OPTION HEDGING FOR HIGH FREQUENCY DATA MODELS

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Abstract

Hedging strategies for contingent claims are studied in a general model for high frequency data. The dynamics of the risky asset price is described through a marked point process Y, whose local characteristics depend on some hidden state variable X. The two processes Y and X may have common jump times, which means that the trading activity may affect the law of X and could be also related to the presence of catastrophic events. Since the market considered is incomplete one has to choose some approach to hedging derivatives. We choose the local risk-minimization criterion. When the price of the risky asset is a general semimartingale, if an optimal strategy exists, the value of the portfolio is computed in the terms of the so-called minimal martingale measure and may be interpreted as a possible arbitrage-free price. In the case where the price of the risky asset is modeled directly under a martingale measure, the computation of the risk-minimizing hedging strategy is given. By using a projection result, we also obtain the risk-minimizing hedging strategy under partial information when the hedger is restricted to observing only the past asset prices and not the exogenous process X which drives their dynamics.

Keywords: High-frequency data, option hedging, marked point processes, jump-diffusions.

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1. Introduction

In models for intraday stock price movements asset prices are used to be described by marked point processes. In fact on a very small time scale, as in high frequency data, real asset prices are piecewise constant and jump in reaction to trades or to significant new information.

In many papers, (see, for instance, [8], [9], [10], [12], [15] and [14]) the asset price process is modeled as a double stochastic Poisson process with marks. In some of them, the local characteristics of this process depend on an unobservable state variable, which may describe the intraday market activity, the activity of other markets, macroeconomics factors or microstructure rules that drive the market.

In this paper we consider a more general model as that introduced in [4]. The behaviour of the asset prices is described via a general marked point process Y, whose local characteristics, in particular the jump-intensity, depend on an exogenous state variable X, which is modeled by a Markov jump-diffusion process. Moreover, the dynamics of Y and X may be strongly dependent, in particular the two processes may have common jump times. Hence our model could take into account also the possibility of catastrophic events. This kind of events, in fact, influence both the asset prices and the hidden state variable which drives their dynamics. We assume that the pair (X, Y) is a solution of a system of stochastic differential equations driven by a Browian motion and a Poisson random measure as a natural way to describe its dynamics.

In this note we are concerned with the hedging of contingent claims. When the given financial market is complete, every claim can be replicated by a self-financing dynamic portfolio strategy which only makes use of the existing assets. In this case, one can reduce to zero the risk of the claim by a suitable strategy. On the other hand, markets modeled by marked point processes, where infinite number of marks are allowed, are incomplete. Then one has to choose some approach to hedging derivatives. Since one cannot ask simultaneously for a perfect replication of a given claim by a portofolio strategy and the self-financing property of this strategy, one has to relax one of these conditions. In this paper we choose the local risk minimization, which keeps the replicability and relax the self-financing condition. In [7] the authors dealt with the study of hedging of contingent claims under market incompleteness by introducing the criterion of risk minimization in the case where the price process is a martingale under the real world probability measure. They proved that the optimal strategy can be obtained by the Kunita-Watanabe decomposition. In the general semimartingale case, since it does not exist in general any risk-minimizing strategy, the weaker concept of locally risk-minimizing strategy was introduced ([16]). In [6], under the further assumption that the risky asset price has continuous trajectories, it has been proved that an optimal strategy exists and that it can be computed by the Kunita-Watanabe decomposition under the minimal martingale measure.

In this paper we first consider the general case where the risky asset price is a semimartingale, but since it is not continuous, the results proved in [6] cannot be applied. However we prove that the value process of an optimal strategy (when it exists) can be again computed in terms of the minimal martingale measure. The explicit expression of the density of the minimal martingale measure is provided for our model where the filtration is generated by the Wiener process and the random Poisson measure. In [13] this has been done in the case of marked point processes with respect to their internal filtration.

In the last section we recall the main results obtained in [5] in the case where the price of the risky asset is a local martingale under the real world probability measure. The risk-minimizing strategy is computed by the Kunita-Watanabe decomposition. Moreover, by using a projection result ([17]), the risk-minimizing strategy when agents have access only to the information contained in the past asset prices (they have not knowledge of the latent state process) can be obtained by solving a filtering problem.
We do not discuss here this filtering problem since it is exhaustively studied in [4] when Y is a discrete valued process and in [5] when Y is a real-valued process.

2. The Model

We consider the same model studied in [4] and [5]. On some underlying filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ we consider a market with two traded assets: a riskless money market account and a risky asset. The risky asset price S is supposed having the form

$$S_t = S_0 e^{Y_t} \tag{2.1}$$

where

$$S_0 \in I\!R^+, \quad Y_t = \sum_{n=0}^{N_t} Z_n, \quad Z_0 = 0, \quad N_t = \sum_{n \geq 1} 1\!\!1_{\{T_n \leq t\}},$$

the random times $\{T_n\}$ represent instants at which a large trade occurs or at which a market maker updates his quotes in reaction to significant new information, Y represents the logreturn process, $Z_n = Y_{T_n} - Y_{T_{n-1}}$ is the size of the nth logreturn change and N is the point process which counts the total number of changes.

Besides the risky asset, there is a risk-free asset traded in our market, whose price is taken equal to 1. This simply means that S is the discounted price of the risky asset and this helps to avoid more complicated notations.

We will consider the case in which the $(P, \mathcal{F}_t)$-local characteristics ([2]) $(\lambda_t, \Phi_t(dz))$ of the marked point process Y may depend on some exogenous process X.

In [8], [9] and [10], the possibility that the jump-times of N and X coincide has been excluded. In this note, we allow common jump times between N and X.

A natural way to describe this kind of behaviour is to suppose that the pair (X, Y) takes values in $I\!R \times I\!R$, and that it is a global solution to the following system

$$X_t = x_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s + \int_0^t \int_Z K_0(s, X_{s^-}; \zeta)\, \mathcal{N}(ds, d\zeta) \tag{2.2}$$

$$Y_t = \int_0^t \int_Z K_1(s, X_{s^-}, Y_{s^-}; \zeta)\, \mathcal{N}(ds, d\zeta) \tag{2.3}$$

where $x_0 \in I\!R$, W_t is a $(P, \mathcal{F}_t)$-standard Brownian motion, $\mathcal{N}(dt, d\zeta)$ is a $(P, \mathcal{F}_t)$-Poisson random measure on $I\!R^+ \times Z$, independent of W_t, with mean measure $dt\,\nu(d\zeta)$, with $\nu(d\zeta)$ a σ-finite measure on a measurable space $(Z, \mathcal{Z})$. The $I\!R$-valued functions $b(x)$, $\sigma(x)$, $K_0(t, x; \zeta)$ and $K_1(t, x, y; \zeta)$ are jointly measurable functions of their arguments.

Overall this paper we assume existence and uniqueness (at least weak uniqueness) to the system (2.2), (2.3) (see [4] and [5] for a discussion on this topic).

In Proposition (2.1) and (2.2) below we recall some results proved in [5]. At first, the $(P, \mathcal{F}_t)$-local characteristics $(\lambda_t, \Phi_t(dz))$ of Y are derived taking into account the representation (2.3). The time-dependency of $(\lambda_t, \Phi_t(dz))$ incorporate seasonality effects, which are typical for high frequency data. In particular λ_t, corresponds to the rate at which new economic information is absorbed by the market.

First, we introduce the sequence of jump times of Y

$$T_1 = inf\{t > 0 : \int_0^t \int_Z K_1(s, X_{s^-}, 0; \zeta)\, \mathcal{N}(ds, d\zeta) \neq 0\}$$

$$T_{n+1} = inf\{t > T_n : \int_{T_n}^t \int_Z K_1(s, X_{s^-}, Y_{T_n}; \zeta)\, \mathcal{N}(ds, d\zeta) \neq 0\}$$

and the sequence of the marks

$$Z_n = Y_{T_n} - Y_{T_{n-1}} = \int_Z K_1(T_n, X_{T_n^-}, Y_{T_{n-1}}; \zeta)\, \mathcal{N}(\{T_n\}, d\zeta).$$

Let us define

$$D_1(t, x, y) = \{\zeta \in Z : K_1(t, x, y; \zeta) \neq 0\} \tag{2.4}$$

and

$$D_0(t, x, y) = \{\zeta \in Z : K_0(t, x; \zeta) \neq 0, \quad K_1(t, x, y; \zeta) \neq 0\}. \tag{2.5}$$

Proposition 2.1 *Let $\forall T > 0$, $\forall t \in [0, T]$, $\forall A \in \mathcal{B}(I\!R)$ (where $\mathcal{B}(I\!R)$ denotes the family of Borel subsets of $I\!R$)*

$$D_1^A(t, x, y) = \{\zeta \in Z : K_1(t, x, y; \zeta) \in A \setminus \{0\}\} \subseteq D_1(t, x, y), \tag{2.6}$$

and denote by m the integer valued random measure associated to Y ([2],[11])

$$m(dt, dz) = \sum_{n \geq 1} \delta_{\{T_n, Z_n\}}(dt, dz)\, 1\!I_{\{T_n < \infty\}}. \tag{2.7}$$

Then, under the assumption

$$E[\int_0^T \nu(D_1(s, X_s, Y_s))\, ds] < \infty \qquad \Big(\int_0^T \nu(D_1(s, X_s, Y_s))\, ds < \infty \quad P-a.s.\Big) \tag{2.8}$$

the $(P, \mathcal{F}_t)$-predictable projection of m is given by

$$m^p(dt, dz) = \lambda_t \Phi_t(dz) dt = \lambda(t, X_{t^-}, Y_{t^-}) \Phi(t, X_{t^-}, Y_{t^-}, dz) dt \tag{2.9}$$

where

$$\lambda_t = \lambda(t, X_{t^-}, Y_{t^-}) = \nu(D_1(t, X_{t^-}, Y_{t^-})) \tag{2.10}$$

provides the $(P, \mathcal{F}_t)$-predictable intensity of the point process $N_t = \sum_{n \geq 1} 1\!I_{\{T_n \leq t\}}$ and on $\{T_n < \infty\}$

$$\Phi_{T_n}(A) = \frac{\nu(D_1^A(T_n, X_{T_n^-}, Y_{T_n^-}))}{\nu(D_1(T_n, X_{T_n^-}, Y_{T_n^-}))}. \tag{2.11}$$

Moreover, whenever there exists a transition function $\mu(t,x,y,A)$ *such that,* $\forall A \in \mathcal{B}(I\!R)$

$$P\big(Z_n \in A \mid \mathcal{F}_{T_n -}\big) = \mu(T_n, X_{T_n^-}, Y_{T_n^-}, A)$$

then on $\{T_n < \infty\}$

$$\Phi_{T_n}(A) = P\big(Z_n \in A \mid \mathcal{F}_{T_n^-}\big). \tag{2.12}$$

By applying Itô formula to (2.3) we derive the joint dynamics of the pair (X, S):

$$X_t = x_0 + \int_0^t b(X_s)\ ds + \int_0^t \sigma(X_s)\ dW_s + \int_0^t \int_Z K_0(s, X_{s^-}; \zeta)\ \mathcal{N}(ds, d\zeta) \tag{2.13}$$

$$S_t = S_0 + \int_0^t \int_Z S_{r^-}(e^{K_1(r, X_{r^-}, log(S_{r^-}/S_0); \zeta)} - 1)\mathcal{N}(dr, d\zeta). \tag{2.14}$$

The pair (X, S) is a Markov process whose generator is given in the next proposition.

Proposition 2.2 *Under the assumptions,* $\forall T > 0$

$$E\big[\int_0^T \sigma^2(X_s)\ ds\ \big] < \infty \tag{2.15}$$

$$E\big[\int_0^T \nu(D_i(s, X_s, Y_s))\ ds\big] < \infty \qquad i = 0, 1 \tag{2.16}$$

for real-valued, bounded functions $f(t,x,s)$ *such that* $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ *are bounded and continuous, the process*

$$f(t, X_t, S_t) - f(0, x_0, S_0) - \int_0^t Lf(r, X_r, S_r)\ dr \tag{2.17}$$

is a $(P, \mathcal{F}_t)$*-martingale , where*

$$\begin{aligned} Lf(t,x,s) &= \frac{\partial f}{\partial t}(t,x,s) + L_t f(t,x,s) = \\ &= \frac{\partial f}{\partial t}(t,x,s) + b(x)\ \frac{\partial f}{\partial x}(t,x,s) + \frac{1}{2}\ \sigma^2(x)\ \frac{\partial^2 f}{\partial x^2}(t,x,s) \\ &+ \int_Z \Big(f\big(t, x + K_0(t,x;\zeta), s e^{K_1(t,x,log(\frac{s}{S_0});\zeta)}\big) - f(t,x,s)\Big)\ \nu(d\zeta). \end{aligned} \tag{2.18}$$

In [5] it has been studied the case where S is a $(P, \mathcal{F}_t)$-local martingale. Instead of this, here we will consider the more general case where S is a $(P, \mathcal{F}_t)$-semimartingale.

Proposition 2.3 *Under (2.8) and the following condition*

$$\int_0^T \int_Z (e^{K_1(t, X_t, Y_t; \zeta)} - 1)^2 \nu(d\zeta) dt < +\infty \quad P - a.s. \tag{2.19}$$

S is a special semimartingale ([11]) with the decomposition

$$S_t = S_0 + M_t + A_t \tag{2.20}$$

where

$$A_t = \int_0^t \int_Z S_{r^-}(e^{K_1(r,X_{r^-},Y_{r^-});\zeta)} - 1)\nu(d\zeta)dr$$

is a predictable process with paths locally of bounded variation,

$$M_t = \int_0^t \int_Z S_{r^-}(e^{K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)(\mathcal{N}(dr, d\zeta) - \nu(d\zeta)dr)$$

is a local martingale, locally square-integrable whose angle process is given by

$$< M >_t = \int_0^t \int_Z S^2_{r^-}(e^{K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)^2\nu(d\zeta)dr. \tag{2.21}$$

Proof.
First notice that (2.8) and (2.19) imply

$$\int_0^T \int_Z | e^{K_1(t,X_t,Y_t;\zeta)} - 1 | \nu(d\zeta)dt < +\infty \quad P - a.s. \tag{2.22}$$

hence

$$R_t = \int_0^t \int_Z (e^{K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)\mathcal{N}(dr, d\zeta)$$

is a semimartingale and by (2.19) square integrable. By (2.14) S is a semimartingale being the stochastic exponential of the semimartingale R. To conclude observe that, since S^2 is also a semimartingale being the stochastic exponential of the semimartingale

$$\int_0^t \int_Z (e^{2K_1(r,X_{r^-},Y_{r^-};\zeta)} - 1)\mathcal{N}(dr, d\zeta),$$

S is locally square-integrable. □

Let us observe that the following representations in terms of the integer valued measure m associated to Y hold

$$S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{r^-}(e^z - 1)m(dr, dz) \tag{2.23}$$

$$M_t = \int_0^t \int_{\mathbb{R}} S_{r^-}(e^z - 1)(m(dr, dz) - \lambda_r\phi_r(dz)) \tag{2.24}$$

$$A_t = \int_0^t \int_{\mathbb{R}} S_{r^-}(e^z - 1)\lambda_r\phi_r(dz), \tag{2.25}$$

and condition (2.19) can be written as

$$\int_0^T \int_{\mathbb{R}} (e^z - 1)^2\lambda_r\phi_r(dz) < +\infty \quad P - a.s.$$

3. Hedging of a contingent claim

3.1. Problem formulation

Since our market is incomplete we have to choose some approach to hedging derivatives. In this paper we will use the criterion of risk minimization. This approach has been proposed in [7] in the martingale case and weakened in local sense in [16] for the general semimartingale case.

We consider a European contingent claim with maturity T whose payoff is given by $H(S_T)$ and such that $E[H^2(S_T)] < \infty$. The simplest example is given by a call option with strike price k where

$$H(S_T) = (S_T - k)^+.$$

We look for a trading strategy which generates the required payoff $H(S_T)$ and at the same time minimizes some measure of riskiness.

A $\{\mathcal{F}_t\}$**-trading strategy** is a pair $(\xi, \eta) = \{(\xi_t, \eta_t) : t \in [0, T]\}$, where ξ_t is an $\{\mathcal{F}_t\}$-predictable process and η_t is a process $\{\mathcal{F}_t\}$-adapted; ξ_t is the number of shares of the risky asset to be held at time t, while η_t is the amount invested in the riskless asset.

The **value** at time t of such a portfolio is given by

$$V_t = V_t(\xi, \eta) = \xi_t S_t + \eta_t.$$

We shall concentrate on strategies, (ξ, η), which are H-admissible in the sense that

$$V_T(\xi, \eta) = H(S_T) \qquad P - a.s.$$

and satisfies

$$E\left(\int_0^T \xi_t^2 d < S >_t\right) < \infty \tag{3.1}$$

$$E((sup_{t\in[0,T]} \mid V_t \mid)^2) < \infty. \tag{3.2}$$

The **cost process** of (ξ, η) is defined by

$$C_t(\xi, \eta) = V_t(\xi, \eta) - \int_0^t \xi_r dS_r \tag{3.3}$$

and provides the cumulative cost up to time t as current value of the portfolio minus total gains from trade. Under (3.1) and (3.2) C is a square integrable process. Moreover a strategy (ξ, η) is called self-financing if its cost process $C_t(\xi, \eta)$ is constant and it is called mean-self-financing if $C_t(\xi, \eta)$ is a martingale.

In an incomplete market perfect duplication is, in general, impossible and so the cost process will not be constant but fluctuate randomly over time. Hence we need a criterion to compare different strategies.
As a measure of riskiness, we introduce for each strategy (ξ, η) the conditional mean square error process

$$R_t(\xi, \eta) = E\Big((C_T(\xi, \eta) - C_t(\xi, \eta))^2 \mid \mathcal{F}_t\Big) \tag{3.4}$$

and the **problem of risk minimization** is formulated as follows

Given $H = H(S_T)$ with $E[H^2(S_T)] < \infty$, we have to find an H-admissible $\{\mathcal{F}_t\}$-strategy minimizing the $\{\mathcal{F}_t\}$-risk process, R_t, over the class of H-admissible $\{\mathcal{F}_t\}$-strategy. This strategy will be called $\{\mathcal{F}_t\}$-risk minimizing strategy.

In [7], in the martingale case, this problem was completely solved by using the Kunita-Watanabe decomposition. While, in the general case of a semimartingale there cannot exist any risk-minimizing strategy hence in [16] the weaker concept of locally risk-minimizing strategy was introduced.
It has been also proved that this definition is equivalent to the following

Definition 3.1 *An H-admissible strategy (ξ^*, η^*) is called optimal if the associated cost process $C(\xi^*, \eta^*)$ defined in (3.3) is a square-integrable $(P, \mathcal{F}_t)$-martingale orthogonal to M under P, that is the angle process $< C(\xi^*, \eta^*), M >= 0$ P-a.s..*

This concept of optimal strategy is related to the existence of the minimal martingale measure as we will see in Proposition (3.4) and (3.5).

3.2. The minimal martingale measure

We recall that absence of arbitrage opportunities is related to the existence of risk-neutral probability measures. That is probability measures Q, equivalent to P, such that S is a local $(Q, \mathcal{F}_t)$-martingale. We concentrate our attention to the minimal martingale measure

Definition 3.2 *A martingale measure P^* equivalent with respect to P is called minimal if any square-integrable $(P, \mathcal{F}_t)$-martingale which is orthogonal to M under P is still a martingale under P^*.*

Existence and uniqueness of the minimal martingale measure for general semimartingales satisfying the structure condition, (SC), has been discussed in [1]. The (SC) condition requires that S assumes the form

$$S_t = S_0 + M_t + \int_0^t c_r d < M >_r$$

where M is a $(P, \mathcal{F}_t)$-local square integrable martingale and the predictable process c is such that

$$\int_0^t c_r^2 d < M >_r < \infty \quad P - a.s.$$

In our context, taking into account (2.20) and (2.19), the (SC) condition is fulfilled with

$$M_t = \int_0^t \int_Z S_{r^-} \left(e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1 \right) (\mathcal{N}(dr, d\zeta) - \nu(d\zeta)dr),$$

$$c_r = \frac{\int_Z (e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1)\nu(d\zeta)}{S_{r^-} \int_Z (e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1)^2 \nu(d\zeta)},$$

and under (2.8), the mean-variance tradeoff process is such that

$$K_t = \int_0^t c_r^2 d < M >_r = \int_0^t \frac{(\int_Z (e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1)\nu(d\zeta))^2}{\int_Z (e^{K_1(r, X_{r^-}, Y_{r^-}; \zeta)} - 1)^2 \nu(d\zeta)} dr \le \int_0^t \nu(D_1(r, X_r, Y_r))dr < \infty \quad P - a.s.$$

where we recall that D_1 is defined in (2.4).

Hence by the result proved [1], we get the following proposition

Proposition 3.3 *Under (2.8), (2.19) if*

$$c_t \Delta M_t < 1 \tag{3.5}$$

the minimal martingale measure P^* exists and is defined on $(\Omega, \mathcal{F}_T)$ by

$$\frac{dP^*}{dP} = L_T^*$$

where L^* is the Doleans-Dade exponential martingale associated to the $(P, \mathcal{F}_t)$-martingale $m_t = -\int_0^t c_r dM_r$.

Let us observe that condition (3.5) can be written as

$$\frac{\int_Z (e^{K_1(r,X_{r-},Y_{r-};\zeta)}) - 1)\nu(d\zeta)}{\int_Z (e^{K_1(r,X_{r-},Y_{r-};\zeta)} - 1)^2\nu(d\zeta)} \int_Z (e^{K_1(r,X_{r-},Y_{r-};\zeta)}) - 1)\mathcal{N}(\{t\}, d\zeta) < 1$$

or equivalently, in terms of the local characteristics of Y, as

$$\frac{\int_{I\!R}(e^z - 1)\Phi_t(dz)}{\int_{I\!R}(e^z - 1)^2\Phi_t(dz)}(e^{\Delta Y_t} - 1) < 1.$$

3.3. Existence of optimal strategies

In [6] it is proved that an optimal strategy corresponds to the Follmer-Schweizer decomposition, more precisely

Proposition 3.4 *The existence of an optimal strategy is equivalent to a decomposition*

$$H(S_T) = H_0 + \int_0^T \xi_r^H dS_r + L_T^H \tag{3.6}$$

with H_0 square-integrable $\mathcal{F}_0$-measurable random variable, ξ^H predictable and satisfying (3.1), L^H square-integrable martingale orthogonal to M.
For such decomposition, the associated optimal strategy (ξ^*, η^*) is given by

$$\xi^* = \xi^H, \quad \eta^* = V(\xi^*, \eta^*) - \xi^* S$$

with

$$V_t(\xi^*, \eta^*) = H_0 + \int_0^t \xi_r^H dS_r + L_t^H.$$

In [6], when S has continuous paths, it has been proved that the above decomposition is uniquely determined and coincides with the Kunita-Watanabe decomposition under the minimal martingale measure. Hence the optimal strategy exists and can be computed in terms of the minimal martingale measure. This result is obtained by using the property that the minimal martingale measure preserves orthogonality (see Theorem 3.5 of [6]), property which is not satisfied in the case where S has discontinuous paths. But, even if S is not continuous, if an optimal strategy exists, the value process associated to it can be computed again as the conditional expectation of the contingent claim $H(S_T)$ under the minimal martingale measure, as we will prove in the following proposition.

Proposition 3.5 *Assume (2.15), (2.16), (2.19) and (3.5).*
If there exists an optimal strategy (ξ^*, η^*), the value process is given by

$$V_t(\xi^*, \eta^*) = E^{P^*}(H(S_T) \mid \mathcal{F}_t) = l(t, X_t, S_t) \tag{3.7}$$

where if $l \in \mathcal{C}_b^{1,2}([0,T] \times I\!R \times I\!R^+)$ it is a solution of the following integro-differential equation

$$L^* l(t,x,y) = \frac{\partial l}{\partial t}(t,x,y) + b(x)\ \frac{\partial l}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma(x)^2\ \frac{\partial^2 l}{\partial x^2}(t,x,y) + \tag{3.8}$$

$$\int_Z \Big(l(t, x + K_0(t,x;\zeta), y e^{K_1(t,x,log(y/S_0);\zeta)}) - l(t,x,y)\Big)(1 + U^*(t,x,y;\zeta))\nu(d\zeta) = 0$$

$$l(T,x,y) = H(y)$$

with

$$U^*(r,x,y;\zeta) = -(e^{K_1(r,x,log(y/S_0);\zeta)} - 1)\frac{\int_Z (e^{K_1(r,x,log(y/S_0);\zeta)} - 1)\nu(d\zeta)}{\int_Z (e^{K_1(r,x,log(y/S_0);\zeta)}) - 1)^2\nu(d\zeta)}. \tag{3.9}$$

Proof.
Since $C(\xi^*,\eta^*)$ is a square-integrable $(P,\mathcal{F}_t)$-martingale orthogonal to M under P we have that it is a $(P^*,\mathcal{F}_t)$-martingale hence we get

$$E^{P^*}(H(S_T)\mid \mathcal{F}_t) = E^{P^*}(V_T(\xi^*,\eta^*)\mid \mathcal{F}_t) = E^{P^*}(C_T(\xi^*,\eta^*) + \int_0^T \xi^*_r dS_r \mid \mathcal{F}_t) = V_t(\xi^*,\eta^*).$$

By a suitable version of Girsanov Theorem ([3]), since

$$-\int_0^t c_r dM_r = \int_0^t U^*(r, X_{r^-}, S_{r^-};\zeta)(\mathcal{N}(dr,d\zeta) - \nu(d\zeta)dr)$$

where $U^*(r,x,y;\zeta)$ is given in (3.9), we get that the $(P^*,\mathcal{F}_t)$- compensator of the integer-valued random measure $\mathcal{N}(dr,d\zeta)$ is given by

$$\nu^{P^*}(dr,d\zeta) = (1 + U^*(r,X_{r^-},S_{r^-};\zeta))\nu(d\zeta)dr.$$

Finally, by Itô formula we get that for any $f \in \mathcal{C}^{1,2}_b([0,T]\times I\!R \times I\!R^+)$

$$f(t,X_t,S_t) = f(0,x_0,S_0) + \int_0^t L^* f(r,X_r,S_r)dr + m_t \tag{3.10}$$

where L^* is given in (3.8). By (2.15) and (2.16)

$$m_t = \int_0^t \sigma(X_r)\,\frac{\partial f}{\partial x}(r,X_r,S_r)dW_r +$$

$$\int_0^t \int_Z \Big(f\big(r, X_{r^-} + K_0(r,X_{r^-};\zeta), S_{r^-} e^{K_1(r,X_{r^-},Y_{r^-};\zeta)}\big) - f(r,X_{r^-},S_{r^-})\Big)(1 + U^*(r,X_{r^-},S_{r^-};\zeta))(\mathcal{N}(dr,d\zeta) - \nu(d\zeta)dr)$$

is a $(P^*,\mathcal{F}_t)$-martingale. To this end it is sufficient to observe that

$$\int_Z \mid \Big(f\big(r, x + K_0(r,x;\zeta), y e^{K_1(r,x,log(y/S_0);\zeta)}\big) - f(r,x,y)\Big)(1 + U^*(r,x,y;\zeta)) \mid \nu(d\zeta) \le$$

$$2\|f\|\Big(\nu\big(D_0(t,x,log(y/S_0))\big) + \nu\big(D_1(t,x,log(y/S_0))\big) + \int_Z \mid U^*(r,x,y;\zeta) \mid \nu(d\zeta)\Big)$$

and

$$\int_Z \mid U^*(r,x,y;\zeta)\mid \nu(d\zeta) \le \frac{(\int_Z \mid e^{K_1(r,x,log(y/S_0);\zeta)} - 1 \mid \nu(d\zeta))^2}{\int_Z (e^{K_1(r,x,log(y/S_0);\zeta)} - 1)^2 \nu(d\zeta)} \le \nu\big(D_1(r,x,log(y/S_0))\big).$$

Hence under P^* the Markovianity of the pair (X,S) is preserved and L^* provides its generator.
Now, for $f = l$ in (3.10), since $l(t,X_t,S_t)$ is a $(P^*,\mathcal{F}_t)$-martingale all finite variations terms have to vanish and this leads to equation (3.8). □

Notice that analytical solutions to equation (3.8) are difficult to find but one could search approximating solutions. Otherwise one could compute the expectation in (3.7) by Monte Carlo simulations. This problem has been mentioned in [10] where related references are given.

3.4. The martingale case

In the sequel we shall assume (2.8), (2.19) and

$$\forall t \in [0,T], x \in I\!R, y \in I\!R \qquad \int_Z (e^{K_1(t,x,y;\zeta)} - 1)\nu(\zeta) = 0 \tag{3.11}$$

which ensure that P is a martingale measure for S.

Taking into account (2.23) condition (3.11) means, when (2.12) holds, that

$$E[e^{Z_n} - 1 \mid \mathcal{F}_{T_n^-}] = \int_{I\!R} (e^z - 1)\Phi_{T_n}(dz) = 0,$$

and this condition should be compared with that given in [8], where it is assumed that $E[e^{Z_n} - 1] = 0$.

The martingale case is discussed in [5]. In the sequel we will recall the main results there proved. Since S is a martingale we have that the decomposition (3.6) is given by the Kunita-Watanabe one and the following hold

Proposition 3.6 *Under the hypotheses (2.15), (2.16) and (2.19), let us define*

$$g(t, X_t, S_t) := E(H(S_T) \mid \mathcal{F}_t). \tag{3.12}$$

If $g \in \mathcal{C}_b^{1,2}([0,T] \times I\!R \times I\!R^+)$ it is a solution of the following integro-differential equation

$$Lg(t,x,y) = \frac{\partial g}{\partial t}(t,x,y) + b(x)\ \frac{\partial g}{\partial x}(t,x,y) + \frac{1}{2}\ \sigma(x)^2\ \frac{\partial^2 g}{\partial x^2}(t,x,y) + \tag{3.13}$$

$$\int_Z \Big(g\big(t, x + K_0(t,x;\zeta), ye^{K_1(t,x,log(y/S_0);\zeta)}\big) - g(t,x,y)\Big)\ \nu(d\zeta) = 0$$

$$g(T,x,y) = H(y).$$

Furthermore, the risk-minimizing hedging strategy (ξ^*, η^*) is given by

$$\xi_t^* = \frac{h(t, X_{t^-}, S_{t^-})}{S_{t^-}\Sigma(t, X_{t^-}, S_{t^-})} \tag{3.14}$$

$$\eta_t^* = g(t, X_{t^-}, S_{t^-}) - \xi_t^* S_t$$

where

$$h(t, X_{t^-}, S_{t^-}) = \int_Z \Big(e^{K_1(t,X_{t^-},Y_{t^-};\zeta)} - 1\Big)\Big(g\big(t, X_{t^-} + K_0(t, X_{t^-};\zeta), S_{t^-}e^{K_1(t,X_{t^-},Y_{t^-};\zeta)}\big) - g(t, X_{t^-}, S_{t^-})\Big)\ \nu(d\zeta) \tag{3.15}$$

$$\Sigma(t, X_{t^-}, S_{t^-}) = \int_Z \big(e^{K_1(t,X_{t^-},Y_{t^-};\zeta)} - 1\big)^2 \nu(d\zeta) \tag{3.16}$$

and $Y_{t^-} = log(S_{t^-}/S_0)$.

The criterion of risk minimization is also well suited to deal with restricted information. We assume now that the hedger has access only to the information given by the past asset price, that is the filtration generated by S, $\mathcal{F}_t^S = \sigma\{S_r : r \leq t\}$ which coincides with the filtration generated by Y, $\mathcal{F}_t^Y = \sigma\{Y_r : r \leq t\}$.
In this framework we restrict our attention to $\{\mathcal{F}_t^S\}$-strategy and, as in [17] and [6], we consider the $\{\mathcal{F}_t^S\}$-risk process of an $\{\mathcal{F}_t^S\}$- strategy defined by

$$R_t^S(\xi,\eta) = E\Big((C_T(\xi,\eta) - C_t(\xi,\eta))^2 \mid \mathcal{F}_t^S\Big). \tag{3.17}$$

In [17] is proved that there exists a unique H-admissible, $\{\mathcal{F}_t^S\}$-risk minimizing strategy (ξ', η'), where ξ' is given by the Radon-Nikodym derivative of $< V(\xi^*, \eta^*), S >^{p,\mathcal{F}_t^S}$, with respect to $< S >^{p,\mathcal{F}_t^S}$:

$$\xi'_t = \frac{d < V(\xi^*, \eta^*), S >_t^{p,\mathcal{F}^S}}{d < S >_t^{p,\mathcal{F}^S}}$$

where (ξ^*, η^*) is the H-admissible, $\{\mathcal{F}_t\}$-risk minimizing strategy and, for any locally integrable process A of finite variation, $A^{p,\mathcal{F}^S}$ denotes the $\{\mathcal{F}_t^S\}$-predictable projection (see [11] for details).
Moreover η' is given by

$$\eta'_t = V_t(\xi', \eta') - \xi'_t S_t$$

and the value process is such that

$$V_t(\xi', \eta') = E[H(S_T) \mid \mathcal{F}_t^S] = E[g(t, X_t, S_t) \mid \mathcal{F}_t^S].$$

Hence, since

$$< V(\xi^*, \eta^*), S >_t = \int_0^t S_{r^-} h(r, X_{r^-}, S_{r^-}) dr, \quad < S >_t = \int_0^t S_{r^-}^2 \Sigma(r, X_{r^-}, S_{r^-}) dr$$

we get

$$\xi'_t = \frac{E[h(t, X_{t^-}, S_{t^-}) \mid \mathcal{F}_{t^-}^S]}{S_{t^-} E[\Sigma(t, X_{t^-}, S_{t^-}) \mid \mathcal{F}_{t^-}^S]} = \frac{\pi_{t^-}(h(t, \cdot, S_{t^-}))}{S_{t^-} \pi_{t^-}(\Sigma(t, \cdot, S_{t^-}))}$$

$$\eta' = \pi_t(g(t, \cdot, S_t)) - \xi'_t S_t$$

where π_{t^-} denotes the left-continuous version of the filter π_t. The filter π_t is the probability measure-valued $\mathcal{F}_t^S$-adapted process such that for any f bounded measurable function on $I\!R$

$$\pi_t(f(\cdot)) = E(f(X_t) \mid \mathcal{F}_t^S).$$

Thus the knowledge of the filter allows us to compute our strategy under restricted information.
For a discussion of the filtering problem see [4] and [5].

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