.
As a consequence of a Girsanov Theorem, under $\widetilde{P}$, for $i = 1, 2$, the intensity of N_t^i is given by

$$\widetilde{\lambda}_i(t) = \lambda_i(t)\ \big(1 + \widetilde{U}_t^i\big), \tag{4.3}$$

thus a sufficient condition of risk neutrality (see for instance [11]) can be obtained by computing

$$\lambda_1(t)\ K_t^1\ (1 + \widetilde{U}_t^1) - \lambda_2(t)\ K_t^2\ (1 + \widetilde{U}_t^2) = 0$$

where

$$(1 + \widetilde{U}_t^1) = \left(\frac{\lambda_1(t)\ K_t^1}{\lambda_2(t)\ K_t^2}\right)^{-\frac{K_t^1}{K_t}} \qquad (1 + \widetilde{U}_t^2) = \left(\frac{\lambda_1(t)\ K_t^1}{\lambda_2(t)\ K_t^2}\right)^{\frac{K_t^2}{K_t}}.$$

Furthermore, denoting by $\widetilde{\mathbb{E}}$ the mean value under $\widetilde{P}$ we get

$$H(\widetilde{P}|P) = \widetilde{\mathbb{E}}\big[\log \frac{d\widetilde{P}}{dP}\big] = \widetilde{\mathbb{E}}\Big[\sum_{i=1}^{2} \int_0^T \lambda_i(t)\ \Big((1 + \widetilde{U}_t^i)\ \log(1 + \widetilde{U}_t^i) - \widetilde{U}_t^i\Big)\ dt\Big] < +\infty$$

since the integrand is bounded under the assumptions made in this note.
□

Remark 4.2 *Setting $\widetilde{\lambda}(t) = \widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)$, which is the intensity of N_t under $\widetilde{P}$, the function f_α defined in (3.10) can be also written as*

$$f_\alpha(t, z_1, z_2) = \frac{\lambda(t)}{\alpha} - \frac{\widetilde{\lambda}(t)}{\alpha}\ \exp\left\{-\alpha\ \frac{K_t^1 z_2 - K_t^2 z_1}{K_t}\right\}. \tag{4.4}$$

Theorem 4.3 *The duality principle (4.2) holds true. Furthermore, the measure Q^B solution to the dual problem has the density*

$$\frac{dQ^B}{dP}\bigg|_{\mathcal{F}_t} = \exp\left\{\sum_{i=1}^{2}\left(\int_0^t \log(1 + U_s^i(\pi^B))\ dN_s^i - \int_0^t \lambda_i(s)\ U_s^i(\pi^B)\ ds\right)\right\} \tag{4.5}$$

with, $U_s^1(\pi^B) = \exp\Big\{-\alpha\big(\pi_s^B K_s^1 - Z_s^{B,1}\big)\Big\} - 1,\ \ U_s^2(\pi^B) = \exp\Big\{\alpha\big(\pi_s^B K_s^2 - Z_s^{B,2}\big)\Big\} - 1.$

Proof.
Since, for $i = 1, 2$

$$1 + U_t^i(\pi^B) = (1 + \widetilde{U}_t^i)\ \exp\left\{-\frac{K_t^1 Z_t^{B,2} - K_t^2 Z_t^{B,1}}{K_t}\right\}$$

it is easy to verify that Q^B is a risk-neutral measure equivalent to P.

Next, following a method described in [12], we introduce the probability measure P^B equivalent to P defined by

$$\frac{dP^B}{dP} = C_B\ e^{\alpha B} \qquad \text{with} \qquad C_B = \frac{1}{\mathbb{E}[e^{\alpha B}]}$$

and the duality relation (4.2) becomes

$$\inf_{\Pi} \mathbb{E}^B\left[\exp\Big\{-\int_0^T \frac{\alpha\ \pi_t}{S_{t^-}}\ dS_t\Big\}\right] = exp\Big\{-\inf_{Q\in\mathcal{M}_f} H(Q|P^B).\Big\} \tag{4.6}$$

Let us note that we did not change the class $\mathcal{M}_f$ because the boundedness of B implies that

$$H(Q|P) < +\infty \quad \Longleftrightarrow \quad H(Q|P^B) < +\infty$$

being

$$H(Q|P) = H(Q|P^B) + E^Q[\alpha B] - \log E[e^{\alpha B}]. \tag{4.7}$$

The solution to the problem (4.6) is the minimal entropy martingale measure under P^B, whose existence and uniqueness is assured by Lemma 4.1 and by a result provided in [20], being the price process S_t locally bounded. We will prove that Q^B is the solution by using Proposition 3.2 in [23]. To this end, recalling (3.14) and (3.16) we have

$$\frac{dQ^B}{dP} = e^{-\alpha Y_0^B} M_t^{\pi^B} = e^{-\alpha Y_0^B} R_t^{\pi^B} = e^{-\alpha Y_0^B} e^{\alpha B} exp\left\{-\int_0^T \frac{\alpha\ \pi_t^B}{S_{t^-}}\ dS_t\right\} \tag{4.8}$$

and

$$\frac{dQ^B}{dP^B} = \frac{dQ^B}{dP}\ \frac{dP}{dP^B} = e^{-\alpha Y_0^B}\ E[e^{\alpha B}] exp\left\{-\int_0^T \frac{\alpha\ \pi_t^B}{S_{t^-}}\ dS_t\right\}.$$

Hence, $H(Q^B|P^B) = \log E[e^{\alpha B}] - \alpha Y_0^B$, and by (4.7), $H(Q^B|P) = E^{Q^B}[\alpha B] - \alpha Y_0^B$. Inserting this value in (4.2), we realize the equality and this in turn implies the validity of duality relation (4.2).
□

Remark 4.4 *A different version of the Duality Principle is given in [4]. There the relation (4.6) is provided by introducing a set Π^B defined by means of loss random variables. Let us denote by $\mathcal{D}$ the set of loss random variables $D \geq 1$, $P-a.s.$ verifying*

$$\begin{array}{ll} (i) & \exists\pi:\ \pi_t \neq 0,\ P-a.s.\ \forall t\in[0,T]\ ;\quad \dfrac{\pi_t}{S_{t^-}} \in \Pi_0\ ,\ \textit{and}\quad \left|\displaystyle\int_0^t \frac{\pi_r}{S_{r^-}}\ dS_r\right| \leq D \\ (ii) & E[e^{cD}] < +\infty \qquad \forall c>0, \end{array}$$

and let us consider the following class of admissible strategies

$$\Pi^B = \left\{\pi\in\Pi_0 \textit{ such that exist } D\in\mathcal{D},\ c\geq 0:\ \int_0^T \frac{\pi_r}{S_{r^-}}\ dS_r \geq -cD\right\}.$$

It easy to see that $\Pi \subseteq \Pi^B$. In fact setting $D = \overline{\pi}\overline{K}\ N_T + 1$ we have, for any $\pi\in\Pi$

$$\left|\int_0^T \frac{\pi_r}{S_{r^-}}\ dS_r\right| \leq D.$$

The result given in the Theorem 4.3 allows us to claim that the infimum over Π and that one over Π^B in the l.h.s. of (4.6) coincide.

As a consequence we find the density of the minimal entropy martingale measure under P. This result can be seen as the dual problem of the utility maximization one presented in (3.19), with $\alpha = 1$.

Corollary 4.5 *The minimal entropy martingale measure for the model discussed in this note is defined by the density*

$$\frac{dQ^*}{dP} = e^{-\widehat{Y}_0^0}\ exp\left\{-\int_0^T \frac{\pi_t^*}{S_{t^-}}\ dS_t\right\}. \tag{4.9}$$

where

$$\pi_t^* = \pi_t^0\Big|_{\alpha=1} = \frac{1}{K_t}\left\{\log\left(\frac{\lambda_1(t)K_t^1}{\lambda_2(t)K_t^2}\right) + \widehat{Z}_t^{0,1} + \widehat{Z}_t^{0,2}\right\} \tag{4.10}$$

and $(\widehat{Y}^0_t, \widehat{Z}^{0,1}_t, \widehat{Z}^{0,2}_t)$ *is the solution to (3.22) for* $\alpha = 1$. *Moreover* $H(Q^*|P) = -\widehat{Y}^0_0$ *and the intensities of* N^i_t, $i = 1, 2$ *under* Q^*, *are given by* $\lambda^*_i(t) = (1 + U^i_t(\pi^*))\ \lambda_i(t)$ *for* $\alpha = 1$, *that is*

$$\lambda^*_1(t) = \exp\Big\{ -\big(\pi^*_t K^1_t - \widehat{Z}^{0,1}_t\big)\Big\}\ \lambda_1(t) = \lambda_1(t)\ \left(\frac{\lambda_1(t) K^1_t}{\lambda_2(t) K^2_t}\right)^{-\frac{K^1_t}{K_t}} \exp\left\{ \frac{K^2_t\ \widehat{Z}^{0,1}_t - K^1_t\ \widehat{Z}^{0,2}_t}{K_t}\right\} \tag{4.11}$$

$$\lambda^*_2(t) = \exp\Big\{ \big(\pi^*_t K^2_t - \widehat{Z}^{0,2}_t\big)\Big\}\ \lambda_2(t) = \lambda_2(t)\ \left(\frac{\lambda_1(t) K^1_t}{\lambda_2(t) K^2_t}\right)^{\frac{K^2_t}{K_t}} \exp\left\{ \frac{K^2_t\ \widehat{Z}^{0,1}_t - K^1_t\ \widehat{Z}^{0,2}_t}{K_t}\right\}.$$

Proof.
By (4.8), for $\alpha = 1$ and $B = 0$, (4.9) follows. The last assertion is a standard consequence of the Girsanov Theorem.
□

Remark 4.6 *In a particular case, similar to that studied in [9, 10], a restrictive hypothesis on the model consists in assuming the existence of a deterministic function* $\Gamma(t)$ *such that*

$$\frac{K^1_t}{K^2_t} = \Gamma(t). \tag{4.12}$$

In this case we are able to give an explicit expression of the value process (3.24).

First, we observe that, as a consequence of (4.12), the intensity $\widetilde{\lambda}$ *is a deterministic function of* t, *and the function* f_α *given in (4.4),*

$$f_\alpha(t, z_1, z_2) = \frac{\lambda(t)}{\alpha} - \frac{\widetilde{\lambda}(t)}{\alpha}\ \exp\left\{\alpha\ \frac{z_1 - \Gamma(t) z_2}{1 + \Gamma(t)}\right\},$$

is a deterministic function of (t, z_1, z_2). *This allows us to claim that the unique bounded solution to the equation (3.22) is given by* $(Y^0, 0, 0)$, *with*

$$Y^0_t = -\frac{1}{\alpha}\int_t^T \big(\lambda(s) - \widetilde{\lambda}(s)\big)\ ds.$$

Thus, the optimal control (see (3.23)) is

$$\pi^0_t = \frac{1}{\alpha K_t}\ \log\left(\frac{\lambda_1(t)}{\lambda_2(t)}\ \Gamma(t)\right)$$

and the value process is a deterministic function given by

$$V^0_t(x) = e^{-\alpha x} \exp\left\{ -\int_t^T \big(\lambda(s) - \widetilde{\lambda}(s)\big)\ ds\right\}.$$

More, by Corollary 4.9 the probability measure $\widetilde{P}$ *coincides with the MEMM.*
If in addition $\lambda_1(t), \lambda_2(t)$ *and* K^i_t, $i = 1, 2$, *are constant we get that* π^0_t *does not depend on time, hence the optimal cash amount invested in the stock is constant as in the Merton model for exponential utility.*

5. Indifference valuation

In this section we introduce the notion of the utility indifference price and hedging strategy for the contingent claim B. The utility indifference value p^α_t process for B is defined at any time $t \in [0, T]$ as the implicit solution to the equation

$$V^0_t(x) = V^B_t(x + p^\alpha_t).$$

This means that starting with the capital x one has the same maximal utility from solely trading on $(t,T]$, as from selling the claim at time t for p_t^α again trading and then paying out B at time T.
The utility indifference hedging strategy Ψ_t^α is defined as the difference of the respective optimal investment strategies

$$\Psi_t^\alpha = \pi_t^B - \pi_t^0.$$

By Theorem (3.6) and Theorem (3.9), $V_t^B(x) = -e^{\alpha(Y_t^B - x)}$ and $V_t^0(x) - e^{\alpha(Y_t^0 - x)}$ where $(Y^B, Z^{B,1}, Z^{B,2})$ and $(Y^0, Z^{0,1}, Z^{0,2})$ are the bounded solutions of the BSDE (3.9) and (3.22), respectively. Then p_t^α does not depend on x and

$$p_t^\alpha = Y_t^B - Y_t^0, \quad \Psi_t^\alpha = \frac{1}{K_t}(Z_t^{B,1} - Z_t^{0,1} + Z_t^{B,2} - Z_t^{0,2}). \tag{5.1}$$

In the next Proposition we will prove that p_t^α is the unique solution to a BSDE under the MEMM Q^*. By this BSDE description we will able to prove that p_t^α converges to $\mathbb{E}^*[B|\mathcal{F}_t]$ as the risk aversion parameter α goes to zero and that the hedging strategies Ψ_t^α converge to the strategy Ψ_t^* which is risk-minimizing in the sense of Follerman and Sondermann, [17], under the MEMM.
These results have been proved for a Brownian filtration in [16], for a general underlying continuous filtration in [27] and in [2] for a noncontinuous filtration (generated by a Brownian motion and an integer-valued random measure) but always for continuous underlying assets.

Proposition 5.1 *Let $(\widetilde{Y}_t^\alpha, \widetilde{Z}_t^{\alpha,1}, \widetilde{Z}_t^{\alpha,2})$ be the unique bounded solution to the following BSDE under Q^**

$$\widetilde{Y}_t^\alpha = B - \int_t^T \widetilde{Z}_s^{\alpha,1}\, dN_s^1 + \int_t^T \widetilde{Z}_s^{\alpha,2}\, dN_s^2 + \int_t^T \frac{\lambda^*(s)}{\alpha}\Big(\exp\big\{-\alpha\ \frac{K_s^1 \widetilde{Z}_s^{\alpha,2} - K_s^2 \widetilde{Z}_s^{\alpha,1}}{K_s}\big\} - 1\Big)ds \tag{5.2}$$

where $\lambda^*(s) = \lambda_1^*(s) + \lambda_2^*(s)$ is the intensity of N_t under Q^* (see (4.11)).
Thus, the exponential utility indifference value process p_t^α coincides with $\widetilde{Y}_t^\alpha$, and $|p_t^\alpha| \le |B|$. Moreover the indifference hedging strategy is given by

$$\Psi_t^\alpha = \frac{1}{K_t}(\widetilde{Z}_t^{\alpha,1} + \widetilde{Z}_t^{\alpha,2}) \tag{5.3}$$

and $\widetilde{Z}_t^i = Z_t^{B,i} - Z_t^{0,i}$, i=1,2.

Proof.
By Proposition 3.8, $p_t^\alpha = Y_t^B - Y_t^0 = \frac{1}{\alpha}(\widehat{Y}_t^B - \widehat{Y}_t^0)$, where $(\widehat{Y}_t^B, \widehat{Z}_t^{B,1}, \widehat{Z}_t^{B,2})$ and $(\widehat{Y}_t^0, \widehat{Z}_t^{0,1}, \widehat{Z}_t^{0,2})$ are the solutions to the BSDE (3.18) with terminal data αB and 0, respectively.
Denote by $\delta\widehat{Y}_t = \widehat{Y}_t^B - \widehat{Y}_t^0$ and $\delta\widehat{Z}_t^i = \widehat{Z}_t^{B,i} - \widehat{Z}_t^{0,i}$, $i = 1, 2$. Since, by (4.11)

$$f_1(t, \widehat{Z}_t^{B,1}, \widehat{Z}_t^{B,2}) - f_1(t, \widehat{Z}_t^{0,1}, \widehat{Z}_t^{0,2}) =$$

$$= -\lambda_1(t)\frac{K_t}{K_t^2}\left(\frac{\lambda_1(t)K_t^1}{\lambda_2(t)K_t^2}\right)^{-\frac{K_t^1}{K_t}} \exp\big\{\frac{K_t^2 \widehat{Z}_t^{0,1} - K_t^1 \widehat{Z}_t^{0,2}}{K_t}\big\}\left(\exp\big\{\frac{K_t^2 \delta\widehat{Z}_t^1 - K_t^1 \delta\widehat{Z}_t^2}{K_t}\big\} - 1\right) =$$

$$-\frac{K_t}{K_t^2}\lambda_1^*(t)\left(\exp\big\{\frac{K_t^2 \delta\widehat{Z}_t^1 - K_t^1 \delta\widehat{Z}_t^2}{K_t}\big\} - 1\right) = -\lambda^*(t)\left(\exp\big\{\frac{K_t^2 \delta\widehat{Z}_t^1 - K_t^1 \delta\widehat{Z}_t^2}{K_t}\big\} - 1\right),$$

where the last equality is a consequence of the risk neutrality of Q^*

$$\lambda_1^*(t)\ K_t^1 - \lambda_2^*(t)\ K_t^2 = 0 \quad P \times dt - a.e. \tag{5.4}$$

we get that

$$\frac{1}{\alpha}\Big(f_1(t, \widehat{Z}_t^{B,1}, \widehat{Z}_t^{B,2}) - f_1(t, \widehat{Z}_t^{0,1}, \widehat{Z}_t^{0,2})\Big) = -\frac{\lambda^*(t)}{\alpha}\Big(\exp\big\{-\alpha\frac{K_t^1 \frac{\delta\widehat{Z}_t^2}{\alpha}{}_t - K_t^2 \frac{\delta\widehat{Z}_t^1}{\alpha}}{K_t}\big\} - 1\Big)$$

and

$$p_t^\alpha = B - \int_t^T \frac{\delta \widehat{Z}_s^1}{\alpha} \, dN_s^1 + \int_t^T \frac{\delta \widehat{Z}_s^2}{\alpha} dN_s^2 + \int_t^T \frac{\lambda^*(s)}{\alpha} \Big(\exp\big\{ -\alpha \, \frac{K_s^1 \frac{\delta \widehat{Z}_s^1}{\alpha} - K_s^2 \frac{\delta \widehat{Z}_s^2}{\alpha}}{K_s} \big\} - 1 \Big) ds. \tag{5.5}$$

Taking into account that $\delta Z_t^i = Z_t^{B,i} - Z_t^{0,i} = \frac{1}{\alpha} \delta \widehat{Z}_t^i$ we have that

$$p_t^\alpha = B - \int_t^T \delta Z_s^1 \, dN_s^1 + \int_t^T \delta Z_s^2 dN_s^2 + \int_t^T \frac{\lambda^*(s)}{\alpha} \Big(\exp\big\{ -\alpha \, \frac{K_s^1 \delta Z_s^1 - K_s^2 \delta Z_s^2}{K_s} \big\} - 1 \Big) ds. \tag{5.6}$$

□

Remark 5.2 *Comparing the result given in the previous Proposition with that one given in Theorem 3.6 we get that*

$$e^{\alpha p_t^\alpha} = ess \inf_{\pi \in \Pi_t} \mathbb{E}^* \left[\exp\left\{ -\alpha \left(\int_t^T \frac{\pi_r}{S_{r^-}} \, dS_r - B \right) \right\} \middle| \, \mathcal{F}_t \right]. \tag{5.7}$$

Here and in what follows $\mathbb{E}^*$ denotes the expectation w.r.t. the minimal entropy martingale measure. Furthermore the optimal control of this exponential utility optimization problem with respect to the MEMM, coincides with the indifference hedging ψ_t^α given in (5.3).

Always in order to prove the mentioned convergence results we need some additional preliminaries.
Since S_t is a locally bounded $(Q^*, \mathcal{F}_t)$-martingale we can apply the Kunita-Watanabe decomposition

$$B = \mathbb{E}^*(B) + \int_0^T \Psi_r^* dS_r + L_T \tag{5.8}$$

where Ψ_t^* is a $(Q^*, \mathcal{F}_t)$-predictable process such that $\mathbb{E}^*(\int_0^T (\Psi_r^*)^2 d\langle S \rangle_r) < +\infty$ and L_T is a square-integrable $(Q^*, \mathcal{F}_t)$-martingale orthogonal to S_t. The integrand, Ψ_t^*, in the Kunita-Watanabe decomposition is risk-minimizing in the sense of Follerman and Sondermann [17] with respect to Q^*. The next Lemma provides a representation of the process Ψ_t^*.

Lemma 5.3 *Let $(Y_t^*, Z_t^{*,1}, Z_t^{*,2})$ be the unique bounded solution to the BSDE*

$$Y_t^* = B - \int_t^T Z_s^{*,1} \, (dN_s^1 - \lambda_1^*(s) ds) + \int_t^T Z_s^{*,2} \, (dN_s^2 - \lambda_2^*(s) ds). \tag{5.9}$$

Then

$$\Psi_t^* = \frac{1}{S_{t^-}} \frac{Z_t^{*,1} + Z_t^{*,2}}{K_t} \qquad P \times dt - a.s. \tag{5.10}$$

Proof.
By (5.8) we deduce that

$$\mathbb{E}^*[B|\mathcal{F}_t] = \mathbb{E}^*(B) + \int_0^t \Psi_r^* dS_r + L_t = B - \int_t^T \Psi_r^* dS_r - (L_T - L_t). \tag{5.11}$$

Let us now observe that in our framework L_t can be written as

$$L_t = \int_0^t L_s^1 \, (dN_s^1 - \lambda_1^*(s) ds) + \int_0^t L_s^2 \, (dN_s^2 - \lambda_2^*(s) ds)$$

where L_t^i are $(Q^*, \mathcal{F}_t)$-predictable processes such that $\mathbb{E}^*(\int_0^T (L_r^i)^2 dr) < +\infty$. Moreover, since L_t is orthogonal to S_t, and since the MEMM is a risk-neutral probability measure, we get that for any $t \in (0, T]$

$$dS_t = S_{t^-} \Big(K_t^1 \, (dN_t^1 - \lambda_1^*(t) \, dt) - K_t^2 \, (dN_t^2 - \lambda_2^*(t) \, dt) \Big),$$

$$0 = \langle L, S \rangle_t = \int_0^t S_{r^-} \left(L_r^1 K_r^1 \lambda_1^*(r) - L_r^2 K_r^2 \lambda_2^*(r) \right) dr$$

which implies together with the risk neutrality condition (5.4) that $P \times dt - a.e.$

$$L_t^1 = L_t^2 = \frac{1}{K_t} \Big(K_t^2 Z_t^{*,1} - K_t^1 Z_t^{*,2} \Big)$$

and

$$L_t = \int_0^t \frac{1}{K_r} \Big(K_r^2 Z_r^{*,1} - K_r^1 Z_r^{*,2} \Big) \; (dN_r - \lambda^*(r) \; dr).$$

Replacing in (5.11) we obtain

$$I\!E^*[B|\mathcal{F}_t] = B - \int_t^T \{\Psi_r^* S_{r^-} K_r^1 + L_r^1\}(dN_r^1 - \lambda_1^*(r)dr) + \int_t^T \{\Psi_r^* S_{r^-} K_r^2 - L_r^1\}(dN_r^2 - \lambda_2^*(r)dr). \tag{5.12}$$

On the other hand, the unique bounded solution to (5.9) is such that $Y_t^* = I\!E^*[B|\mathcal{F}_t]$.
By a comparison between (5.9) and (5.12) we finally obtain (5.10).
□

In the next Proposition we will prove the convergence, for vanishing risk aversion, of the exponential utility indifference value to the MEMM price and of the exponential utility indifference hedging strategy to the Q^*-risk-minimizing strategy. This means, loosely speaking that, in small risk aversion limit, exponential indifference hedging converges to risk-minimization under the MEMM.

Lemma 5.4 *Let $(\widetilde{Y}_t^\alpha, \widetilde{Z}_t^{\alpha,1}, \widetilde{Z}_t^{\alpha,2})$ be the unique bounded solution to BSDE (5.2) and $(Y_t^*, Z_t^{*,1}, Z_t^{*,2})$ be the unique bounded solution to the BSDE (5.9).
Then there is a constant $C > 0$ such that, for all $\alpha \in (0, 1]$*

$$I\!E^*[\sup_{t\in[0,T]} |\widetilde{Y}_t^\alpha - Y_t^*|^2 + \int_0^T |\widetilde{Z}_t^{\alpha,1} - Z_t^{*,1}|^2 dt + \int_0^T |\widetilde{Z}_t^{\alpha,1} - Z_t^{*,1}|^2 dt] \leq \alpha^2 C. \tag{5.13}$$

Proof.
To prove (5.13) we apply the estimate (2.5)

$$I\!E^*[\sup_{t\in[0,T]} |\widetilde{Y}_t^\alpha - Y_t^*|^2 + \int_0^T |\widetilde{Z}_t^{\alpha,1} - Z_t^{*,1}|^2 dt + \int_0^T |\widetilde{Z}_t^{\alpha,2} - Z_t^{*,2}|^2 dt] \leq C I\!E^*\big[\int_0^T |\delta f(t, \widetilde{Z}_t^{\alpha,1}, \widetilde{Z}_t^{\alpha,2})|^2 dt\big]$$

where

$$\delta f(t, \widetilde{Z}_t^{\alpha,1}, \widetilde{Z}_t^{\alpha,2}) = -\frac{\lambda^*(t)}{\alpha} \Big(\exp\big\{ - \alpha \; \frac{K_t^1 \widetilde{Z}_t^{\alpha,1} - K_s^2 \widetilde{Z}_t^{\alpha,2}}{K_t} \big\} - 1 \Big) + \lambda_1^*(t) \widetilde{Z}_t^{\alpha,1} - \lambda_2^*(t) \widetilde{Z}_t^{\alpha,2}.$$

By the risk neutrality condition (5.4) we have that

$$\lambda_1^*(t) = \frac{K_t^2}{K_t} \lambda^*(t), \qquad \lambda_2^*(t) = \frac{K_t^1}{K_t} \lambda^*(t).$$

Since $\widetilde{Z}_t^{\alpha,i}$, $i = 1, 2$ are bounded uniformly in α, more precisely $|\widetilde{Z}_t^{\alpha,i}| \leq 2C_1$, $i = 1, 2$, (see (2.14)), finally we get that

$$|\delta f(t, Z_t^{\alpha,1}, Z_t^{\alpha,2})|^2 \leq const. \alpha^2. \tag{5.14}$$

and the thesis.
□

Proposition 5.5 *For vanishing risk aversion we get*

$$\sup_{t\in[0,T]} |p_t^\alpha - I\!E^*[B|\mathcal{F}_t]|^2 \leq \alpha^2 C \quad \alpha \in (0, 1], \qquad \textit{and} \qquad \lim_{\alpha\to 0} \sup_{t\in[0,T]} |p_t^\alpha - I\!E^*[B|\mathcal{F}_t]| = 0 \;\; \textit{in } L^\infty.$$

Furthermore

$$\lim_{\alpha\to 0}\int_0^t \frac{\Psi_r^\alpha}{S_{r^-}} dS_r = \int_0^t \Psi_r^* dS_r \quad \text{in } \mathcal{H}^2(Q^*).$$

Here $\mathcal{H}^2(Q^*)$ denotes the space of $(Q^*, \mathcal{F}_t)$-square integrable martingales.

Proof.
By (5.2) and (5.9) we can write

$$p_t^\alpha - I\!E^*[B|\mathcal{F}_t] = \widetilde{Y}_t^\alpha - Y_t^* = I\!E^*\left[\int_t^T \delta f(r, \widetilde{Z}_r^{\alpha,1}, \widetilde{Z}_r^{\alpha,2})\ dr \mid \mathcal{F}_t\right]$$

Thus, by (5.14), $\forall t \in [0,T]$,

$$|p_t^\alpha - I\!E^*[B|\mathcal{F}_t]| \le I\!E^*\left[\int_t^T |\delta f(r, \widetilde{Z}_r^{\alpha,1}, \widetilde{Z}_r^{\alpha,2})|\ dr \mid \mathcal{F}_t\right] \le const.\alpha$$

and we obtain the first claim. More, $\forall t \in [0,T]$,

$$|p_t^\alpha - I\!E^*[B|\mathcal{F}_t]|^2 \le T\ I\!E^*\left[\int_t^T |\delta f(r, \widetilde{Z}_r^{\alpha,1}, \widetilde{Z}_r^{\alpha,2})|^2\ dr \mid \mathcal{F}_t\right] \le const.\alpha^2.$$

Finally, recalling (5.3)

$$\lim_{\alpha\to 0}\int_0^t \frac{\Psi_r^\alpha}{S_{r^-}} dS_r = \int_0^t \Psi_r^* dS_r \quad \text{in } \mathcal{H}^2(Q^*)$$

is a consequence of the previous Lemma,

$$I\!E^*\left[\int_0^t \left|\frac{\Psi_r^\alpha}{S_{r^-}} - \Psi_r^*\right|^2 d\langle S\rangle_r\right] \le I\!E^*\left[\int_0^t \frac{(K_r^1)^2\lambda_1^*(r) + (K_r^2)^2\lambda_2^*(r)}{K_r^2}\left\{\left|\widetilde{Z}_r^{\alpha,1} - Z_r^{*,1}\right|^2 + \left|\widetilde{Z}_r^{\alpha,2} - Z_r^{*,2}\right|^2\right\} dr\right] \le const.\alpha^2$$

since, under Q^*

$$d\langle S\rangle_r = S_{r^-}^2\left\{(K_r^1)^2\lambda_1^*(r) + (K_r^2)^2\lambda_2^*(r)\right\}dr.$$

□

6. Markovian case

In this section we consider a Markovian setting. More precisely we assume that the dynamics of the traded stock price is given by

$$dS_t = S_{t^-}\Big(K^1(t, S_{t^-}, Z_{t^-})dN_t^1 - K^2(t, S_{t^-}, Z_{t^-})\ dN_t^2\Big), \tag{6.1}$$

with $S_0 = s_0 \in I\!R^+$, $K^i(t,y,z), i = 1,2$, jointly measurable and positive functions, $K^2(t,y,x) < 1$.
Here Z_t is an $\mathcal{F}_t$-adapted marked point process which may be considered as a stochastic factor as, for instance, in [34], describing the amount of information received by the traders related to intraday market activity, the activity of other markets, macroeconomics factors or microstructure rules. Alternatively, it may represents the level of a nontradable asset as in [11, 30, 1]. In this latter situation the agent expects to receive or pay out the claim depending on the nontradable asset and trades on the correwlated asset S to manage his risk. Examples include option on basket of stocks where the basket is illiquid, executive stock options and weather derivatives.

We assume that the process Z_t is the solution to

$$dZ_t = H^1(t, Z_{t^-})dN_t^1 - H^2(t, Z_{t^-})\ dN_t^2, \tag{6.2}$$

with $Z_0 = z_0 \in I\!R$, and $H^i(t,z), i = 1,2$, bounded jointly measurable functions.

In this model S and Z are correlated since common jump times are allowed. More precisely, the quadratic variation of S and Z is given by

$$[S,Z]_t = \int_0^t S_{r^-}\,\left[K^1(t,S_{t^-},Z_{t^-})H^1(t,Z_{t^-}) + K^2(t,S_{t^-},Z_{t^-})H^2(t,Z_{t^-})\right]\,dr$$

and, for $H^1,\ H^2$ both nonnegative, $[S,Z]_t \geq 0$, while for $H^1,\ H^2$ both nonpositive, $[S,Z]_t \leq 0$. Both cases make sense from an economic point of view.

We consider a contingent claim with the payoff at time T of the form $B = B(S_T, Z_T)$, where $B(y,z)$ is a measurable and bounded function. As in Section 3 the agent's objective is to maximize his expected utility from terminal wealth given in (3.3)

$$\mathbb{E}\Big[-\exp\big\{-\alpha(X_T - B)\big\}\Big] = \mathbb{E}\Big[-\exp\big\{-\alpha(x_0 + \int_0^T \frac{\pi_r}{S_{r^-}} dS_r \ - B)\big\}\Big].$$

Since we are in a markovian setting we introduce the value function

$$v^B(t,x,y,z) = \sup_{\pi\in\Pi_t} \mathbb{E}\Big[-\exp\big\{-\alpha(X_T - B)\big\} \mid X_t = x, S_t = y, Z_t = z\Big] = -e^{-\alpha x}\ w^B(t,y,z) \tag{6.3}$$

where

$$w^B(t,y,z) = \inf_{\pi\in\Pi_t} \mathbb{E}\Big[\exp\big\{-\alpha\big(\int_t^T \frac{\pi_r}{S_{r^-}} dS_r \ - B)\big\}\Big) \mid S_t = y, Z_t = z\Big]. \tag{6.4}$$

Clearly,

$$v^B(t,x,S_t,Z_t) = V_t^B(x), \quad w^B(t,S_t,Z_t) = W_t^B.$$

Proposition 6.1 *Let $(Y^B, Z^{B,1}, Z^{B,2})$ be the unique bounded solution to the BSDE (3.9)*

$$Y_t^B = B - \int_t^T Z_s^{B,1}\ dN_s^1 + \int_t^T Z_s^{B,2}\ dN_s^2 - \int_t^T f_\alpha(s, Z_s^{B,1}, Z_s^{B,2})\ ds$$

with the function $f_\alpha(t,z_1,z_2)$ given by (4.4)

$$f_\alpha(t,z_1,z_2) = \frac{\lambda(t)}{\alpha} - \frac{\widetilde{\lambda}(t)}{\alpha}\ \exp\left\{-\alpha\ \frac{K^1(t,S_{t^-},Z_{t^-})z_2 - K^2(t,S_{t^-},Z_{t^-})z_1}{K(t,S_{t^-},Z_{t^-})}\right\}. \tag{6.5}$$

In the setting of this section we obtain the markovian property of $(Y^B, Z^{B,1}, Z^{B,2})$. This means that there exist measurable functions $u^B(t,y,z)$, $d_1^B(t,y,z)$, $d_2^B(t,y,z)$ such that

$$Y_t^B = u^B(t,S_t,Z_t), \qquad Z_t^{B,1} = d_1^B(t,S_{t^-},Z_{t^-}), \qquad Z_t^{B,2} = d_2^B(t,S_{t^-},Z_{t^-}).$$

The optimal control (3.11) is markovian and given by $\pi^B(t,S_{t^-},Z_{t^-})$ where

$$\pi^B(t,y,z) = \frac{1}{K(t,y,z)}\left\{\frac{1}{\alpha}\ \log\left(\frac{\lambda_1(t)K^1(t,y,z)}{\lambda_2(t)K^2(t,y,z)}\right) + d_1^B(t,y,z) + d_2^B(t,y,z)\right\}, \tag{6.6}$$

and the value function is

$$v^B(t,x,y,z) = -e^{\alpha(u^B(t,y,z)-x)}. \tag{6.7}$$

Proof.
By Theorem 3.6 the value process is given by $V_t^B(x) = -e^{\alpha(Y_t^B - x)}$, hence

$$Y_t^B = \frac{1}{\alpha}\ \log W_t^B = \frac{1}{\alpha}\ \log w^B(t,S_t,Z_t) = u^B(t,S_t,Z_t),$$

with

$$u^B(t,y,z) = \frac{1}{\alpha}\ \log w^B(t,y,z).$$

Moreover, setting

$$
\begin{aligned}
d_1^B(t,y,z) &= u^B\big(t, y(1+K^1(t,y,z)), z+H^1(t,z)\big) - u^B(t,y,z) \\
d_2^B(t,y,z) &= -u^B\big(t, y(1-K^2(t,y,z)), z-H^2(t,z)\big) + u^B(t,y,z),
\end{aligned}
\tag{6.8}
$$

the processes

$$
d_i^B(t, S_{t^-}, Z_{t^-}), \quad i=1,2,
$$

verify (2.6) in Lemma 2.4 and are predictable, thus

$$
Z_t^{B,i} = d_i^B(t, S_{t^-}, Z_{t^-}), \quad i=1,2.
$$

Finally (6.6) is a consequence of Theorem 3.6.
□

The Markov property of the solution to BSDEs with Markovian coefficients has been proved in a Brownian filtrations setting under standard assumptions in [15] and in the case of generators with quadratic growth in [1].

Remark 6.2 *Let us observe that the BSDE approach allows us to solve the situation described in Remark 3.2 in [11] in the particular case where the diffusive component of the nontradable asset dynamics is equal to zero.*

In what follows, we deal with the control problem we are discussing in this Section by the classical approach that consists in writing down the Hamilton-Jacobi- Bellman equation. In [11] the assumption on the model allowed us to deduce a linear equation providing an explicit expression of the value function by Feynman-Kac formula and of the optimal control. This is not the case of the model described by (6.1) and (6.2). Nevertheless useful results can be obtained, that is an implicit definition of the function $u(t,y,z)$.

First, let us observe that since $\widetilde{\lambda}(t)$ is a markovian process also the function $f_\alpha(t,z_1,z_2)$ in (6.5) is markovian, that is we can define a measurable deterministic function $g_\alpha(t,y,z,z_1,z_2)$ such that

$$
f_\alpha(t,z_1,z_2) = g_\alpha(t, S_{t^-}, Z_{t^-}, z_1, z_2).
$$

Proposition 6.3 *If the function $u^B(t,y,z)$ is absolutely continuous w.r.t. t, it is defined implicitly by the equation*

$$
u^B(t,y,z) = B(y,z) - \int_t^T g_\alpha\big(s,y,z,d_1^B(s,y,z), d_2^B(s,y,z)\big)\, ds \tag{6.9}
$$

where $d_i^B(s,y,z)$, $i=1,2$, are defined in (6.8).

Proof.
For a constant π, the process (X_t, S_t, Z_t) is a Markov process with a generator that for a bounded measurable function $f(t,x,y,z)$ is given by

$$
\begin{aligned}
Lf(t,x,y,z) &= \frac{\partial}{\partial t} f(t,x,y,z) + L_t f(t,x,y,z) \\
L_t f(t,x,y,z) &= \lambda_1(t)\Big[f\big(t, x+\pi K^1(t,y,z), y(1+K^1(t,y,z)), z+H^1(t,y,z)\big) - f(t,x,y,z)\Big] + \\
&+ \lambda_1(t)\Big[f\big(t, x-\pi K^2(t,y,z), y(1+K^2(t,y,z)), z-H^2(t,y,z)\big) - f(t,x,y,z)\Big]
\end{aligned}
$$

thus the Hamilton-Jacobi- Bellman equation that the value function satisfies is

$$
\sup_\pi \Big\{ \frac{\partial}{\partial t} v^B(t,x,y,z) + L_t v^B(t,x,y,z) \Big\} = 0, \quad \forall t \in [0,T), \qquad v^B(T,x,y,z) = -e^{-\alpha x}\ e^{\alpha B(y,z)}. \tag{6.10}
$$

Since $v^B(t,x,y,z) = -e^{-\alpha x}\ e^{\alpha u^B(t,y,z)}$, replacing this expression in (6.10) we obtain that $u^B(t,y,z)$ solves $\forall t \in [0,T)$

$$\begin{aligned}\inf_{\pi} \Big\{ \alpha \frac{\partial}{\partial t} u^B(t,y,z) \quad &+ \quad \lambda_1(t) \exp\big\{ -\alpha\pi K^1(t,y,z) + \alpha d_1^B(t,y,z)\big\} \\ &+ \quad \lambda_2(t) \exp\big\{\alpha\pi K^2(t,y,z) - \alpha d_2^B(t,y,z)\big\} - \lambda(t)\Big\} = 0\end{aligned} \tag{6.11}$$

with the final condition $u^B(T,y,z) = B(y,z)$.
By the latter equation it is easy to find that the infimum in (6.11) is achieved in $\pi^B(t,y,z)$ defined in (6.6) (compare with (3.11)). Finally, inserting this expression in (6.11), we get that $u^B(t,y,z)$ solves the following nonlinear equation

$$\frac{\partial}{\partial t} u^B(t,y,z) - g_\alpha\big(t,y,z,d_1^B(t,y,z), d_2^B(t,y,z)\big) = 0, \quad \forall t \in [0,T), \qquad u^B(T,x,y,z) = B(y,z). \tag{6.12}$$

□

The Hamilton-Jacobi-Bellman approach under a nontrivial assumption could provide the following verification result. Moreover the BSDE could be derived by Ito Formula.

Corollary 6.4 *If there exists a bounded measurable function $u^B(t,y,z)$ verifying (6.9), then the value function is*

$$v^B(t,x,y.z) = -e^{-\alpha x}\ e^{\alpha u^B(t,y,z)} \tag{6.13}$$

and an optimal feedback control is given by (6.6). The process $(u^B(t,S_t,Z_t), d_1^B(t,S_{t^-},Z_{t^-}), d_2^B(t,S_{t^-},Z_{t^-}))$, with $d_i^B(t,y,z),\ i=1,2$, defined in (6.8), is a bounded solution to the BSDE

$$\begin{aligned}u^B(t,S_t,Z_t) \quad &= \quad B(S_t,Z_t) - \int_t^T d_1^B(r,S_{r^-},Z_{r^-})\ dN_r^1 + \int_t^T d_2^B(r,S_{r^-},Z_{r^-})\ dN_r^2 \\ &- \quad \int_t^T g_\alpha\big(r,S_r,Z_r,d_1^B(r,S_r,Z_r),d_2^B(r,S_r,Z_r)\big)\ ds.\end{aligned}$$

Proof.
By a verification result, the function defined in (6.13) is a solution to the HJB-equation (6.10), then it coincides with the value function. Next, by the Ito Formula,

$$\begin{aligned}u^B(t,S_t,Z_t) \quad &= \quad u^B(0,S_0,Z_0) + \int_0^t \frac{\partial}{\partial r} u^B(r,S_r,Z_r)\ dr + \sum_{0<r\leq t}\Big(u^B(r,S_r,Z_r) - u^B(r,S_{r^-},Z_{r^-})\Big)\ 1\!\!1_{\Delta N_r^1 \neq 0} + \\ &+ \quad \sum_{0<r\leq t}\Big(u^B(r,S_r,Z_r) - u^B(r,S_{r^-},Z_{r^-})\Big)\ 1\!\!1_{\Delta N_r^2 \neq 0}\end{aligned}$$

taking into account (6.9) and the definition of the functions $d_i,\ i=1,2$, the conclusion.
□

Remark 6.5 *Let us observe that the reduced HJB-equation (6.12) can be written as*

$$\frac{\partial}{\partial t} u^B(t,y,z) - L_t u^B(t,y,z) - \widetilde{g}_\alpha\big(t,y,z,d_1^B(t,y,z),d_2^B(t,y,z)\big) = 0, \quad \forall t \in [0,T), \qquad u^B(T,x,y,z) = B(y,z),$$

where L_t denotes the generator of the pair (S,Z) and

$$\widetilde{g}_\alpha\big(t,y,z,d_1^B(t,y,z),d_2^B(t,y,z)\big) = g_\alpha\big(t,y,z,d_1^B(t,y,z),d_2^B(t,y,z)\big) + \lambda_1(t) d_1^B(t,y,z) - \lambda_2(t) d_2^B(t,y,z).$$

By the BSDE representation of $u^B(t,S_t,Z_t)$ we get the following generalized Feynman-Kac formula

$$u^B(t,y,z) = I\!\!E[B(S_T,Z_T) - \int_t^T \widetilde{g}_\alpha\big(r,S_r,Z_r,d_1^B(r,S_r,Z_r),d_2^B(r,S_r,Z_r)\big) dr \mid S_t = y, Z_t = z].$$

Proposition 6.1 implies, moreover, the Markovian property of the utility indifference price and the indifference hedging strategy, whose expression has been found in (5.1) that here recall

$$p_t^\alpha = Y_t^B - Y_t^0, \quad \Psi_t^\alpha = \frac{1}{K_t}(Z_t^{B,1} - Z_t^{0,1} + Z_t^{B,2} - Z_t^{0,2}).$$

Thus, setting, with a little abuse of notations

$$p^\alpha(t,y,z) = u^B(t,y,z) - u^0(t,y,z), \qquad \Psi^\alpha(t,y,z) = \frac{1}{K(t,y,z)}\big(d_1^B(t,y,z) - d_1^0(t,y,z) + d_2^B(t,y,z) - d_2^0(t,y,z)\big)$$

where $K(t,y,z) = K^1(t,y,z) + K^2(t,y,z)$, one easily gets

$$p_t^\alpha = p^a(t, S_t, Z_t) \qquad \Psi_t^\alpha = \Psi^\alpha(t, S_{t^-}, Z_{t^-}).$$

On the other hand we recall two facts. The first one is that $p_t^\alpha = \widetilde{Y}_t^\alpha$ and that the indifference hedging strategy is given by

$$\Psi_t^\alpha = \frac{1}{K_t}(\widetilde{Z}_t^{\alpha,1} + \widetilde{Z}_t^{\alpha,2})$$

where $(\widetilde{Y}_t^\alpha, \widetilde{Z}_t^{\alpha,1}, \widetilde{Z}_t^{\alpha,2})$ is the unique bounded solution to the BSDE under Q^* (5.2).
The second one is that p_t^α is involved in the optimization problem defined by (5.7) in Remark 5.2, under Q^*. Then, following the same procedure adopted in Proposition 6.3 we obtain a result analogous to (6.9) under an analogous regularity assumption, that is

$$p^\alpha(t,y,z) = B(y,z) - \int_t^T \frac{\lambda^*(t,y.z)}{\alpha}\ \Big[\exp\Big\{ - \alpha\ \Big(\frac{K^2(t,y,z)}{K(t,y,z)}\widetilde{d}_1(t,y,z) - \frac{K^1(t,y,z)}{K(t,y,z)}\widetilde{d}_2(t,y,z).\Big)\Big]$$

Moreover

$$\Psi^\alpha(t,y,z) = \frac{1}{K(t,y,z)}\big(\widetilde{d}_1(t,y,z) + \widetilde{d}_2(t,y,z)\big) \tag{6.14}$$

where, the expression of $\lambda^*(t,y,z)$ can be deduced by (4.11) and $\widetilde{d}_i,\ i = 1,2$, have to be defined in analogy with (6.8)

$$\begin{aligned} \widetilde{d}_1(t,y,z) &= p^\alpha\big(t, y(1 + K^1(t,y,z)), z + H^1(t,z)\big) - p^\alpha(t,y,z) \\ \widetilde{d}_2(t,y,z) &= -p^\alpha\big(t, y(1 - K^2(t,y,z)), z - H^2(t,z)\big) + p^\alpha(t,y,z), \end{aligned} \tag{6.15}$$

Remark 6.6 *In the case where $K^i(t,y,z)$, $i = 1,2$, are only function on (t,z) we get that the value functions $v^B(t,x,y,z)$, $w^B(t,y,z)$, the optimal strategy $\pi^B(t,y,z)$, the indifference price $p^\alpha(t,y,z)$ and the indifference hedging strategy $\Psi^\alpha(t,y,z)$ do not depend on the variable y but only on (t,z), that is are functions on time and on the value of the stochastic factor (or the nontradable level) at time t.*
In this situation (6.14) can be compared with the results in [1, 34] in a continuous frame where the hedging strategy is expressed in terms of the partial derivative of $p^\alpha(t,z)$ w.r.t. z.

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