OPTIMAL INVESTMENT PROBLEMS
WITH MARKED POINT STOCK DYNAMICS

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Abstract

Optimal investment problems in an incomplete financial market with pure jump stock dynamics are studied. An investor with Constant Relative Risk Aversion (CRRA) preferences, including the logarithmic utility, wants to maximize her/his expected utility of terminal wealth by investing in a bond and in a risky asset. The risky asset price is modeled as a geometric marked point process, whose dynamics is driven by two independent doubly stochastic Poisson processes, describing upwards and downwards jumps. A stochastic control approach allows us to provide optimal investment strategies and closed formulas for the value functions associated to the utility optimization problems. Moreover, the solution to the dual problems associated to the utility maximization problems are derived. The case when intermediate consumption is allowed is also discussed.

Keywords: Utility Maximization, Pure Jump processes.
2000 Mathematical Subject Classification: 91B70; 60J75; 91B16; 93E20.

1. Introduction

In this paper we deal with the dynamic optimization problem of a portfolio in an incomplete financial market with respect to Constant Relative Risk Aversion (CRRA) utility functions.

The fundamental stochastic model of optimal investment and consumption was first introduced by Merton ([?]) who exhibited closed form solutions under the assumption that the stock price follows a geometric Brownian motion and for special utility functions, in particular of CCRA type. A general diffusion case, where the coefficients of the underlying stock price are non-linear functions of the current stock level, has been analyzed in [?]. In [?] and [?] a correlated stochastic factor has been considered. In [?] and in [?] the wealth optimization problem has been studied in incomplete markets driven by asset prices which may exhibit a jumping behaviour.

The contribution of this paper is to provide explicit solutions in an incomplete market model with a general pure jump stock dynamics. More precisely, a pure jump unidimensional market driven by doubly stochastic independent Poisson processes with coefficients depending on time and on the current stock level is studied. An agent with Constant Relative Risk Aversion (CRRA) preferences, including the logarithmic utility, wants to maximize her/his expected utility of terminal wealth by investing in a bond and in a risky asset which is modeled as a geometric point process. In [?] a similar pure jump model with stochastic factor has been treated for an agent with exponential utility function. In [?] optimal portfolio problems have been studied in a pure jump multidimensional market driven by independent Poisson processes and in [?] for unidimensional jump-diffusion stock prices. In both these papers, the assumption of constant coefficients of the underlying stock prices has been made. In the present note, a non-linear pure jump stock dynamics is considered and to the author's knowledge it is the first time that the utility maximization problem is explicitly solved in such a model.

We work in a Markovian setting and we treat the utility maximization problems by stochastic control methods ([?], [?], [?], [?], [?]). Other approaches are proposed in literature by using the convex duality theory ([?], [?], [?], [?] and references therein).

The paper is organized as follows. The model is described in Section 2. In Section 3, we define the optimization problems and we write down the associated Hamilton-Jacobi-Bellman (HJB) equation. The aim is to apply Verification Theorems in order to find the value functions and optimal investment strategies. In the case of a logarithmic utility function, the portfolio optimization problem can be solved (as usual) easier than in the case of a power law utility. In fact, in the logarithmic utility case, making an ansatz for the value function we reduce the associated HJB-equation to a linear equation whose solution can be obtained by the Feymnan-Kac formula. Closed form solutions for the value function and an optimal investment policy are obtained. Whereas, in the case of a power utility, we are able to derive explicit forms for the value function and an optimal strategy only when the coefficients of the underlying stock price are deterministic functions on time (linear stock dynamics). For the non-linear stock dynamics we give a verification result which requires additional assumptions. In both the cases, the optimal investment rules obtained by Verification Theorems, are Markovian and linear in the wealth variable. This is fulfilled also in the diffusion model studied in [?] and in the jump-diffusion model analysed in [?] in the case of CRRA preferences. In particular, when the coefficients of the underlying stock price and the intensities of the point processes which drive its dynamics are constant, the optimal strategy dictates to keep a fixed proportion of the current total wealth as in the Merton's original problem with CRRA preferences.

Section 4 is devoted to derive the solutions of the dual problems associated to the wealth optimization problems. We consider non-linear stock dynamics for the logarithmic utility and linear for the power utility. The solutions to the utility maximization problems, obtained in Section 4 by stochastic control techniques, allow us to obtain explicit solutions to the associated dual problems.

Section 5 studies the case when intermediate consumption is allowed. The object of the agent is to choose a portfolio-consumption strategy in a such way as to maximize his total utility over a finite time interval. We consider the instantaneous utility function for consumption of the same type of the utility function for the final wealth. Explicit solutions are obtained in the logarithmic case for the non-linear stock dynamics and in the power utility case for the simplified linear stock dynamics. In both the cases, the optimal consumption rules are Markovian and linear in the wealth variable. In particular, for the logarithmic utility the optimal consumption rules as the same form as in the Merton's original problem.

2. The Model

We consider a finite time horizon investment model on $[0,T]$ with one riskless money market account and a risky asset. The price of the bond or cash account, B, solves

$$dB_t = rB_t dt, \quad B_0 \in \mathbb{R}^+$$

where $r \geq 0$ is the risk-free interest rate. The stock price, S, satisfies the following non-linear equation

$$dS_t = S_{t^-}\Big(K_1(t,S_{t^-})dN^1_t - K_2(t,S_{t^-})dN^2_t\Big), \quad S_0 \in \mathbb{R}^+ \tag{2.1}$$

where $K_i(t,y)$, $i=1,2$, are positive jointly measurable functions and N^i_t, $i=1,2$, are independent doubly stochastic Poisson processes defined on a probability space $(\Omega,\mathcal{F},P)$ equipped with the filtration

$$\mathcal{F}_t = \sigma\{N^i_u, i=1,2,\ u \leq t\}.$$

The process N^1_t counts the number of upwards jumps and N^2_t of downwards jumps, occurred in $[0,t]$.

The $\{P,\mathcal{F}_t\}$-intensities of N^i_t, $i=1,2$, are denoted by $\lambda_i(t)$, $i=1,2$, and $\lambda(t)=\lambda_1(t)+\lambda_2(t)$ is the intensity of the point process, $N_t = N^1_t + N^2_t$, which counts the total number of changes occurred in $[0,t]$.

We assume that

$$K_2(t,y) < 1, \quad t \in [0,T], y > 0.$$

By the Doléans-Dade exponential formula we get that

$$S_t = S_0 e^{Y_t}, \tag{2.2}$$

where the logreturn process Y is given by

$$Y_t = \int_0^t \log\,(1+K_1(r,S_{r^-}))dN^1_r + \int_0^t \log\,\big(1-K_2(r,S_{r^-})\big)dN^2_r. \tag{2.3}$$

From now on, we shall assume the existence of positive constants A_1, A_2 such that, $\forall t \in [0,T], y > 0$

$$A_1 \le \lambda_i(t) \le A_2, \quad A_1 \le K_i(t,y) \le A_2 \quad i = 1, 2. \tag{2.4}$$

Notice that these conditions ensure that the coefficients of the stochastic differential equation (**??**) satisfy a Lipschitz and a sublinear growth condition on y, uniformly in t. Hence, both strong existence and uniqueness to the equation (**??**) hold.

In the next proposition we will give the semimartingale structure for the risky asset S_t.

Proposition 2.1 *S_t is a special locally bounded semimartingale with the decomposition*

$$S_t = S_0 + M_t + A_t$$

where

$$A_t = \int_0^t S_{r^-}\Big(K_1(r, S_{r^-})\lambda_1(r) - K_2(r, S_{r^-})\lambda_2(r)\Big)dr$$

is a predictable process with bounded variation paths,

$$M_t = \int_0^t S_{r^-}K_1(r, S_{r^-})(dN_r^1 - \lambda_1(r)dr) - \int_0^t S_{r^-}K_2(r, S_{r^-})(dN_r^2 - \lambda_2(r)dr)$$

is a square-integrable martingale whose angle process is given by

$$< M >_t = \int_0^t S_{r^-}^2\Big(K_1(r, S_{r^-})^2\lambda_1(r) + K_2(r, S_{r^-})^2\lambda_2(r)\Big)dr.$$

Proof.
Let us denote by $\{T_n\}$ the sequence of jump times of N_t. By assumption (**??**) there exists a constant $C > 0$ such that, $\forall n \ge 1$, $S_{T_n} \le S_0 e^{nC}$ $P-a.s.$, hence S_t is locally bounded.

Again by assumption (**??**) the process

$$R_t = \int_0^t \big(K_1(r, S_{r^-})dN_r^1 - K_2(r, S_{r^-})dN_r^2\big)$$

is a semimartingale and since

$$dS_t = S_{t^-}dR_t$$

S_t is a semimartingale being the stochastic exponential of a semimartingale.

Finally, by assumption (**??**), there exists a constant $C > 0$ such that

$$S_t^2 = S_0^2 e^{2Y_t} \le S_0^2 e^{CN_t} \quad P-a.s.$$

and, taking into account that

$$\mathbb{E}[e^{CN_t}] = e^{(e^C - 1)\int_0^t \lambda(s)ds} \tag{2.5}$$

the square integrability of S_t is proved.
□

Proposition 2.2 *The stock price S_t is a Markov process whose generator is given by*

$$\mathcal{L}^S f(t,y) = \frac{\partial f}{\partial t}(t,y) + \mathcal{L}^S_t f(t,y) = \tag{2.6}$$

$$\frac{\partial f}{\partial t}(t,y) + \Big(f\big(t, y(1+K_1(t,y))\big) - f(t,y)\Big)\lambda_1(t) + \Big(f\big(t, y(1-K_2(t,y))\big) - f(t,y)\Big)\lambda_2(t).$$

More precisely, for bounded measurable functions $f(t,y)$ absolutely continuous w.r.t. t

$$f(t,S_t) - f(0,S_0) - \int_0^t \mathcal{L}^S f(r,S_r)dr$$

is a $\{P,\mathcal{F}_t\}$-martingale.

Proof.
It is a direct consequence of Ito formula and (**??**), since

$$f(t,S_t) = f(0.S_0) - \int_0^t \frac{\partial f}{\partial r}(r,S_r)dr +$$

$$\int_0^t \Big(f\big(r, S_{r^-}(1+K_1(r,S_{r^-}))\big) - f(r,S_{r^-})\Big)dN^1_r + \int_0^t \Big(f\big(t, S_{r^-}(1-K_2(r,S_{r^-}))\big) - f(r,S_{r^-})\Big)dN^2_r$$

and

$$I\!\!E\Big(\int_0^t \mid f\big(r, S_{r^-}(1+K_1(r,S_{r^-}))\big) - f(r,S_{r^-}) \mid \lambda_1(r) + \mid f\big(t, S_{r^-}(1-K_2(r,S_{r^-}))\big) - f(r,S_{r^-}) \mid \lambda_2(r)dr\Big) \leq 4\|f\|A_2.$$

□

3. The utility maximization problem. The Hamilton-Jacobi-Bellman approach

In this section we discuss the utility optimization problems. We assume to live in a world where continuous trading and unlimited short selling are possible. An investor starts with initial capital $z_0 > 0$ and invests at any time $t \in [0,T]$ the amount $\theta_t \frac{S_t}{S_{t^-}}$ in the risky asset and his remaining wealth, $Z_t - \theta_t \frac{S_t}{S_{t^-}}$, in the bond. Restricting to self-financing investment strategies, the following differential equation describes the dynamics of the wealth process controlled by the investment process θ_t.

$$dZ_t = \frac{\theta_t}{S_{t^-}} dS_t + (Z_t - \theta_t \frac{S_t}{S_{t^-}})\frac{dB_t}{B_t} = \theta_t\Big(K_1(t,S_{t^-})dN^1_t - K_2(t,S_{t^-})dN^2_t\Big) + (Z_t - \theta_t)rdt, \quad Z_0 = z_0 \tag{3.1}$$

For a given strategy θ_t, the solution process Z_t to (**??**) will of course depend on the chosen investment policy θ_t. To be precise we should therefore denote the process Z_t by Z^θ_t, but sometimes we will suppress θ.

A strategy θ_t is said to be admissible if it is a $I\!\!R$-valued $(P,\mathcal{F}_t)$-predictable process such that the following integrability condition is satisfied

$$\int_0^T |\theta_t|dt < +\infty \quad P - a.s. \tag{3.2}$$

and there exists a unique solution to equation (**??**) satisfying $I\!\!E|Z_t| < +\infty$, $t \in [0,T]$, and the state constraint

$$Z_t > 0 \quad a.e. t \in [0,T].$$

We denote by Θ the set of admissible policies.

We consider an agent with Constant Relative Risk Aversion (CRRA) utility function

$$
\begin{cases} U_\alpha(t,x,z) = \frac{z^\alpha}{\alpha} & 0<\alpha<1, \\ \\ U_\alpha(t,x,z) = \log z & \alpha = 0. \end{cases} \tag{3.3}
$$

The investor's objective is to maximize his expected utility from terminal wealth

$$
I\!\!E\Big[U_\alpha(Z_T)\Big]. \tag{3.4}
$$

By considering the utility maximization problem as a stochastic control problem with only final reward, we introduce the associated value function

$$
V_\alpha(t,z,y) = \sup_{\theta\in\Theta} I\!\!E\Big(U_\alpha(Z_T) \mid Z_t = z, S_t = y\Big).
$$

In the next Lemma we prove that the class of admissible investment strategies is not empty.

Lemma 3.1 *The set of admissible investment strategies Θ contains the following set of Markovian policies*

$$
\Theta_1 = \Big\{\theta_t = \tilde\theta(t,S_{t^-})Z_{t^-} : \tilde\theta(t,y) \in \Big(\frac{-1}{K_1(t,y)}, \frac{1}{K_2(t,y)}\Big)\Big\}. \tag{3.5}
$$

Moreover the wealth associated to such strategies is strictly positive and given by

$$
Z_t = z_0 exp\{\int_0^t r(1-\tilde\theta(s,S_{s^-}))ds + \int_0^t \big[\log\big(1+\tilde\theta(s,S_{s^-})K_1(s,S_{s^-})\big)dN^1_s + \log\big(1-\tilde\theta(s,S_{s^-})K_2(s,S_{s^-})\big)dN^2_s\big]\}. \tag{3.6}
$$

Proof.
Let us observe that the wealth associated to Markov control policies of the form $\theta_t = \tilde\theta(t,S_{t^-})Z_{t^-}$ satisfies

$$
dZ_t = Z_{t^-}dM_t
$$

where

$$
M_t = \int_0^t \tilde\theta(u,S_{u^-})\Big(K_1(u,S_{u^-})dN^1_u - K_2(u,S_{u^-})dN^2_u\Big) + \int_0^t (1-\tilde\theta(u,S_{u^-}))rdu.
$$

Hence, by the Doléans-Dade exponential formula, Z_t is well defined on $t\in[0,T]$ and given by

$$
Z_t = z_0 e^{M_t}\Pi_{s\le t}(1+\Delta M_s)e^{-\Delta M_s} =
$$

$$
= z_0 e^{M_t}\Pi_{s\le t}\Big(\big(1+\tilde\theta(s,S_{s^-})K_1(s,S_{s^-})\big)\Delta N^1_s + \big(1-\tilde\theta(s,S_{s^-})K_2(s,S_{s^-})\big)\Delta N^2_s + 1\!\!1_{\{\Delta N^1_s=0,\Delta N^2_s=0\}}\Big)e^{-\Delta M_s} > 0.
$$

Moreover, Z_t can be written as in (**??**) and since $\tilde\theta(t,y)$ is a bounded function by assumption (**??**) we get that there exist constants $C_i>0$, $i=1,2$, such that

$$
Z_t \le z_0 e^{C_1(T+N_T)} \quad P-a.s.
$$

which in turn implies $I\!\!E(Z_t) < +\infty$ and

$$
\int_0^T |\theta_t|dt \le C_2 e^{C_1(T+N_T)}T < +\infty \quad P-a.s.
$$

□

A classical approach in stochastic control theory consists in examining the Hamilton-Jacobi-Bellman (HJB) equation that the value function is expected to satisfy. This equation is given by

$$\frac{\partial u}{\partial t}(t,z,y)+\sup_\theta \mathcal{L}_t^\theta u(t,z,y)=0 \quad t\in(0,T), y>0, z>0 \tag{3.7}$$

with the boundary conditions $u(T,z,y)=U_\alpha(z)$, where $\mathcal{L}^\theta$ denotes the generator of the controlled Markov process (Z_t,S_t) associated to the constant strategy θ

$$\mathcal{L}^\theta f(t,z,y)=\frac{\partial f}{\partial t}(t,z,y)+\mathcal{L}_t^\theta f(t,z,y)=\frac{\partial f}{\partial t}(t,z,y)+\frac{\partial f}{\partial z}(t,z,y)(z-\theta)r+ \tag{3.8}$$

$$\Big(f\big(t,z+\theta K_1(t,y),y(1+K_1(t,y))\big)-f(t,z,y)\Big)\lambda_1(t)+\Big(f\big(t,z-\theta K_2(t,y),y(1-K_2(t,y))\big)-f(t,z,y)\Big)\lambda_2(t).$$

In general the fact that the value function of a stochastic control problem solves, in the classical sense, the HJB-equation requires the knowledge a priori that the value function has enough regularity. Conversely, Verifications results yield that, if there exist a function $F(t,z,y)$ classical solution of (**??**) and the supremum in (**??**) is attained by $\theta^*(t,z,y)$, then F coincides with the value function and if $\theta_t^*=\theta^*(t,Z_{t-}^*,S_{t-})$ is admissible then it is an optimal feedback control (with Z_t^* being the wealth process given by (**??**) when the policy θ_t^* is being used)

3.1. The logarithmic utility

In this subsection we deal with the logarithmic utility. By applying Verification results we will derive closed form solutions for the value function and the optimal policy.

Theorem 3.2 *There exists an optimal feedback strategy, $\theta_t^*=\tilde\theta^*(t,S_{t-})Z_{t-}^*\in\Theta_1$, where $\tilde\theta^*(t,y)$ is defined in Lemma **??** below with $\alpha=0$.*

The value function is given by

$$V_0(t,z,y)=\log z+I\!E\Big[\int_t^T H(\tilde\theta^*)(r,S_{r-})dr \mid S_t=y\Big] \tag{3.9}$$

where

$$H(\tilde\theta^*)(t,y)=r(1-\tilde\theta^*(t,y))+\log(1+\tilde\theta^*(t,y)K_1(t,y))\lambda_1(t)+\log(1-\tilde\theta^*(t,y)K_2(t,y))\lambda_2(t). \tag{3.10}$$

The optimal final wealth is given by

$$Z_T^*=z_0 exp\{\int_0^T r(1-\tilde\theta^*(t,S_{t-}))dt+\int_0^T\big[\log\big(1+\tilde\theta^*(t,S_{t-})K_1(t,S_{t-})\big)dN_t^1+\log\big(1-\tilde\theta^*(t,S_{t-})K_2(t,S_{t-})\big)dN_t^2\big]\}. \tag{3.11}$$

Proof.
The associated HJB-equation is given by

$$\frac{\partial u}{\partial t}(t,z,y)+\sup_\theta \mathcal{L}_t^\theta u(t,z,y)=0 \tag{3.12}$$

with the terminal condition

$$u(T,z,y)=\log z. \tag{3.13}$$

We look for a candidate solution of (**??**) in the form

$$u(t,z,y)=\log z+h(t,y), \tag{3.14}$$

and (**??**) yields that $h(t,y)$ solves,

$$\frac{\partial h}{\partial t}(t,y)+\Big(h\big(t,y(1+K_1(t,y))\big)-h(t,y)\Big)\lambda_1(t)+\Big(h\big(t,y(1-K_2(t,y))\big)-h(t,y)\Big)\lambda_2(t)+ \tag{3.15}$$

$$\sup_{\tilde\theta}\Big(r(1-\tilde\theta)+\log(1+\tilde\theta K_1(t,y))\lambda_1(t)+\log(1-\tilde\theta K_2(t,y))\lambda_2(t)\Big)=0$$

with the terminal condition $h(T,y)=0$, where the control $\tilde\theta$ corresponds to $\frac{\theta}{z}$, with θ being the control variable appearing in (**??**).

Observe that (**??**) can be written as

$$\frac{\partial h}{\partial t}(t,y)+\mathcal{L}_t^S h(t,y)+\sup_{\tilde\theta} H(\tilde\theta)(t,y)=0,\quad h(T,y)=0 \tag{3.16}$$

where H is defined in (**??**) and $\mathcal{L}_t^S$ denotes the generator of the Markov process S defined in (**??**).

We get that the maximum of $H(\tilde\theta)$ is achieved at $\tilde\theta^*(t,y)$ defined in Lemma **??** below, since $H'(\tilde\theta)=\Phi_\alpha(\tilde\theta)$ with $\alpha=0$.

Hence $h(t,y)$ solves,

$$\frac{\partial h}{\partial t}(t,y)+\mathcal{L}_t^S h(t,y)+H(\tilde\theta^*)(t,y)=0,\quad h(T,y)=0. \tag{3.17}$$

whose solution, by Lemma **??** below, is given by

$$h(t,y)=I\!E\big[\int_t^T H(\tilde\theta^*)(r,S_{r^-})dr\mid S_t=y\big].$$

Finally, by Lemma **??**, $\theta_t^*=\tilde\theta^*(t,S_{t^-})Z_{t^-}^*\in\Theta$ and Verification results allows us to conclude that $V_0(t,z,y)$ given in (**??**) is the value function and θ_t^* is an optimal markovian investment strategy.
□

Lemma 3.3 *There exists a unique solution, $\tilde\theta^*(t,y)\in(\frac{-1}{K_1(t,y)},\frac{1}{K_2(t,y)}\Big)$, to the following equation* $\forall t\in[0,T], y>0, 0\le\alpha<1$

$$(1+\tilde\theta(t,y)K_1(t,y))^{\alpha-1}\lambda_1(t)K_1(t,y)-(1-\tilde\theta(t,y)K_2(t,y))^{\alpha-1}\lambda_2(t)K_2(t,y)=r. \tag{3.18}$$

Proof.
It is sufficient to observe that, for any fixed $t\in[0,T], y>0$, the function

$$\Phi_\alpha(\theta)=(1+\theta K_1(t,y))^{\alpha-1}\lambda_1(t)K_1(t,y)+(1-\theta K_2(t,y))^{\alpha-1}\lambda_2(t)K_2(t,y)-r,\quad 0\le\alpha<1$$

is continuous, strictly decreasing in $\Big(\frac{-1}{K_1(t,y)},\frac{1}{K_2(t,y)}\Big)$ and

$$lim_{\theta\to\frac{-1}{K_1(t,y)}}\Phi_\alpha(\theta)=+\infty,\quad lim_{\theta\to\frac{1}{K_2(t,y)}}\Phi_\alpha(\theta)=-\infty.$$

□

Remark 3.4 *Notice that for the logarithmic case, corresponding to $\alpha=0$, explicit solutions to (**??**) can be obtained. More precisely, for $r=0$*

$$\tilde\theta^*(t,y)=\frac{K_1(t,y)\lambda_1(t)-K_2(t,y)\lambda_2(t)}{K_1(t,y)K_2(t,y)\lambda(t)} \tag{3.19}$$

and for $r\neq0$, $\tilde\theta^*(t,y)$ is the smallest root of the following second order equation

$$rK_1(t,y)K_2(t,y)\tilde\theta(t,y)^2-\big[r(K_1(t,y)-K_2(t,y))+\lambda(t)K_1(t,y)K_2(t,y)\big]\tilde\theta(t,y)-r+\lambda_1(t)K_1(t,y)-\lambda_2(t)K_2(t,y)=0. \tag{3.20}$$

Lemma 3.5 *The following linear equation with final condition*

$$\frac{\partial v}{\partial t}(t,y) + \mathcal{L}_t^S v(t,y) + H(\tilde{\theta}^*)(t,y) = 0, \quad v(T,y) = 0 \tag{3.21}$$

admits a unique measurable bounded solution, $h(t,y)$, which is absolutely continuous with respect to t. Then, for any y and for a.a. t, there exists $\frac{\partial h}{\partial t}(t,y)$ and is bounded.
Moreover its Feynman-Kac representation is given by

$$h(t,y) = I\!\!E\Big[\int_t^T H(\tilde{\theta}^*)(r, S_{r^-})dr \mid S_t = y\Big]. \tag{3.22}$$

Proof.
Let us observe that equation (**??**) can be written as

$$\frac{\partial v}{\partial t}(t,y) - \lambda(t)v(t,y) + v\big(t, y(1+K_1(t,y))\big)\lambda_1(t) + v\big(t, y(1-K_2(t,y))\big)\lambda_2(t) + H(\tilde{\theta}^*)(t,y) = 0, \quad v(T,y) = 0$$

taking $v = g - 1$, we study the following problem

$$\frac{\partial g}{\partial t}(t,y) - \lambda(t)g(t,y) + g\big(t, y(1+K_1(t,y))\big)\lambda_1(t) + g\big(t, y(1-K_2(t,y))\big)\lambda_2(t) + H(\tilde{\theta}^*)(t,y) = 0, \quad g(T,y) = 1 \tag{3.23}$$

which is equivalent to

$$g(t,y) = e^{-\int_t^T \lambda(s)ds} + \int_t^T \Big[g\big(s, y(1+K_1(s,y))\big)\lambda_1(s) + g\big(s, y(1-K_2(s,y))\big)\lambda_2(s) + H(\tilde{\theta}^*)(s,y)\Big] e^{-\int_t^s \lambda(r)dr} ds. \tag{3.24}$$

In fact, differentiating both sides w.r.t. t, we obtain an equation that, joint with (**??**) reproduces (**??**).

Equation (**??**) has a unique bounded solution. If g_1, g_2 are two bounded solutions, setting

$$F(t) = \sup_y |g_1(t,y) - g_2(t,y)|$$

we get

$$F(t) \leq 2A_2 \int_t^T \Gamma(s)ds$$

and the assertion follows by a slight modification of Gronwall Lemma.
By a classical recursive method, we obtain existence of a bounded solution absolutely continuous w.r.t. t.
Define

$$g_0(t,y) = e^{-\int_t^T \lambda(s)ds} + \int_t^T H(\tilde{\theta}^*)(s,y) e^{-\int_t^s \lambda(r)dr} ds$$

$$g_{k+1}(t,y) = e^{-\int_t^T \lambda(s)ds} + \int_t^T \Big[g_k\big(s, y(1+K_1(s,y))\big)\lambda_1(s) + g_k\big(s, y(1-K_2(s,y))\big)\lambda_2(s) + H(\tilde{\theta}^*)(s,y)\Big] e^{-\int_t^s \lambda(r)dr} ds$$

we have that

$$||g_1 - g_0|| \leq 2A_2(T-t)||H(\tilde{\theta}^*)||, \quad ||g_{k+1} - g_k|| \leq \frac{(2A_2 T)^k}{k!} ||g_1 - g_0||$$

and the conclusion follows by standard arguments.

Finally, by Proposition **??**

$$h(T,S_T)=h(t.S_t)+\int_t^T\Big(\frac{\partial h}{\partial r}(r,S_r)+\mathcal{L}_r^S h(r,S_r)\Big)dr+m_T-m_t$$

where m_t is a $\{P,\mathcal{F}_t\}$-martingale and taking the expectation conditioned to $\mathcal{F}_t$ by equation (**??**) we obtained (**??**).
□

In the power law utility case, in order to exhibit closed-form solutions for the value function and the optimal strategy, we will consider linear dynamics for the stock price, by assuming that the functions $K_i(t,y)$, $i=1,2$, are dependent only on t. The general non-linear stock dynamics will be analysed in Subsection 3.3.

3.2. The linear stock dynamics case

In this subsection the functions $K_i(t,y)$ will replaced by $K_i(t)$, $i=1,2$. Then we consider the following stock dynamics

$$dS_t=S_{t^-}\Big(K_1(t)dN_t^1-K_2(t)dN_t^2\Big). \tag{3.25}$$

The value function is now given by

$$V_\alpha(t,z)=\sup_{\theta\in\Theta}I\!\!E\Big(U_\alpha(Z_T)\mid Z_t=z\Big).$$

Observe that it has been possible to absorb the stock price variable y in the wealth variable z, being the wealth dynamics given by

$$dZ_t=\frac{\theta_t}{S_{t^-}}dS_t+(Z_t-\theta_t\frac{S_t}{S_{t^-}})\frac{dB_t}{B_{t^-}}=\theta_t\big(K_1(t)dN_t^1-K_2(t)\big)dN_t^2\big)+(Z_t-\theta_t)rdt,\quad Z_0=z_0>0. \tag{3.26}$$

Main results are outlined below. For sake of completeness we consider both the power law and the logarithmic utilities for the simplified stock dynamics given in (**??**).

Theorem 3.6 *There exists an optimal strategy θ_t^*, $\theta_t^*=\tilde\theta^*(t)Z_{t^-}^*\in\Theta_1$ (see (**??**)), where $\tilde\theta^*(t)$ is the unique solution to equation (**??**).*

The value function is, for the power utility ($0<\alpha<1$), given by

$$V_\alpha(t,z)=\frac{z^\alpha}{\alpha}e^{\int_t^T(H_\alpha(\tilde\theta^*)(s)-\lambda(s))ds} \tag{3.27}$$

where

$$H_\alpha(\tilde\theta^*)(t)=\alpha r(1-\tilde\theta^*(t))+\big(1+\tilde\theta^*(t)K_1(t)\big)^\alpha\lambda_1(t)+\big(1-\tilde\theta^*(t)K_2(t)\big)^\alpha\lambda_2(t) \tag{3.28}$$

and, for the logarithmic utility ($\alpha=0$), given by

$$V(t,z)=\log z+\int_t^T H(\tilde\theta^*)(s)ds. \tag{3.29}$$

*where $H(\tilde\theta^*)$ is defined in (**??**).*

*Moreover, the final wealth process is given in both the cases by (**??**).*

Proof.
First, let us denote by $\mathcal{L}^\theta$ the generator of the controlled Markov process Z_t associated to the constant strategy θ

$$\mathcal{L}^\theta f(t,z) = \frac{\partial f}{\partial t}(t,z) + \mathcal{L}^\theta_t f(t,z) = \frac{\partial f}{\partial t}(t,z) + \frac{\partial f}{\partial z}(t,z)(z-\theta)r + \tag{3.30}$$
$$\Big(f\big(t, z+\theta K_1(t)\big) - f(t,z)\Big)\lambda_1(t) + \Big(f\big(t, z-\theta K_2(t)\big) - f(t,z)\Big)\lambda_2(t).$$

The Hamilton-Jacobi-Bellman equation is given by

$$\frac{\partial u}{\partial t}(t,z) + \sup_\theta \mathcal{L}^\theta_t u(t,z) = 0 \tag{3.31}$$

with the terminal condition

$$u(T,z) = U_\alpha(z). \tag{3.32}$$

When $0 < \alpha < 1$ we look for a candidate solution of (**??**) in the form

$$u(t,z) = \frac{z^\alpha}{\alpha} h(t), \tag{3.33}$$

and (**??**) yields that $h(t)$ solves,

$$\frac{dh}{dt}(t) - h(t)\lambda(t) + \sup_{\tilde\theta}\Big(\alpha r(1-\tilde\theta) + (1+\tilde\theta K_1(t))^\alpha \lambda_1(t) + (1-\tilde\theta K_2(t))^\alpha \lambda_2(t)\Big)h(t) = 0 \tag{3.34}$$

with the terminal conditions $h(T) = 1$, where the control $\tilde\theta = \frac{\theta}{z}$, with θ being the control variable appearing in (**??**).

By Lemma **??**, since $H'_\alpha(\tilde\theta) = \alpha\Phi_\alpha(\tilde\theta)$, we get that the maximum of $H_\alpha(\tilde\theta)(t)$ is achieved, for each $t \in [0,T]$, at the unique $\tilde\theta^*(t)$ solution to equation (**??**), replacing $K_i(t,y)$ by $K_i(t)$, $i = 1, 2$.

Hence $h(t)$ solves,

$$\frac{dh}{dt}(t) + h(t)(H_\alpha(\tilde\theta^*)(t) - \lambda(t)) = 0, \quad h(T) = 1 \tag{3.35}$$

By Lemma **??**, $\theta^*_t \in \Theta$ and Verification results imply that

$$V_\alpha(t,z) = \frac{z^\alpha}{\alpha} e^{\int_t^T (H_\alpha(\tilde\theta^*)(s) - \lambda(s))ds}$$

is the value function, and $\theta^*_t = \tilde\theta^*(t) Z^*_{t-}$, is an optimal markovian investment strategy.

In the logarithmic case, as in the previous subsection, we look for a candidate solution of (**??**) in the form

$$u(t,z) = \log z + h(t), \tag{3.36}$$

and (**??**) yields that $h(t)$ solves,

$$\frac{dh}{dt}(t) + H(\tilde\theta^*)(t) = 0, \quad h(T) = 0. \tag{3.37}$$

Then Verification results allows us to conclude that

$$V_0(t,z) = \log z + \int_t^T H(\tilde\theta^*)(s)ds \tag{3.38}$$

is the value function and $\theta^*_t = \tilde\theta^*(t) Z^*_{t-}$ is an optimal investment strategy. □

Corollary 3.7 *Explicit forms for the optimal strategy can be obtained when $r = 0$ and $0 \leq \alpha < 1$*

$$\tilde{\theta}^*(t) = \frac{1 - \Gamma(t)^{\frac{1}{1-\alpha}}}{K_2(t) + \Gamma(t)^{\frac{1}{1-\alpha}} K_1(t)} \tag{3.39}$$

where

$$\Gamma(t) = \frac{K_2(t)\lambda_2(t)}{K_1(t)\lambda_1(t)},$$

*and in the logarithmic case ($\alpha = 0$) also when $r \neq 0$ being $\tilde{\theta}^*(t)$ the smallest root of the second order equation (**??**) replacing $K_i(t, y)$ by $K_i(t)$, $i = 1, 2$.*

Proof.
It is a direct consequence of Theorem **??** and Lemma **??**.
□

Remark 3.8 *Let us observe that when $K_i(t)$ and $\lambda_i(t)$, $i = 1, 2$ are not time dependent the optimal strategy dictates that is optimal to keep a fixed proportion of the current total wealth, as in the Merton's original problem with CRRA preferences.*

3.3. The power law utility

We will now go back to the general non-linear stock dynamics described in Section 2.

First, let us recall a suitable version of Girsanov Theorem for our model.
A probability measure Q is equivalent to P iff

$$\frac{dQ}{dP} |_{\mathcal{F}_T} = L_T = \mathcal{E}(M_T), \quad E[L_T] = 1 \tag{3.40}$$

where

- M_t is a $\{P, \mathcal{F}_t\}$-local martingale given by

$$M_t = \sum_{i=1}^{2} \int_0^T U_s^i (dN_s^i - \lambda_i(s) ds)$$

- U_s^i, for $i = 1, 2$, are $\{P, \mathcal{F}_t\}$-predictable process such that

$$U_s^i + 1 > 0, \qquad \text{and} \qquad \int_0^T |U_s^i + 1| \lambda_i(s) ds < +\infty \quad P - a.s.$$

Under Q, N_t^i, $i = 1, 2$, are point processes with $(Q, \mathcal{F}_t)$-intensities given by

$$\lambda_t^{i,Q} = (U_t^i + 1)\lambda_i(t), \quad j = 1, 2, .., m \tag{3.41}$$

respectively.

Moreover L_T can be written as

$$L_T = exp\Big\{ \sum_{i=1}^{2} \Big[\int_0^T log(1 + U_s^i) dN_s^i - \int_0^T U_s^i \lambda_i(s) ds \Big] \Big\}. \tag{3.42}$$

Lemma 3.9 *Let $\tilde{\theta}_t = \tilde{\theta}(t, S_{t^-})$, with $\tilde{\theta}(t, y) \in \big(\frac{-1}{K_1(t,y)}, \frac{1}{K_2(t,y)}\big)$.*
Then it is well defined the probability measure $P^{\tilde{\theta}}$ as

$$\frac{dP^{\tilde{\theta}}}{dP} \mid_{\mathcal{F}_T} = L_T^{P^{\tilde{\theta}}}, \tag{3.43}$$

with $L_T^{P^{\tilde{\theta}}} = \mathcal{E}(M_T^{\tilde{\theta}})$, and

$$U_t^{\tilde{\theta},1} = (1 + \tilde{\theta}_t K_1(t, S_{t^-}))^{\alpha} - 1$$

$$U_t^{\tilde{\theta},2} = (1 - \tilde{\theta}_t K_2(t, S_{t^-}))^{\alpha} - 1.$$

Under $P^{\tilde{\theta}}$, N_t^1 and N_t^2 are point processes with intensities

$$\lambda^{\tilde{\theta},1}(t, S_{t^-}) = (1 + \tilde{\theta}_t K_1(t, S_{t^-}))^{\alpha} \lambda_1(t), \quad \lambda^{\tilde{\theta},2}(t, S_{t^-}) = (1 - \tilde{\theta}_t K_2(t, S_{t^-}))^{\alpha} \lambda_2(t) \tag{3.44}$$

respectively and S_t is a Markov process whose generator, for bounded measurable functions $f(t, y)$, absolutely continuous w.r.t. t, is given by

$$\mathcal{L}^{\tilde{\theta},S} f(t, y) = \frac{\partial f}{\partial t}(t, y) + \mathcal{L}_t^{\tilde{\theta},S} f(t, y) = \tag{3.45}$$

$$\frac{\partial f}{\partial t}(t, y) + \Big(f\big(t, y(1 + K_1(t, y))\big) - f(t, y)\Big)\lambda^{\tilde{\theta},1}(t, y) + \Big(f\big(t, y(1 - K_2(t, y))\big) - f(t, y)\Big)\lambda^{\tilde{\theta},2}(t, y).$$

Proof.
It is sufficient to observe that $U_t^{\tilde{\theta},i}$, $i = 1, 2$, are bounded. In fact, recalling that $\mathbb{E}[L_T^{P^{\tilde{\theta}}}] \leq 1$ and that

$$L_t^{\tilde{\theta}} = 1 + \int_0^t L_{s^-}^{\tilde{\theta}} \sum_{i=1}^{2} U_s^{\tilde{\theta},i}(dN_s^i - \lambda_i(s)ds)$$

we get that $L_t^{\tilde{\theta}}$ is a $(P, \mathcal{F}_t)$-martingale by

$$\mathbb{E}[\int_0^t L_{s^-}^{\tilde{\theta}} \sum_{i=1}^{2} \mid U_s^{\tilde{\theta}} \mid \lambda_i(s)ds] < +\infty.$$

By Girsanov Theorem and Ito formula, (**??**) and (**??**) can be obtained.
□

Theorem 3.10 *(i) When the class of admissible investment strategies reduces to Θ_1 defined in (**??**), then the associated value function is of the form*

$$V_a^1(t, y, z) = \frac{z^a}{\alpha} h(t, y), \tag{3.46}$$

where $h(t, y)$ is a bounded function given by

$$h(t, y) = \sup_{\tilde{\theta}} \mathbb{E}^{\tilde{\theta}} \Big[e^{\int_t^T (H_\alpha(\tilde{\theta})(s, S_s) - \lambda(s))ds} \mid S_t = y \Big], \tag{3.47}$$

*where $\mathbb{E}^{\tilde{\theta}}$ denotes the expected value under $P^{\tilde{\theta}}$, that is the probability measure defined in Lemma **??** and $H_\alpha(\tilde{\theta})$ is given in (**??**) replacing $\tilde{\theta}^*(t)$ by $\tilde{\theta}(t, y)$ and $K_i(t)$ by $K_i(t, y)$, $i = 1, 2$.*

(ii) (Verification result) If there exists a bounded solution $h(t, y)$, absolutely continuous w.r.t. t, to

$$\frac{\partial h}{\partial t}(t, y) + \sup_{\tilde{\theta}} \Big(\mathcal{L}_t^{\tilde{\theta},S} h(t, y) + \big(H_\alpha(\tilde{\theta})(t, y) - \lambda(t)\big)h(t, y) \Big) = 0, \quad h(T, y) = 1, \tag{3.48}$$

where $\mathcal{L}_t^{\tilde\theta,S}$ *is defined in (??), then the value function is given by* $V_\alpha(t,y,z) = \frac{z^\alpha}{\alpha} h(t,y)$ *and there exists an optimal strategy* $\theta^*_t = \tilde\theta^*(t, S_{t-}) Z^*_{t-} \in \Theta_1$*, with* $\tilde\theta^*(t,y)$*, for any* $t \in [0,T], y > 0$*, the unique solution to*

$$\big(1+\tilde\theta(t,y)K_1(t,y)\big)^{\alpha-1}\lambda_1(t)K_1(t,y)\ h\big(t, y(1+K_1(t,y))\big) - \tag{3.49}$$

$$\big(1-\tilde\theta(t,y)K_2(t,y)\big)^{\alpha-1}\lambda_2(t)K_2(t,y)\ h\big(t, y(1-K_2(t,y))\big) = r\ h(t,y)$$

Moreover the following Feynman-Kac representation holds

$$h(t,y) = I\!E^{\tilde\theta^*}\Big[e^{\int_t^T (H_\alpha(\tilde\theta^*)(s,S_s)-\lambda(s))ds} \mid S_t = y\Big]. \tag{3.50}$$

Proof.
(i) Since the wealth associated to strategies belonging to Θ_1 is given by (??) we get that

$$V_\alpha(t,z,y) = \sup_{\theta\in\Theta_1} I\!E\Big(\frac{Z_t^\alpha}{\alpha} \mid Z_t = z, S_t = y\Big) = \frac{z^\alpha}{\alpha}\times$$

$$\sup_{\tilde\theta} I\!E\Big(exp\{\int_t^T \alpha(r-\tilde\theta(s,S_{s^-}))ds + \int_t^T \alpha\big[\log\big(1+\tilde\theta(s,S_{s^-})K_1(s,S_{s^-})\big)dN_s^1 + \log\big(1-\tilde\theta(s,S_{s^-})K_2(s,S_{s^-})\big)dN_s^2\big]\} \mid S_t = y\Big),$$

then the value function is in form

$$V_\alpha(t,z,y) = \frac{z^\alpha}{\alpha} h(t,y),$$

where

$$h(t,y) = \tag{3.51}$$

$$\sup_{\tilde\theta} I\!E\Big(exp\{\int_t^T \alpha(r-\tilde\theta(s,S_{s^-}))ds + \int_t^T \alpha\big[\log\big(1+\tilde\theta(s,S_{s^-})K_1(s,S_{s^-})\big)dN_s^1 + \log\big(1-\tilde\theta(s,S_{s^-})K_2(s,S_{s^-})\big)dN_s^2\big]\} \mid S_t = y\Big).$$

Moreover, by assumptions (??) there exists a constant $C > 0$ such that

$$exp\{\int_t^T \alpha(r-\tilde\theta(s,S_{s^-}))ds + \int_t^T \alpha\big[\log\big(1+\tilde\theta(s,S_{s^-})K_1(s,S_{s^-})\big)dN_s^1 + \log\big(1-\tilde\theta(s,S_{s^-})K_2(s,S_{s^-})\big)dN_s^2\big] \le e^{C(T+N_T)}$$

which implies that $h(t,y)$ is a bounded function. Finally, notice that

$$I\!E\Big(exp\{\int_t^T \alpha(r-\tilde\theta(s,S_{s^-}))ds + \int_t^T \alpha\big[\log\big(1+\tilde\theta(s,S_{s^-})K_1(s,S_{s^-})\big)dN_s^1 + \log\big(1-\tilde\theta(s,S_{s^-})K_2(s,S_{s^-})\big)dN_s^2\big]\} \mid S_t = y\Big) =$$

$$= I\!E\Big(\frac{L_T^{\tilde\theta}}{L_t^{\tilde\theta}} e^{\int_t^T (H_\alpha(\tilde\theta)(s,S_s)-\lambda(s))ds} \mid S_t = y\Big) = I\!E^{\tilde\theta}\Big[e^{\int_t^T (H_\alpha(\tilde\theta)(s,S_s)-\lambda(s))ds} \mid S_t = y\Big],$$

where $I\!E^{\tilde\theta}$ denotes the expected value under $P^{\tilde\theta}$, that is the probability measure defined in Lemma ??, hence (??) follows.

(ii) We have that $V_\alpha(t,y,z) = \frac{z^\alpha}{\alpha} h(t,y)$ is a classical solution to the HJB-equation

$$\frac{\partial V_\alpha}{\partial t}(t,z,y) + \sup_\theta \mathcal{L}_t^\theta V_\alpha(t,y,z) = 0, \quad t \in (0,T), y > 0, z > 0 \tag{3.52}$$

with the terminal condition $V_\alpha(T,z,y) = \frac{z^\alpha}{\alpha}$ and where the control $\theta = \tilde\theta z$, with $\tilde\theta$ being the control variable appearing in (??). Moreover $\theta^*(t,y,z) = \tilde\theta^*(t,y)z$ realizes the supremum in (??). By Verification results $V_\alpha(t,y,z)$ is the value function and $\theta^*_t = \tilde\theta^*(t,S_{t-})Z^*_{t-} \in \Theta_1$ is an optimal investment strategy.
Finally, by applying Feynman-Kac formula we have the representation (??). $\square$

Remark 3.11 *Let us observe that $\tilde\theta^*(t,y)$, defined in (**??**), depends on the function h. In order to implement the optimal investment plan we need to know h, that is the solution to equation (**??**).*
For instance, explicit formulas depending on h can be obtained when $r = 0$

$$\tilde\theta^*(t,y) = \frac{1 - \tilde\Gamma(t,y)^{\frac{1}{1-\alpha}}}{K_2(t,y) + \tilde\Gamma(t,y)^{\frac{1}{1-\alpha}} K_1(t,y)} \tag{3.53}$$

where

$$\tilde\Gamma(t,y) = \frac{K_2(t,y)\lambda_2(t)}{K_1(t,y)\lambda_1(t)} \frac{h(t, y(1 - K_2(t,y))}{h(t, y(1 + K_1(t,y))}.$$

*Substituting this expression into (**??**) we get a non-linear equation. Whereas, in the logarithmic case we obtained a linear equation, see (**??**) .*

Remark 3.12 *When the Verification result (ii) can not use one has to relax the notion of solutions to equation (**??**) by introducing viscosity solutions (see [**?**], [**?**], [**?**]. [**?**] and references therein). Herein we do not deal with this topic.*

4. The dual problem

In this section we provide the solution to the dual problem associated to our utility maximization problems.

First, we characterize the set, $\mathcal{M}_e$, of the martingale measures, consisting of all probability measures P', equivalent to P, such that the discounted stock price, $\tilde S_t = \frac{S_t}{B_t}$, is a local $(P', \mathcal{F}_t)$-martingale.

Proposition 4.1 *A probability measure P' equivalent to P, defined as in (**??**), is a risk-neutral measure iff*

$$\sum_{i=1}^{2} \int_0^T K_i(s, S_{s^-})(1 + U_s^i)\lambda_i(s) ds < +\infty \qquad P - a.s. \tag{4.1}$$

$$K_1(t, S_{t^-})(1 + U_t^1)\lambda_1(t) - K_2(t, S_{t^-})(1 + U_t^2)\lambda_2(t) = r \qquad \text{for a.a. } t \in [0,T], \quad y > 0, \quad P - a.s. \tag{4.2}$$

Proof.
The dynamics of $\tilde S_t$, under the probability measure P is given by

$$d\tilde S_t = \tilde S_{t^-}\Big(K_1(t, S_{t^-})dN_t^1 - K_2(t, S_{t^-})dN_t^2 - r dt\Big)$$

Recalling (**??**) and Girsanov Theorem we can write

$$\tilde S_t = \tilde S_0 + \int_0^t \tilde S_{s^-}\Big(K_1(s, S_{s^-})(1 + U_s^1)\lambda_1(s) - K_2(s, S_{s^-})(1 + U_s^2)\lambda_2(s) - r\Big)ds +$$

$$\int_0^t \tilde S_{s^-} \sum_{i=1}^{m} (-1)^{i-1} K_i(s, S_{s^-})\big(dN_s^i - (1 + U_s^i)\ \lambda_i(s) ds\big)$$

Thus, $\tilde S$ is a special semimartingale under P', and a local martingale iff (**??**) and (**??**) hold.
□

In this section we consider the general non linear dynamics (**??**) for the logarithmic utility and the simplified linear (**??**) for the power law case. We will denote by K_t^i, $i = 1, 2$, the coefficients appear either in (**??**) or (**??**) and by $\tilde\theta_t^*$, the strategy $\tilde\theta^*(t, S_{t^-})$ defined in Lemma **??** (which does not depend on S_{t^-} in the power law case).

Proposition 4.2 *The probability measure Q^* defined as*

$$\frac{dQ^*}{dP} \mid_{\mathcal{F}_T} = L_T^{Q^*}, \tag{4.3}$$

with $L_T^{Q^*} = \mathcal{E}(M_T^*)$, and

$$U_t^{*1} = (1 + \tilde{\theta}_t^* K_t^1)^{\alpha-1} - 1$$

$$U_t^{*2} = (1 - \tilde{\theta}_t^* K_t^2)^{\alpha-1} - 1$$

with $0 \leq \alpha < 1$, is a risk-neutral probability measure.

Proof.
It is sufficient to observe that the risk neutral condition is a consequence of (**??**) and that U_t^{*i}, $i = 1, 2$, are bounded.
The proof that $L_t^{Q^*}$ is a $(P, \mathcal{F}_t)$-martingale is along the lines of that of Lemma **??**.
□

Using the theory of convex duality we introduce the dual problem associated to our utility maximization problem. More precisely the following duality relation holds ([**?**])

$$\sup_{\theta} I\!\!E\Big[U_\alpha(Z_T)\Big] = \inf_{P' \in \mathcal{M}_e} \inf_{\gamma > 0} \Big(\gamma z + I\!\!E\Big[\Psi_\alpha\Big(\gamma L_T^{P'}\Big(\frac{B_T}{B_0}\Big)^{-1}\Big)\Big]\Big), \tag{4.4}$$

where Ψ_α is the conjugate convex function associated to U_α, defined by

$$\Psi_\alpha(y) = sup_{x \in I\!\!R}[U(x) - yx] \quad y > 0.$$

The conjugate of the power law utility is given by

$$\Psi_\alpha(y) = \frac{1-\alpha}{\alpha} y^{\frac{\alpha}{\alpha-1}}, \quad 0 < \alpha < 1$$

and that of the logarithmic is

$$\Psi_\alpha(y) = -\log y - 1.$$

Theorem 4.3 *The probability measure Q^* solves the dual problem.*

Proof.
We have that the right hand side of (**??**) is given by

$$\log z_0 + rT - \inf_{P' \in \mathcal{M}_e} I\!\!E[\log L_T^{P'}], \quad \alpha = 0. \tag{4.5}$$

$$\inf_{P' \in \mathcal{M}_e} \frac{(z_0)^\alpha}{\alpha} e^{\alpha r T} I\!\!E\Big[(L_T^{P'})^{\frac{\alpha}{\alpha-1}}\Big]^{1-\alpha}, \quad 0 < \alpha < 1 \tag{4.6}$$

By (**??**) we get that

$$I\!\!E[\log L_T^{Q^*}] = -I\!\!E\Big[\int_0^T \big[\log(1 + \tilde{\theta}_t^* K_t^1) dN_t^1 + \log(1 - \tilde{\theta}_t^* K_t^2) dN_t^2\big]\Big] +$$

$$-I\!\!E\Big[\int_0^T \big[\big((1 + \tilde{\theta}_t^* K_t^1)^{-1} - 1\big)\lambda_1(t) + \big((1 - \tilde{\theta}_t^* K_t^2)^{-1} - 1\big)\lambda_2(t)\big] dt\Big].$$

Then by the definition of intensity of point processes and by (**??**)

$$\mathbb{E}[\log L_T^{Q^*}] = -\mathbb{E}\Big[\int_0^T \big[\log(1+\tilde{\theta}_t^* K_t^1)\lambda_1(t) + \log(1-\tilde{\theta}_t^* K_t^2)\lambda_2(t) - \tilde{\theta}_t^* r\big]dt\Big].$$

Finally, since by Theorem **??**

$$\sup_\theta \mathbb{E}\Big[\log Z_T\Big] = \log z_0 + rT + \mathbb{E}\Big[\int_0^T \big[\log(1+\tilde{\theta}_t^* K_t^1)\lambda_1(t) + \log(1-\tilde{\theta}_t^* K_t^2)\lambda_2(t) - \tilde{\theta}_t^* r\big]dt\Big],$$

the assertion, in the $\alpha = 0$ case, is proved.

In the $0 < \alpha < 1$ case, recalling that K_t^i, $i = 1, 2$, and $\tilde{\theta}_t^*$ are deterministic functions, by (**??**) we get that

$$\mathbb{E}\Big[(L_T^{Q^*})^{\frac{\alpha}{\alpha-1}}\Big] = \mathbb{E}\Big[exp\Big\{\int_0^T \big[\ln(1+\tilde{\theta}_t^* K_t^1)^\alpha dN_t^1 + \ln(1-\tilde{\theta}_t^* K_t^2)^\alpha dN_t^2\big]\Big\}\Big]\times$$

$$exp\Big\{\frac{\alpha}{1-\alpha}\int_0^T \big[(1+\tilde{\theta}_t^* K_t^1)^{\alpha-1}\lambda_1(t) + (1-\tilde{\theta}_t^* K_t^2)^{\alpha-1}\lambda_2(t) - \lambda(t)\big]dt\Big\}.$$

By Lemma **??** below (recalling that N_t^1 and N_t^2 are independent) we obtain

$$\mathbb{E}\Big[exp\{\int_0^T \big[\ln(1+\tilde{\theta}_t^* K_t^1)^\alpha dN_t^1 + \ln(1-\tilde{\theta}_t^* K_t^2)^\alpha dN_t^2\big]\}\Big] = exp\Big\{\int_0^T \big[((1+\tilde{\theta}_t^* K_t^1)^\alpha - 1)\lambda_1(t) + ((1-\tilde{\theta}_t^* K_t^2)^\alpha - 1)\lambda_2(t)\big]dt\Big\},$$

and taking into account (**??**)

$$\mathbb{E}\Big[(L_T^{Q^*})^{\frac{\alpha}{\alpha-1}}\Big]^{1-\alpha} = exp\{\int_0^T \big[(1+\tilde{\theta}_t^* K_t^1)^\alpha\lambda_1(t) + (1-\tilde{\theta}_t^* K_t^2)^\alpha\lambda_2(t) - \alpha\tilde{\theta}_t^* - \lambda(t)\big]dt\}.$$

Finally, by Theorem **??**

$$\sup_\theta \mathbb{E}\Big[\frac{Z_T^\alpha}{\alpha}\Big] = \frac{(z_0)^\alpha}{\alpha}e^{\alpha rT} exp\{\int_0^T \big[(1+\tilde{\theta}_t^* K_t^1)^\alpha\lambda_1(t) + (1-\tilde{\theta}_t^* K_t^2)^\alpha\lambda_2(t) - \alpha\tilde{\theta}_t^* - \lambda(t)\big]dt\}$$

and this concludes the proof.

□

Lemma 4.4 *Let be N_t a double stochastic Poisson process with intensity $\lambda(t)$. Then for any bounded deterministic process $c(t)$*

$$\mathbb{E}\Big[e^{\int_0^T c(t)dN_t}\Big] = exp\{\int_0^T (e^{c(t)} - 1)\lambda(t)dt\}.$$

Proof.

It is sufficient to consider $c(t) = 1\!\!1_{(t_1,t_2]}$, with $t_1 < t_2 < T$.

$$\mathbb{E}\Big[e^{\int_0^T c(t)dN_t}\Big] = \mathbb{E}\Big[e^{N_{t_2} - N_{t_1}}\Big] = \sum_k \frac{e^k}{k!}\Big(\int_{t_1}^{t_2} \lambda(t)dt\Big)^k e^{-\int_{t_1}^{t_2}\lambda(t)dt} =$$

$$= e^{\int_{t_1}^{t_2}(e-1)\lambda(t)dt} = exp\{\int_0^T (e^{c(t)} - 1)\lambda(t)dt\}.$$

Finally the assertion follows by dominated convergence results.

□

5. Investment models with intermediate consumption

In this section we examine the combined problem of portfolio selection and consumption rules. We assume that the individual preferences are modeled through a CRRA utility and a bequest function for the consumption of the same risk aversion.

A single agent manages his portfolio by investing in a bond and in a stock account. The processes, that the prices of the two assets follow, are the same as in Section 2.
The investor starts with initial capital $z_0 > 0$ and invests at any time $t \in [0,T]$ the amount $\theta_t \frac{S_t}{S_{t-}}$ in the risky asset and his remaining wealth, $Z_t - \theta_t \frac{S_t}{S_{t-}}$, in the bond. He also consumes out of his bond holdings at the rate C_t. The wealth process Z_t evolves according to

$$dZ_t = \frac{\theta_t}{S_{t^-}} dS_t + (Z_t - \theta_t \frac{S_t}{S_{t^-}}) \frac{dB_t}{B_t} - C_t dt = \theta_t \Big(K_1(t, S_{t^-}) dN_t^1 - K_2(t, S_{t^-}) dN_t^2 \Big) + (Z_t - \theta_t) r dt - C_t dt, \quad Z_0 = z_0 \tag{5.1}$$

The pair of control processes (θ_t, C_t) is said to be admissible if θ_t is a $I\!R$-valued $(P, \mathcal{F}_t)$-predictable process and C_t a non negative $\mathcal{F}_t$-progressively measurable process such that

$$\int_0^T |\theta_t| dt < +\infty, \qquad \int_0^T C_t dt < +\infty \quad P - a.s. \tag{5.2}$$

and there exists a unique solution to equation (**??**) satisfying $I\!E|Z_t| < +\infty, \forall t \in [0,T]$, and the state constraint

$$Z_t > 0 \quad a.e. t \in [0,T].$$

We denote by $\mathcal{A}$ the set of admissible policies.

Lemma 5.1 *The set of admissible investment-consumption strategies $\mathcal{A}$ contains the following set of Markovian policies*

$$\mathcal{A}_1 = \Big\{ (\theta_t, C_t) = (\tilde{\theta}(t, S_{t^-}) Z_{t^-}, \tilde{C}(t, S_t) Z_t) : \tilde{\theta}(t,y) \in \Big(\frac{-1}{K_1(t,y)}, \frac{1}{K_2(t,y)} \Big), \tilde{C}(t,y) \geq 0 \text{ and bounded} \Big\}. \tag{5.3}$$

Moreover, the wealth associated to such strategies is strictly positive and is given by

$$Z_t^{\theta, C} = Z_t^{\theta} e^{-\int_0^t \tilde{C}(u, S_u) du} \tag{5.4}$$

*where Z_t^θ is the wealth associated to the investment strategy $\theta_t = \tilde{\theta}(t, S_{t^-}) Z_{t^-}$ given in (**??**).*

Proof.
We have that the wealth associated to Markov control policies in $\mathcal{A}_1$ satisfies

$$dZ_t^{\theta, C} = Z_{t^-}^{\theta, C} d\tilde{M}_t$$

where

$$\tilde{M}_t = \int_0^t \tilde{\theta}(u, S_{u^-}) \Big(K_1(u, S_{u^-}) dN_u^1 - K_2(u, S_{u^-}) dN_u^2 \Big) + \int_0^t \Big((1 - \tilde{\theta}(u, S_{u^-})) r - \tilde{C}(u, S_u) \Big) du.$$

Hence, by the Doléans-Dade exponential formula, $Z_t^{\theta, C}$ is well defined on $t \in [0,T]$ and given by

$$Z_t^{\theta, C} = Z_t^{\theta} e^{-\int_0^t \tilde{C}(u, S_u) du} > 0$$

where $Z_t^{\tilde{\theta}}$ is the wealth associated to the investment strategy $\theta_t = \tilde{\theta}(t, S_{t^-}) Z_{t^-}$ given in (**??**).
Since $Z_t^{\theta, C} \leq Z_t^\theta$ Lemma **??** implies $I\!E(Z_t) < +\infty$ and

$$\int_0^T |\theta_t| dt < +\infty \qquad \int_0^T C_t dt < +\infty \quad P - a.s.$$

and this concludes the proof. □

The investor's objective is to choose a portfolio-consumption strategy in a such way to maximize

$$\mathbb{E}\Big[\int_0^T U_\alpha(C_t)dt + U_\alpha(Z_T)\Big], \tag{5.5}$$

where U_α is a CRRA utility function defined in (**??**).
The associated value function is

$$V(t,z,y) = \sup_{(\theta.C)\in\mathcal{A}} \mathbb{E}\Big(\int_0^T U_\alpha(C_s)ds + U_\alpha(Z_T) \mid Z_t = z, S_t = y\Big)$$

and the Hamilton-Jacobi-Bellman equation for the optimal investment/consumption problem is given by

$$\frac{\partial V}{\partial t}(t,z,y) + \sup_{\theta,C}\{\mathcal{L}_t^{\theta,C} V(t,y,z) + U_\alpha(C)\} = 0, \quad t\in(0,T), y>0, z>0 \tag{5.6}$$

with the terminal condition $V(T,z,y) = U_\alpha(z)$, where, for constants (θ,C), $\mathcal{L}^{\theta,C}$ denotes the generator of the Markov process (Z_t, S_t)

$$\mathcal{L}_t^{\theta,C} = \frac{\partial V}{\partial z}(t,z,y)\big[(z-\theta)r - C\big] +$$

$$\Big(V\big(t, z+\theta K_1(t,y), y(1+K_1(t,y))\big) - V(t,z,y)\Big)\lambda_1(t) + \Big(V\big(t, z-\theta K_2(t,y), y(1-K_2(t,y))\big) - V(t,z,y)\Big)\lambda_2(t).$$

We first deal with the logarithmic case.

Theorem 5.2 *There exists an optimal strategy (θ_t^*, C_t^*), where $\theta_t^* = \tilde{\theta}^*(t, S_{t^-})Z_{t^-}^*$, with $\tilde{\theta}^*(t,y)$ unique solution to (**??**) and $C_t^* = \frac{Z_t^*}{1+T-t}$.*
The value function is given by

$$V_0^C(t,z,y) = (1+T-t)\ln z + h^C(t,y) \tag{5.7}$$

where

$$h^C(t,y) = \mathbb{E}\big[\int_t^T \big(H\tilde{(}\theta^*)(u, S_{u^-})(1+T-u) - 1 - \ln(1+T-u)\big)du | S_t = y\big] \tag{5.8}$$

*with $H(\tilde{\theta}^*)(t,y)$ defined in (**??**) .*

Proof.
The Hamilton-Jacobi-Bellman equation for the optimal investment/consumption problem is given by

$$\frac{\partial V}{\partial t}(t,z) + \sup_{\theta,C}\{\mathcal{L}_t^{\theta,C} V(t,z) + \log C)\} = 0, \quad t\in(0,T), y\in\mathbb{R}, z>0 \tag{5.9}$$

with the terminal condition $V(T,z) = \log z$
We look for a candidate solution in the form $(1+T-t)\log z + h^C(t,y)$, hence $h^C(t,y)$ solves

$$\frac{\partial h^C}{\partial t}(t,y) + \mathcal{L}_t^S h^C(t,y) + \sup_{\tilde{\theta},\tilde{C}}\Big[\big(H(\tilde{\theta})(t,y) - \tilde{C}\big)(1+T-t) + \ln\tilde{C}\Big] = 0 \quad t\in(0,T), \quad h(T,y) = 1. \tag{5.10}$$

where $H(\tilde{\theta})(t,y)$ is defined in (**??**), $\mathcal{L}_t^S$ is the generator of the Markov process S_t given in (**??**) and the control $(\tilde{\theta},\tilde{C})$ corresponds to $(\frac{\theta}{z}, \frac{C}{z})$ with (θ, C) being the control variable appearing in (**??**).

The maximum over $\tilde{\theta}$ is achieved at $\tilde{\theta}^*(t,y)$, unique solution to (**??**) and the maximum over $\tilde{C}$ at

$$\tilde{C}^*(t) = \frac{1}{1+T-t}$$

Inserting this expression into equation (**??**) yields

$$\frac{\partial h}{\partial t}(t,y) + \mathcal{L}_t^S h(t,y) + K(\tilde{\theta}^*)(t,y) = 0, \quad h(T,y) = 0. \tag{5.11}$$

with

$$K(\tilde{\theta}^*)(t,y) = (1+T-t)\ H(\tilde{\theta}^*)(t,y) - 1 - \log(1+T-t)$$

whose solution, by Lemma **??**, is given by

$$h^C(t,y) = \mathbb{E}\Big[\int_t^T K(\tilde{\theta}^*)(r, S_{r^-})dr \mid S_t = y\Big].$$

By Lemma **??**, $(\theta_t^*, C_t^*) \in \mathcal{A}$, where

$$\theta_t^* = \tilde{\theta}^*(t, S_{t^-})Z_{t^-}^*, \quad C_t^* = \frac{Z_t^*}{1+T-t}$$

Finally, Verification results allows us to conclude that $V_0^C(t,z,y)$ given in (**??**) is the value function and (θ_t^*, C_t^*) is an optimal markovian investment-consumption strategy.
□

We now discuss the power law utility case.

Theorem 5.3 *(i) When the class of admissible investment strategies reduces to $\mathcal{A}_1$ defined in (**??**), then the associated value function is of the form*

$$V_\alpha^{1,C}(t,y,z) = \frac{z^\alpha}{\alpha} h^{1,C}(t,y), \tag{5.12}$$

where $h^{1,C}(t,y)$ is a bounded function given by

$$h^{1,C}(t,y) = \sup_{(\tilde{\theta},\tilde{C})} \mathbb{E}^{\tilde{\theta}}\Big[\int_t^T \tilde{C}(u,S_u)^\alpha\ e^{\int_t^u G_\alpha(\tilde{\theta},\tilde{C})(s,S_s)ds} + e^{\int_t^T G_\alpha(\tilde{\theta},\tilde{C})(s,S_s)ds} \mid S_t = y\Big], \tag{5.13}$$

*where $\mathbb{E}^{\tilde{\theta}}$ denotes the expected value under $P^{\tilde{\theta}}$, that is the probability measure defined in Lemma **??**,*

$$G_\alpha(\tilde{\theta},\tilde{C})(t,y) = H_\alpha(\tilde{\theta})(t,y) - \lambda(t) - \alpha\tilde{C}(t,y) \tag{5.14}$$

*and $H_\alpha(\tilde{\theta})(t,y)$ given in (**??**) replacing $\tilde{\theta}^*(t)$ by $\tilde{\theta}(t,y)$ and $K_i(t)$ by $K_i(t,y)$, $i = 1,2$.*

(ii) (Verification result) If there exists a bounded solution $h^C(t,y)$, absolutely continuous w.r.t. t, to

$$\frac{\partial h}{\partial t}(t,y) + \sup_{(\tilde{\theta},\tilde{C})}\Big(\mathcal{L}_t^{\tilde{\theta},S} h(t,y) + G_\alpha(\tilde{\theta},\tilde{C})(t,y)\ h(t,y) + \tilde{C}^\alpha\Big) = 0, \quad h(T,y) = 1, \tag{5.15}$$

*where $\mathcal{L}_t^{\tilde{\theta},S}$ is defined in (**??**), then the value function is given by*

$$V_\alpha^C(t,y,z) = \frac{z^\alpha}{\alpha} h^C(t,y)$$

*and there exists an optimal strategy (θ_t^*, C_t^*), where $\theta_t^* = \tilde{\theta}^*(t,S_{t^-})Z_{t^-}^*$, with $\tilde{\theta}^*(t,y)$ unique solution to (**??**) and $C_t^* = \tilde{C}^*(t,S_t)Z_t^*$ with $\tilde{C}^*(t,y) = h(t,y)^{\frac{1}{\alpha-1}}$.*

Moreover the following Feynman-Kac representation holds

$$h^C(t,y) = \mathbb{E}^{\tilde{\theta}^*}\Big[\int_t^T \tilde{C}^*(u,S_u)^\alpha\ e^{\int_t^u G_\alpha(\tilde{\theta}^*,\tilde{C}^*)(s,S_s)ds}du + e^{\int_t^T G_\alpha(\tilde{\theta}^*,\tilde{C}^*)(s,S_s)ds} \mid S_t = y\Big]. \tag{5.16}$$

Proof.
(i) From Lemma (**??**), the wealth associated to Markov control policies in $\mathcal{A}_1$ satisfies

$$Z_T^{\theta,C} = Z_t^{\theta,C} e^{R_T - R_t}$$

where

$$R_t = \int_0^t \Big(\log\big(1+\tilde{\theta}(u,S_{u^-})K_1(u,S_{u^-})\big)dN_u^1 + \log\big(1-\tilde{\theta}(u,S_{u^-})K_2(u,S_{u^-})\big)dN_u^2\Big) + \int_0^t \big((1-\tilde{\theta}(u,S_{u^-}))r - \tilde{C}(u,S_u)\big)du.$$

Moreover, $\forall u \in [t,T]$

$$C_u = \tilde{C}(u,S_u)\; Z_t^{\theta,C} e^{R_u - R_t}.$$

Therefore, we have

$$I\!E\Big(\int_t^T \frac{C_u^\alpha}{\alpha}du + \frac{Z_T^\alpha}{\alpha} \mid Z_t = z, S_t = y\Big) = \frac{z^\alpha}{\alpha} I\!E\Big(\int_t^T \tilde{C}(u,S_u)^\alpha\; e^{\alpha(R_u - R_t)}du + e^{\alpha(R_u - R_t)} \mid S_t = y\Big) =$$

$$= I\!E\Big(\int_t^T \tilde{C}(u,S_u)^\alpha\; \frac{L_u^{\tilde{\theta}}}{L_t^{\tilde{\theta}}}\; e^{\int_t^u G_\alpha(\tilde{\theta},\tilde{C})(s,S_s)ds} du + \frac{L_T^{\tilde{\theta}}}{L_t^{\tilde{\theta}}}\; e^{\int_t^T G_\alpha(\tilde{\theta},\tilde{C})(s,S_s)ds} \mid S_t = y\Big)$$

which implies (**??**) and (**??**).

(ii) The proof follows the same lines of that of Theorem **??**. We have that $V_\alpha^C(t,y,z) = \frac{z^\alpha}{\alpha} h^C(t,y)$ is a classical solution to the HJB-equation

$$\frac{\partial V_\alpha^C}{\partial t}(t,z,y) + \sup_{(\theta,C)} \Big(\mathcal{L}_t^{\theta,C} V_\alpha^C(t,y,z) + \frac{C^\alpha}{\alpha}\Big) = 0, \quad t \in (0,T), y > 0, z > 0 \tag{5.17}$$

with the terminal condition $V_\alpha^C(T,z,y) = \frac{z^\alpha}{\alpha}$ and where the control (θ, C), corresponds to $(\tilde{\theta}z, \tilde{C}z)$, with $(\tilde{\theta}, \tilde{C})$ being the control variable appearing in (**??**).
Moreover, the pair $(\theta^*(t,y,z) = \tilde{\theta}^*(t,y)z, C^*(t,y,z) = \tilde{C}^*(t,y)z)$ realizes the supremum in (**??**). By Verification results $V_\alpha^C(t,y,z)$ is the value function and $(\theta_t^* = \tilde{\theta}^*(t,S_{t^-})Z_{t^-}^*,\ C_t^* = \tilde{C}^*(t,S_t)Z_t^*) \in \mathcal{A}_1$ is an optimal investment-consumption strategy.
Finally, by applying Feynman-Kac formula we have the representation (**??**). □

As in Section 3, to exhibit closed-form solution for the value function we will consider linear dynamics for the stock price given in (**??**).
The value function is now given by

$$V_\alpha^C(t,z) = \sup_{(\theta,C)\in\mathcal{A}} I\!E\Big(\int_0^T U_\alpha(C_s)ds + U_\alpha(Z_T) \mid Z_t = z\Big).$$

and the wealth dynamics

$$dZ_t = \frac{\theta_t}{S_{t^-}}dS_t + (Z_t - \theta_t\frac{S_t}{S_{t^-}})\frac{dB_t}{B_{t^-}} = \theta_t\big(K_1(t)dN_t^1 - K_2(t)\big)dN_t^2) + (Z_t - \theta_t)rdt - C_t dt, \quad Z_0 = z_0 > 0. \tag{5.18}$$

Theorem 5.4 *There exists an optimal strategy (θ_t^*, C_t^*), where $\theta_t^* = \tilde{\theta}^*(t)Z_{t^-}^*$, with $\tilde{\theta}^*(t)$ the unique solution to (**??**) and $C_t^* = \tilde{C}^*(t)Z_t^*$ with $\tilde{C}^*(t) = \frac{1}{p(t)}$.*
The value function is given by

$$V_C(t,z) = \frac{z^\alpha}{\alpha}\; p(t)^{1-\alpha}, \tag{5.19}$$

$$p(t) = e^{\int_t^T a(r)dr}[1 + \int_t^T e^{-\int_s^T a(r)dr} ds], \quad a(t) = \frac{1}{1-\alpha}(H_\alpha(\theta^*)(t) - \lambda(t)) \tag{5.20}$$

with $H_\alpha(\theta^*)(t)$ *defined in* (**??**) .

Proof.

The Hamilton-Jacobi-Bellman equation for the optimal investment/consumption problem is given by

$$\frac{\partial V}{\partial t}(t,z) + \sup_{\theta,C}\{\mathcal{L}_t^{\theta,C} V(t,z) + U_\alpha(C)\} = 0, \quad t \in (0,T), y \in I\!R, z > 0 \tag{5.21}$$

with the terminal condition $V(T,z) = U_\alpha(z)$

We look for a candidate solution in the form $\frac{z^\alpha}{\alpha}h(t)$ hence $h(t)$ solves

$$\frac{dh}{dt}(t) - h(t)\lambda(t) + \sup_{\tilde{\theta},\tilde{C}}\Big[\big(H_\alpha(\tilde{\theta})(t) - \alpha\tilde{C}\big)h(t) + \tilde{C}^\alpha\Big] = 0 \quad t \in (0,T), \quad h(T) = 1. \tag{5.22}$$

where $H_\alpha(\tilde{\theta})(t)$ is defined in (**??**). Note that the control $(\tilde{\theta}, \tilde{C})$ corresponds to $(\frac{\theta}{z}, \frac{C}{z})$ with (θ, C) being the control variable appearing in (**??**).

The maximum over $\tilde{\theta}$ is achieved at $\tilde{\theta}^*(t)$, unique solution to (**??**), and the maximum over $\tilde{C}$ at $\tilde{C}^*(t) = h(t)^{\frac{1}{\alpha-1}}$. Using the form of $\tilde{C}^*(t)$ in equation (**??**) yields

$$\frac{dh}{dt}(t) + (H_\alpha(\tilde{\theta}^*)(t) - \lambda(t))h(t) + (1-\alpha)h(t)^{\frac{\alpha}{\alpha-1}} = 0 \quad t \in (0,T), \quad h(T) = 1. \tag{5.23}$$

We now make the classical transformation $h(t) = p(t)^{1-\alpha}$, which gives

$$\frac{dp}{dt}(t) + \frac{1}{1-\alpha}(H_\alpha(\tilde{\theta}^*)(t) - \lambda(t))p(t) + 1 = 0 \quad t \in (0,T), \quad p(T) = 1, \tag{5.24}$$

whose solution is given by (**??**). Finally admissibility of (θ_t^*, C_t^*) follows by Lemma **??**. □

References

[1] N. Bellamy, *Wealth optimization in an incomplete market driven by a jump-diffusion process*, Journal of Mathematical Economics, **35**, (2001) 259-287

[2] F. Bellini and M. Frittelli, On the existence of minimax martingale measures, Mathematical Finance, **12**, (2002) 1-21.

[3] S. Biagini and M. Frittelli, Utility maximization in incomplete markets for unbounded processes, Finance and Stochastics, **9**, (2005) 493-517.

[4] F.E. Benth, K.H. Karlsen and K. Reikvam, *Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: a viscosity solution approach*, Finance and Stochastics **5** (3) (2001) 275-303.

[5] C. Ceci, *An HJB approach to exponential utility maximization for jump processes*, Tech.Report R-2007-001 Dip.di Scienze, Univ. di Chieti-Pescara, http://www.sci.unich.it/tecrep/. To appear in International Journal of Risk Assessment and Management (2008).

[6] C. Ceci and B. Bassan, *Mixed optimal stopping and stochastic control problems with semicontinuous final reward for diffusion processes*, Stochastics and Stochastics Reports **76** (4) (2004) 323-337.

[7] M.G. Crandall and P.L.Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. A.M.S.,**277** (1983) 1-42.

[8] G.Callegaro and T.Vargiolu, *Optimal portfolio for HARA utility functions in a pure jump multidimensional incomplete market*, To appear in International Journal of Risk Assessment and Management (2008).

[9] C. Doléans-Dade, *Quelques applications de la formule de changement de variables pour le semi-martingales*, Z. fur W. **16** (1970) 181-194.

[10] W. Flemig and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, New York, Springer (1993).

[11] D.G. Luenberger, *Optimization by vector space methods*, Wiley New York (1969).

[12] R. Merton, *Optimal consumption and portfolio rules in a continuous time model*, Journal of Economic Theory, **3** (1971) 373-413.

[13] H. Pham, *Optimal stopping of controlled jump diffusion processes: a viscosity solution approach*, Journal of Mathematical System, Estimation, and Control, **8** (1) (1998) 1-27.

[14] H. Pham, *Smooth solution to optimal investment models with stochastic volatilities and portfolio constraints*, Appl. Math. Optim., **78** (2002) 55-78.

[15] W. Schachermayer, *Utility maximization in incomplete market* in: Stochastic Methods in Finance, M.Frittelli and W.J. Runggaldier eds., Springer-Verlag (2004) 255-293.

[16] T. Zariphopoulou, *Consumption investment models with constraints*, SIAM J. Control and Optimization, **30** (1994) 59-84.

[17] T. Zariphopoulou, *A solution approach to valuation with unhedgeable risks*, Finance and Stochastics, **5** (2001) 61-82.