AN HJB APPROACH TO EXPONENTIAL UTILITY MAXIMIZATION FOR JUMP PROCESSES

Claudia Ceci
Dipartimento di Scienze, Facolta' di Economia
Universita' di Chieti-Pescara, I-65127-Pescara, Italy

Abstract

This paper deals with the problem of exponential utility maximization in a model where the risky asset price S is a geometric marked point process whose dynamics depend on another process X, referred to as the stochastic factor. The process X is modeled as a jump diffusion process which may have common jump times with S. The classical dynamic programming approach leads to characterize the value function as a solution of the Hamilton-Jacobi-Bellman equation. The solution together with the optimal strategy can be computed under suitable assumptions. Moreover, an explicit representation of the density of the minimal entropy measure (MEMM) and a duality result which gives a relationship between the utility maximization problem and the MEMM are given. This duality result is obtained for a class of strategies greater than those usually considered in literature. A discussion on the pricing of a European claim by the utility indifference approach and its asymptotic variant is performed.

Keywords: Utility Maximization, Minimal Entropy Measure, Marked Point Processes, Jump-Diffusions.

1. Introduction

A basic problem in Mathematical Finance is the problem of an investor who maximizes his expected utility of terminal wealth. This optimal investment model was introduced by Merton ([14]) and later studied by several authors (see for example [20, 21, 15, 16] and references therein). Most of the literature in this subject is based on the assumption that the prices of the underlying assets follow a diffusion-type process. The contribution of this paper in the area of optimal portfolio management is to solve a utility maximization problem in a model where asset prices may exhibit a jumping behaviour and are affected by correlated stochastic factors.

Recently, with the advent of intraday information on financial asset price quotes, a research in finance has been devoted to models with jumps ([18, 19, 8, 9, 12, 17]). In fact real asset prices, on a very small time scale, are piecewise constant and jump only at discrete points in times in reaction to trades or to significant new informations. Therefore, it is sensible to suppose that prices are described by marked point processes. Moreover their dynamics can be directed by another process, which may describe the activity of other markets, macroeconomics factors or microstructure rules that drive the market. This latent process may be considered as an unobservable variable ([4, 8, 9]) or as a stochastic factor fully observable across time, which is the point of view of this paper.

In this note we consider an agent with exponential utility function which invests in a bond and a risky asset. The stock price S of the risky asset is modeled as a geometric marked point process whose dynamics may depend by a stochastic factor X, described by a Markov jump diffusion process, correlated with S. More precisely, the two processes may have common jump times, which means that the trading activity may effect the law of X and could be also related to the presence of catastrophic events. The motivation to study this class of investment models comes from their wide applicability. Our model fits into the general frameworks with nonlinear stock where the stochastic factor can be identified with the nonlinear component or with the presence of a non-traded asset which price dynamics is correlated with the stock price. In [21] the case where the risky asset price is given by a diffusion process with coefficients depending on a stochastic factor described by a correlated diffusion process has been studied for Constant Relative Risk Aversion individual preferences. In [15] a similar model has been treated to hedge a contingent claim written on a non-traded asset correlated with the stock price. As in [21] we will consider X as a stochastic factor fully observed across time.

When the stock price is a locally bounded semimartingale, via a duality result, the utility maximization problem is in relationship with the minimal entropy martingale (MEMM) measure ([1, 6, 13] and references therein). Two different approaches are so proposed in literature to solve the optimal investment problem. The first approach relies on the theory of stochastic control, which is that chosen in this note, the second studies the duality problem investigating on the MEMM. In Markovian settings, which is our case, one may use the two methodologies.

Herein, as just said before, we treat our utility maximization problem by stochastic control methods. We write down the Hamilton-Jacobi-Bellman equation and under suitable assumptions we are able to compute explicitly the value function and the optimal strategy. Moreover we give an explicit representation for the density of the MEMM and the stochastic control approach allows us to state a duality relation even if the asset price is not locally bounded and for a class of admissible strategies which is in general greater than those considered in literature (see [6]). In [5] an explicit representation for the MEMM has been provided under stronger assumptions which imply that the asset price is locally bounded. In [11] the MEMM has been investigated when the asset price is given by a geometric Lévy process. Our model could not be viewed as a particular case of that discussed in [11] since in our context the jump intensity is not constant and the jump size is a stochastic process.
A discussion about the pricing of a European derivative, according to the so called indifference valuation, is performed in the last section. An asymptotic variant of this approach leads to choose the MEMM as pricing measure.

2. The Model

The model here considered is a particular case of that studied in [3]. On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, we consider a finite time horizon model on $[0, T]$ with one riskless money market account and a risky asset. The price of the risk-free asset is taken equal to 1, i.e. the risk-free interest rate is assumed equal to zero. The price S of the risky asset is a geometric marked point process given by

$$S_t = S_0 e^{Y_t} \qquad S_0 \in I\!R^+. \tag{2.1}$$

The logreturn process Y is defined as

$$Y_t = \sum_{n=0}^{N_t} Z_n, \quad Z_0 = 0, \quad N_t = \sum_{n \geq 1} 1\!\!1_{\{T_n \leq t\}}$$

where $Z_n = Y_{T_n} - Y_{T_{n-1}}$ is the size of the n^{th} logreturn change and N is the point process which counts the total number of changes.

We assume that the process Y, described by the double sequence $\{T_n, Z_n\}_{n \geq 0}$, is a marked point process whose dynamics are affected by another process X which will be referred to as the stochastic factor. The dynamics of the pair (X, Y) are governed by the following system

$$X_t = x_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \int_Z K_0(s, X_{s^-}; \zeta)\, \mathcal{N}(ds, d\zeta) \tag{2.2}$$

$$Y_t = \int_0^t \int_Z K_1(s, X_{s^-}, Y_{s^-}; \zeta)\, \mathcal{N}(ds, d\zeta) \tag{2.3}$$

where $x_0 \in I\!R$, W_t is a $(P, \mathcal{F}_t)$-standard Brownian motion, $\mathcal{N}(dt, d\zeta)$ is a $(P, \mathcal{F}_t)$-Poisson random measure on $I\!R^+ \times Z$, independent of W_t, with mean measure $dt\, \nu(d\zeta)$, with $\nu(d\zeta)$ a σ-finite measure on a measurable space $(Z, \mathcal{Z})$. The $I\!R$-valued functions $b(x)$, $\sigma(x)$, $K_0(t, x; \zeta)$ and $K_1(t, x, y; \zeta)$ are jointly measurable functions of their arguments.

The dynamics proposed in (2.2) and (2.3) allow common jump times between X and Y, which implies that the law of the process X can be affected by the actual trading activity. Moreover, it could also be related

to the presence of catastrophic events. This kind of events, in fact, influences both the asset prices and the hidden state variable which drives their dynamics.
More precisely, in our setting the quadratic covariation of Y and X is given by

$$[X,Y]_t = \sum_{s\leq t} \Delta X_s \Delta Y_s = \int_0^t \int_Z K_0(s, X_{s^-}; \zeta) K_1(s, X_{s^-}, Y_{s^-}; \zeta) \nu(d\zeta) ds$$

and it is different from zero if there are common jump times between X and Y.

Suitable assumptions (see [3], [4] and references therein) can be done on the model, in order to assure existence and uniqueness to the system (2.2), (2.3). Overall this paper we shall assume existence and uniqueness to that system.

Let us define

$$D_1(t,x,y) = \{\zeta \in Z : K_1(t,x,y;\zeta) \neq 0\}, \quad D_0(t,x,y) = \{\zeta \in Z : K_0(t,x;\zeta) \neq 0. \quad K_1(t,x,y;\zeta) = 0\} \tag{2.4}$$

and assume in the sequel

$$I\!\!E \int_0^T \nu\big(D_i(t, X_{t^-}, Y_{t^-})\big) dt < +\infty \quad i = 0,1. \quad I\!\!E \int_0^T \sigma(X_t)^2 dt < +\infty. \tag{2.5}$$

In our frame the pair (X,Y) is a Markov process whose generator is given by

$$\mathcal{L}f(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + \mathcal{L}_t f(t,x,y) = \tag{2.6}$$

$$\frac{\partial f}{\partial t}(t,x,y) + b(x)\ \frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 f}{\partial x^2}(t,x,y) +$$

$$\int_Z \Big(f\big(t, x + K_0(t,x;\zeta), y + K_1(t,x,y;\zeta)\big)\big) - f(t,x,y) \Big)\ \nu(d\zeta).$$

More precisely, by Itô formula we get that, for real-valued, bounded continuous functions $f(t,x,y)$ such that $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ are bounded and continuos

$$f(t, X_t, Y_t) - f(0, x_0, 0) - \int_0^t \mathcal{L}f(s, X_s, Y_s) ds$$

is a $\{P, \mathcal{F}_t\}$-martingale.

We shall consider a particular model for the dynamics of Y, assuming $K_1(t,x,y;\zeta)$ of the following form

$$K_1(t,x,y;\zeta) = K_1^+(t,x,y)\ 1\!\!1_{D_1^+(t)}(\zeta) - K_1^-(t,x,y)\ 1\!\!1_{D_1^-(t)}(\zeta) \tag{2.7}$$

where K_1^+ and K_1^- are real valued positive measurable functions, and that $\lambda^+(t) = \nu(D_1^+(t))$ and $\lambda^-(t) = \nu(D_1^-(t))$ are both positive.
Under (2.7) and (2.5), the logreturn process Y_t has a simplified but non trivial structure given by

$$Y_t = \int_0^t K_1^+(s, X_{s^-}, Y_{s^-})\ dN_s^+ - \int_0^t K_1^-(s, X_{s^-}, Y_{s^-})\ dN_s^- \tag{2.8}$$

where the processes

$$N_t^+ = \mathcal{N}\big((0,t), D_1^+(t)\big) \quad \text{and} \quad N_t^- = \mathcal{N}\big((0,t), D_1^-(t)\big)$$

are independent double stochastic Poisson processes (or conditional Poisson) with $\{P, \mathcal{F}_t\}$-intensities $\lambda^+(t)$ and $\lambda^-(t)$ respectively, that is, for every $s \leq t$, $k \in N$

$$P(N_t - N_s = k \mid \mathcal{F}_s) = e^{-\int_s^t \lambda(r) dr} \frac{(\int_s^t \lambda(r) dr)^k}{k!}$$

with $N_t = N_t^+$, $\lambda(r) = \lambda^+(r)$ and $N_t = N_t^-$, $\lambda(r) = \lambda^-(r)$. Observe that the process which counts the total number of jumps is given by $N_t = N_t^+ + N_t^-$ and that N_t is also a double stochastic Poisson process with $\{P, \mathcal{F}_t\}$-intensity $\lambda^+(t) + \lambda^-(t)$.

By applying Itô formula taking into account (2.8) we get that the price of the stock satisfies

$$dS_t = S_{t^-}\Big((e^{K_1^+(t,X_{t-},log(S_{t-}/S_0)} - 1)dN_t^+ + (e^{-K_1^-(t,X_{t-},log(S_{t-}/S_0)} - 1)dN_t^-\Big). \tag{2.9}$$

As a particular case of Proposition 2.3 in [5] we have the following semimartingale structure for risky asset price process S.

Proposition 2.1 *Under the condition*

$$\int_0^T \Big((e^{K_1^+(t,X_t,Y_t)} - 1)^2\lambda^+(t) + (e^{-K_1^-(t,X_t,Y_t)} - 1)^2\lambda^-(t)\Big)dt < + + \infty \quad P-a.s. \tag{2.10}$$

S is a special semimartingale with the decomposition

$$S_t = S_0 + M_t + A_t \tag{2.11}$$

where

$$A_t = \int_0^t S_{r^-}\Big((e^{K_1^+(r,X_{r-},Y_{r-})} - 1)\lambda^+(r) + (e^{-K_1^-(r,X_{r-},Y_{r-})} - 1)\lambda^-(r)\Big)dr$$

is a predictable process with locally bounded variation paths,

$$M_t = \int_0^t S_{r^-}(e^{K_1^+(r,X_{r-},Y_{r-};\zeta)} - 1)(dN_r^+ - \lambda^+(r)dr) + \int_0^t S_{r^-}(e^{-K_1^-(r,X_{r-},Y_{r-};\zeta)} - 1)(dN_r^- - \lambda^-(r)dr)$$

is a locally square-integrable martingale whose angle process is given by

$$< M >_t = \int_0^t S_{r^-}^2\Big((e^{K_1^+(r,X_{r-},Y_{r-})} - 1)^2\lambda^+(r) + (e^{-K_1^-(r,X_{r-},Y_{r-})} - 1)^2\lambda^-(r)\Big)dr. \tag{2.12}$$

Observe that condition (2.10) in particular implies

$$\int_0^T\int_Z \mid e^{K_1(t,X_t,Y_t;\zeta)} - 1 \mid \nu(d\zeta)dt = \int_0^T\Big(e^{K_1^+(t,X_t,Y_t)} - 1)\lambda^+(t) + (1 - e^{-K_1^-(t,X_t,Y_t)} - 1)\lambda^-(t)\Big)dt < + + \infty \quad P-a.s. \tag{2.13}$$

From now on we shall assume all the hypotheses made in this section.

3. Exponential utility maximization

We consider the following expected utility problem. An agent with exponential preferences and initial capital $c_0 > 0$ invests at any time $t \in [0, T]$ the amount θ_t in the risky asset S and his remaining wealth in the bond. The wealth process Z_t evolves according to

$$dZ_t = \theta_t \frac{dS_t}{S_{t^-}} = \theta_t\Big((e^{K_1^+(t,X_{t-},Y_{t-})} - 1)dN_t^+ + (e^{-K_1^-(t,X_{t-},Y_{t-})} - 1)dN_t^-\Big), \quad Z_0 = c_0. \tag{3.1}$$

The equation (3.1) describes the dynamics of the wealth process controlled by the proportion-investment process θ_t. A strategy θ_t is said admissible if it is $(P, \mathcal{F}_t)$-predictable and $\frac{\theta_t}{S_{t^-}}$ is S-integrable, that is the following integrability condition is satisfied

$$E\Big(\int_0^T \mid \theta_t \mid \Big((e^{K_1^+(t,X_{t-},Y_{t-})} - 1)\lambda^+(t) + (1 - e^{-K_1^-(t,X_{t-},Y_{t-})})\lambda^-(t)\Big)dt\Big) < + + \infty.$$

We denote by Θ the set of admissible policies.

We consider an agent with exponential utility function given by

$$U_\alpha(x) = 1 - e^{-\alpha x},$$

with the risk aversion parameter $\alpha \in I\!R^+$. The investor's objective is to maximize his expected utility of terminal wealth

$$E\Big[1 - \exp\big\{ -\alpha(c_0 + \int_0^T \frac{\theta_r}{S_{r-}} dS_r)\big\}\Big] = E\Big[1 - \exp\big\{ -\alpha Z_T\big\}\Big]$$

By considering the utility maximization problem as a stochastic control problem with only final reward, we introduce the associated value function

$$V(t,x,y,z) = sup_{\theta\in\Theta} E\Big(1 - \exp\big\{ -\alpha Z_T\big\} \mid X_t = x, Y_t = y, Z_t = z\Big) = 1 - W(t,x,y,z) \tag{3.2}$$

where

$$W(t,x,y,z) = inf_{\theta\in\Theta} E\Big(\exp\big\{ -\alpha Z_T\big\}\Big) \mid X_t = x, Y_t = y, Z_t = z\Big). \tag{3.3}$$

Observe that we need to work with the triple (X_t, Y_t, Z_t) in order to ensure, for any constant policy, a markovian dynamics. In fact, the dynamics of Z depends explicitly on both the processes X and Y. We outline below the main result of this section.

Proposition 3.1 *Assume (2.5), (2.13) and*

$$\Gamma(t,x,y) = \frac{1 - e^{-K_1^-(t,x,y)}}{e^{K_1^+(t,x,y)} - 1} \tag{3.4}$$

only function of t denoted by $\Gamma(t)$, such that

$$\int_0^T \lambda^-(t)\Gamma(t)dt < +\infty, \qquad \int_0^T \frac{\lambda^+(t)}{\Gamma(t)} dt < +\infty. \tag{3.5}$$

Then

$$W(t,x,y,z) = e^{-\alpha z} e^{-\int_t^T b^*(r)dr}$$

where b^* is given by

$$b^*(t) = \left[1 - \left(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}}\right]\ \lambda^+(t) + \left[1 - \left(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}}\right]\ \lambda^-(t). \tag{3.6}$$

Moreover if

$$\int_0^T \mid \log\left\{\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right\} \mid (\lambda^+(t) + \lambda^-(t))dt < +\infty \tag{3.7}$$

an optimal admissible strategy θ^*_t is given in the feedback form $\theta^*_t = \theta^*(t, X_{t^-}, Y_{t^-})$ where

$$\theta^*(t,x,y) = -\frac{1}{\alpha[e^{K_1^+(t,x,y)} - e^{-K_1^-(t,x,y)}]}\ \log\left\{\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right\}. \tag{3.8}$$

Proof
First, let us denote by $\mathcal{L}^\theta$ the generator of the controlled Markov process (X, Y, Z) associated to the constant strategy θ

$$\mathcal{L}^\theta f(t,x,y,z) = \frac{\partial f}{\partial t}(t,x,y,z) + \mathcal{L}_t^\theta f(t,x,y,z) = \tag{3.9}$$

$$\frac{\partial f}{\partial t}(t,x,y,z) + b(x)\ \frac{\partial f}{\partial x}(t,x,y,z) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 f}{\partial x^2}(t,x,y,z) +$$

$$\int_Z \Big(f\big(t, x + K_0(t,x;\zeta), y + K_1(t,x,y;\zeta), z + \theta(e^{K_1(t,x,y;\zeta)} - 1)\big) - f(t,x,y,z)\Big)\ \nu(d\zeta).$$

A classical approach in stochastic control theory consists in examining the Hamilton-Jacobi-Bellman (HJB) that the value function is expected to satisfy. This equation is given by

$$\frac{\partial u}{\partial t}(t,x,y,z) + inf_\theta \mathcal{L}_t^\theta u(t,x,y,z) = 0 \tag{3.10}$$

with the terminal condition

$$u(T,x,y,z) = e^{-\alpha z}. \tag{3.11}$$

We find a candidate solution of (3.10) of the form

$$u(t,x,y,z) = e^{-\alpha z} h(t,x,y). \tag{3.12}$$

Direct substitution in (3.10) yields that $h(t,x,y)$ solves

$$\frac{\partial h}{\partial t}(t,x,y) + b(x)\ \frac{\partial h}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 h}{\partial x^2}(t,x,y) + \tag{3.13}$$

$$inf_\theta \Big(\int_Z e^{-\theta(e^{K_1(t,x,y;\zeta)}-1)} h(t, x + K_0(t,x;\zeta), y + K_1(t,x,y;\zeta))\nu(d\zeta)\Big) - h(t,x,y)\nu(D_1(t,x,y)) = 0$$

with the terminal condition $h(T,x,y) = 1$.

Taking into account (2.7), we have that

$$\int_Z e^{-\alpha\theta(e^{K_1(t,x,y;\zeta)}-1)} h(t, x + K_0(t,x;\zeta), y + K_1(t,x,y;\zeta))\nu(d\zeta) =$$

$$e^{-\alpha\theta(e^{K_1^+(t,x,y)}-1)} \int_{D_1^+(t)} h(t, x + K_0(t,x;\zeta), y + K_1^+(t,x,y))\nu(d\zeta) +$$

$$e^{-\alpha\theta(e^{-K_1^-(t,x,y)}-1)} \int_{D_1^-(t)} h(t, x + K_0(t,x;\zeta), y - K_1^-(t,x,y))\nu(d\zeta)$$

and the minimum is achieved in

$$\theta^*(t,x,y) = -\frac{1}{\alpha[e^{K_1^+(t,x,y)} - e^{-K_1^-(t,x,y)}]}\ \log\left\{ \frac{\int_{D_1^-(t)} h(t, x + K_0(t,x;\zeta), y - K_1^-(t,x,y))\nu(d\zeta)}{\int_{D_1^+(t)} h(t, x + K_0(t,x;\zeta), y + K_1^+(t,x,y))\nu(d\zeta)}\ \Gamma(t)\right\}. \tag{3.14}$$

Next we suppose h be only a function of t, hence (3.14) reduces to (3.8). Substituting (3.8) in the equation (3.13), we get

$$\frac{\partial h}{\partial t}(t,x,y) + b(x)\ \frac{\partial h}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 h}{\partial x^2}(t,x,y) + \tag{3.15}$$

$$\left(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}} \int_{D_1^+(t)} h(t, x + K_0(t,x;\zeta), y + K_1^+(t,x,y))\nu(d\zeta) +$$

$$\left(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}} \int_{D_1^-(t)} h(t, x+K_0(t,x;\zeta), y-K_1^-(t,x,y))\nu(d\zeta) - h(t,x,y)\nu(D_1(t,x,y)) = 0$$

This is a linear parabolic integro-differential equation which can be written as

$$\frac{\partial h}{\partial t}(t,x,y) + \mathcal{L}_t^* h(t,x,y) - b^*(t)h(t,x,y) = 0 \tag{3.16}$$

together with $h(T,x,y)=1$, where $b^*(t)$ is given in (3.6) and

$$\mathcal{L}_t^* f = b(x)\ \frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 f}{\partial x^2}(t,x,y) + \tag{3.17}$$

$$\int_Z \left(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right)^{\frac{e^{K_1(t,x,y;\zeta)}-1}{e^{K_1^+(t,x,y)}-e^{-K_1^-(t,x,y)}}} \Big(f(t,x+K_0(t,x;\zeta), y+K_1(t,x,y;\zeta)) - f(t,x,y)\Big)\nu(d\zeta).$$

Finally

$$h(t,x,y) = e^{-\int_t^T b^*(r)dr} \tag{3.18}$$

provides the solution to equation (3.16) and our conjecture that h to be only a function of t is fulfilled. Observe that the integrability of $b^*(r)$ over $[0,T]$ is implied by assumptions (2.5) and (3.5), in fact

$$\mid b^*(t) \mid \leq 3(\lambda^-(t)+\lambda^+(t)) + \lambda^-(t)\Gamma(t) + \frac{\lambda^+(t)}{\Gamma(t)}.$$

To conclude, $e^{-\alpha z}e^{-\int_t^T b^*(r)dr}$ is a smooth solution to the HJB-equation (3.10) satisfying (3.11) and $\theta^*(t,x,y)$ given in (3.8) realizes the infimum. By well-known Verification Theorems (see for example [7]) we get that $W(t,x,y,z) = e^{-\alpha z}e^{-\int_t^T b^*(r)dr}$ and $\theta_t^* = \theta^*(t, X_{t-}, Y_{t-})$ provides an optimal admissible strategy. The admissibility of θ_t^* follows by (3.7)

$$E\Big(\int_0^T \mid \theta_t^* \mid \Big((e^{K_1^+(t,X_{t-},Y_{t-})}-1)\lambda_t^+ + (1-e^{-K_1^-(t,X_{t-},Y_{t-})})\lambda_t^-\Big)dt\Big) =$$

$$= \int_0^T \log\left\{\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right\}\Big(\frac{\lambda_1^+(t)}{1+\Gamma(t)} + \frac{\lambda_1^-(t)\Gamma(t)}{1+\Gamma(t)}\Big)dt < +\infty.$$

□

4. The MEMM and the duality relation

Markets where the underlying asset prices are described by pure jump processes or jump-diffusion processes with an infinite number of marks (an infinite number of sources of randomness) are in general not complete. Absence of arbitrage opportunities is equivalent to existence of a risk-neutral probability measure. In incomplete market neither existence nor uniqueness of a risk-neutral probability measure are assured. The set of martingale measures may have an infinite number of elements and therefore different choices has been proposed in literature for valuation of derivatives. Among them we will focus our attention to the minimal entropy martingale measure which plays an important role in the expected exponential utility maximization problem.

The aim of this section is to prove the existence of the MEMM for our model and to derive a duality relation which gives a relationship between the expected utility maximization problem and the minimization of the relative entropy.

First, we are going to characterize the set, $\mathcal{M}_e$, of martingale measures for our model, i.e. the set consisting of all probability measures Q, equivalent to P, such that S is a local $(Q, \mathcal{F}_t)$-martingale. We recall that S agrees with the discounted price of the stock price since our numeraire is taken equal to 1.

From now on, we shall consider as filtration that generated by the Wiener process and the random Poisson measure

$$\mathcal{F}_t = \sigma\{W_u,\ \mathcal{N}((0,u]\times A),\ u \le t,\ A \in \mathcal{Z}\}. \tag{4.1}$$

Let us recall that by a suitable version of Girsanov Theorem ([2]), a probability measure Q is equivalent to P iff

$$\frac{dQ}{dP}\,|_{\mathcal{F}_T} = L_T = \mathcal{E}(M)_T, \quad E[L_T] = 1 \tag{4.2}$$

where $\mathcal{E}(M)$ denotes the stochastic exponential of a $\{P, \mathcal{F}_t\}$-martingale. In our frame, since the filtration is given by (4.1), every martingale can be written as

$$M_t = \int_0^t \Gamma_s\ dW_s + \int_0^t \int_Z U_s(\zeta)\ \big(\mathcal{N}(ds, d\zeta) - \nu(d\zeta)\ ds\big)$$

where Γ is a $\{P, \mathcal{F}_t\}$-adapted process such that $\int_0^T |\Gamma_s|^2\ ds < +\infty \quad P-a.s.$, $U_s(\zeta)$ is $\{P, \mathcal{F}_t\}$-predictable process such that

$$\int_0^T \int_Z |U_s(\zeta)|\ \nu(d\zeta)\ ds < +\infty, \quad U_s(\zeta) + 1 > 0, \quad \int_0^T \int_Z |U_s(\zeta) + 1|\ \nu(d\zeta)\ ds < +\infty \quad P-a.s. \tag{4.3}$$

The exponential of a martingale M, $L_t = \mathcal{E}(M)_t$, is characterized as the unique solution of the following equation

$$L_t = 1 + \int_0^t L_{s-} dM_s$$

and by Doléans-Dade formula it can be written as

$$L_t = exp\Big(\int_0^t \Gamma_{s-} dW_s - \frac{1}{2}\int_0^T |\Gamma_s|^2\ ds + \int_0^t \int_Z log(1+U_s(\zeta))\mathcal{N}(ds, d\zeta) - \int_0^t \int_Z U_s(\zeta)\nu(d\zeta)\ ds\Big). \tag{4.4}$$

Since $\Delta M_t > -1$, $P-a.s.$ (this is implied by $U_s(\zeta)+1 > 0$, $P-a.s.$) L_t is a strictly positive supermartingale and if $E[L_T] = 1$ is a martingale. Hence the probability measure Q is well defined by (4.2).

Under the new probability measure Q we have that the compensator of the integer-valued random measure $\mathcal{N}(ds, d\zeta)$ is given by

$$\nu^q(ds, d\zeta) = \big(1 + U_s(\zeta)\big)\ \nu(d\zeta)\ ds$$

and there exists a Q-Wiener process W_t^q such that

$$dW_t = \Gamma_t\ dt + dW_t^q.$$

As a particular case of Proposition (3.2) in [5] we have the following characterization of martingale measures for our model.

Proposition 4.1 *A probability measure Q equivalent to P is a risk-neutral measure iff*

$$\int_0^T \int_Z |\ e^{K_1(t, X_{t^-}, Y_{t^-}; \zeta)} - 1\ |\ \big(1 + U_t(\zeta)\big)\ \nu(d\zeta)dt < +\infty \qquad P-a.s. \tag{4.5}$$

$$\int_Z (e^{K_1(t, X_{t^-}, Y_{t^-}; \zeta)} - 1)\ \big(1 + U_t(\zeta)\big)\ \nu(d\zeta) = 0 \qquad a.a.t \in [0,T] \quad P-a.s. \tag{4.6}$$

Next, we recall that the MEMM is the martingale measure, P^*, which minimizes the relative entropy in $\mathcal{M}$, consisting of the probability measures, absolutely continuous w.r.t. P, under which the discounted price process S is a local martingale.
The relative entropy of a probability measure P' w.r.t. P is defined by

$$H(P'|P) = \begin{cases} I\!\!E^P\Big[\frac{dP'}{dP}\ log\Big(\frac{dP'}{dP}\Big)\Big] & P' \ll P \\ + + \infty & otherwise \end{cases} \tag{4.7}$$

When the price process is a general locally bounded semimartingale and there exists a risk neutral measure with finite entropy the MEMM ([10]) exists, is unique and the following duality relation yields ([1, 6])

$$\sup_{\theta \in \bar\Theta} E\Big[1 - \exp\big\{-\int_0^T \theta_r dS_r\big\}\Big] = 1 - \exp\Big\{-\inf_{Q\in\mathcal{M}} H(Q|P)\Big\} = 1 - \exp\Big\{-H(P^*|P)\Big\}. \tag{4.8}$$

This duality result is robust for various choices of the class $\bar\Theta$ of admissible strategies subset of $L(S)$, consisting of all $(P, \mathcal{F}_t)$-predictable and S-integrable processes. The standard definition is that a trading strategy is admissible if there exists a constant $c \in I\!\!R$ such that $\forall t \in [0,T]$, $\int_0^T \theta_r dS_r \geq -c$ (c is a finite credit line which the investor must respect in his trading).

Let us observe that our optimal strategy does not belong to this class and so we can not apply the duality result even when S is locally bounded . But, by applying Proposition 6.2 in [5], we shall get the existence of MEMM even if S is not locally bounded (the jump sizes K_1^+ and K_1^- are not necessarily bounded) and by the results of the previous section we shall derive the duality relation for the class of all trading strategies.

Proposition 4.2 *Assume the hypotheses of Proposition 3.1 and $\nu(Z) < +\infty$ then the probability measure P^* defined as*

$$\frac{dP^*}{dP}\,|_{\mathcal{F}_T} = exp\{-\int_0^T \frac{\theta_r^*}{S_{r-}} dS_r + \int_0^T b^*(r)dr\}, \tag{4.9}$$

where $\theta_t^* = \theta^*(t, X_{t^-}, Y_{t^-})$ and $\theta^*(t,x,y)$ is given in (3.8) with $\alpha = 1$, is a risk-neutral probability measure such that under P^* the point processes N_t^+ and N_t^- have intensities

$$\lambda_t^{*,+} = \Big(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\Big)^{\frac{1}{1+\Gamma(t)}} \lambda^+(t), \quad \lambda_t^{*,-} = \Big(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\Big)^{\frac{-\Gamma(t)}{1+\Gamma(t)}} \lambda^-(t).$$

Moreover if

$$\int_0^T |\log\Big\{\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\Big\}|\ \Gamma(t)dt < +\infty \qquad \int_0^T |\log\Big\{\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\Big\}|\ \frac{1}{\Gamma(t)}dt < +\infty \tag{4.10}$$

the probability measure P^* is the minimal entropy measure and the following duality relation holds

$$\sup_{\theta\in L(S)} E\Big[1 - \exp\big\{-\int_0^T \theta_r dS_r\big\}\Big] = \sup_{\theta\in\Theta} E\Big[1 - \exp\big\{-\int_0^T \frac{\theta_r}{S_{r^-}} dS_r\big\}\Big] = 1 - \exp\Big\{-H(P^*|P)\Big\}. \tag{4.11}$$

Proof
Denoting

$$L_T^* = exp\{-\int_0^T \frac{\theta_r^*}{S_{r-}} dS_r + \int_0^T b^*(r)dr\}$$

by Proposition 3.1, taking $\alpha = 1$, we get $E[L_T^*] = 1$.
Moreover, by (4.4)

$$L_T^* = \mathcal{E}(M_T^*) = \mathcal{E}\Big(\int_0^T \int_Z U^*(s, X_s, Y_s; \zeta)\ \big(\mathcal{N}(ds, d\zeta) - \nu(d\zeta)\ ds\big)\Big)$$

where

$$U^*(s,x,y;\zeta) = exp\{-\theta^*(t,x,y)(e^{K_1(t,x,y;\zeta)}-1)\} - 1 =$$

$$\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}} - 1\right]\ 1\!\!\mathrm{I}_{D_1^+(t)}(\zeta) + \left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}} - 1\right]\ 1\!\!\mathrm{I}_{D_1^-(t)}(\zeta).$$

First, observe that conditions (4.3) are satisfied, in particular

$$\int_0^T \int_Z (U^*(s,x,y;\zeta)+1)\nu(d\zeta)dt = -\int_0^T b^*(t)dt + \nu(Z)T < +\infty.$$

By Girsanov Theorem, under P^* the intensities of N_t^+ and N_t^- are given by $\lambda_t^{*,+}$ and $\lambda_t^{*,-}$ respectively. Moreover, by a direct computation we have that also the risk-neutral conditions (4.5) and (4.6) are verified, in fact

$$\int_Z (e^{K_1(t,X_{t-},Y_{t-};\zeta)}-1)\big(1+U^*(t,X_t,Y_t;\zeta)\big)\nu(d\zeta) = (e^{K_1^+(t,X_{t-},Y_{t-})}-1)\lambda_t^{*,+} + (e^{-K_1^-(t,X_{t-},Y_{t-})}-1)\lambda_t^{*,-} = \tag{4.12}$$

$$= (e^{K_1^+(t,X_{t-},Y_{t-})}-1)\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}}\lambda^+(t) + (e^{-K_1^-(t,X_{t-},Y_{t-})}-1)\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}}\lambda^-(t) = 0$$

and

$$\int_0^T \int_Z |\, e^{K_1(t,X_{t-},Y_{t-};\zeta)} - 1\,|\ \ \big(1+U^*(t,X_t,Y_t;\zeta)\big)\ \nu(d\zeta)dt \le$$

$$2\int_0^T \big((e^{K_1^+(t,X_{t-},Y_{t-})}-1)\lambda^+(t) + (1-e^{-K_1^-(t,X_{t-},Y_{t-})})\lambda^-(t)\big)dt < +\infty.$$

By Proposition 4.1, P^* is a risk-neutral probability measure.
To conclude, let us observe that

$$\int_0^T -\frac{\theta_r^*}{S_{r-}}dS_r = -\int_0^T \theta_r^*\Big((e^{K_1^+(t,X_{t-},Y_{t-})}-1)dN_t^+ + (e^{K_1^-(t,X_{t-},Y_{t-})}-1)dN_t^-\Big) =$$

$$\int_0^T \Big(log\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}}\right]dN_t^+ + log\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}}\right]dN_t^-\Big)dt$$

and by (4.10)

$$E^{P^*}\int_0^T \Big(\,|\, log\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}}\right]|\ \lambda_t^{*,+} + |\, log\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\,\Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}}\right]|\ \lambda_t^{*,-}\Big)dt < +\infty.$$

Hence, taking into account the risk neutral condition (4.12) we have that

$$\int_0^T -\frac{\theta_r^*}{S_{r-}}dS_r = -\int_0^T \theta_r^*\Big((e^{K_1^+(t,X_{t-},Y_{t-})}-1)(dN_t^+ - \lambda_t^{*,+}dt) + (e^{K_1^-(t,X_{t-},Y_{t-})}-1)(dN_t^- - \lambda_t^{*,-}dt)\Big)$$

is a $(P^*,\mathcal{F}_t)$-martingale and the relative entropy of P^* w.r.t. P is finite and given by

$$H(P^* \mid P) = E^{P^*}[-\int_0^T \frac{\theta_r^*}{S_{r-}} dS_r + \int_0^T b^*(r)dr] = \int_0^T b^*(r)dr.$$

Finally, by Proposition 4.4 in [5] we have that P^* is the MEMM and the duality relation (4.11) follows by Proposition 3.1. □

Remark 4.3 *In [5] has been proved that P^* provides the MEMM under the stronger assumptions*

$$\exists \quad a, b > 0 \quad a \leq K_1^+(t, x, y) \leq b, \quad a \leq K_1^-(t, x, y) \leq b, \quad \lambda^+(t) \geq a, \quad \lambda^-(t) \geq a$$

In fact these hypotheses imply (3.5) and (4.10). Let us observe that this assumptions imply that S is locally bounded.

5. Discussion on hedging and pricing of a contingent claim

Consider now a European derivative to be priced with the payoff at time T of the form $B = B(Y_T)$, where B is a bounded function. A popular by now pricing methodology is based on utility maximization criteria. More precisely, the so called indifference valuation is based on the comparison of maximal expected utilities corresponding to investment opportunities with and without involving the contingent claim. The latter problem has been studied in Section 3. We now consider the problem involving the contingent claim. The seller after receiving the premium has to hedge to reduce his risk exposure. Its the final net wealth is given by

$$c_0 + \int_0^T \frac{\theta_r}{S_{r-}} dS_r - B(Y_T)$$

(he sells the option at time 0 for the price c_0, pays out to the buyer the payoff $B(Y_T)$ at time T and accumulates the profits and losses arising from the self-financing hedging strategy θ_r). We suppose him risk averse with exponential preferences. His objective consists in maximize the expected utility of the terminal wealth

$$E\Big[1 - \exp\big\{ - \alpha(c_0 + \int_0^T \frac{\theta_r}{S_r} dS_r - B(Y_T))\big\}\Big] = E\Big[1 - \exp\big\{ - \alpha(Z_T - B(Y_T))\big\}\Big].$$

The associated value function is given by

$$V_1(t, x, y, z) = sup_{\theta \in \Theta} E\Big(1 - \exp\big\{ - \alpha(Z_T - B(Y_T))\big\}\Big) \mid X_t = x, Y_t = y, Z_t = z\Big) = 1 - W_1(t, x, y, z) \quad (5.1)$$

where

$$W_1(t, x, y, z) = inf_{\theta \in \Theta} E\Big(\exp\big\{ - \alpha(Z_T - B(S_T))\big\}\Big) \mid X_t = x, Y_t = y, Z_t = z\Big). \quad (5.2)$$

The indifference writer's price $p^\alpha(x, y, z, t)$ is defined implicitly by the following equation

$$V(t, x, y, z) = V_1(t, x, y, z + p^\alpha(x, y, z, t)) \quad (5.3)$$

This means that the agent is indifferent between optimize the expected utility without employing the contingent claim and optimize it taking into account the payoff derivative $B(Y_T)$ at time T with the compensation $p^\alpha(x, y, z, t)$ at time of inscription t. Since in our frame

$$W(t, x, y, z) = e^{-\alpha z} inf_\theta I\!\!E[\exp\big\{ - \alpha \int_t^T \frac{\theta_r}{S_{r-}} dS_r\big\} \mid X_t = x, Y_t = y]$$

$$W_1(t,x,y,z) = e^{-\alpha z} inf_\theta I\!E\Big[\exp\big\{-\alpha\big(\int_t^T \frac{\theta_r}{S_{r-}} dS_r - B(Y_T)\big)\big\} \mid X_t = x, Y_t = y\Big],$$

the utility indifference price does not depend on z and is given by

$$p^\alpha(t,x,y) = \frac{1}{\alpha} ln\Big(\frac{inf_\theta I\!E\big[\exp\big\{-\alpha(\int_t^T \frac{\theta_r}{S_{r-}} dS_r - B(Y_T))\big\} \mid X_t = x, Y_t = y\big]}{inf_\theta I\!E\Big[\exp\big\{-\alpha \int_t^T \frac{\theta_r}{S_{r-}} dS_r\big\} \mid X_t = x, Y_t = y\Big]}\Big). \tag{5.4}$$

Next, we see that an asymptotic variant of the utility indifference price approach leads to choose the MEMM as pricing measure. First, by the results obtained in the previous section, under the assumption required in Proposition 3.1, we have

$$inf_\theta I\!E\Big[\exp\big\{-\alpha \int_0^T \frac{\theta_r}{S_{r-}} dS_r\big\} \mid X_t = x, Y_t = y\Big] = I\!E\Big[\exp\big\{-\int_0^T \frac{\theta^*_r}{S_{r-}} dS_r\big\} \mid X_t = x, Y_t = y\Big] = e^{-\int_t^T b^*(r)dr} \tag{5.5}$$

where $\theta^*_t = \theta^*(t, X_{t^-}, Y_{t^-})$ and $\theta^*(t,x,y)$ is given in (3.8) with $\alpha = 1$, and $b^*(t)$ is given in (3.6). Taking into account (5.4), (5.5) and (4.9) we get that

$$p^\alpha(t,x,y) \leq \frac{1}{\alpha} ln\Big(\frac{I\!E\big[\exp\big\{-(\int_t^T \frac{\theta^*_r}{S_{r-}} dS_r - B(Y_T))\big\} \mid X_t = x, Y_t = y\big]}{I\!E\Big[\exp\big\{-\int_t^T \frac{\theta^*_r}{S_{r-}} dS_r\big\} \mid X_t = x, Y_t = y\Big]}\Big) = \frac{1}{\alpha} ln I\!E^{P^*}\big[e^{-\alpha B(Y_T)} \mid X_t = x, Y_t = y\big].$$

Finally, since $p^\alpha(t,x,y) \geq I\!E^{P^*}\big[B(Y_T) \mid X_t = x, Y_t = y\big]$ (see for example [13]) we have the following known result

$$\lim_{\alpha \to 0} p^\alpha(t,x,y) = I\!E^{P^*}\big[B(Y_T) \mid X_t = x, Y_t = y\big]. \tag{5.6}$$

Let us observe that the behavior of the exponential utility indifference price as the risk aversion parameter goes to zero corresponds to a risk neutral valuation, which is expressed as conditional expectation w.r.t. the MEMM.

Remark 5.1 *Under the probability measure P^* the pair (X,Y) is a Markov process with generator $\mathcal{L}^*_t$ given in (3.17). Denoting by $H(t,x,y)$ the MEMM price of the contingent claim $B(Y_T)$*

$$H(t,x,y) = I\!E^{P^*}\big[B(Y_T) \mid X_t = x, Y_t = y\big] \tag{5.7}$$

by Itô formula we get that H has to satisfy the problem

$$\mathcal{L}^*_t H(t,x,y) = 0, \quad H(T,x,y) = B(y). \tag{5.8}$$

Therefore to compute the MEMM price we may either compute the expectation in (5.7) or solve the PDE given in (5.8). Exact solutions are in general difficult to obtain and so one is led to search for numerical approximate solutions.

Remark 5.2 *In the simple particular case (without the presence of the stochastic factor) when K_1^+, K_1^-, λ^+ and λ^- are positive constants, the risky asset price is given by*

$$S_t = S_0 e^{(K_1^+ N_t^+ - K_1^- N_t^-)}$$

where N_t^+ and N_t^- are Poisson processes with intensities λ^+ and λ^- respectively.
Under P^*, N_t^+ and N_t^- are also Poisson processes with intensities $\lambda^{*,+}$, and $\lambda^{*,-}$ given by

$$\lambda^{*,+} = \lambda^+ \Big[\frac{\lambda^-(1-e^{-K_1^-})}{\lambda^+(e^{K_1^+}-1)}\Big]^{\frac{e^{K_1^+}-1}{e^{K_1^+}-e^{-K_1^-}}}, \quad \lambda^{*,-} = \lambda^- \Big[\frac{\lambda^-(1-e^{-K_1^-})}{\lambda^+(e^{K_1^+}-1)}\ \Big]^{\frac{e^{-K_1^-}-1}{e^{K_1^+}-e^{-K_1^-}}}.$$

Therefore, we have that

$$I\!E^{P^*}\big[B(Y_T) \mid Y_t = y\big] = \sum_{h,k\geq 0} B(y + K_1^+ h - K_1^- k) \frac{(\lambda^{*,+}(T-t))^h}{h!} \frac{(\lambda^{*,-}(T-t))^k}{k!} e^{-(\lambda^{*,+}+\lambda^{*,-})(T-t)}.$$

If we wish to compute $p^\alpha(t,x,y)$ we have to solve

$$inf_\theta I\!E\big[\exp\big\{-\alpha(\int_0^T \frac{\theta_r}{S_{r-}} dS_r - B(Y_T))\big\} \mid X_t = x, Y_t = y\big] = e^{\alpha z} W_1(t,x,y,z) \tag{5.9}$$

We could examining the HJB-equation that W_1 is expected to satisfy and we get the same equation given in (3.10) but with the terminal condition

$$u(t,x,y,z) = e^{-\alpha z} e^{\alpha B(y)}. \tag{5.10}$$

We look for a candidate solution of the form $e^{-\alpha z} h_1(t,x,y)$. Direct substitution in the HJB equation (3.10) yields that $h_1(t,x,y)$ solves equation (3.13) together with the terminal condition $h_1(T,x,y) = e^{\alpha B(y)}$.
We get that the minimum in equation (3.13) is achieved in (3.14) where h is replaced by h_1. Finally, substituting (3.14) in (3.13) we have that the $h_1(t,x,y)$ satisfies

$$\begin{gathered}
\frac{\partial h_1}{\partial t}(t,x,y) + b(x)\ \frac{\partial h_1}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 h_1}{\partial x^2}(t,x,y) + \\
\Big(\Gamma(t) \int_{D_1^-(t)} h_1(t, x + K_0(t,x;\zeta), y - K_1^-(t,x,y)) \nu(d\zeta)\Big)^{\frac{1}{1+\Gamma(t)}} \times \\
\times \Big(\int_{D_1^+(t)} h_1(t, x + K_0(t,x;\zeta), y + K_1^+(t,x,y)) \nu(d\zeta)\Big)^{\frac{\Gamma(t)}{1+\Gamma(t)}} \big(1 + \Gamma(t)^{-1}\big) \\
- h_1(t,x,y) \nu(D_1(t,x,y)) = 0
\end{gathered} \tag{5.11}$$

Therefore we get

Proposition 5.3 *If equation (5.11) admits a smooth solution h_1 then $W_1(t,x,y,z) = e^{-\alpha z} h_1(t,x,y)$ and the optimal strategy θ_t^* is given in feedback form, $\theta_t^* = \theta^*(t, X_{t-}, Y_{t-})$, where the function $\theta^*(t,x,y)$ is defined by*

$$\theta^*(t,x,y) = -\frac{1}{\alpha[e^{K_1^+(t,x,y)} - e^{-K_1^-(t,x,y)}]}\ \log\left\{\frac{\int_{D_1^-(t)} h_1(t, x + K_0(t,x;\zeta), y - K_1^-(t,x,y)) \nu(d\zeta)}{\int_{D_1^+(t)} h_1(t, x + K_0(t,x;\zeta), y + K_1^+(t,x,y)) \nu(d\zeta)}\ \Gamma(t)\right\}.$$

Moreover, the utility indifference price is given by

$$p^\alpha(t,x,y) = \frac{1}{\alpha} ln\big(h_1(t,x,y) e^{\int_t^T b^*(r) dr}\big).$$

Remark 5.4 *In general the value function of a stochastic control problem solves, in the classical sense, the HJB equation under the knowledge a priori that the value function has enough regularity. Conversely, Verification Theorems yield that if the HJB equation has a smooth solution then it coincides with the value function. The HJB equation (3.10) is a second-order fully nonlinear equation and therefore might not have a unique smooth solution. Hence one has to relax the notion of solutions to the HJB equation by introducing viscosity solutions. Herein we do not deal with this subject .*

References

[1] F. Bellini and M. Frittelli, On the existence of minimax martingale measures, Mathematical Finance, **12**, (2002) 1-21.

[2] T. Bjork, Y. Kabanov and W. Runggaldier, *Bond market structure in presence of marked point processes*, Mathematical Finance **7** (2) (1997) 211-223.

[3] C. Ceci, *Risk minimizing hedging for a partially observed high frequency data model*, Stochastics: An International Journal of Probability and Stochastic Processes **78** (1) (2006) 13-31.

[4] C. Ceci and A. Gerardi, *A model for high frequency data under partial information: a filtering approach*, International Journal of Theoretical and Applied Finance **9** (4) (2006) 1-22.

[5] C. Ceci and A. Gerardi, *Pricing for geometric marked point processes under partial information: entropy approach*, Tech.Report R-2006-007 Dip.di Scienze, Univ. di Chieti-Pescara, http://www.sci.unich.it/tecrep/.

[6] F. Delbaen, P. Grandits, T. Rheinlander, D. Samperi, M. Schweizer and C. Stricker, *Exponential hedging and entropic penalties*, Mathematical Finance **12** (2002) 99-123.

[7] W. Flemig and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, New York, Springer (1993).

[8] R. Frey, *Risk minimization with incomplete information in a model for high-frequency data*, Mathematical Finance **10** (2) (2000) 215-22.

[9] R. Frey and W. Runggaldier, *Risk-minimization hedging strategies under restricted information: The case of stochastic volatility models observed only at discrete random times*, Mathematical Methods of Operations Research **50** (1999) 339-350.

[10] M. Frittelli, *the minimal entropy martingale measure and the valuation problem in incomplete markets*, Mathematical Finance **10** (1) (2000) 39-52.

[11] T. Fujiwara and Y. Miyahara, *The minimal entropy martingale measures for geometric Lévy processes*, Finance and Stochastics **7** (2003) 509-531.

[12] M.Kirch and W.J. Runggaldier, *Efficient hedging when asset prices follow a geometric Poisson process with unknown intensities*, SIAM J. on Control and Optimization, **43** (2004) 1174-1195.

[13] M. Mania and M. Schweizer, *Dynamic exponential utility indifference valuation*, The Annals of Applied Probability, **15** (3) (2005) 2113-2143.

[14] R. Merton, *Optimal consumption and portfolio rules in a continuous time model*, Journal of Economic Theory, **3** (1971) 373-413.

[15] M. Musiela and T. Zariphopoulou, *An example of indifference prices under exponential preferences*, Finance and Stochastics, **8** (2004) 229-239.

[16] H. Pham, *Smooth solution to optimal investment models with stochastic volatilities and portfolio constraints*, Appl. Math. Optim., **78** (2002) 55-78.

[17] J.L. Prigent, Option pricing with a general marked point process, Mathematics of Operations Research, **26** (2001) 50-66.

[18] L. C. G. Rogers and O. Zane, *Designing models for high-frequency data*, Preprint, University of Bath (1998).

[19] T. Rydberg and N. Shephard, *A modeling framework for prices and trades made at the New York stock exchange*, Nuffield College working paper series 1999-W14, Oxford, *Nonlinear and Nonstationary Signal Processing*, eds. W.J. Fitzgerald *et al.*, Cambridge University Press (2000) 217-246.

[20] T. Zariphopoulou, *Consumption investment models with constraints*, SIAM J. Control and Optimization, **30** (1994) 59-84.

[21] T. Zariphopoulou, *A solution approach to valuation with unhedgeable risks*, Finance and Stochastics, **5** (2001) 61-82.