PRICING FOR GEOMETRIC MARKED POINT PROCESSES UNDER PARTIAL INFORMATION: ENTROPY APPROACH

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Abstract

The problem of the arbitrage-free pricing of a European contingent claim B is considered in a general model for intraday stock price movements in the case of partial information. The dynamics of the risky asset price is described through a marked point process Y, whose local characteristics depend on some unobservable jump diffusion process X. The processes Y and X may have common jump times, which means that the trading activity may affect the law of X and could be also related to the presence of catastrophic events. Risk-neutral measures are characterized and in particular, the minimal entropy martingale measure is studied. The problem of pricing under restricted information is discussed, and the arbitrage-free price of the claim B w.r.t. the minimal entropy martingale measure is computed by using filtering techniques.

Keywords: Pricing under restricted information, minimal entropy martingale measure, marked point processes, jump-diffusions, filtering.

MSC Classification: 91B24, 91B70, 60J75, 60J60.

1. Introduction

Valuation of contingent claims is a central problem in mathematical finance. This problem can be easily solved in complete markets, thanks to the existence of a unique equivalent martingale measure. Hence the no-arbitrage price of a contingent claim is identified to the conditional expectation with respect to this probability measure.
In the case of incomplete markets, as it happens for the model presented in this paper, the set of martingale measures has in general an infinite number of elements. In [27] the case of full information for a marked point process has been studied and the special case of the minimal martingale measure has been considered. In this note, in Section 3, we discuss the characterization of martingale measures in the frame of the model described in Section 2.

On the other hand, the existence of many martingale measures implies that the first question that one has to face is concerned with the appropriate equivalent measure to be chosen. For a chosen risk-neutral probability measure the no-arbitrage price of a contingent claim is again the conditional expectation with respect to this probability measure.
Different choices have been proposed according to some measure of the choice, as the minimal martingale measure ([12], [32]), the mean-variance martingale measure ([9], [31]) and the minimal entropy measure (MEMM) ([16], [18], [19]). Moreover, there is a correspondence between hedging criteria and martingale measures and the choice of a specific pricing measure can be based on the choice of a specific hedging criterion.

In this paper, we will adopt the relative entropy as a measure of the choice.
The MEMM plays an important role in the utility indifference approach to valuation of derivatives ([1], [2], [8], [25] and references therein). In fact, we recall that, via a duality result the MEMM is in relationship with the utility maximization problem and its explicit representation provides the optimal strategy for this problem in a particular case.

Furthermore, an asymptotic result leads to choosing the MEMM as pricing measure since the MEMM price is the limit of the dynamic utility indifference price as the risk aversion parameter goes to zero ([25], Theorem 17).

In Section 4 we recall the definition and the basic properties of the MEMM and we discuss its existence for our model with a structure that preserves the Markovianity of the model. It is also shown that the case of geometric marked point processes with respect to their internal filtration can be viewed as a particular case of a full information model.
Let us notice that, in [18] the existence of the MEMM is proven and an explicit representation is given when the asset price is described by a geometric Lévy process. As we will see later on, our model could not be viewed as a particular case of that discussed in [18] since the local characteristics of S are stochastic processes.

Next, we describe more precisely the model presented in Section 2 of this note. Most models that have been proposed to describe the dynamics of prices consider processes with continuous trajectories, while, recently, with the advent of intraday information on financial asset price quotes, many papers have been devoted to models with jumps ([13], [14], [15], [29], [30]). In fact real asset prices are piecewise constant and jump in reaction to trades or to significant new information, hence it is sensible to suppose that prices are described by marked point processes. Moreover their dynamics can be directed by another unobservable process, which may describe the activity of other markets, macroeconomics factors or microstructure rules that drive the market ([6], [7], [13], [14], [15]).

In this paper we consider the same model introduced in [5] and [6], more precisely the behavior of the asset prices is described by a geometric marked point process S, whose local characteristics may depend by an exogenous process X modeled by a Markov jump-diffusion. Moreover, the two processes may have common jump times, which means that the trading activity may affect the law of X and could be also related to the presence of catastrophic events. Agents have access only to the information contained in past asset prices.

Thus we deal with a partially observed model. As a consequence we need to choose an appropriate approach for the valuation of contingent claims. Many choices are possible. As we shall discuss in Section 3, taking into account the discussion performed in [26] for the totally observed case, we will adopt the following approach.

We study the set of the risk-neutral measures with respect to a filtration greater than that generated by the marked point process S and the price with partial observation is computed as the conditional expectation with respect to the chosen risk-neutral probability measure given the past asset prices.

When the chosen martingale measure preserves the Markovianity of the pair (X, S) the valuation of the contingent claim leads to a filtering problem similar to that studied in [6] when S takes values in a discrete space and in [5] when S takes values in $I\!R$. In Section 5 we discuss the filtering problem related to our approach of the pricing problem and we perform some explicit computation in a particular case in which an explicit representation of the MEMM is provided.

A different approach could consist in looking for a risk-neutral measure with respect to the filtration generated by the past asset price, but this needs the computation of the minimal local characteristics of S again by introducing a filtering problem and in order to preserve Markovianity the pair asset price and filter has to be considered, processes whose joint dynamics is not easy to handle.

2. The Model

The general model here considered is the same studied in [6]. In this section, first, we shall recall the main results given there for later use, then we shall introduce a simplified model.

2.1. Preliminaries

On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ we consider a financial market consisting of two traded assets: a riskless money market account and a risky asset. We assume that the price S of the risky asset is a pure jump process given by

$$S_t = S_0 e^{Y_t} \qquad S_0 \in I\!R^+. \tag{2.1}$$

The logreturn process Y is defined as

$$Y_t = \sum_{n=0}^{N_t} Z_n, \quad Z_0 = 0, \quad N_t = \sum_{n \geq 1} \mathbb{I}_{\{T_n \leq t\}}$$

where $Z_n = Y_{T_n} - Y_{T_{n-1}}$ is the size of the n^{th} logreturn change and N the point process which counts the total number of changes.

The price of the risk-free asset is taken equal to 1. This simply means that S is the discounted price of the risky asset and this helps to avoid more complicated notations.

We assume that the process Y described by the double sequence $\{T_n, Z_n\}$ is a marked point process whose $(P, \mathcal{F}_t)$-local characteristics ([4]) $(\lambda_t, \Phi_t(dz))$ may depend on some unobservable process X, and common jump times between X and Y are allowed. The presence of common jump times implies that the law of the latent process X can be affected by the actual trading activity, represented by the point process N and also that catastrophic events can be considered.

The pair (X, Y) takes values in $I\!\!R \times \mathcal{Y}$ ($\mathcal{Y}$ discrete subset of $I\!\!R$), and it is a global solution to the following stochastic differential equations

$$X_t = x_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \int_Z K_0(s, X_{s^-}; \zeta)\, \mathcal{N}(ds, d\zeta) \tag{2.2}$$

$$Y_t = \int_0^t \int_Z K_1(s, X_{s^-}, Y_{s^-}; \zeta)\, \mathcal{N}(ds, d\zeta) \tag{2.3}$$

where $x_0 \in I\!\!R$, W_t is a $(P, \mathcal{F}_t)$-standard Brownian motion, $\mathcal{N}(dt, d\zeta)$ is a $(P, \mathcal{F}_t)$-Poisson random measure on $I\!\!R^+ \times Z$, independent of W_t, with mean measure $dt\, \nu(d\zeta)$, with $\nu(d\zeta)$ a σ-finite measure on a measurable space $(Z, \mathcal{Z})$. The $I\!\!R$-valued functions $b(x)$, $\sigma(x)$, $K_0(t, x; \zeta)$ and the $\mathcal{Y}$-valued function $K_1(t, x, y; \zeta)$ are jointly measurable functions of their arguments.

We notice that, by applying Itô formula to e^{Y_t}, taking into account (2.3) we get that X and S, the price of the risky asset, solve (2.2) and

$$S_t = S_0 + \int_0^t \int_Z S_{r^-} (e^{K_1(r, X_{r^-}, log(S_{r^-}/S_0); \zeta)} - 1) \mathcal{N}(dr, d\zeta). \tag{2.4}$$

Remark 2.1 *Suitable assumptions can be done on the model, in order to assure existence and uniqueness (at least in a weak sense) to the system* (2.2), (2.3). *We do not study here this topic. We refer to [5], [6] and references therein for a complete discussion of it.*
Overall this paper we shall assume existence and uniqueness (at least in a weak sense) to that system.

2.2. A particular case

We shall also consider a simplified model.

First, we define, $\forall T > 0$, $\forall t \in [0, T]$, the set

$$D_1(t, x, y) = \{\zeta \in Z : K_1(t, x, y; \zeta) \neq 0\}, \tag{2.5}$$

and assume that

$$E[\int_0^T \nu(D_1(s, X_s, Y_s))\, ds] < \infty \qquad (\int_0^T \nu(D_1(s, X_s, Y_s))\, ds < \infty \quad P - a.s.). \tag{2.6}$$

Then we assume that the sets

$$D_1^+(t, x, y) = \{\zeta \in Z : K_1(t, x, y; \zeta) > 0\} \qquad D_1^-(t, x, y) = \{\zeta \in Z : K_1(t, x, y; \zeta) < 0\} \tag{2.7}$$

are both non-empty, and that $\nu\big(D_1^+(t,X_{t^-},Y_{t^-})\big)$ and $\nu\big(D_1^-(t,X_{t^-},Y_{t^-})\big)$, which are finite P-a.s. by (2.6), are both positive (see Remark 3.7 below).

Finally we set

$$K_1(t,x,y;\zeta) = K_1^+(t,x,y)\ 1\!\!1_{D_1^+(t,x,y)}(\zeta) - K_1^-(t,x,y)\ 1\!\!1_{D_1^-(t,x,y)}(\zeta) \tag{2.8}$$

where K_1^+ and K_1^- are $\mathcal{Y}$-valued positive functions.

Under (2.8), the logreturn process Y_t has a simplified but non trivial structure. In fact, the processes

$$N_t^+ = \mathcal{N}\big((0,t),D_1^+(t,X_{t^-},Y_{t^-})\big) \quad \text{and} \quad N_t^- = \mathcal{N}\big((0,t),D_1^-(t,X_{t^-},Y_{t^-})\big)$$

are independent counting processes with $\{P,\mathcal{F}_t\}$-intensities, respectively,

$$\nu\big(D_1^+(t,X_{t^-},Y_{t^-})\big)\big) \quad \text{and} \quad \nu\big(D_1^-(t,X_{t^-},Y_{t^-})\big)$$

and Y_t and S_t satisfy, respectively,

$$Y_t = \int_0^t K_1^+(r,X_{r^-},Y_{r^-})\ dN_r^+ - \int_0^t K_1^-(r,X_{r^-},Y_{r^-})\ dN_r^-, \qquad N_t = N_t^+ + N_t^-,$$

$$S_t = S_0 + \int_0^t S_{r^-}\ \big(e^{K_1^+(r,X_{r^-},Y_{r^-})} - 1\big)\ dN_r^+ + \int_0^t S_{r^-}\ \big(e^{-K_1^-(r,X_{r^-},Y_{r^-})} - 1\big)\ dN_r^-.$$

These last representations suggest that this particular model can be seen as a generalization of that proposed in [21].

2.3. Local characteristics of Y and semimartingale representation of S

The $(P,\mathcal{F}_t)$-local characteristics $(\lambda_t,\Phi_t(dz))$ of Y are derived in [6], taking into account the representation (2.3). Defining the sequence of the jump times and the sequence of the marks of Y by

$$T_1 = inf\{t>0: \int_0^t \int_Z K_1(s,X_{s^-},0;\zeta)\ \mathcal{N}(ds,d\zeta) \neq 0\}$$

$$T_{n+1} = inf\{t>T_n: \int_{T_n}^t \int_Z K_1(s,X_{s^-},Y_{T_n};\zeta)\ \mathcal{N}(ds,d\zeta) \neq 0\}$$

$$Z_n = Y_{T_n} - Y_{T_{n-1}} = \int_Z K_1(T_n,X_{T_n^-},Y_{T_{n-1}};\zeta)\ \mathcal{N}(\{T_n\},d\zeta),$$

we recall te following result.

Proposition 2.2 *Denote by m the integer valued random measure associated to Y ([4],[24])*

$$m(dt,dz) = \sum_{n\geq 1} \delta_{\{T_n,Z_n\}}(dt,dz) 1\!\!1_{\{T_n<\infty\}} = \sum_{h\in\mathcal{Y}\setminus\{0\}} m(dt,\{h\})\ \delta_h(dz) \tag{2.9}$$

$$m(dt,\{h\}) = \sum_{n\geq 1} 1\!\!1_{\{Z_n=h\}}\ \delta_{T_n}(dt)\ 1\!\!1_{\{T_n<\infty\}}.$$

Then, under the assumption (2.6) the $(P,\mathcal{F}_t)$-predictable projection of m is given by

$$m^p(dt,\{h\}) = \lambda_t\Phi_t(\{h\})\ dt = \lambda(t,X_{t^-},Y_{t^-})\ \Phi(t,X_{t^-},Y_{t^-},\{h\})\ dt. \tag{2.10}$$

where

$$\lambda_t = \lambda(t,X_{t^-},Y_{t^-}) = \nu(D_1(t,X_{t^-},Y_{t^-})) \tag{2.11}$$

provides the $(P,\mathcal{F}_t)$-predictable intensity of the point process $N_t = \sum_{n\geq 1} 1\!\!1_{\{T_n\leq t\}}$ and on $\{T_n<\infty\}$

$$\Phi_{T_n}(\{h\}) = \frac{\nu(D_1^h(T_n,X_{T_n^-},Y_{T_n^-}))}{\nu(D_1(T_n,X_{T_n^-},Y_{T_n^-}))}. \tag{2.12}$$

with, $\forall h \in \mathcal{Y},\ h \neq 0$

$$D_1^h(t,x,y) = \{\zeta \in Z : K_1(t,x,y;\zeta) = h\} \subseteq D_1(t,x,y). \tag{2.13}$$

Moreover, whenever there exists a transition function $\mu(t,x,y,\{h\})$ *such that,* $\forall h \in \mathcal{Y},\ h \neq 0$

$$P\big(Z_n = h \mid \mathcal{F}_{T_n-}\big) = \mu(T_n, X_{T_n^-}, Y_{T_n^-}, \{h\})$$

then on $\{T_n < \infty\}$

$$\Phi_{T_n}(\{h\}) = P\big(Z_n = h \mid \mathcal{F}_{T_n^-}\big). \tag{2.14}$$

Summing up, the $(P,\mathcal{F}_t)$-local characteristics $(\lambda_t, \Phi_t(dz)) = (\lambda(t, X_{t^-}, Y_{t^-}), \Phi(t, X_{t^-}, Y_{t^-}, dz))$ of Y depend on t and on X_t. The dependence on t takes account of seasonality effects, typical for high frequency data.

In [5] it has been studied the case where S is a $(P,\mathcal{F}_t)$-local martingale. Instead of this, here we will consider the more general case where S is a $(P,\mathcal{F}_t)$-semimartingale.

Proposition 2.3 *Under (2.6) and the following condition*

$$\int_0^T \int_Z (e^{K_1(t,X_{t^-},Y_{t^-};\zeta)} - 1)^2 \nu(d\zeta)\ dt < +\infty \quad P-a.s. \tag{2.15}$$

S is a special semimartingale ([24]) with the decomposition

$$S_t = S_0 + M_t + A_t \tag{2.16}$$

where

$$A_t = \int_0^t \int_Z S_{r^-}(e^{K_1(r,X_{r^-},Y_{r^-});\zeta)} - 1)\nu(d\zeta)dr$$

is a predictable process with bounded variation paths,

$$M_t = \int_0^t \int_Z S_{r^-}(e^{K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)(\mathcal{N}(dr,d\zeta) - \nu(d\zeta)dr)$$

is a locally square-integrable local martingale whose angle process is given by

$$< M >_t = \int_0^t \int_Z S_{r^-}^2 (e^{K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)^2 \nu(d\zeta)dr. \tag{2.17}$$

Proof.
First notice that (2.6) and (2.15) imply

$$\int_0^T \int_Z \mid e^{K_1(t,X_{t^-},Y_{t^-};\zeta)} - 1 \mid \nu(d\zeta)\ dt < +\infty \quad P-a.s. \tag{2.18}$$

hence

$$R_t = \int_0^t \int_Z (e^{K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)\mathcal{N}(dr,d\zeta)$$

is a semimartingale and by (2.15) it is square integrable.
Finally, since by (2.4),

$$dS_t = S_{t^-}\ dR_t$$

S is a semimartingale being the stochastic exponential of a semimartingale. The square-integrability of M follows by noticing that S^2 is the stochastic exponential of the semimartingale

$$R'_t = \int_0^t \int_Z (e^{2K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)\mathcal{N}(dr,d\zeta).$$

□

Remark 2.4 *From now on, we shall assume conditions (2.6) and (2.15).*

3. Pricing under partial information

We deal with the main topic of this paper, that is the valuation of a contingent claim under partial information.
Setting $\mathcal{F}_t^S = \sigma\{S_u,\ u \le t\}$, let B be a square integrable $\mathcal{F}_T^S$-measurable random variable representing the payoff of a European contingent claim with maturity T. The problem of pricing consists in finding the value of B at each time t, avoiding arbitrage opportunities.
We recall that absence of arbitrage opportunities is related to the existence of risk-neutral probability measures. that is probability measures Q, locally equivalent to P, such that S is a local $(Q, \mathcal{F}_t)$-martingale (in our framework the numeraire has been taken equal to 1). We shall denote by $\mathcal{M}_e$ this class of measures, whose characterization will be discussed in Subsection 3.2.

3.1. Arbitrage free price under partial observation

We suppose that investors can observe only the behavior of the stock price. This situation is referred to as a partial information case, in contrast with the case of full information. In partially observed models different approaches are possible. Herein we propose as price the conditional expectation of B with respect to the observations $\mathcal{F}_t^Y = \mathcal{F}_t^S$, under a chosen risk-neutral probability measure . We will give a motivation of our choice in Propositions 3.2 and 3.4 below. First we recall some basic results in the case of full information (see, for instance, [26]).

Let $Q \in \mathcal{M}_e$, a risk neutral measures for S_t.
A self-financing trading strategy θ_t, $t \in [0, T]$, is called $(Q, \mathcal{F}_t)$-admissible if it is a $\mathcal{F}_t$-predictable process such that the gains process $\int_0^t \theta_r dS_r$ follows a martingale under Q.

It is well known that the arbitrage price of a contingent claim B, settles at time T, attainable by a Q-admissible strategy agrees with the Q-conditional expectation of B given $\mathcal{F}_t$.

Since the hedger has access only to the information given by the filtration $\mathcal{F}_t^S$ we modify the classical definition by restricting our attention to strategies adapted to $\mathcal{F}_t^S$, and we give the following definition.

Definition 3.1 *A self-financing trading strategy θ_t, $t \in [0, T]$, is called $(Q, \mathcal{F}_t^S)$-admissible if it is a $\mathcal{F}_t^S$-predictable process such that the gains process $\int_0^t \theta_r dS_r$ is a $(Q, \mathcal{F}_t)$-martingale.*

Similarly to the full information case we have

Proposition 3.2 *For any European contingent claim B which settles at time T and is attainable by a $(Q, \mathcal{F}_t^S)$-admissible strategy θ_t, the associated wealth process $V_t(\theta)$ agrees with $E^Q\left[B \middle| \mathcal{F}_t^S\right]$.*

Proof.
It is sufficient to observe that since $\int_0^t \theta_r dS_r$, $t \in [0, T]$, is a $(Q, \mathcal{F}_t)$-martingale adapted to $\mathcal{F}_t^S$ it is also a $(Q, \mathcal{F}_t^S)$-martingale. Recalling that θ_t is a self-financing replicating strategy we have

$$V_t(\theta) = V_0(\theta) + \int_0^t \theta_r dS_r, \quad V_T(\theta) = B$$

and

$$V_t(\theta) = E^Q\left[V_T(\theta) | \mathcal{F}_t^S\right] = E^Q\left[B \middle| \mathcal{F}_t^S\right].$$

□

Definition 3.3 *A self-financing trading strategy θ_t, $t \in [0, T]$, is called $\mathcal{F}_t^S$-admissible if it is $(Q, \mathcal{F}_t)$-admissible for some $Q \in \mathcal{M}_e$, and the associated wealth process is bounded from below, that is $V_t(\theta) \ge m$, $m \in I\!R$. The class of all $\mathcal{F}_t^S$-admissible strategies will be denoted by Θ^S.*

Proposition 3.4 *Assume that $\mathcal{M}_e$ is not an emptyset. The market (S, Θ^S) is arbitrage-free. The arbitrage price of any contingent claim attainable by a $\mathcal{F}_t^S$-admissible strategy is well-defined. If a European contingent claim B, which settles at time T, is attainable by $(Q^i, \mathcal{F}_t^S)$-admissible strategies, $i = 1, 2$, then for every $t \in [0, T]$*

$$E^{Q^1}\left[B \middle| \mathcal{F}_t^S\right] = E^{Q^2}\left[B \middle| \mathcal{F}_t^S\right].$$

Proof.
It is a well known result that the market (S, Θ), where Θ is the class of all $\mathcal{F}_t$-admissible strategies, is arbitrage-free then also (S, Θ^S) is arbitrage-free since $\Theta^S \subset \Theta$.
To prove the second statement, consider a Θ^S-attainable claim B with maturity T. Let θ^i, $i = 1, 2$, be two strategies such that $V_T(\theta^i) = B$. Let θ^i, Q^i-admissible for some $Q^i \in \mathcal{M}_e$. Then

$$V_t(\theta^i) = E^{Q^i}\left[B \middle| \mathcal{F}_t^S\right], \quad i = 1, 2.$$

On the other hand,

$$V_T(\theta^1) = V_t(\theta^1) + \int_t^T \theta_r^1 dS_r$$

where $\int_0^t \theta_r^1 dS_r$ is a $(Q^2, \mathcal{F}_t^S)$-local martingale bounded from below. Since, by Fatou's lemma, it is a $(Q^2, \mathcal{F}_t^S)$-supermartingale, we have

$$V_t(\theta^2) = E^{Q^2}\left[B \middle| \mathcal{F}_t^S\right] = E^{Q^2}\left[V_T(\theta^1) \middle| \mathcal{F}_t^S\right] \leq V_t(\theta^1).$$

Interchanging the roles, we find $V_t(\theta^1) \leq V_t(\theta^2)$ and thus the equality.
□

In incomplete market not every claim can be replicated by a self-financing strategy. Then one has to choose some approach to hedging and pricing derivatives. Different choices have been proposed according to some measure of the choice. This choice may be dependent by agents' preferences and it is related to some risk neutral measure.
In the next subsection, we shall characterize the risk neutral measures and we shall consider, in particular, the Markovian case.

3.2. Martingale measures and Markov property

Existence and uniqueness of martingale measures is one of the fundamental problems in the theory of mathematical finance. Since our financial market is incomplete, neither existence nor uniqueness are assured. From now on, with a little abuse of notations, we set $\mathcal{F}_t = \sigma\{W_u,\ \mathcal{N}((0, u] \times A),\ u \leq t,\ A \in Z\}$, $\mathcal{F} = \mathcal{F}_T$, for a fixed time horizon $T > 0$.
Let us observe that the local characteristics of the logreturn process does not change, being adapted to this filtration. Moreover let us recall that, in a finite time horizon, measures Q, locally equivalent to P are indeed equivalent to P, then the class $\mathcal{M}_e$ reduces to the set of probability measures Q, equivalent to P, such that S is a local $(Q, \mathcal{F}_t)$-martingale and we shall characterize this set for this model in the sequel.

An essential tool for our purposes is a suitable version of the Girsanov theorem that we recall in the form we will use later on, based on classical results given in [10] and in [24], (Theorem 6.2).

In our frame, a local $\{P, \mathcal{F}_t\}$-martingale can be written as

$$M_t = \int_0^t \Gamma_s \, dW_s + \int_0^t \int_Z U_s(\zeta) \, \big(\mathcal{N}(ds, d\zeta) - \nu(d\zeta) \, ds\big) \tag{3.1}$$

where Γ is a $\{P, \mathcal{F}_t\}$-adapted process and $U_s(\zeta)$ is a joint measurable predictable process such that, for any $t \in [0, T]$

$$\int_0^t |\Gamma_s|^2 \, ds < \infty \quad \text{and} \quad \int_0^t \int_Z |U_s(\zeta)| \, \nu(d\zeta) \, ds < \infty \quad P - a.s. \tag{3.2}$$

Theorem 3.5 *Under (3.2) and*

$$U_s(\zeta) + 1 > 0 \quad P - a.s. \tag{3.3}$$

the process $\mathcal{L} = \mathcal{E}(M)$ *given by*

$$\mathcal{L}_t = \exp\left\{\int_0^t \Gamma_s \; dW_s - \frac{1}{2}\int_0^t |\Gamma_s|^2 \; ds + \int_0^t \int_Z log\big(1 + U_s(\zeta)\big) \; \mathcal{N}(ds, d\zeta) - \int_0^t \int_Z U_s(\zeta) \; \nu(d\zeta) \; ds\right\} \tag{3.4}$$

is a strictly positive local $\{P, \mathcal{F}_t\}$*-martingale. When*

$$I\!E[\mathcal{L}_T] = 1 \tag{3.5}$$

and

$$\int_0^T \int_Z \big(1 + U_s(\zeta)\big) \; \nu(d\zeta) \; ds < \infty \quad P - a.s. \tag{3.6}$$

there exists a probability measure Q *equivalent to* P*, with*

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \mathcal{L}_t, \tag{3.7}$$

and

(i) there exists a Q*-Wiener process* W^q_t *such that*

$$dW_t = \Gamma_t \; dt + dW^q_t,$$

(ii) the $(Q, \mathcal{F}_t)$*-compensator of the integer-valued random measure* $\mathcal{N}(ds, d\zeta)$ *is*

$$\nu^q(ds, d\zeta) = \big(1 + U_s(\zeta)\big) \; \nu(d\zeta) \; ds,$$

(iii) every probability measure equivalent to P *has the structure above, in the sense that, on* $\mathcal{F}_t$*, its density* $\mathcal{L}_t$ *w.r.t.* P *is given by (3.4), (3.5) holds, the pair* Γ_s*,* $U_s(\zeta)$ *verify (3.2), (3.3), (3.6) and conditions (i) and (ii) are fulfilled.*

Let us remark that the last claim strictly depends on the choice of the internal filtration.

At this point we are able to find conditions ensuring that the price process S_t is a local $(Q, \mathcal{F}_t)$-martingale.

Proposition 3.6 *The probability measure* Q *equivalent to* P *is a risk-neutral measure iff*

$$\int_0^T \int_Z \left|e^{K_1(t, X_{t^-}, Y_{t^-}; \zeta)} - 1\right| \; \big(1 + U_t(\zeta)\big) \; \nu(d\zeta) dt < +\infty \qquad P - a.s. \tag{3.8}$$

and for a.a. $t \in [0, T]$

$$\int_Z (e^{K_1(t, X_{t^-}, Y_{t^-}; \zeta)} - 1) \; \big(1 + U_t(\zeta)\big) \; \nu(d\zeta) = 0 \qquad P - a.s. \tag{3.9}$$

Proof.
Recalling (2.4), since Q is equivalent to P, by Theorem 3.5, we can write

$$\begin{aligned}
S_t &= S_0 + \int_0^t \int_Z S_{r^-} (e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1) \mathcal{N}(dr, d\zeta) = \\
&= S_0 + \int_0^t \int_Z S_{r^-} (e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1) \left(\mathcal{N}(dr, d\zeta) - \nu^q(dr, d\zeta)\right) + \\
&+ \int_0^t \int_Z S_{r^-} (e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1) \big(1 + U_r(\zeta)\big) \; \nu(d\zeta) \; dr
\end{aligned}$$

and the thesis is achieved, since, as in Proposition 2.3, we get that S is a special semimartingale under Q. Then S is a local martingale under Q iff (3.8) and (3.9) hold.
□

Remark 3.7 *We observe that, as a consequence of condition (3.3), when, for any fixed* $t \in [0,T]$ *and* $x \in I\!R,\ y \in \mathcal{Y}$,

$$K_1(t,x,y;\zeta) \geq 0,\ \forall \zeta \in Z \qquad or \qquad K_1(t,x,y;\zeta) \leq 0,\ \forall \zeta \in Z$$

condition (3.9) cannot be fulfilled. In such a case the model does not admit risk neutral measures. Then, from now on, we assume that the sets $D_1^+(t,x,y)$ *and* $D_1^-(t,x,y)$ *defined in (2.7) are both non-empty, as we did in Subsection 2.2 for the particular model.*

The final part of this section will be devoted to discuss the conditions under which the process (X_t, Y_t) is a $(Q, \mathcal{F}_t)$-Markov process.

In our frame, the pair (X, Y), under P, is a Markov process whose generator is given by,

$$\begin{aligned} Lf(t,x,y) &= \frac{\partial f}{\partial t}(t,x,y) + L_t f(t,x,y) = \\ &= \frac{\partial f}{\partial t}(t,x,y) + b(x)\ \frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 f}{\partial x^2}(t,x,y) \\ &+ \int_Z \Big(f\big(t, x + K_0(t,x;\zeta), y + K_1(t,x,y;\zeta)\big) - f(t,x,y) \Big)\ \nu(d\zeta). \end{aligned} \tag{3.10}$$

More precisely, in [5], [6] it is proven that for real-valued, bounded continuous functions $f(t,x,y)$ such that $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ are bounded and continuous,

$$f(t, X_t, Y_t) - f(0, x_0, 0) - \int_0^t Lf(r, X_r, Y_r)\ dr \tag{3.11}$$

is a $(P, \mathcal{F}_t)$-martingale under the assumptions,

$$E\Big[\int_0^T \sigma^2(X_s)\ ds\ \Big] < \infty \tag{3.12}$$

$$E\Big[\int_0^T \nu(D_i(s, X_s, Y_s))\ ds\Big] < \infty \qquad i = 0, 1 \tag{3.13}$$

where, $\forall t \in [0,T]$

$$D_0(t,x,y) = \{\zeta \in Z : K_0(t,x;\zeta) \neq 0, \quad K_1(t,x,y;\zeta) = 0\} \tag{3.14}$$

Remark 3.8 *As a consequence of the Remark 2.1, we can claim that the Martingale Problem associated to the operator* L *with initial conditions* $(0, x_0, 0)$ *is well posed.*

Then choosing a probability measure Q defined as in (3.7) under the assumptions of Theorem 3.5, Itô formula allows us to write, for a suitable f

$$\begin{aligned} & f(t, X_t, Y_t) = f(0, X_0, Y_0) + \\ & + \int_0^t \Big\{ \frac{\partial f}{\partial s}(s, X_s, Y_s) + \big(b(X_s) + \sigma(X_s)\Gamma_s\big)\ \frac{\partial f}{\partial x}(s, X_s, Y_s) + \frac{1}{2}\ \sigma^2(X_s)\ \frac{\partial^2 f}{\partial x^2}(s, X_s, Y_s) \Big\}\ ds\ + \\ & + \int_0^t \sigma(X_s)\ \frac{\partial f}{\partial x}(s, X_s, Y_s)\ dW_s^q\ + \\ & + \int_0^t \int_Z \Big\{ f(s, X_{s^-} + K_0(s, X_{s^-};\zeta), Y_{s^-} + K_1(s, X_{s^-}, Y_{s^-};\zeta)) - f(s, X_{s^-}, Y_{s^-}) \Big\} \big(\mathcal{N}(ds,d\zeta) - \nu^q(ds,d\zeta)\big)\ + \\ & + \int_0^t \int_Z \Big\{ f(s, X_{s^-} + K_0(s, X_{s^-};\zeta), Y_{s^-} + K_1(s, X_{s^-}, Y_{s^-};\zeta)) - f(s, X_{s^-}, Y_{s^-}) \Big\}\ \nu^q(ds,d\zeta) \end{aligned}$$

and recalling the expression of $\nu^q(ds, d\zeta)$ given in Theorem 3.5, we can deduce, at first, that the Markov property cannot hold for the process (X_t, Y_t) under Q unless measurable functions $\gamma(t,x,y)$, $U(t,x,y,z)$ exist, such that

$$\Gamma_s = \gamma(s, X_{s^-}, Y_{s^-}) \qquad \text{and} \qquad U_s(\zeta) = U(s, X_{s^-}, Y_{s^-}, \zeta). \tag{3.15}$$

In this case

Proposition 3.9 *Under (3.12), (3.13), (3.15) and*

$$\mathbb{E}\left[\int_0^T \int_Z \big(1 + U(s, X_{s^-}, Y_{s^-}, \zeta)\big)\ \nu(d\zeta)\ ds\right] < \infty, \tag{3.16}$$

for any real-valued bounded function f such that $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ are bounded and continuous, the process

$$M_f^q(t) = f(t, X_t, Y_t) - f(0, x_0, 0) - \int_0^t L^q f(s, X_s, Y_s)\ ds$$

is a $\{Q, \mathcal{F}_t\}$-martingale, where

$$L^q f(s,x,y) = \frac{\partial f}{\partial s}(s,x,y) + \big(b(x) + \sigma(x)\gamma(s,x,y)\big)\ \frac{\partial f}{\partial x}(s,x,y) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 f}{\partial x^2}(s,x,y) + \\ + \int_Z \Big\{f\big(s, x + K_0(s,x;\zeta), y + K_1(s,x,y;\zeta)\big) - f(s,x,y)\Big\}\ \big(1 + U(s,x,y;\zeta)\big)\ \nu(d\zeta). \tag{3.17}$$

Remark 3.10 *The Martingale Problem for the operator L^q and initial condition $(0, x_0, 0)$ is well posed. In fact, recalling the Remark 3.8, it is sufficient to notice that this Martingale Problem is well posed once such is the Martingale Problem for the operator L defined in (3.10), with the same initial conditions.*

In particular, this implies that the process (X_t, Y_t) is a Markov process under Q while this is no more true for X_t alone. For further details on this topic, see, for example, [11].

We will refer to this situation as a Markovian change of probability measure. In such a case Proposition 3.6 provides the following sufficient condition.

Proposition 3.11 *Let Q be a probability measure equivalent to P, defined by a pair Γ_s, $U_s(\zeta)$ verifying (3.15). Under (3.8) and*

$$\forall t \in [0,T], \quad x \in \mathbb{R},\ y \in \mathcal{Y} \qquad \int_Z (e^{K_1(t,x,y;\zeta)} - 1)\ \big(1 + U(t,x,y;\zeta)\big)\ \nu(d\zeta) = 0 \tag{3.18}$$

then the risky asset price S is a local $(Q, \mathcal{F}_t)$-martingale.

As a final remark, let us notice that, choosing a martingale measure preserving the Markov property of the model, the pricing problem can be dealt with by filtering techniques. This will be done in Section 5 for the particular case, under the minimal entropy martingale measure, whose properties are discussed in the next Section.

4. Minimal entropy martingale measure.

In this section we discuss the problem of the existence of the minimal entropy martingale measure in the model described in this note. In a general setting, ([16] and references therein) we can give the following

Definition 4.1 *Let P' any probability measure on $(\Omega, \mathcal{F}, P)$. The relative entropy of P' w.r.t. P is defined as*

$$H(P'|P) = \begin{cases} \mathbb{E}^P\Big[\dfrac{dP'}{dP}\ log\Big(\dfrac{dP'}{dP}\Big)\Big] & P' \ll P \\ +\infty & otherwise \end{cases} \tag{4.1}$$

and it verifies the following

(i) $H(P'|P) \geq 0$ and $H(P'|P) = 0$ iff $P' = P$.

(ii) The functional $P' \longrightarrow H(P'|P)$ is strictly convex.

Definition 4.2 *The minimal entropy martingale measure (MEMM) is a probability measure $P^* \in \mathcal{M}$ such that*

$$H(P^*|P) = \min_{Q \in \mathcal{M}} H(Q|P) \tag{4.2}$$

where $\mathcal{M}$ denotes the set of probability measure Q, absolutely continuous w.r.t. P, such that S is a local $(Q, \mathcal{F}_t)$-martingale.

If the MEMM exists, by definition is unique. Moreover, ([16], Theorem 2.2, [19], Theorem 3.1) under the assumption

$$\inf_{Q \in \mathcal{M}_e} H(Q \mid P) < +\infty \tag{4.3}$$

it is equivalent to P, and, ([16], Theorem 2.1 and Remark 2.1), when the price process S_t is locally bounded, the assumption

$$\inf_{Q \in \mathcal{M}} H(Q \mid P) < +\infty \tag{4.4}$$

is a necessary and sufficient condition for its existence.

This measure plays an important role in the utility indifference approach to valuation of derivatives. This role in a hedging problem is highlighted by a duality result, exhaustively discussed in [8] and [20], that in a particular case becomes

$$\sup_{\theta \in \Theta} E\Big[- \exp\big\{ - \int_0^T \theta_r dS_r \big\}\Big] = - \exp\Big\{ - \inf_{Q \in \mathcal{M}} H(Q|P)\Big\} = - \exp\Big\{ - H(P^*|P)\Big\}. \tag{4.5}$$

and the supremum in the left hand side is attained choosing the predictable process θ^* such that

$$\frac{dP^*}{dP} = c\; e^{- \int_0^T \theta_t^* \; dS_t} \qquad c \in I\!R^+. \tag{4.6}$$

4.1. A representation for the MEMM in this model

In the sequel the function $K_1(t, x, y; \zeta)$ is assumed to be bounded. In such a case the price process S_t is locally bounded and a localizing sequence is given by its jump times. Then, under (4.4), there exists a unique MEMM and we are looking for its representation in our frame, preserving the Markov property of the process (X, Y). Its density will be characterized as the solution to an exponential equation driven by a martingale defined as in (3.1) with $\Gamma = 0$, that is

$$M_t^* = \int_0^t \int_Z U^*(s, X_{s^-}, Y_{s^-}; \zeta) \; \big(\mathcal{N}(ds, d\zeta) - \nu(d\zeta) \; ds\big) \tag{4.7}$$

with $U^*(t, x, y; \zeta)$ real valued measurable function such that

$$\int_0^T \int_Z |U^*(s, X_{s^-}, Y_{s^-}; \zeta)| \; \nu(d\zeta) \; ds < \infty \quad P - a.s. \tag{4.8}$$

Then we have to determine the structure of the function U^*. A first result, related to the risk-neutrality condition (3.9) in Proposition 3.6, is given by the following Lemma.

Lemma 4.3 *There exists a unique real valued measurable function $\beta^*(t, x, y)$ such that*

$$\int_Z (U^*(t, x, y; \zeta) + 1\big) \; \big(e^{K_1(t,x,y;\zeta)} - 1\big) \; \nu(d\zeta) = 0. \tag{4.9}$$

for any $t \in [0, T]$, $x \in I\!R$, $y \in \mathcal{Y}$, with

$$U^*(t, x, y; \zeta) = \exp\Big\{\beta^*(t, x, y)\big(e^{K_1(t,x,y;\zeta)} - 1\big)\Big\} - 1. \tag{4.10}$$

Proof.
Let us consider the functions of the real variable β, for any fixed $t \in [0,T],\ x,y \in I\!R$.

$$F(\beta) = \int_{D_1(t,x,y)} \exp\left\{\beta\ \left(e^{K_1(t,x,y;\zeta)} - 1\right)\right\}\ \left(e^{K_1(t,x,y;\zeta)} - 1\right)\ \nu(d\zeta),$$

$$I(\beta) = \int_{D_1(t,x,y)} \exp\left\{\beta\ \left(e^{K_1(t,x,y;\zeta)} - 1\right)\right\}\ \left|e^{K_1(t,x,y;\zeta)} - 1\right|\ \nu(d\zeta).$$

The function $F(\beta)$ is well defined in the set

$$\mathcal{D}(t,x,y) = \{\beta \in I\!R \text{ such that } I(\beta) < +\infty\}.$$

The set $\mathcal{D}(t,x,y)$ is non-empty, since it contains $\beta = 0$. Moreover, according with Remark 3.7, the sets $D_1^+(t,x,y)$ and $D_1^-(t,x,y)$ defined in (2.7) are both non-empty, then setting

$$I^+(\beta) = \int_{D_1^+(t,x,y)} \exp\left\{\beta\ \left(e^{K_1(t,x,y;\zeta)} - 1\right)\right\}\ \left|e^{K_1(t,x,y;\zeta)} - 1\right|\ \nu(d\zeta)$$

$$I^-(\beta) = \int_{D_1^-(t,x,y)} \exp\left\{\beta\ \left(e^{K_1(t,x,y;\zeta)} - 1\right)\right\}\ \left|e^{K_1(t,x,y;\zeta)} - 1\right|\ \nu(d\zeta)$$

we have that $I^+(\beta)\ \big(I^-(\beta)\big)$ is a strictly increasing (strictly decreasing) function of β. Thus we can identify $\mathcal{D}(t,x,y)$ with the not necessarily bounded interval $(\underline{\beta},\overline{\beta})$, where

$$\overline{\beta} = \sup\{\beta \in I\!R \text{ such that } I^+(\beta) < +\infty\} > 0 \qquad \underline{\beta} = \inf\{\beta \in I\!R \text{ such that } I^-(\beta) < +\infty\} < 0.$$

As a consequence, we have that $F(\beta)$ is a strictly increasing continuous function defined on $(\underline{\beta},\overline{\beta})$ and

$$\lim_{\beta\to\underline{\beta}} F(\beta) = -\infty \qquad \lim_{\beta\to\overline{\beta}} F(\beta) = +\infty.$$

□

Then, choosing U^* as in (4.10), we get that, by Theorem 3.5, $\mathcal{L}^* = \mathcal{E}(M^*)$ can be written as

$$\mathcal{L}_t^* = \exp\left\{\int_0^t \frac{1}{S_{r^-}}\ \beta^*(r,X_{r^-},Y_{r^-})\ dS_r - \int_0^t \int_Z U^*(r,X_{r^-},Y_{r^-};\zeta)\ \nu(d\zeta)\ dr\right\}$$

Next we can claim that

Proposition 4.4 *Assuming*

$$\int_0^T \int_Z \big(U^*(r,X_{r^-},Y_{r^-};\zeta) + 1\big)\ \nu(d\zeta)\ dr < +\infty \qquad P - a.s., \tag{4.11}$$

the measure P^* defined by

$$\frac{dP^*}{dP} = \mathcal{L}_T^* \qquad \textit{with} \qquad I\!E[\mathcal{L}_T^*] = 1 \tag{4.12}$$

is a probability measure equivalent to P and $P^* \in \mathcal{M}_e$.

Furthermore, setting

$$F_t = \int_0^t \frac{1}{S_{r^-}}\ \beta^*(r,X_{r^-},Y_{r^-})\ dS_r = \int_0^t \int_Z log\big(1 + U^*(r,X_{r^-},Y_{r^-};\zeta)\big)\ \mathcal{N}(dr,d\zeta). \tag{4.13}$$

if $F_T \in L^1(P^*)$ and there exists a real valued measurable function $b^*(t)$ integrable on $[0,T]$ such that $\forall t \in [0,T]$, $\forall x \in I\!R$, $y \in \mathcal{Y}$

$$\int_Z U^*(t,x,y;\zeta)\ \nu(d\zeta) = b^*(t), \tag{4.14}$$

then P^* is the MEMM for this model.

Proof.
First we observe that (4.10), (4.11) and (2.6) imply (4.8). Therefore M_t^* is a local $\{P, \mathcal{F}_t\}$-martingale. By Theorem 3.5, the structure of U^* and (4.11) guarantee that (4.12) defines a probability measure P^* equivalent to P and, by Proposition 3.11 and Lemma 4.3, $P^* \in \mathcal{M}_e$.

When (4.14) holds true, we have, by (4.9)

$$\begin{aligned} H(P^*|P) &= I\!E^{P^*}\Big[\int_0^T \int_Z \beta^*(s, X_{s^-}, Y_{s^-})\ \big(e^{K_1(s,X_{s^-},Y_{s^-};\zeta)} - 1\big)\ \big(U^*(s, X_{s^-}, Y_{s^-};\zeta) + 1\big)\ \nu(d\zeta)\ ds\Big] - \\ &- I\!E^{P^*}\Big[\int_0^T \int_Z U^*(s, X_{s^-}, Y_{s^-};\zeta)\ \nu(d\zeta)\ ds\Big] = -\int_0^T b^*(s)\ ds \end{aligned} \tag{4.15}$$

This means that (4.4) is verified and the MEMM exists.
Moreover we observe that, for any given $Q \in \mathcal{M}$, F_t is a local $\{Q, \mathcal{F}_t\}$-martingale and let $\{\tau_n\}$ be a localizing sequence. Then, P-a.s., $\{\tau_n\}$ is a non decreasing sequence converging to T and $F_{\tau_n \wedge t}$ is a $\{Q, \mathcal{F}_t\}$-martingale null at 0. By the Optional Sampling Theorem ([11])

$$\frac{dP^*}{dP}\bigg|_{\mathcal{F}_{\tau_n}} = \mathcal{L}_{\tau_n} \qquad \text{and} \qquad \log\frac{dP^*}{dP}\bigg|_{\mathcal{F}_{\tau_n}} = -\int_0^{\tau_n} b^*(t)dt + F_{\tau_n}$$

thus $\log \frac{dP^*}{dP}\Big|_{\mathcal{F}_{\tau_n}}$ is Q-integrable and

$$I\!E^Q\left[\log\frac{dP^*}{dP}\bigg|_{\mathcal{F}_{\tau_n}}\right] = -I\!E^Q\left[\int_0^{\tau_n} b^*(t)dt\right] \qquad \lim_{n\to\infty} I\!E^Q\left[\log\frac{dP^*}{dP}\bigg|_{\mathcal{F}_{\tau_n}}\right] = -\int_0^T b^*(t)dt$$

Finally the thesis is achieved since, as a consequence of Lemma 2.1 in [18], denoting by $R\big|_{\mathcal{B}}$ the restriction of any probability measure R to the sub-σ-field $\mathcal{B} \subseteq \mathcal{F}$, we have

$$H(Q|P) \geq H(Q\big|_{\mathcal{F}_{\tau_n}}|P\big|_{\mathcal{F}_{\tau_n}}) \geq I\!E^Q\left[\log\frac{dP^*}{dP}\bigg|_{\mathcal{F}_{\tau_n}}\right] = -\int_0^T b^*(t)dt = H(P^*|P).$$

□

Under the assumptions of Proposition 4.4 we get the following representation of the optimal strategy for the control problem (4.5)

$$\theta_t^* = -\frac{\beta^*(t, X_{t^-}, Y_{t^-})}{S_{t^-}}.$$

Remark 4.5 *Let us observe that, when K_1 is only a function of t and ζ, the local characteristics of Y_t under P are deterministic functions of the time. In such a case, (4.14) is trivially true, and it can be seen as a definition of $b^*(t)$. This last particular situation can be compared with that described in [18].*

4.2. The MEMM for marked point processes under full information

In this subsection we study the MEMM for totally observed geometric marked point processes with respect to their internal filtration. We shall consider these processes as a particular case of the model proposed in this paper, when the jump size of Y does not depend on the process X, that is $K_1(t, x, y; \zeta) = K_1(t, y; \zeta)$.

Let us observe that, in this case, the $(P, \mathcal{F}_t)$-local characteristics $(\lambda_t, \Phi_t(dz))$, given in Proposition 3.11, are $\mathcal{F}_t^Y$-adapted hence they provide the minimal local characteristics of Y.

In the sequel, for any probability measures Q defined on $\mathcal{F}_T$, we shall denote by $\widehat{Q} = Q\mid_{\mathcal{F}_T^Y}$ its restriction to $\mathcal{F}_T^Y = \mathcal{F}_T^S$. Then we give the following natural definition

Definition 4.6 *Let Q any probability measure on $(\Omega, \mathcal{F}_T^Y, \widehat{P})$. The relative entropy of Q w.r.t. $\widehat{P}$ is defined as*

$$H_S(Q|\widehat{P}) = \begin{cases} I\!E^P\Big[\dfrac{dQ}{d\widehat{P}}\ log\Big(\dfrac{dQ}{d\widehat{P}}\Big)\Big] & Q \ll \widehat{P} \\ +\infty & otherwise \end{cases} \tag{4.16}$$

Since the price process S_t is locally bounded, under the condition

$$\inf_{Q \in \mathcal{M}_S} H_S(Q \mid \widehat{P}) < +\infty \tag{4.17}$$

there exists the (unique) minimal entropy martingale measure P^*_S on $(\Omega, \mathcal{F}^Y_T, \widehat{P})$, that is the probability measure equivalent to $\widehat{P}$ such that

$$H_S(P^*_S \mid \widehat{P}) = \min_{Q \in \mathcal{M}_S} H_S(Q \mid \widehat{P}).$$

where $\mathcal{M}_S$ is the set of probability measures Q defined on $\mathcal{F}^Y_T$, absolutely continuous with respect to $\widehat{P}$ such that S_t is a local $(Q, \mathcal{F}^Y_t)$-martingale.

Remark 4.7 *First, let us observe that if $Q \in \mathcal{M}$ then $\widehat{Q} \in \mathcal{M}_S$. In fact if $Q \ll P$ on $(\Omega, \mathcal{F}, P)$, obviously $\widehat{Q} \ll \widehat{P}$ on $(\Omega, \mathcal{F}^Y_T, \widehat{P})$. Moreover, for $u \le t$*

$$\mathbb{E}^{\widehat{Q}}\big[S_t \big| \mathcal{F}^Y_u\big] = \mathbb{E}^{Q}\big[S_t \big| \mathcal{F}^Y_u\big] = \mathbb{E}^{Q}\Big[\mathbb{E}^{Q}\big[S_t \big| \mathcal{F}_u\big] \big| \mathcal{F}^Y_u\Big] = S_u.$$

Of course the converse claim is not true.

Moreover, in the general partially observed model, for Q any probability measure on $(\Omega, \mathcal{F}, P)$ such that $Q \ll P$, we have that $\widehat{Q} \ll \widehat{P}$ and that

$$\frac{d\widehat{Q}}{d\widehat{P}} = \frac{dQ}{dP}\bigg|_{\mathcal{F}^Y_T} = \mathbb{E}\left[\frac{dQ}{dP}\bigg| \mathcal{F}^Y_T\right].$$

It is easy to see, by the Jensen inequality, that

$$H_S(\widehat{Q}|\widehat{P}) \le H(Q|P).$$

Besides,

$$\inf_{Q \in \mathcal{M}_S} H_S(Q \mid \widehat{P}) \le \inf_{Q \in \mathcal{M}} H_S(\widehat{Q} \mid \widehat{P}) \le \inf_{Q \in \mathcal{M}} H(Q|P).$$

Consequently, condition (4.4) imply (4.17) and under (4.4) both P^*_S on $(\Omega, \mathcal{F}^Y_T, \widehat{P})$ and P^* on $(\Omega, \mathcal{F}, P)$ there exist. Moreover the following inequalities hold

$$H_S(P^*_S \mid \widehat{P}) \le H_S(\widehat{P}^* \mid \widehat{P}) \le H(P^* \mid P). \tag{4.18}$$

We turn now to the setup stated in this subsection. The discussion performed in Subsection 6.1 leads to the following

Proposition 4.8 *When K_1 does not depend on x, under the assumptions (4.11), (4.12) and (4.14), the probability measure P^* defined in (4.12) restricted to $\mathcal{F}^Y_T$ is the MEMM for the totally observed geometric marked process.*

Proof.
Lemma 4.3 and Proposition 4.4 provide a function $\beta^*(t, y)$ such that, defining $U^*(t, y)$ as in (4.10), the MEMM P^* has a density with respect to P given by

$$\frac{dP^*}{dP} = \mathcal{L}^*_T = c^* e^{F^*}$$

where

$$c^* = \exp\{-\int_0^T b^*(t)\ dt\} \qquad F^* = \int_0^T \frac{\beta^*(t, Y_{t^-})}{S_{t^-}}\ dS_t$$

hence it is $\mathcal{F}^Y_T$-measurable, coincides with $\frac{d\widehat{P}^*}{d\widehat{P}}$ and $H(P^*|P) = H_S(\widehat{P}^*|\widehat{P})$.

To prove that $\widehat{P}^* = P^*_S$ the same argument in the proof of Proposition 4.4 could be used, but we prefer to provide a different approach.

We shall prove, in the following Lemma, that, for any $Q \in \mathcal{M}_S$, there exists a $Q_1 \in \mathcal{M}$ such that

$$\frac{dQ_1}{dP} = \frac{dQ}{d\widehat{P}} \qquad \text{hence} \qquad H(Q_1|P) = H_S(Q|\widehat{P}).$$

Recalling Remark 4.7, this allows us to claim that

$$H_S(P^*_S|\widehat{P}) = \inf_{Q\in\mathcal{M}_S} H_S(Q \mid \widehat{P}) = \inf_{Q\in\mathcal{M}} H(Q|P) = H(P^*|P) = H_S(\widehat{P}^*|\widehat{P}).$$

Uniqueness of the MEMM completes the proof.

□

Lemma 4.9 *For any $Q \in \mathcal{M}_S$, there exists a $Q_1 \in \mathcal{M}$ such that*

$$\frac{dQ_1}{dP} = \frac{dQ}{d\widehat{P}} \qquad \textit{hence} \qquad H(Q_1|P) = H_S(Q|\widehat{P}).$$

Proof.
A suitable version of the Girsanov Theorem ([3]) implies that

$$\frac{dQ}{d\widehat{P}} = \mathcal{L}_T = \mathcal{E}(M)_T$$

with respect to the $\{\widehat{P}, \mathcal{F}^Y_t\}$-martingale

$$M_t = \int_0^t V_s(z)\big(m(ds, dz) - \lambda(s, Y_{s^-})\ \Phi(s, Y_{s^-}, dz)ds\big).$$

Since $Q \in \mathcal{M}_S$ and

$$S_t = S_0 + \int_0^t \int_{I\!R} S_{r^-}(e^z - 1)\ m(dr, dz)$$

as in Proposition 3.6, we get

$$\text{for a.a. } r \in [0,T], \qquad \int_{I\!R} (e^z - 1)\big(1 + V_r(z)\big)\lambda(r, Y_{r^-})\ \Phi(r, Y_{r^-}, dz)dr = 0 \qquad \widehat{P} - a.s. \tag{4.19}$$

Setting

$$U_t(\zeta) = \begin{cases} V_t(K_1(t, Y_{t^-}; \zeta)) & \text{if} \quad K_1(t, Y_{t^-}; \zeta) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

the martingale M_t can be written as

$$M_t = \int_0^t \int_Z U_s(\zeta)\ \big(\mathcal{N}(ds, d\zeta) - \nu(d\zeta)ds\big),$$

condition (4.19) implies (3.9), then the measure Q_1 required by the thesis is given by

$$\frac{dQ_1}{dP} = \mathcal{L}_T = \mathcal{E}(M)_T.$$

□

5. Pricing in a particular model: filtering approach

5.1. Filtering equation for the general model

We recall that the hedger is restricted to observing past asset prices, thus our choice consists in computing the conditional expectation of B with respect to the observations $\mathcal{F}^Y_t = \mathcal{F}^S_t$, under the the minimal entropy martingale measure P^*. Assuming $B = B(S_T)$, by the Markov property, there exists a measurable function $h(t,x,y)$ such that

$$I\!\!E^*\big[B(S_T)\big|\mathcal{F}_t\big] = h(t, X_t, Y_t).$$

Since by definition, $h(t, X_t, Y_t)$ is a $\{P^*, \mathcal{F}_t\}$-martingale, one can easily see, by Itô Formula, that the function $h(t,x,y)$ has to satisfy the problem

$$L^* h(t,x,y) = 0 \qquad\qquad h(T,x,y) = B(S_0 e^y)$$

where the operator L^*, recalling (3.17), is

$$\begin{aligned} L^* f(s,x,y) &= \frac{\partial f}{\partial s}(s,x,y) + b(x)\,\frac{\partial f}{\partial x}(s,x,y) + \frac{1}{2}\,\sigma^2(x)\,\frac{\partial^2 f}{\partial x^2}(s,x,y) + \\ &+ \int_Z \Big\{ f\big(s, x + K_0(s,x;\zeta), y + K_1(s,x,y;\zeta)\big) - f(s,x,y) \Big\}\,\big(1 + U^*(s,x,y;\zeta)\big)\,\nu(d\zeta) \end{aligned} \tag{5.1}$$

since

$$\nu^*(d\zeta) = \big(1 + U^*(s, X_{s^-}, Y_{s^-}, \zeta)\big)\,\nu(d\zeta)$$

for U^* defined in (4.10).

Hence we have that

$$I\!\!E^*\big[B(S_T)\big|\mathcal{F}^Y_t\big] = I\!\!E^*\big[I\!\!E^*\big[B(S_T)\big|\mathcal{F}_t\big]\big|\mathcal{F}^Y_t\big] = I\!\!E^*\big[h(t, X_t, Y_t)\big|\mathcal{F}^Y_t\big].$$

Thus it is possible to deal with the problem of pricing by filtering techniques. Since the observations process Y is a pure-jump process, there exists a probability measure-valued cadlag process π^* (the filter) such that, for any bounded measurable real valued function $f(t,x)$

$$\pi^*(f) = I\!\!E^*\big[f(t, X_t)\big|\mathcal{F}^Y_t\big].$$

Remark 5.1 *A different approach could consist in looking for the risk-neutral measures on the space $(\Omega, \mathcal{F}^Y_T, P)$. On this space, first, the minimal local characteristics of Y have to be computed, again by the filter. Moreover, in order to preserve the Markovianity of the model, the process (Y_t, π^*_t) has to be considered, process whose joint dynamics is not so easy to handle.*

We turn then to the problem of computing $\pi^*\big(h(\cdot, Y_t)\big)$.
It is well known that the filter is a solution of the Kushner-Stratonovich equation (KS-equation) ([4], VIII, Theorem T9). In Proposition 5.2 this equation will be written down. The procedure used is a slight modification of that used in [6], thus the proof will be omitted.

First, we introduce the point processes $\{v^h\}$

$$v^h_t = \int_0^t \int_Z 1\!\!\mathrm{I}_{\{(s,\zeta): K_1(s, X_{s^-}, Y_{s^-}; \zeta) = h\}}\,\mathcal{N}(ds, d\zeta), \qquad h \in \mathcal{Y},\ h \neq 0.$$

The process v^h counts the jump times T_n of Y up to time t, when $Z_n = Y_{T_n} - Y_{T_{n-1}} = h$ and under the assumption (2.6), its $(P, \mathcal{F}_t)$-predictable intensity is

$$\lambda_h(t) = \nu(D^h_1(t, X_{t^-}, Y_{t^-})) = \lambda_h(t, X_{t-}, Y_{t-})$$

(where the set $D^h_1(t,x,y)$ has been defined in (2.13) as $D^h_1(t,x,y) = \{\zeta \in Z : K_1(t,x,y;\zeta) = h\}$), while its $(P^*, \mathcal{F}_t)$-predictable intensity is given by

$$\lambda^*_h(t) = \nu^*(D^h_1(t, X_{t^-}, Y_{t^-})) = \int_{D^h_1(t, X_{t^-}, Y_{t^-})} \big(1 + U^*(t, X_{t^-}, Y_{t^-}; \zeta)\big)\,\nu(d\zeta) = \lambda^*_h(t, X_{t-}, Y_{t-}) \tag{5.2}$$

By noticing that, for any $h \neq 0$, v^h_t is $\mathcal{F}^Y_t$-adapted and

$$Y_t = \sum_{h \neq 0,\ h \in \mathcal{Y}} h v_t^h$$

we have

$$\mathcal{F}_t^Y = \mathcal{F}_t^v = \bigvee_{h \neq 0,\ h \in \mathcal{Y}} \mathcal{F}_t^{v^h} = \bigvee_{h \neq 0,\ h \in \mathcal{Y}} \sigma\{v_s^h\ ,\ s \leq t\}.$$

and

$$\pi_t^*(f) = E^*(f(t, X_t) \mid \mathcal{F}_t^Y) = E^*(f(t, X_t) \mid \mathcal{F}_t^v). \tag{5.3}$$

Finally we write down the Kushner-Stratonovich equation.

Proposition 5.2 *Assuming*

$$E^*[\int_0^T \nu(D_0(s, X_s, Y_s))\ ds] < \infty \qquad E^*[\int_0^T \nu(D_1(s, X_s, Y_s))\ ds] < \infty \quad \textit{and} \quad E^*\big[\int_0^T \sigma^2(X_s)\ ds\ \big] < \infty,$$

the filter is a solution of the Kushner-Stratonovich equation, $(KS - equation)$, that, for any function $f(t,x)$ in $\mathcal{C}_b^{1,2}(I\!\!R^+ \times I\!\!R)$, is given by

$$\begin{aligned}\pi_t^*(f) &= f(0, x_0) + \int_0^t \pi_s^*(L_X^* f)\ ds + \qquad (5.4)\\ &+ \sum_{h \neq 0,\ h \in \mathcal{Y}} \int_0^t \pi_{s-}^*\big(\lambda_h^*(\cdot, Y_{s-})\big)^+ \Big\{\pi_{s-}^*\big(\lambda_h^*(\cdot, Y_{s-})\ f\big) - \pi_{s-}^*\big(\lambda_h^*(\cdot, Y_{s-})\big)\ \pi_{s-}^*(f) + \pi_{s-}^*\big(R_h^* f(\cdot, Y_{s-})\big)\Big\} \times\\ &\qquad \times \big(dv_s^h - \pi_{s-}^*\big(\lambda_h^*(\cdot, Y_{s-})\big)\ ds\big)\end{aligned}$$

where $a^+ = \frac{1}{a}\ 1\!\!I_{a>0}$,

$$R_h^* f(t, x, y) = \int_{D_1^h(t,x,y)} \Big[f\big(t, x + K_0(t, x; \zeta)\big) - f\big(t, x\big)\Big]\ \big(1 + U^*(s, x, y; \zeta)\big)\ \nu(d\zeta)$$

and L_X^*, the restriction of the operator L^* on a function $f(t,x)$ is given by

$$\begin{aligned}L_X^* f(t, x, y) &= \frac{\partial f}{\partial s}(s, x) + b(x)\ \frac{\partial f}{\partial x}(s, x) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 f}{\partial x^2}(s, x) +\\ &+ \int_Z \Big\{f\big(s, x + K_0(s, x; \zeta)\big) - f(s, x, y)\Big\}\ \big(1 + U^*(s, x, y; \zeta)\big)\ \nu(d\zeta).\end{aligned} \tag{5.5}$$

A discussion about uniqueness for the solutions to the Kushner-Stratonovich equation can be performed following the same lines as in [5] and [6]. In fact, the assertions in Remarks 2.1, 3.8, 3.10, allow us to use the Filtered Martingale Problem approach ([22]).

A more handle expression for the filter can be obtained, as in [6], by noticing that the KS-equation has a natural recursive structure, following the jump times $\{T_i\}$, and that at any jump time T_i, $\pi_{T_i}^*(f)$ is completely determined by the observed data $Y_{T_{i-1}}$, $Z_i = Y_{T_i} - Y_{T_{i-1}}$ and by the knowledge of $\pi_t^*(f)$ in the interval $[T_{i-1}, T_i)$, while it is easy to see that $\pi_t^*(f)$ has a deterministic behavior between two consecutive jumps times of Y. In fact, at any jump time T_i we have that

$$\begin{aligned}&\pi_{T_i}^*(f) - \pi_{T_i^-}^*(f) = \qquad (5.6)\\ &= \pi_{T_i^-}^*\big(\lambda_h^*(\cdot, Y_{T_i^-})\big)^+ \Big\{\pi_{T_i^-}^*\big(\lambda_h^*(\cdot, Y_{T_i^-})\ f\big) - \pi_{T_i^-}^*\big(\lambda_h^*(\cdot, Y_{T_i^-})\big)\ \pi_{T_i^-}^*(f) + \pi_{T_i^-}^*\big(R_h^* f(\cdot, Y_{T_i^-})\big)\Big\}\Big|_{Y_{T_i} - Y_{T_i^-} = h.}\end{aligned}$$

For $t \in [T_i, T_{i+1})$

$$\pi_t^*(f) = \pi_{T_i}^*(f) + \int_{T_i}^t \Big\{ \pi_s^*\big(\widetilde{L}_X^* f(\cdot, Y_{T_i})\big) - \pi_s^*\big(\lambda^*(\cdot, Y_{T_i})\ f\big) + \pi_s^*\big(\lambda^*(\cdot, Y_{T_i})\big)\pi_s^*(f)\Big\}\ ds \tag{5.7}$$

where

$$\lambda^*(t,x,y) = \sum_{h\neq 0,\ h\in\mathcal{Y}} \lambda_h^*(t,x,y)$$

$$\begin{aligned}\widetilde{L}_X^* f(t,x) &= L_X^* f(t,x) - \sum_{h\neq 0,\ h\in\mathcal{Y}} R_h^* f(t,x,y) = \\ &= \frac{\partial f}{\partial t}(t,x) + b(x)\ \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\ \sigma(x)^2\ \frac{\partial^2 f}{\partial x^2}(t,x) + \\ &+ \int_{D_0(t,x,y)} \Big(f(t, x + K_0(t,x;\zeta)) - f(t,x)\Big)\ \big(1 + U^*(t,x,y;\zeta)\big)\ \nu(d\zeta).\end{aligned}$$

Then, as in [5] and [6], the computation of the filter for $t \in [T_i, T_{i+1})$ can be reduced to the evaluation of ordinary expectations. To this end we need that the Martingale Problem associated to the operator $\widetilde{L}_X^* f$ and with deterministic initial conditions (s,x) is well posed, for any $y \in I\!\!R$, and this last result can be obtained taking into account Proposition 4.2 in [5] with the same argument used in Remark 3.10.

In the next Subsection we shall give more detailed results referring to the particular case introduced in Proposition 5.4 below.

5.2. The particular model

Over all this Subsection we consider the model defined by (2.8). We shall see that in this particular case, we can get a more explicit expression for the density of the MEMM under the additional assumption

$$\nu\big(D_1(t,x,y)\big) < +\infty \qquad \forall t \in [0,T],\quad x \in I\!\!R,\quad y \in \mathcal{Y}. \tag{5.8}$$

First, we shall see, in the next proposition, that in this particular case, the computation of the function $\beta^*(t,x,y)$ can be explicitly performed.

Proposition 5.3 *Under (5.8)*

$$\beta^*(t,x,y) = \frac{1}{e^{K_1^+(t,x,y)} - e^{-K_1^-(t,x,y)}}\ \log\left\{ \frac{\nu\big(D_1^-(t,x,y)\big)}{\nu\big(D_1^+(t,x,y)\big)}\ \frac{1 - e^{-K_1^-(t,x,y)}}{e^{K_1^+(t,x,y)} - 1}\right\}. \tag{5.9}$$

Proof.
Condition (4.9) becomes

$$\begin{aligned}&\exp\Big\{\beta^*(t,x,y)\big(e^{K_1^+(t,x,y)} - 1\big)\Big\}\ \big(e^{K_1^+(t,x,y)} - 1\big)\ \nu(D_1^+(t,x,y)) + \\ &\qquad + \exp\Big\{\beta^*(t,x,y)\big(e^{-K_1^-(t,x,y)} - 1\big)\Big\}\ \big(e^{-K_1^-(t,x,y)} - 1\big)\ \nu(D_1^-(t,x,y)) = 0.\end{aligned}$$

Thus (5.9) can be easily deduced.
□

Next, we shall consider a more particular model.

Proposition 5.4 *Under (5.8), (4.14) is implied by the existence of real valued measurable functions $\lambda^+(t)$, $\lambda^-(t)$, $\Gamma(t)$ such that*

$$\lambda^+(t) = \nu(D_1^+(t,x,y)), \qquad \lambda^-(t) = \nu(D_1^-(t,x,y)), \qquad \Gamma(t) = \frac{1 - e^{-K_1^-(t,x,y)}}{e^{K_1^+(t,x,y)} - 1}.$$

If in addition $\nu(Z) < +\infty$ and there exist positive constants k, K such that

$$\lambda^+(t), \lambda^-(t) \geq k, \quad k \leq K_1^+(t,x,y) \leq K, \quad k \leq K_1^-(t,x,y) \leq K \tag{5.10}$$

then $F_T \in L^1(P^*)$, condition (4.11) holds true and $I\!\!E[\mathcal{L}_T^*] = 1$.

Proof.
A direct computation provides

$$U^*(t,x,y;\zeta)=\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}}-1\right]1\!\!1_{D_1^+(t,x,y)}(\zeta)+\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}}-1\right]1\!\!1_{D_1^-(t,x,y)}(\zeta) \qquad (5.11)$$

and

$$b^*(t)=\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}}-1\right]\ \lambda^+(t)+\left[\left(\frac{\lambda^-(t)}{\lambda^+(t)}\ \Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}}-1\right]\ \lambda^-(t).$$

By $\nu(Z)<+\infty$ and (5.10) we get that $F_T\in L^1(P^*)$ and (4.11) is satisfied. In fact, the quantities

$$\int_Z |U^*(r,x,y;\zeta)|\ \nu(d\zeta),\qquad \int_Z log(1+U^*(r,x,y;\zeta))(1+U^*(r,x,y;\zeta))\nu(d\zeta)$$

are bounded from above by $C(k,K)\ \nu(Z)$, where $C(k,K)>0$ is a suitable computable constant.
Thus, recalling that

$$L_t^*=1+\int_0^t\int_Z L_{r-}^*U^*(r,X_{r-},Y_{r-};\zeta)(\mathcal{N}(dr,d\zeta)-\nu(d\zeta)\ dr),\quad I\!\!E[L_t^*]\le 1$$

we get

$$I\!\!E\left[\int_0^t\int_Z L_{r-}^*\ |U^*(r,X_{r-},Y_{r-};\zeta)|\ \nu(d\zeta)\ dr\right]<+\infty$$

which implies that L_t^* is a $\{P,\mathcal{F}_t\}$- martingale.
□

Finally we will give a more explicit description of the filter's behavior under the assumption of Proposition 5.4. Recalling (5.11) and setting

$$U^+(t)=\left(\frac{\lambda^-(t)}{\lambda^+(t)}\Gamma(t)\right)^{\frac{1}{1+\Gamma(t)}}\qquad U^-(t)=\left(\frac{\lambda^-(t)}{\lambda^+(t)}\Gamma(t)\right)^{\frac{-\Gamma(t)}{1+\Gamma(t)}}$$

we can write, successively

$$U^*(t,x,y;\zeta)=\left[U^+(t)-1\right]1\!\!1_{D_1^+(t,x,y)}(\zeta)+\left[U^-(t)-1\right]1\!\!1_{D_1^-(t,x,y)}(\zeta)$$

$$\lambda_h^*(t,X_{t-},Y_{t-}))=\lambda_h(t,X_{t-},Y_{t-}))\ \left[U^+(t)\ 1\!\!1_{\{h>0\}}+U^-(t)\ 1\!\!1_{\{h<0\}}\right]$$

$$\pi_{t-}^*\big(\lambda_h^*(\cdot,Y_{t-})\big)=\pi_{t-}^*\big(\lambda_h(\cdot,Y_{t-})\big)\ \left[U^+(t)\ 1\!\!1_{\{h>0\}}+U^-(t)\ 1\!\!1_{\{h<0\}}\right]$$

and then, at every jump time t of Y, the filter has a jump given by

$$\pi_t^*(f)-\pi_{t-}^*(f)=\pi_{t-}^*\big(\lambda_h(\cdot,Y_{t-})\big)^+\Big\{\pi_{t-}^*\big(\lambda_h(\cdot,Y_{t-})\ f\big)-\pi_{t-}^*\big(\lambda_h(\cdot,Y_{t-})\big)\ \pi_{t-}^*(f)+\pi_{t-}^*\big(R_hf(\cdot,Y_{t-})\big)\Big\}\Big|_{Y_t-Y_{t-}=h}$$

where

$$R_hf(t,x,y)=\int_{D_1^h(t,x,y)}\Big[f\big(t,x+K_0(t,x;\zeta)\big)-f\big(t,x\big)\Big]\ \nu(d\zeta)$$

For $t\in[T_i,T_{i+1})$ we get that (5.7) has a very simple form. In fact

$$\lambda^*(t,x,y) = U^+(t) \sum_{h>0,\ h\in\mathcal{Y}} \lambda^*_h(t,x,y) + U^-(t) \sum_{h<0,\ h\in\mathcal{Y}} \lambda^*_h(t,x,y) = U^+(t)\ \lambda^+(t) + U^-(t)\ \lambda^-(t),$$

$$\widetilde{L}^*_X f(t,x) = \frac{\partial f}{\partial t}(t,x) + b(x)\ \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\ \sigma(x)^2\ \frac{\partial^2 f}{\partial x^2}(t,x) + \\ + \int_{D_0(t,x,y)} \Big(f(t, x + K_0(t,x;\zeta)) - f(t,x) \Big)\ \nu(d\zeta) = \widetilde{L}_X f(t,x),$$

and, for $t \in [T_i, T_{i+1})$ we can write

$$\pi^*_t(f) = \pi^*_{T_i}(f) + \int_{T_i}^t \pi^*_s\big(\widetilde{L}_X f(\cdot, Y_{T_i})\big)\ ds \tag{5.12}$$

As a consequence of Lemma 3.2 in [6], we can assert that, under suitable assumptions, this generator determines a unique process on $D_{[0,+\infty)}(I\!R)$, the space of cadlag functions from $[0,+\infty)$ into $I\!R$, for any deterministic initial condition.

Proposition 5.5 *Denoting by ξ_t the process determined by the generator $\widetilde{L}_X$ with initial condition (s,x) and by $P^y_{(s,x)}$ its law on $D_{[0,+\infty)}(I\!R)$, setting for any real-valued bounded measurable f,*

$$\Phi_t(s,x,y)(f) = I\!E^y_{(s,x)}\big[f(t,\xi_t)\big],$$

a solution of (5.12) is given by

$$\pi^*_t(f) = \int_{I\!R} \Phi_t(T_i, x, Y_{T_i})(f)\ \pi_{T_i}(dx). \tag{5.13}$$

Proof.
First we observe that $f \geq 0$ ($f > 0$) implies $\Phi_t(s,x,y)(f) \geq 0$ ($\Phi_t(s,x,y)(f) > 0$), that $\Phi_t(s,x,y)(f)$, as a function of t is a cadlag function, and finally that it is jointly measurable w.r.t. (s,x,y) (the last claim is a consequence of Theorem 4.6, Chap. 4 in [11]). Thus (5.13) makes sense and can be easily obtained by Itô Formula.
□

As a conclusion we shall provide a recursive algorithm for the filter. To this end we consider a suitable family of finite state, discrete time Markov chains $\{\xi^n_k\}_{k\geq 0}$ with state space $\{x^n_1, \ldots, x^n_{m(n)}\}$ and transitions probabilities $\{p^n_{ij}\}_{i,j=1,\ldots,m(n)}$, constructed as in [23] where it is also proven that the processes

$$\xi^n_t = \sum_{k=0}^{n-1} \xi^n_k\ 1\!\!1_{\{t^n_k \leq t < t^n_{k+1}\}}$$

for $t^n_k = s + \frac{k(T-s)}{n}$, converges in law to ξ_t as $n \to \infty$.

Proposition 5.6 *Let $P^{n,y}_{(s,x)}$ be the law of ξ^n_t on $D_{[0,T]}(I\!R)$ and set*

$$\Phi^n_t(s,x,y)(f) = I\!E^{n,y}_{(s,x)}\Big[f(t,\xi^n_t)\Big],$$

When f is a bounded function, continuous w.r.t. x, $\Phi^n_t(s,x,y)(f)$ converges to $\Phi_t(s,x,y)(f)$, for a.a. $t \in [0,T]$ and

$$\Phi^n_t(s,x,y)(f) = \sum_{i=1}^{m(n)} f(t,x^n_i) \sum_{k=0}^{n-1} I\!I_{\{t^n_k \leq t < t^n_{k+1}\}}\ P\big(\xi^n_k = x^n_i\big). \tag{5.14}$$

Proof.
For the convergence result, it is sufficient to observe that

$$f(t,\cdot) \quad : \quad D_{[0,T]}(I\!R) \longrightarrow I\!R$$

is a bounded continuous functional for a.a. $t \in [0,T]$, while (5.14) is obtained by a direct computation. The recursion is a consequence of the Markov property

$$P\big(\xi^n_{k+1} = x^n_j\big) = \sum_{i=1}^{m(n)} p^n_{ij}\; P\big(\xi^n_k = x^n_i\big).$$

□

Acknowledgment

The authors wish to thank Professor W. Runggaldier for his helpful comments.

References

[1] D. Becherer, *Rational hedging and valuation of integrated risks under constant absolute risk aversion*, Insurance Math. Econom. **33** (2003) 1-28.

[2] F. Bellini and M. Frittelli, *On the existence of minimax martingale measures*, Mathematical Finance **12** (2002) 1-21.

[3] T. Bjork, Y. Kabanov and W. Runggaldier, *Bond market structure in presence of marked point processes*, Mathematical Finance **7** (2) (1997) 211-223.

[4] P. Brémaud, *Point Processes and Queues*, Springer-Verlag (1980).

[5] C. Ceci, *Risk minimizing hedging for a partially observed high frequency data model*, Stochastics: An International Journal of Probability and Stochastic Processes **78** (1) (2006) 13-31.

[6] C. Ceci and A. Gerardi, *A model for high frequency data under partial information: a filtering approach*, International Journal of Theoretical and Applied Finance **9** (4) (2006) 1-22.

[7] S. Centanni and M. Minozzo. *Estimation and Filtering by Reversible Jump MCMC for a Doubly Stochastic Poisson Model for Ultra-High-Frequency Financial Data*, Proceedings of *II Workshop on Correlated Data Modeling)*, (2004), *Franco Angeli editor, Milano.* To appear in Statistical Modeling.

[8] F. Delbaen, P. Grandits, T. Rheinlander, D. Samperi, M. Schweizer and C. Stricker, *Exponential hedging and entropic penalties*, Mathematical Finance **12** (2002) 99-123.

[9] F. Delbaen and M. Schachermayer, *The variance-optimal martingale measure for continuous processes*, Bernoulli 2 (1996) 81-105.

[10] C. Doléans-Dade, *Quelques applications de la formule de changement de variables pour le semi-martingales*, Z. fur W. **16** (1970) 181-194

[11] S.N. Ethier and T.G. Kurtz, *Markov Processes, Characterization and Convergence*, J. Wiley & sons (1986).

[12] H. Follmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, in Applied Stochastic Analysis, Stochastic Monographs, 5, Gordon & Breach, London/New York, (1991) 389-414.

[13] R. Frey, *Risk minimization with incomplete information in a model for high-frequency data*, Mathematical Finance **10** (2) (2000) 215-22.

[14] R. Frey and W. Runggaldier, *Risk-minimization hedging strategies under restricted information: The case of stochastic volatility models observed only at discrete random times*, Mathematical Methods of Operations Research **50** (1999) 339-350.

[15] R. Frey and W. Runggaldier, *A nonlinear filtering approach to volatility estimation with a view towards high frequency data*, International Journal of Theoretical and Applied Finance **4** (2) (2001) 199-210.

[16] M. Frittelli, *the minimal entropy martingale measure and the valuation problem in incomplete markets*, Mathematical Finance **10** (1) (2000) 39-52.

[17] M. Frittelli, *Introduction to a theory of value coherent with no-arbitrage principle*, Finance and Stochastics **4** (2000) 275-297.

[18] T. Fujiwara and Y. Miyahara, *The minimal entropy martingale measures for geometric Lévy processes*, Finance and Stochastics **7** (2003) 509-531.

[19] P. Grandits and T. Rheinlander, *On the minimal entropy martingale measures*, The Annals of Probability **30** (3) (2002) 1003-1038.

[20] Y. Kabanov and C. Stricker, *On the optimal portfolio for the exponential utility maximization: remark to the six-author paper*, Mathematical Finance **12** (2002) 125-134.

[21] M.Kirch and W.J. Runggaldier, *Efficient hedging when asset prices follow a geometric Poisson process with unknown intensities*, SIAM J. on Control and Optimization **43** (2004) 1174-1195.

[22] T. G. Kurtz and D. Ocone, *Unique characterization of conditional distribution in nonlinear filtering*, Annals of Probability **16** (1988) 80-107.

[23] H. J. Kushner and P. G. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer-Verlag (1993).

[24] J. Jacod, *Calcul stochastique et problemes de martingales*, Springer-Verlag (1979).

[25] M. Mania and M. Schweizer, *Dynamic exponential utility indifference valuation*, The Annals of Applied Probability, **15** (3) (2005) 2113-2143.

[26] M. Musiela and M. Rutkowski, *Martingale methods in financial modeling*, Applications of Mathematics, Stochastic Modelling and Applied Probability **36**, Springer-Verlag (1997).

[27] J. L. Prigent. *Option pricing with a General Marked Point Process*, *Mathematics of Operations Research*, **26** (1) (2001) 50-66.

[28] T. Rheinlander, *An entropy approach to the Stein and Stein model with correlation*, Finance and Stochastics **9** (2005) 399-413.

[29] L. C. G. Rogers and O. Zane, *Designing models for high-frequency data*, Preprint, University of Bath (1998).

[30] T. Rydberg and N. Shephard, *A modelling framework for prices and trades made at the New York stock exchange*, Nuffield College working paper series 1999-W14, Oxford, *Nonlinear and Nonstationary Signal Processing*, eds. W.J. Fitzgerald *et al.*, Cambridge University Press (2000) 217-246.

[31] M. Schweizer, *Approximation pricing and the variance-optimal martingale measure*, Annals of Probability 24 (1996) 206-236.

[32] M. Schweizer, *A guided tour through quadratic hedging approaches*, in *Option Pricing, Interest Rate and Risk Management*, E. Jouini, J. Cvitanic, M. Musiela (eds.), Cambridge University Press (2001) 538-574.