BEYOND BOLTZMANN-GIBBS-SHANNON IN PHYSICS, MATHEMATICS AND ELSEWHERE

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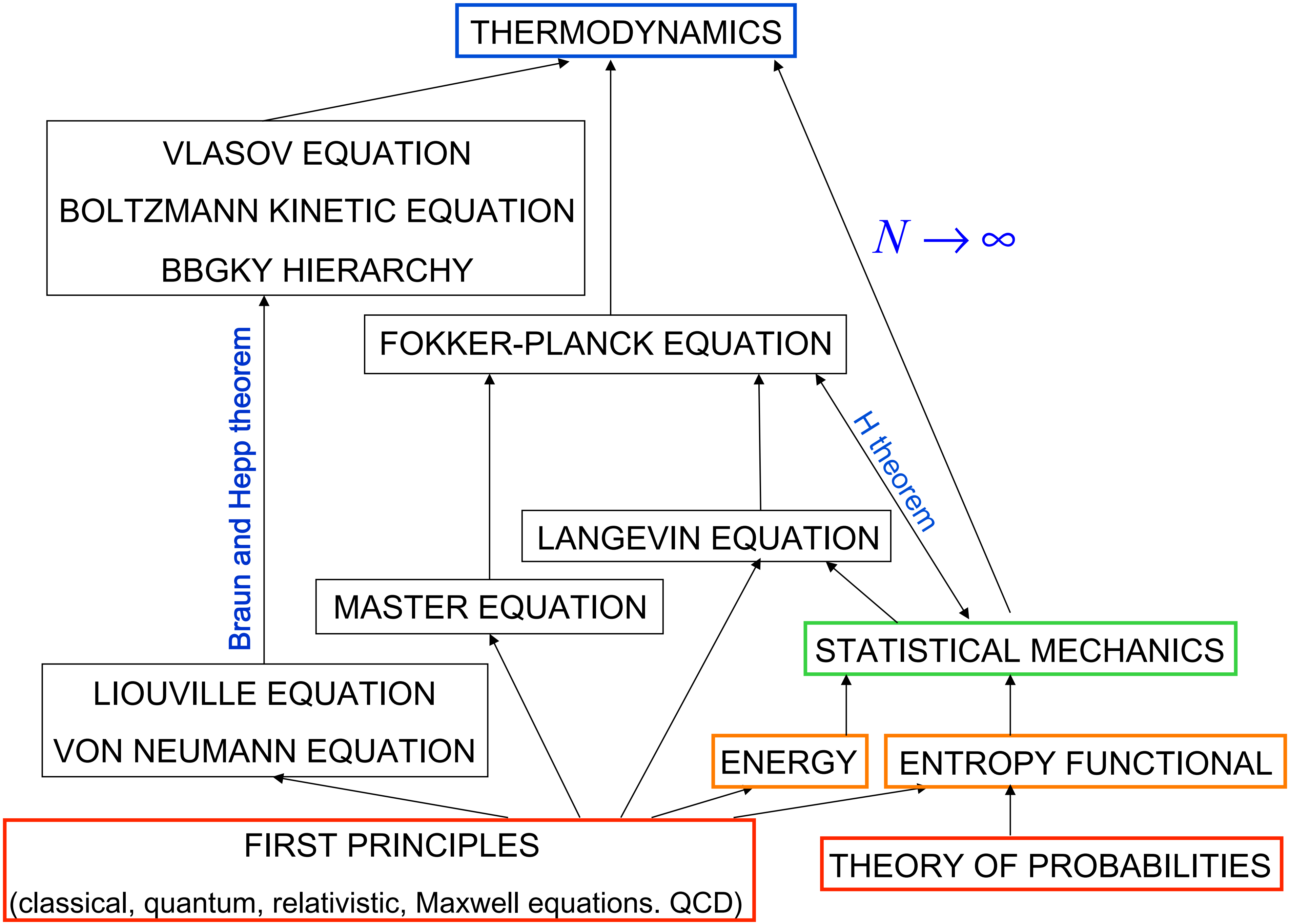
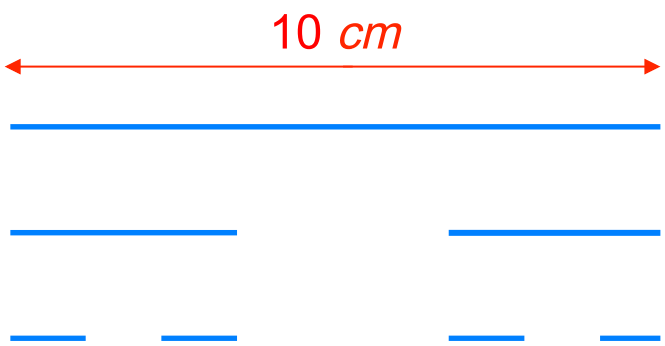
Complexity Science Hub Vienna, Austria

Pescara, July 2019



THERMODYNAMICS
VLASOV EQUATION
BOLTZMANN KINETIC EQUATION
BBGKY HIERARCHY
$N \to \infty$
FOKKER-PLANCK EQUATION
Braun and Hepp theorem
H theorem
LANGEVIN EQUATION
MASTER EQUATION
STATISTICAL MECHANICS
LIOUVILLE EQUATION
VON NEUMANN EQUATION
ENERGY
ENTROPY FUNCTIONAL
FIRST PRINCIPLES
(classical, quantum, relativistic, Maxwell equations. QCD)
THEORY OF PROBABILITIES

TRIADIC CANTOR SET:



$$d_F = \frac{\ln 2}{\ln 3} = 0.6309...$$

Hence the interesting measure is

$$(10\ cm)^{0.6309...} \cong 4.275\ cm^{0.6309}$$

It is the natural (or artificial or social) system itself which, through its geometrical-dynamical properties, mandates the specific informational tool --- entropy --- to be meaningfully used for the study of its thermostatistical and thermodynamical properties.

Enrico FERMI *Thermodynamics* (Dover, 1936)

The entropy of a system composed of several parts is very often equal to the sum of the entropies of all the parts. This is true if the energy of the system is the sum of the energies of all the parts and if the work performed by the system during a transformation is equal to the sum of the amounts of work performed by all the parts. Notice that these conditions are not quite obvious and that in some cases they may not be fulfilled. Thus, for example, in the case of a system composed of two homogeneous substances, it will be possible to express the energy as the sum of the energies of the two substances only if we can neglect the surface energy of the two substances where they are in contact. The surface energy can generally be neglected only if the two substances are not very finely subdivided; otherwise, it can play a considerable role.

ENTROPIC FUNCTIONALS

	$p_i = \frac{1}{W} \quad (\forall i)$ equiprobability	$\forall p_i \; (0 \le p_i \le 1)$ $\left(\sum_{i=1}^{W} p_i = 1 \right)$
BG entropy ($q = 1$)	$k \ln W$	$-k \sum_{i=1}^{W} p_i \ln p_i$
Entropy *Sq* (q *real*)	$k \frac{W^{1-q} - 1}{1-q}$	$k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}$

additive

- Concave
- Extensive
- Lesche-stable
- Finite entropy production per unit time
- Pesin-like identity (with largest entropy production)
- Composable (**unique** trace form; Enciso-Tempesta)
- Topsoe-factorizable (unique)
- Amari-Ohara-Matsuzoe conformally invariant geometry (unique)
- Biro-Barnafoldi-Van thermostat universal independence (unique)

nonadditive (if $q \neq 1$)

Possible generalization of Boltzmann-Gibbs statistical mechanics

C.T., J. Stat. Phys. **52**, 479 (1988)

DEFINITIONS: *q-logarithm:* $\ln_q x \equiv \frac{x^{1-q}-1}{1-q} \quad (x>0;\ \ln_1 x = \ln x)$

q-exponential: $e_q^x \equiv [1+(1-q)\,x]^{\frac{1}{1-q}} \quad (e_1^x = e^x)$

Hence, the entropies can be rewritten:

	equal probabilities	*generic probabilities*
BG entropy $(q=1)$	$k\ \ln W$	$k\sum_{i=1}^{W} p_i \ln\frac{1}{p_i}$
entropy S_q $(q\in R)$	$k\ \ln_q W$	$k\sum_{i=1}^{W} p_i \ln_q\frac{1}{p_i}$

TYPICAL SIMPLE SYSTEMS:

$$W(N) \propto \mu^N \;\; (\mu > 1)$$

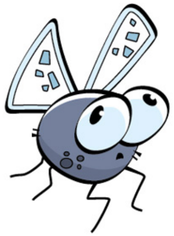
Short-range space-time correlations

Markovian processes (short memory), Additive noise

Strong chaos (positive maximal Lyapunov exponent), **Ergodic**, Riemannian geometry

Short-range many-body interactions, weakly quantum-entangled subsystems

Linear and homogeneous Fokker-Planck equations, Gaussians



→ **Boltzmann-Gibbs entropy (additive)**

→ **Exponential dependences (Boltzmann-Gibbs weight, ...)**

TYPICAL COMPLEX SYSTEMS:

$$\text{e.g., } W(N) \propto N^\rho \;\; (\rho > 0)$$

Long-range space-time correlations

Non-Markovian processes (long memory), Additive and multiplicative noises

Weak chaos (zero maximal Lyapunov exponent), **Nonergodic**, Multifractal geometry

Long-range many-body interactions, strongly quantum-entangled sybsystems

Nonlinear and/or inhomogeneous Fokker-Planck equations, *q*-Gaussian

→ **Entropy S_q (nonadditive)**

→ ***q*-exponential dependences (asymptotic power-laws)**

ADDITIVITY: O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon, Oxford, 1970), page 167

An entropy is additive if, for any two probabilistically independent systems A and B,

$$S(A+B) = S(A) + S(B)$$

Therefore, since $$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\,\frac{S_q(A)}{k}\,\frac{S_q(B)}{k}$$

S_{BG} and S_q^{Renyi} $(\forall q)$ are additive, and S_q $(\forall q \neq 1)$ is nonadditive .

EXTENSIVITY:

Consider a system $\Sigma \equiv A_1 + A_2 + ... + A_N$ made of N (not necessarily independent) identical elements or subsystems A_1 and A_2, ..., A_N.

An entropy is extensive if

$$0 < \lim_{N\to\infty} \frac{S(N)}{N} < \infty \text{ , } i.e., \;\; S(N) \propto N \;\; (N \to \infty)$$

EXTENSIVITY OF THE ENTROPY $(N \to \infty)$

$W \equiv$ total number of possibilities with nonzero probability,

assumed to be equally probable

If $W(N) \sim \mu^N \quad (\mu > 1)$

$$\Rightarrow S_{BG}(N) = k_B \ln W(N) \propto N \qquad \text{OK!}$$

If $W(N) \sim N^\rho \quad (\rho > 0)$

$$\Rightarrow S_q(N) = k_B \ln_q W(N) \propto [W(N)]^{1-q} \propto N^{\rho(1-q)}$$

$$\Rightarrow S_{q=1-1/\rho}(N) \propto N \qquad \text{OK!}$$

If $W(N) \sim \nu^{N^\gamma} \quad (\nu > 1;\ 0 < \gamma < 1)$

$$\Rightarrow S_\delta(N) = k_B \left[\ln W(N)\right]^\delta \propto N^{\gamma\,\delta}$$

$$\Rightarrow S_{\delta=1/\gamma}(N) \propto N \qquad \text{OK!}$$

IMPORTANT: $\boxed{\mu^N >> \nu^{N^\gamma} >> N^\rho \quad \text{if} \quad N >> 1}$

All happy families are alike; each unhappy family is unhappy in its own way.

Leo Tolstoy (*Anna Karenina*, 1875-1877)

SYSTEMS $W(N)$ *(equiprobable)*	ENTROPY S_{BG} (ADDITIVE)	ENTROPY S_q $(q \neq 1)$ (NONADDITIVE)	ENTROPY S_δ $(\delta \neq 1)$ (NONADDITIVE)
$e.g., \mu^N$ $(\mu > 1)$	EXTENSIVE	NONEXTENSIVE	NONEXTENSIVE
$e.g., N^\rho$ $(\rho > 0)$	NONEXTENSIVE	EXTENSIVE $(q = 1 - 1/\rho)$	NONEXTENSIVE
$e.g., \nu^{N^\gamma}$ $(\nu > 1;$ $0 < \gamma < 1)$	NONEXTENSIVE	NONEXTENSIVE	EXTENSIVE $(\delta = 1/\gamma)$

King Thutmosis I
18th Dynasty
circa 1500 BC

A theory is the more impressive the greater the simplicity of its premises is, the more different kinds of things it relates, and the more extended is its area of applicability. Therefore the deep impression that classical thermodynamics made upon me. It is the only physical theory of universal content concerning which I am convinced that, within the framework of applicability of its basic concepts, it will never be overthrown.

Albert Einstein (1949)

COMPOSITION OF VELOCITIES OF INERTIAL SYSTEMS (d=1)

$$v_{13} = v_{12} + v_{23} \qquad \text{(Galileo)}$$

$$v_{13} = \frac{v_{12} + v_{23}}{1 + \frac{v_{12}}{c}\frac{v_{13}}{c}} \qquad \text{(Einstein)}$$

Newton mechanics:
It satisfies Galilean additivity **but** violates Lorentz invariance (hence mechanics can not be unified with Maxwell electromagnetism)

Einstein mechanics (Special relativity):
It satisfies Lorentz invariance (hence mechanics is unified with Maxwell electromagnetism) **but** violates Galilean additivity

Question: which is physically more fundamental, the additive composition of velocities **or** the unification of mechanics and electromagnetism?

Special relativity recovers Newtonian/Galilean mechanics as particular case:

$$v_{13} = \frac{v_{12} + v_{23}}{1 + \frac{v_{12}}{c}\frac{v_{23}}{c}} \sim v_{12} + v_{23}$$

if $\quad 1/c \to 0, \ \forall v \qquad$ or $\qquad \forall \ 1/\mathrm{c} \neq 0$ with $v/c \to 0$

---

q-statistics recovers Boltzmann-Gibbs statistics as particular case:

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\,\frac{S_q(A)}{k}\,\frac{S_q(B)}{k} \sim \frac{S_{BG}(A)}{k} + \frac{S_{BG}(B)}{k}$$

and

$$\mathrm{e}_q^{-\beta E} \equiv \frac{1}{\left[1+(q-1)\beta E\right]^{\frac{1}{q-1}}} \sim e^{-\beta E}$$

if $\quad (q\text{-}1) \to 0, \ \forall \beta E \qquad$ or $\qquad \forall \ (\mathrm{q}\text{-}1) \neq 0$ with $\beta E \to 0$

Prediction of the q - triplet: C. T., Physica A **340**,1 (2004)

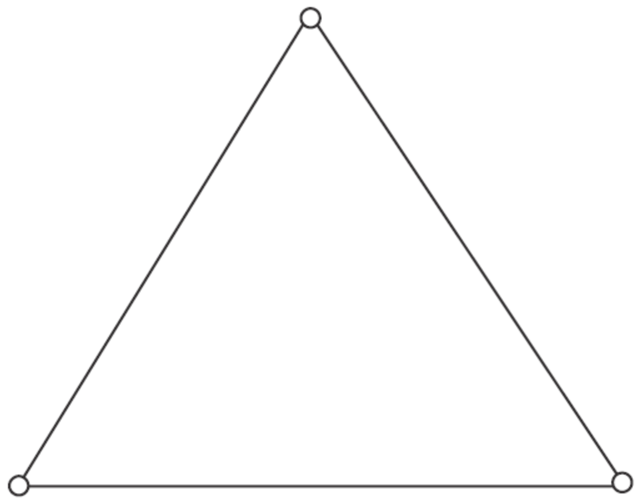


Fig. 2. The triangle of the basic values of q, namely those associated with sensitivity to the initial conditions, relaxation and stationary state. For the most relevant situations we expect $q_{sen} \leqslant 1$, $q_{rel} \geqslant 1$ and $q_{stat} \geqslant 1$. These indices are presumably inter-related since they all descend from the particular dynamical exploration that the system does of its full phase space. For example, for long-range Hamiltonian systems characterized by the decay exponent α and the dimension d, it could be that q_{stat} decreases from a value above unity (e.g., 2 or $\frac{3}{2}$) to unity when α/d increases from zero to unity. For such systems one expects relations like the (particularly simple) $q_{stat} = q_{rel} = 2 - q_{sen}$ or similar ones. In any case, it is clear that, for $\alpha/d > 1$ (i.e., when BG statistics is known to be the correct one), one has $q_{stat} = q_{rel} = q_{sen} = 1$. All the weakly chaotic systems focused on here are expected to have well defined values for q_{sen} and q_{rel}, but only those associated with a Hamiltonian are expected to *also* have a well defined value for q_{stat}.

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Physica A 356 (2005) 375–384

PHYSICA A

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Triangle for the entropic index q of non-extensive statistical mechanics observed by Voyager 1 in the distant heliosphere

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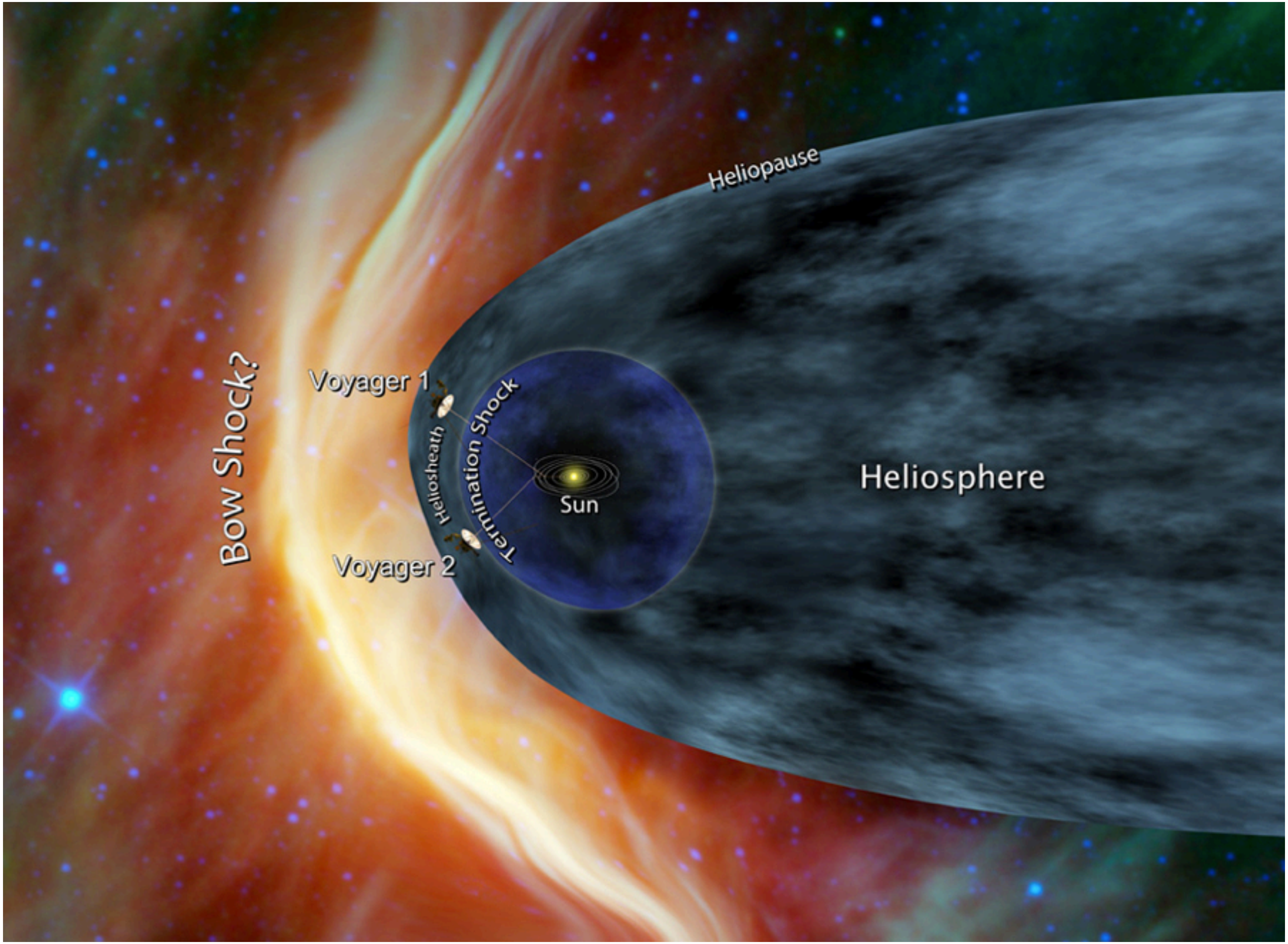
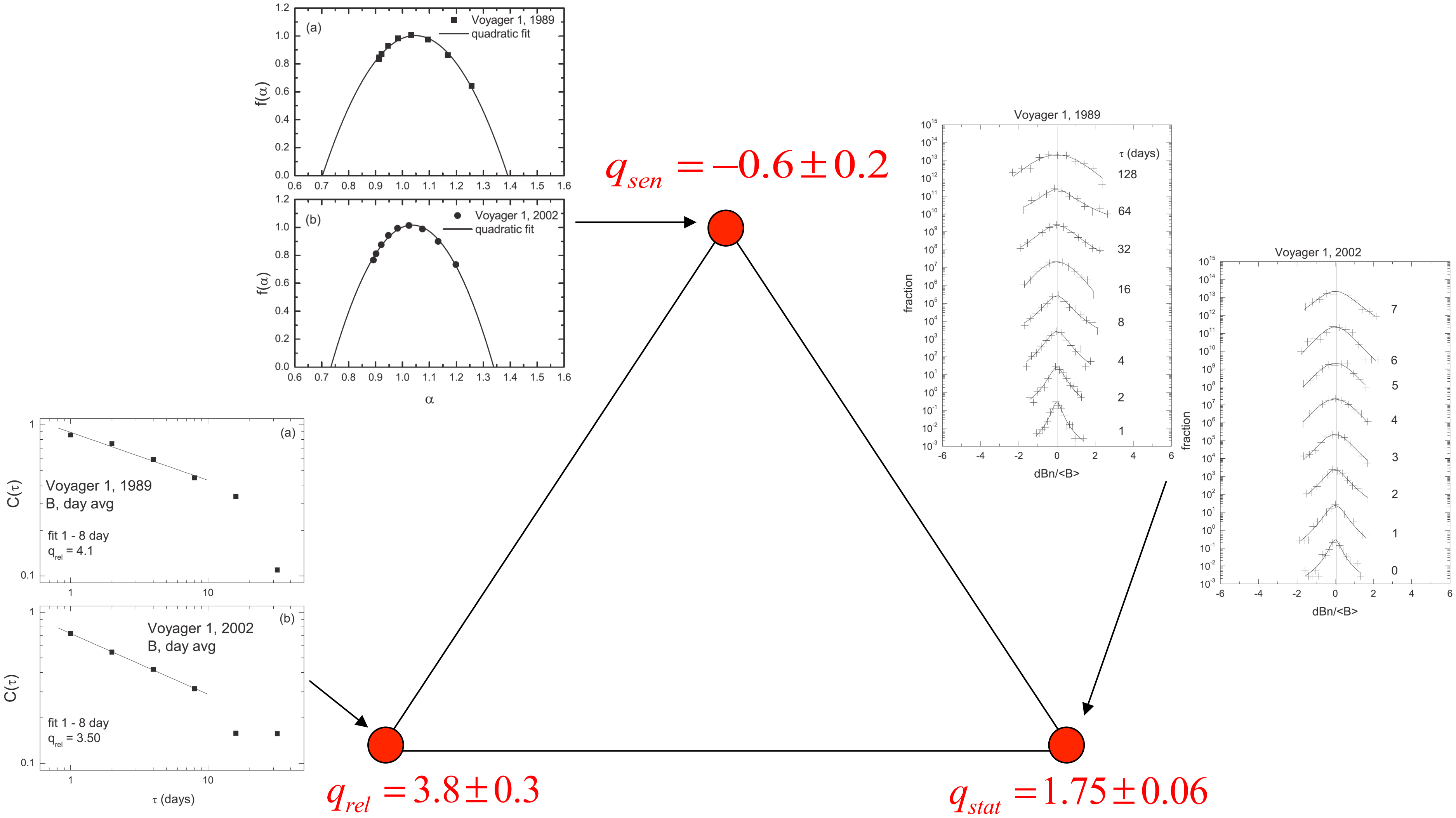
Received 10 June 2005
Available online 11 July 2005


Heliopause
Bow Shock?
Voyager 1
Heliosheath
Termination Shock
Sun
Voyager 2
Heliosphere

SOLAR WIND: Magnetic Field Strength

L.F. Burlaga and A. F.-Vinas (2005) / NASA Goddard Space Flight Center; Physica A **356**, 375 (2005)

[Data: Voyager 1 spacecraft (1989 and 2002); 40 and 85 AU; daily averages]



$$q_{sens} < 1 < q_{stat} < q_{rel}$$

Asymptotically scale-invariant occupancy of phase space makes the entropy S_q extensive

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Contributed by Murray Gell-Mann, July 25, 2005

Phase space can be constructed for N equal and distinguishable subsystems that could be probabilistically either *weakly* correlated or *strongly* correlated. If they are locally correlated, we expect the Boltzmann–Gibbs entropy $S_{BG} = -k\sum_i p_i \ln p_i$ to be *extensive*, i.e., $S_{BG}(N) \propto N$ for $N \to \infty$. In particular, if they are independent, S_{BG} is *strictly additive*, i.e., $S_{BG}(N) = NS_{BG}(1)$, $\forall N$. However, if the subsystems are globally correlated, we expect, for a vast class of systems, the entropy $S_q \equiv k[1 - \sum_i p_i^q]/(q-1)$ (with $S_1 = S_{BG}$) for some special value of $q \neq 1$ to be the one which is extensive [i.e., $S_q(N) \propto N$ for $N \to \infty$]. Another concept which is relevant is strict or asymptotic *scale-freedom* (or *scale-invariance*), defined as the situation for which all marginal probabilities of the N-system coincide or asymptotically approach (for $N \to \infty$) the joint probabilities of the $(N-1)$-system. If each subsystem is a binary one, scale-freedom is guaranteed by what we hereafter refer to as the *Leibnitz rule*, i.e., the sum of two successive joint probabilities of the N-system coincides or asymptotically approaches the corresponding joint probability of the $(N-1)$-system. The kinds of interplay of these various concepts are illustrated in several examples. One of them justifies the title of this paper. We conjecture that these mechanisms are deeply related to the very frequent emergence, in natural and artificial complex systems, of scale-free structures and to their connections with nonextensive statistical mechanics. Summarizing, we have shown that, for asymptotically scale-invariant systems, it is S_q with $q \neq 1$, and not S_{BG}, the entropy which matches standard, clausius-like, prescriptions of classical thermodynamics.

continuous variables ($N = 1, 2, 3$). In both cases, certain correlations that are scale-invariant in a suitable limit can create an intrinsically inhomogeneous occupation of phase space. Such systems are strongly reminiscent of the so called scale-free networks (24, 25), with their hierarchically structured hubs and spokes and their nearly forbidden regions.

Discrete Models

Some Basic Concepts. The most general probabilistic sets for N equal and distinguishable binary subsystems are given in Fig. 1 with

$$\sum_{n=0}^{N} \frac{N!}{(N-n)!}\,\pi_{N,n} = 1$$

$$(\pi_{N,n} \in [0, 1];\ N = 1, 2, 3, \ldots;\ n = 0, 1, \ldots, N). \qquad [2]$$

Let us from now on call *Leibnitz rule* the following recursive relation:

$$\pi_{N,n} + \pi_{N,n+1} = \pi_{N-1,n}\ (n = 0, 1, \ldots, N-1;\ N = 2, 3, \ldots). \qquad [3]$$

This relation guarantees what we refer to as *scale-invariance* (or *scale-freedom*) in this article. Indeed, it guarantees that, for any value of N, the associated *joint probabilities* $\{\pi_{N,n}\}$ produce *marginal probabilities* which coincide with $\{\pi_{N-1,n}\}$. Assuming $\pi_{10} + \pi_{11} =$

PHYSICS

PNAS

Playing with additive duality $(q \to 2-q)$

and with multiplicative duality $(q \to 1/q)$

(*and using numerical results related to the* $q-$*generalized central limit theorem*)

we conjecture

$$q_{rel} + \frac{1}{q_{sen}} = 2 \qquad \text{and} \qquad q_{stat} + \frac{1}{q_{rel}} = 2$$

hence $$1 - q_{sen} = \frac{1 - q_{stat}}{3 - 2\, q_{stat}}$$

hence only one independent!

Burlaga and Vinas (*NASA*) *most precise value of the* $q-$*triplet is*

$$q_{stat} = 1.75 = 7/4$$

hence $q_{sen} = -0.5 = -1/2$ (*consistent with* $q_{sen} = -0.6 \pm 0.2$!)

and $q_{rel} = 4$ (*consistent with* $q_{rel} = 3.8 \pm 0.3$!)

C.T., M. Gell-Mann and Y. Sato, Proc Natl Acad Sc USA **102**, 15377 (2005)

$$\varepsilon_{sen} \equiv 1 - q_{sen} = 1 - (-1/2) = 3/2$$

$$\varepsilon_{rel} \equiv 1 - q_{rel} = 1 - 4 = -3$$

$$\varepsilon_{stat} \equiv 1 - q_{stat} = 1 - 7/4 = -3/4$$

We verify

$$\varepsilon_{stat} = \frac{\varepsilon_{sen} + \varepsilon_{rel}}{2}$$ (arithmetic mean!)

$$\varepsilon_{sen} = \sqrt{\varepsilon_{stat} \ \varepsilon_{rel}}$$ (geometric mean!)

$$\varepsilon_{rel}^{-1} = \frac{\varepsilon_{sen}^{-1} + \varepsilon_{stat}^{-1}}{2}$$ (harmonic mean!)

N.O. Baella (2008)

Eur. Phys. J. Special Topics **226**, 455–466 (2017)

DOI: 10.1140/epjst/e2016-60159-x

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Generalization of the possible algebraic basis of *q*-triplets

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[2] Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA

$$q_a(q) = \frac{(a+2) - aq}{a - (a-2)q} \quad (a \in \mathscr{R}) \qquad q_0 = 1/q \qquad q_2 = 2 - q$$

Statistical mechanics for complex systems: On the structure of *q*-triplets

Constantino Tsallis

S. Duarte et al. (eds.), *Physical and Mathematical Aspects of Symmetries*,
https://doi.org/10.1007/978-3-319-69164-0_7

MÖBIUS TRANSFORMS, CYCLES AND q-TRIPLETS IN STATISTICAL MECHANICS

JEAN-PIERRE GAZEAU[A,B] AND CONSTANTINO TSALLIS[A,C,D]

(2019)

	q_{sens}	q_{stat}	q_{rel}	q_{aux}	x_1
Solar wind (observations)	-0.6 ± 0.2	1.75 ± 0.06	3.8 ± 0.3	0.5158	0.0316
Solar wind (conjectural)	-1/2	7/4	4	0.5	0
Feigenbaum point (calculations)	0.2444877...	1.65 ± 0.05	2.2497841	0.50375	0.0075
Ozone layer (observations)	-8.1	1.32	1.89	0.805	0.61
Bitcoin (observations)	0.14	1.54	2.25	0.6088	0.2176
Brazos river (observations)	0.244	1.65	2.25	0.5203	0.0406
Standard map (calculations)	0	1.935	1.4	0.71985	0.4397
Solar activity/SN (observations)	-0.71 ± 0.10	1.31 ± 0.07	1	0.725	0.725
Solar activity/MF (observations)	-0.44 ± 0.07	1.21 ± 0.06	1	0.803	0.803
Solar activity/TSI (observations)	-0.52 ± 0.10	1.54 ± 0.03	1	0.544	0.544

PHYSICAL REVIEW E **78**, 021102 (2008)

Nonadditive entropy reconciles the area law in quantum systems with classical thermodynamics

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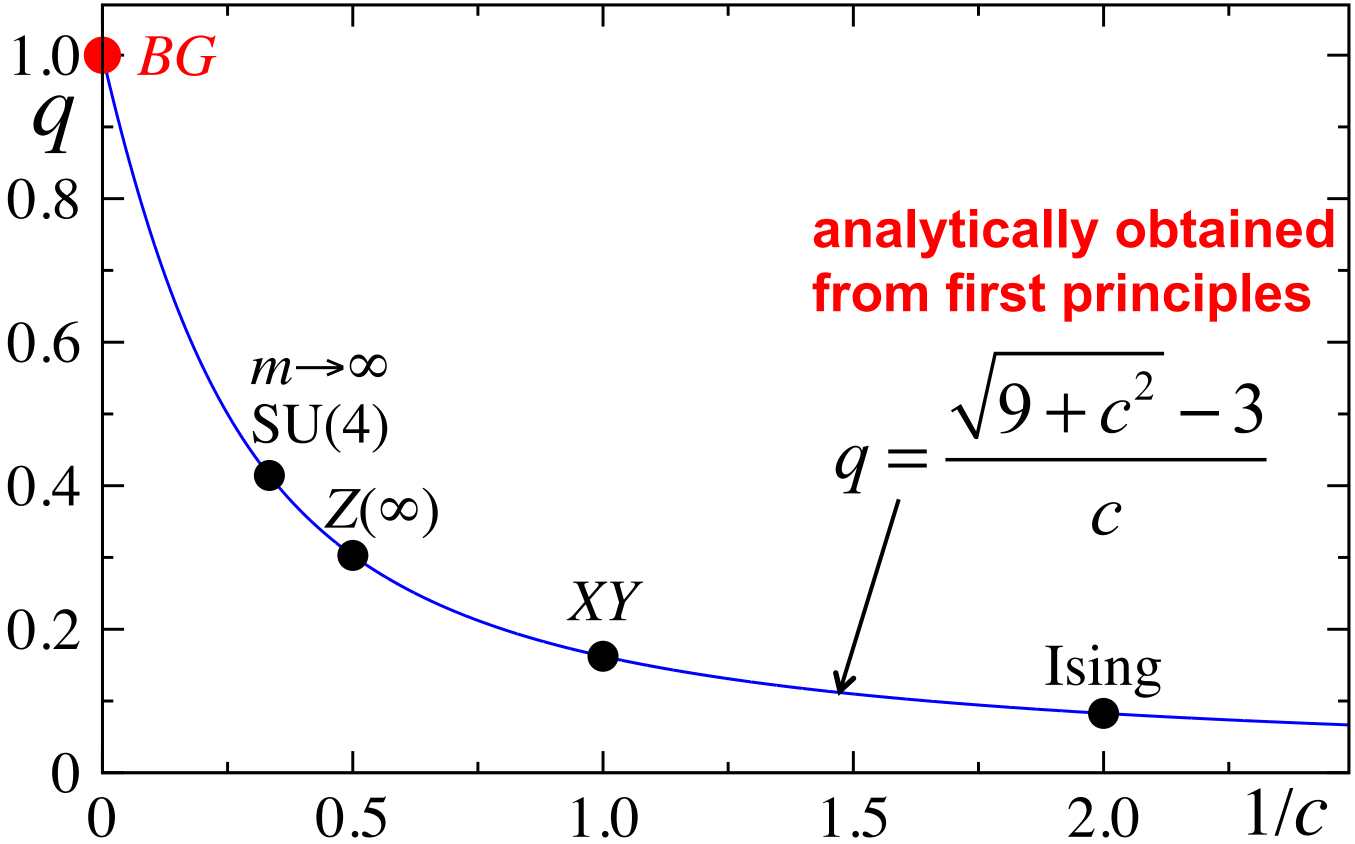
(Received 16 March 2008; revised manuscript received 16 May 2008; published 5 August 2008)

The Boltzmann–Gibbs–von Neumann entropy of a large part (of linear size L) of some (much larger) d-dimensional quantum systems follows the so-called area law (as for black holes), i.e., it is proportional to L^{d-1}. Here we show, for $d=1,2$, that the (nonadditive) entropy S_q satisfies, for a special value of $q \neq 1$, the classical thermodynamical prescription for the entropy to be extensive, i.e., $S_q \propto L^d$. Therefore, we reconcile with classical thermodynamics the area law widespread in quantum systems. Recently, a similar behavior was exhibited in mathematical models with scale-invariant correlations [C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sci. U.S.A. **102** 15377 (2005)]. Finally, we find that the system critical features are marked by a maximum of the special entropic index q.

Block entropy for the *d=1+1* model, with central charge *c*, at its quantum phase transition at *T=0* and critical transverse "magnetic" field



Self-dual $Z(n)$ magnet $(n = 1,2,\ldots)$ [FC Alcaraz, JPA 20 (1987) 2511]

$$\to c = \frac{2(n-1)}{n+2} \in [0,2]$$

$SU(n)$ magnets $(n = 1,2,\ldots;\ m = 2,3,\ldots)$ [FC Alcaraz and MJ Martins, JPA 23 (1990) L1079]

$$\to c = (n-1)\left[1 - \frac{n(n+1)}{(m+n-2)(m+n-1)}\right] \in [0, n-1]$$

EDGE OF CHAOS OF THE LOGISTIC MAP:

(Using result in http://pi.lacim.uqam.ca/piDATA/feigenbaum.txt)

$$q = 1 - \frac{\ln 2}{\ln \alpha_F} =$$

M.L. Lyra and C. T. , Phys Rev Lett **80**, 53 (1998)

0.24448770134128206619877042340468040523444693549005767367036509863277496727665586657551562268575407062883496403827283060636001937303318189645513410812778097921943860270831944900524658135215031745349520749404481654609490874483340567236224664880833330721423189871458729926815484967746078648218345690633702059468204618990216753214575461174383050084968604088469694917043674789915060166464910602178348278899938183825225545823380381131180318054482367579449903970743954661463408155531687885350301138214914112662463289401303701523549365714712699179210216226888330296754057806307068223688104320157903521237407354446029700060552504231420280891935788112397319779748442351524560409264467095795703046586141295664796666877436832404920227573930047508953118551795587204839926968968275558524450244365268256094237801280330948779544035425248590433797618027118300045735855507389411367587844006291356304216745416940921356986032078590881998593590073193368010699674967079044560924186321120541305473939857955444103476122225921368462193460093600...

(1018 meaningful digits)

Rendiconti Lincei. Scienze Fisiche e Naturali
https://doi.org/10.1007/s12210-018-0749-9

Accademia dei Lincei, Rome (24-25 June 2019)

THE QUANTUM WORLD OF MOLECULES

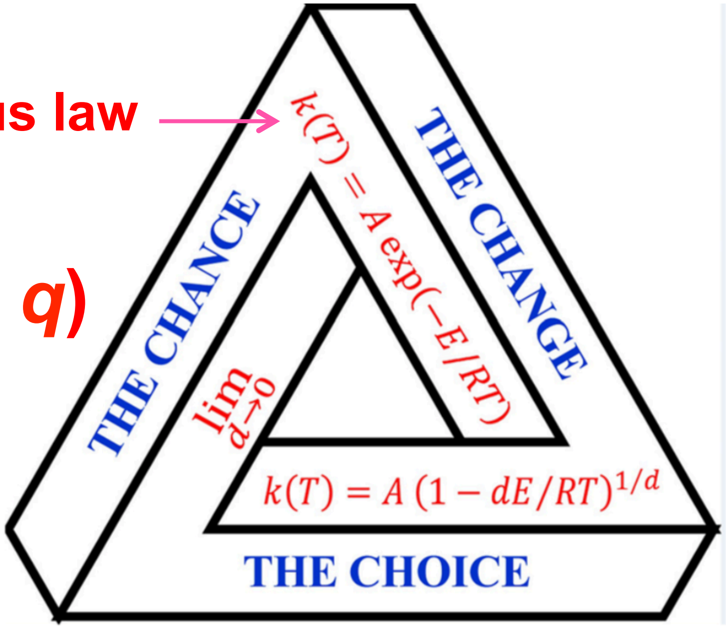
From statistical thermodynamics to molecular kinetics: the change, the chance and the choice

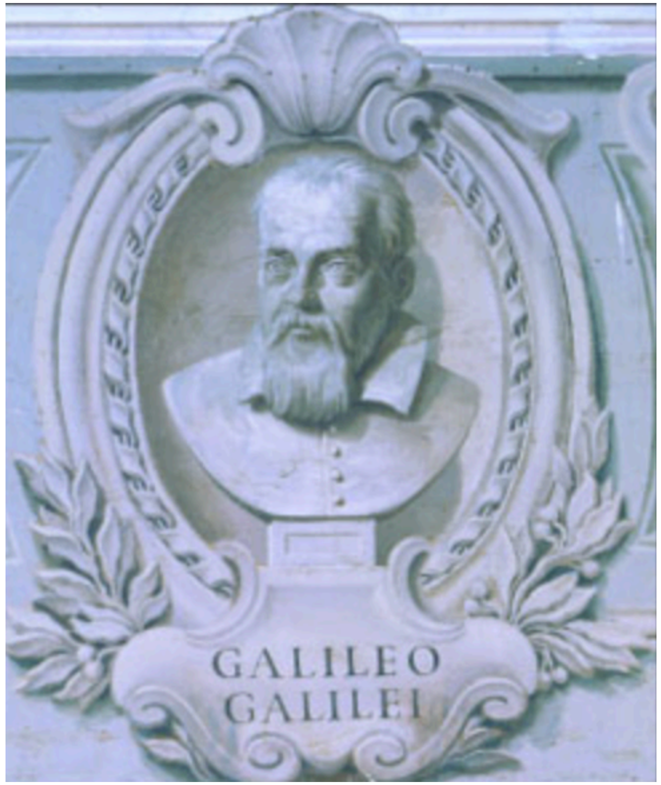
Vincenzo Aquilanti[1,2] · Ernesto P. Borges[3] · Nayara D. Coutinho[1] · Kleber C. Mundim[4] · Valter H. Carvalho-Silva[5]

Received: 1 August 2018 / Accepted: 12 October 2018

Arrhenius law

(d = 1 - q)





PHYSICAL REVIEW E **84**, 021121 (2011)

Group entropies, correlation laws, and zeta functions

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(Received 15 February 2011; revised manuscript received 3 May 2011; published 10 August 2011)

The notion of group entropy is proposed. It enables the unification and generaliztion of many different definitions of entropy known in the literature, such as those of Boltzmann-Gibbs, Tsallis, Abe, and Kaniadakis. Other entropic functionals are introduced, related to nontrivial correlation laws characterizing universality classes of systems out of equilibrium when the dynamics is weakly chaotic. The associated thermostatistics are discussed. The mathematical structure underlying our construction is that of formal group theory, which provides the general structure of the correlations among particles and dictates the associated entropic functionals. As an example of application, the role of group entropies in information theory is illustrated and generalizations of the Kullback-Leibler divergence are proposed. A new connection between statistical mechanics and zeta functions is established. In particular, Tsallis entropy is related to the classical Riemann zeta function.

$$S_q \leftrightarrow \frac{1}{(1-q)^{s-1}} \varsigma(s) \quad (q<1)$$

$$\text{with } \varsigma(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ \text{prime}} \frac{1}{1-p^{-s}}$$

$$= \frac{1}{1-2^{-s}} \frac{1}{1-3^{-s}} \frac{1}{1-5^{-s}} \frac{1}{1-7^{-s}} \frac{1}{1-11^{-s}} \cdots$$

q – PRODUCT:

L. Nivanen, A. Le Mehaute and Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003)
E.P. Borges, Physica A **340**, 95 (2004)

The *q* - product is defined as follows:

$$x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1 \right]^{\frac{1}{1-q}}$$

Properties :

$i) \quad x \otimes_1 y = x\, y$

$ii) \quad \ln_q (x \otimes_q y) = \ln_q x + \ln_q y$ (extensivity of Sq)

$[\textit{whereas} \;\; \ln_q (x\, y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y)]$

(nonadditivity of Sq)

q - GENERALIZED CENTRAL LIMIT THEOREM:

S. Umarov, C.T. and S. Steinberg, Milan J Math **76,** 307 (2008)

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\, dx = \int_{-\infty}^{\infty} e_q^{ix\xi[f(x)]^{q-1}} f(x)\, dx$$

$(q \geq 1)$

(nonlinear!)

For q<1 see K.P. Nelson and S. Umarov, Physica A **389**, 2157 (2010)

$$q-FourierTransform\left[\frac{\sqrt{\beta}}{C_q}e_q^{-\beta\, t^2}\right]=e_{q_1}^{-\beta_1\,\omega^2}$$

where $$q_1=\frac{1+q}{3-q}$$

invertible

and $$\beta_1=\frac{3-q}{8\beta^{2-q}C_q^{2(1-q)}}\quad\Leftrightarrow\quad(\beta_1)^{\frac{1}{\sqrt{2-q}}}\,\beta^{\sqrt{2-q}}=\left[\frac{3-q}{8C_q^{2(1-q)}}\right]^{\frac{1}{\sqrt{2-q}}}\equiv K(q)$$

with $$C_q=\begin{cases}\dfrac{2\sqrt{\pi}\,\Gamma\left(\dfrac{1}{q-1}\right)}{(3-q)\sqrt{(1-q)}\,\Gamma\left(\dfrac{3-q}{2(1-q)}\right)} & if\quad q<1\\ \sqrt{\pi} & if\quad q=1\\ \dfrac{\sqrt{\pi}\,\Gamma\left(\dfrac{3-q}{2(q-1)}\right)}{\sqrt{q-1}\,\Gamma\left(\dfrac{1}{q-1}\right)} & if\quad 1<q<3\end{cases}$$

Milan j. math. 76 (2008), 307–328

1424-9286/010307-22, *published online* 14.3.2008
DOI 10.1007/s00032-008-0087-y

Milan Journal of Mathematics

On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

JOURNAL OF MATHEMATICAL PHYSICS **51**, 033502 (2010)

Generalization of symmetric α-stable Lévy distributions for $q>1$

Sabir Umarov,[1,a)] Constantino Tsallis,[2,3,b)] Murray Gell-Mann,[3,c)] and Stanly Steinberg[4,d)]
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[2]*Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*
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See also:

H.J. Hilhorst, JSTAT P10023 (2010)

M. Jauregui and C. T., Phys Lett A **375**, 2085 (2011)

M. Jauregui, C. T. and E.M.F. Curado, JSTAT P10016 (2011)

A. Plastino and M.C. Rocca, Physica A and Milan J Math (2012)

A. Plastino and M.C. Rocca, Physica A **392**, 3952 (2013)

S. Umarov and C. T., J Phys A **49**, 415204 (2016)

CENTRAL LIMIT THEOREM

$N^{1/[\alpha(2-q)]}$ *-scaled attractor* $\mathrm{F}(x)$ *when summing* $N\to\infty$ q*-independent identical random variables*

with symmetric distribution $f(x)$ *with* $\sigma_Q \equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q$ $\left(Q \equiv 2q-1,\ q_1 = \dfrac{1+q}{3-q} \right)$

	$q=1$ [*independent*]	$q\neq 1$ (*i.e.*, $Q\equiv 2q-1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathrm{F}(x) = $ *Gaussian* $G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathrm{F}(x) = G_q(x) \equiv G_{(3q_1-1)/(1+q_1)}(x)$, *with same* σ_Q *of* $f(x)$ $G_q(x) \sim \begin{cases} G(x) & \text{if } \lvert x\rvert \ll x_c(q,2) \\ f(x) \sim C_q/\lvert x\rvert^{2/(q-1)} & \text{if } \lvert x\rvert \gg x_c(q,2) \end{cases}$ *with* $\lim_{q\to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg, Milan J Math 76, 307 (2008)
$\sigma_Q \to \infty$ $(0<\alpha<2)$	$\mathrm{F}(x) = $ *Levy distribution* $L_\alpha(x)$, *with same* $\lvert x\rvert \to\infty$ *behavior* $\mathrm{L}_\alpha(x) \sim \begin{cases} G(x) \\ \quad \text{if } \lvert x\rvert \ll x_c(1,\alpha) \\ f(x) \sim C_\alpha/\lvert x\rvert^{1+\alpha} \\ \quad \text{if } \lvert x\rvert \gg x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha\to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathrm{F}(x) = L_{q,\alpha}$, *with same* $\lvert x\rvert \to \infty$ *asymptotic behavior* $L_{q,\alpha} \sim \begin{cases} G_{\frac{2(1-q)-\alpha(1+q)}{2(1-q)-\alpha(3-q)},\,\alpha}(x) \sim C^*_{q,\alpha}/\lvert x\rvert^{\frac{2(1-q)-\alpha(3-q)}{2(1-q)}} \\ \quad (\textit{intermediate regime}) \\ G_{\frac{2\alpha q-\alpha+3}{\alpha+1},\,2}(x) \sim C^L_{q,\alpha}/\lvert x\rvert^{(1+\alpha)/(1+\alpha q-\alpha)} \\ \quad (\textit{distant regime}) \end{cases}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg J Math Phys 51, 033502 (2010)

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SCIENTIFIC REPORTS

OPEN

The standard map: From Boltzmann-Gibbs statistics to Tsallis statistics

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Ugur Tirnakli[1,*] & Ernesto P. Borges[2,3,*]

As well known, Boltzmann-Gibbs statistics is the correct way of thermostatistically approaching ergodic systems. On the other hand, nontrivial ergodicity breakdown and strong correlations typically drag the system into out-of-equilibrium states where Boltzmann-Gibbs statistics fails. For a wide class of such systems, it has been shown in recent years that the correct approach is to use Tsallis statistics instead. Here we show how the dynamics of the paradigmatic conservative (area-preserving) standard map exhibits, in an exceptionally clear manner, the crossing from one statistics to the other. Our results unambiguously illustrate the domains of validity of both Boltzmann-Gibbs and Tsallis statistical distributions. Since various important physical systems from particle confinement in magnetic traps to autoionization of molecular Rydberg states, through particle dynamics in accelerators and comet dynamics, can be reduced to the standard map, our results are expected to enlighten and enable an improved interpretation of diverse experimental and observational results.

STANDARD MAP (Chirikov 1969)

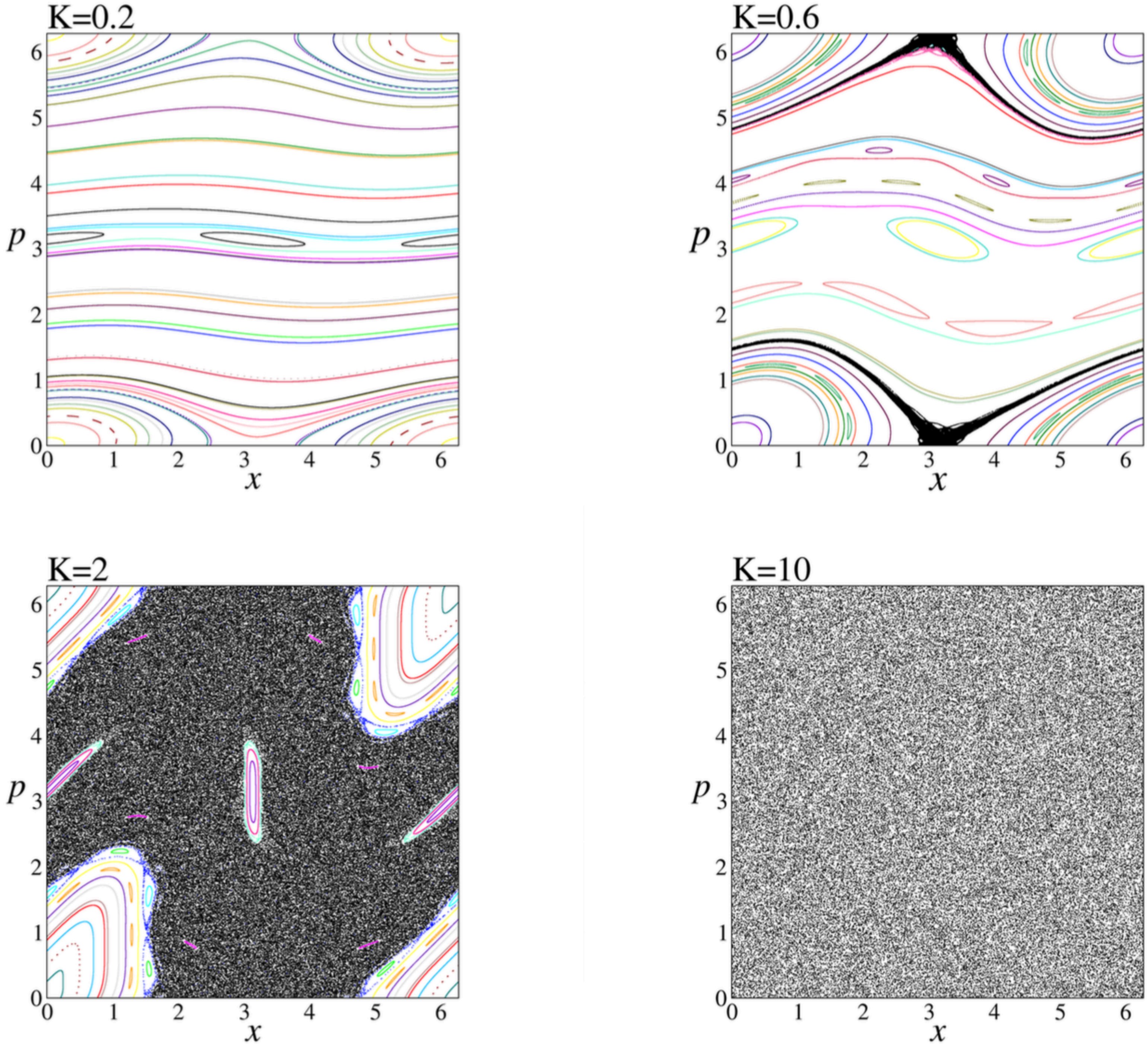
$$p_{i+1} = p_i - K \sin x_i \pmod{2\pi}$$

$$x_{i+1} = x_i + p_{i+1} \pmod{2\pi}$$

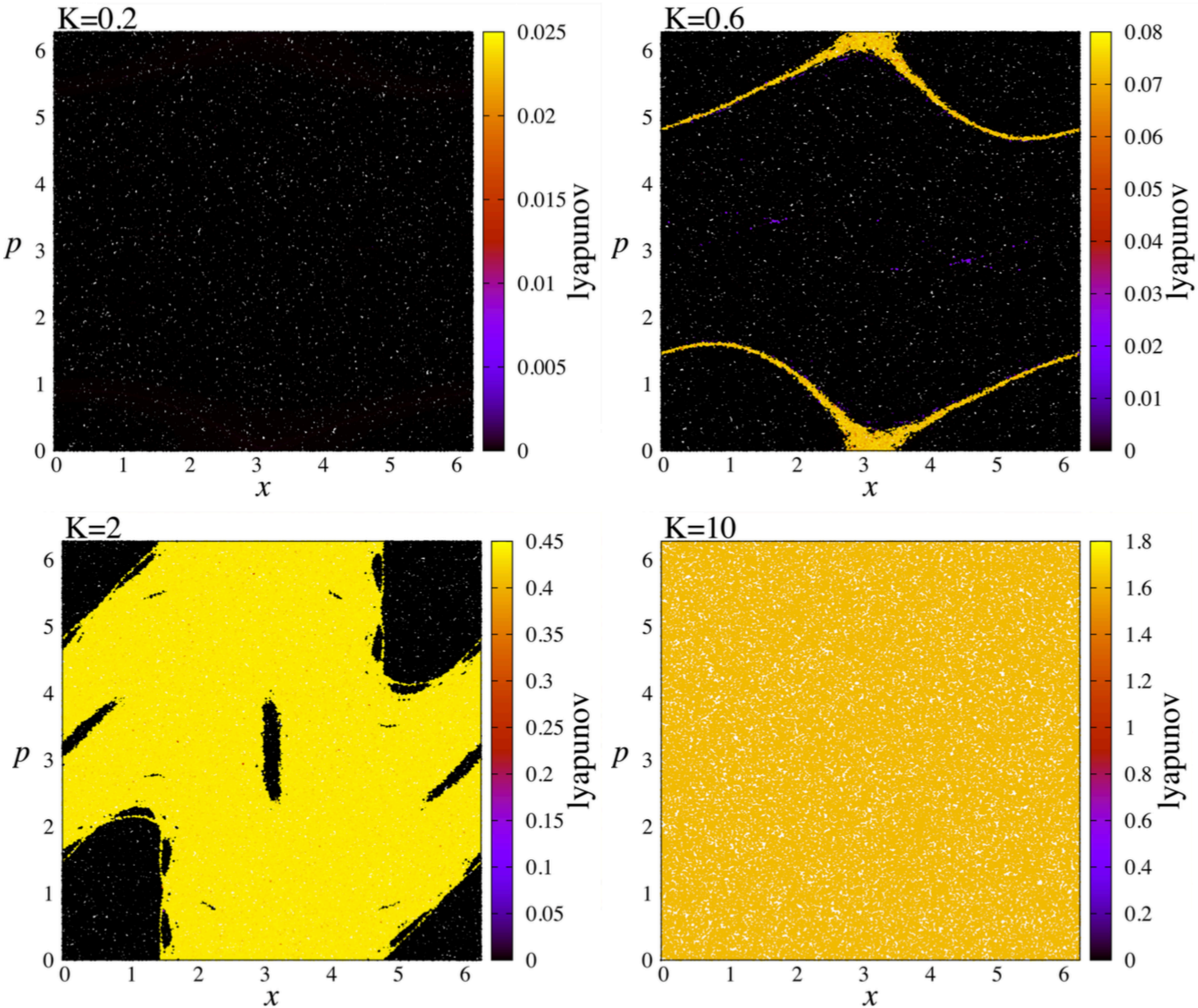
$$(i = 0,1,2,\ldots)$$

(area-preserving)

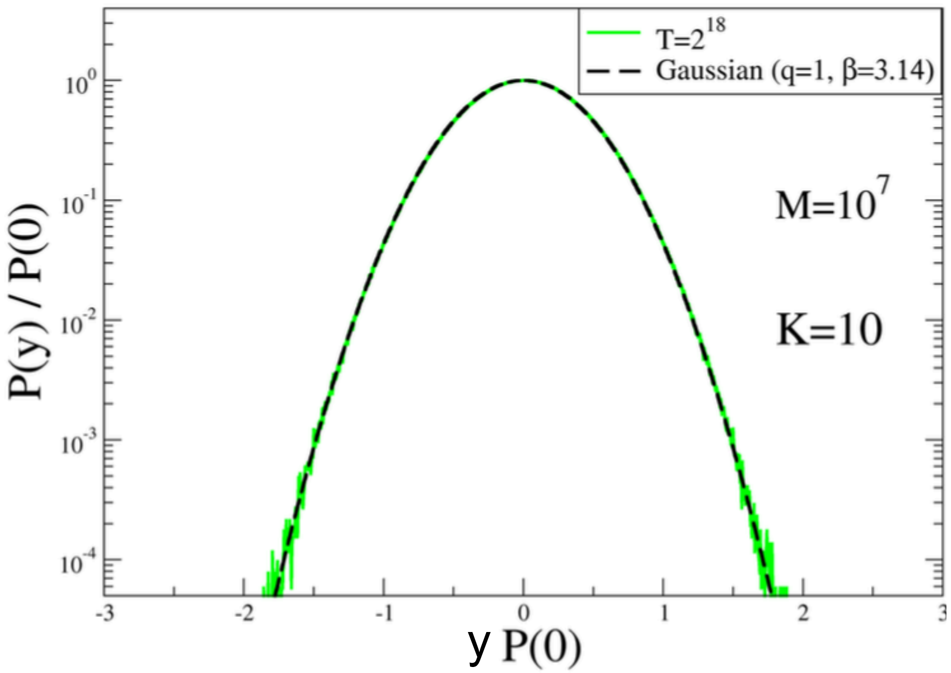
Particle confinement in magnetic traps,
particle dynamics in accelerators,
comet dynamics,
ionization of Rydberg atoms,
electron magneto-transport

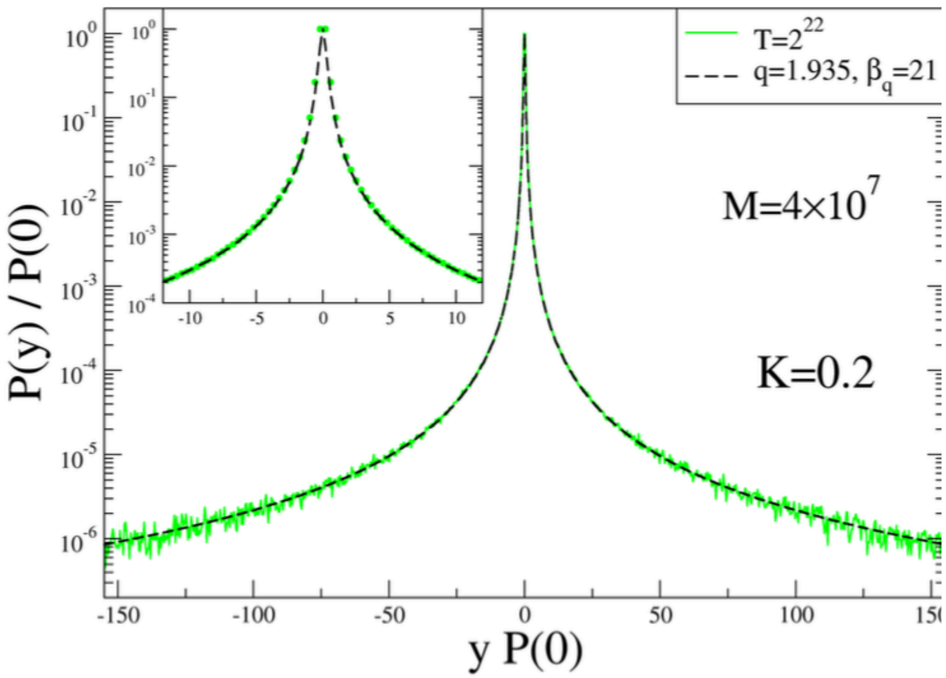


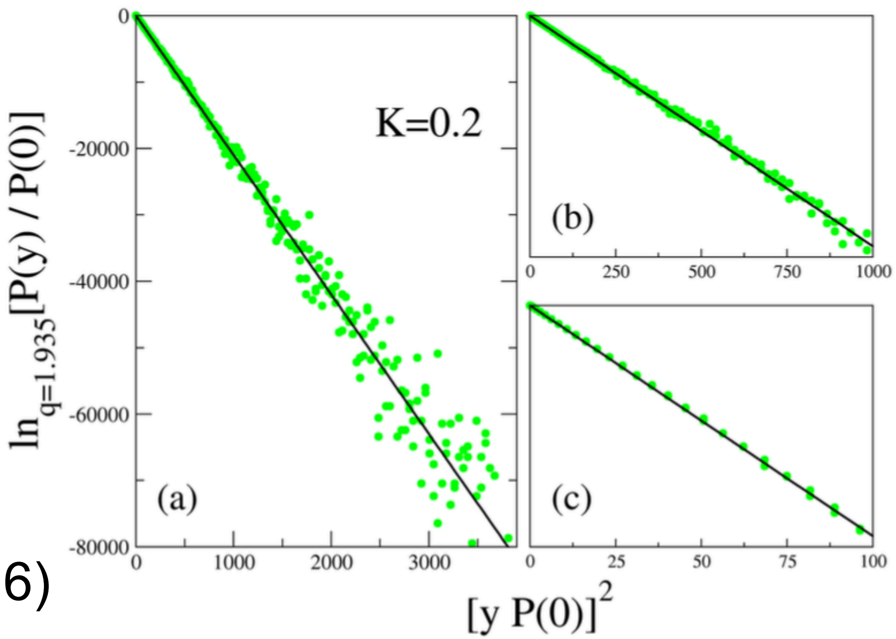
Tirnakli and Borges
Nature / Scientific Reports **6**, 23644 (2016)



Tirnakli and Borges
Nature / Scientific Reports **6**, 23644 (2016)







Tirnakli and Borges

Nature / Scientific Reports **6**, 23644 (2016)

Eur. Phys. J. B (2018) 91: 1
https://doi.org/10.1140/epjb/e2017-80535-3

THE EUROPEAN
PHYSICAL JOURNAL B

Regular Article

Evidence for criticality in financial data

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Abstract. We provide evidence that cumulative distributions of absolute normalized returns for the 100 American companies with the highest market capitalization, uncover a critical behavior for different time scales Δt. Such cumulative distributions, in accordance with a variety of complex – and financial – systems, can be modeled by the cumulative distribution functions of q-Gaussians, the distribution function that, in the context of nonextensive statistical mechanics, maximizes a non-Boltzmannian entropy. These q-Gaussians are characterized by two parameters, namely (q, β), that are uniquely defined by Δt. From these dependencies, we find a monotonic relationship between q and β, which can be seen as evidence of criticality. We numerically determine the various exponents which characterize this criticality.

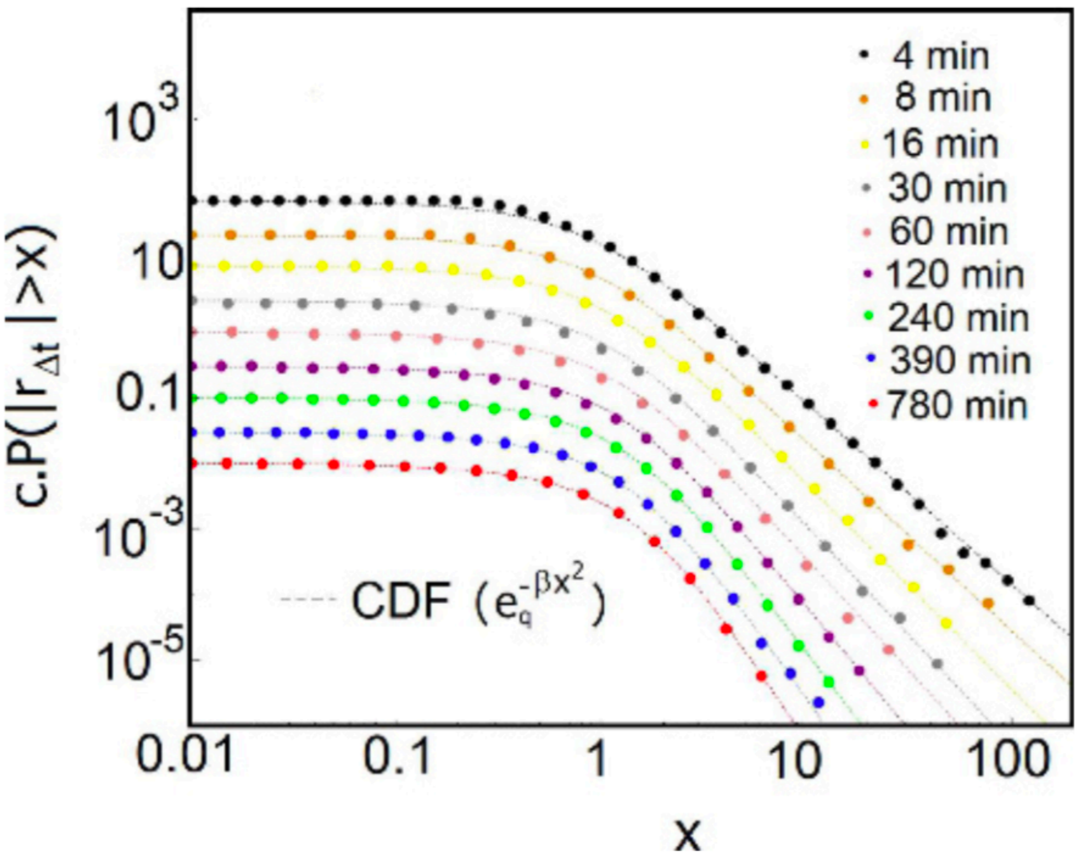


Fig. 1. Cumulative distributions of absolute normalized returns that correspond to different time scales Δt for the 100 American companies with the highest market capitalization (points), and the fitted cumulative q-Gaussian distributions (lines). In order to better visualize the results, each q-Gaussian CDF and the respective experimental data have been multiplied by a positive factor, $c \neq 1$.

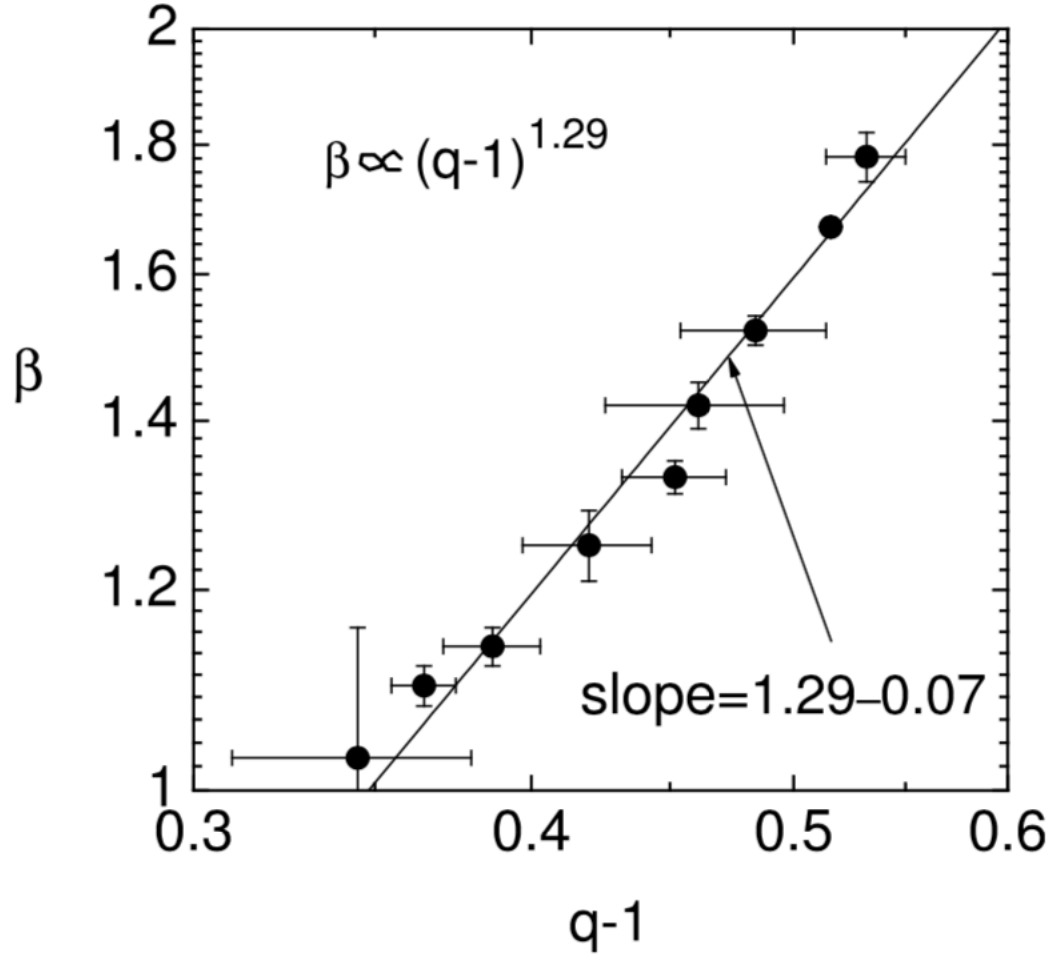
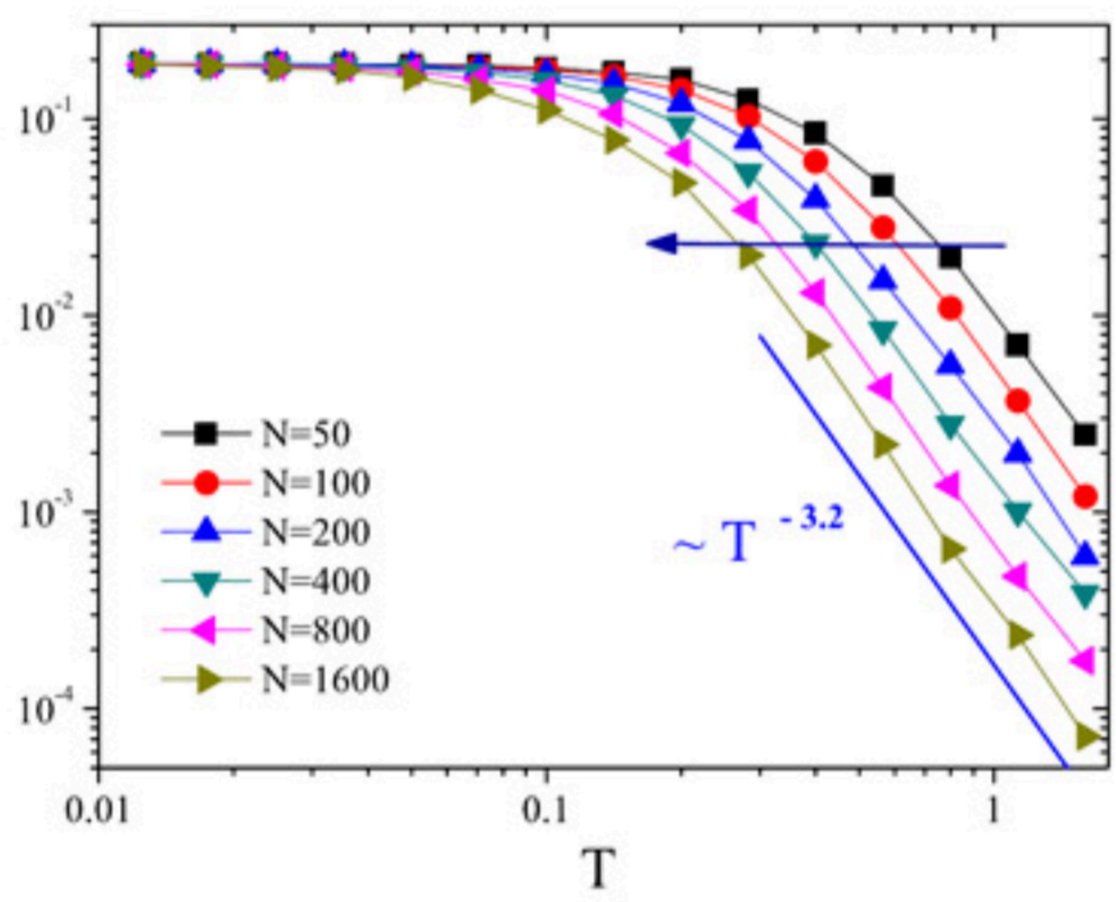


Fig. 6. Log–log representation of the re-normalized inverse temperature β versus $q-1$, for the estimated q-Gaussian pdfs of normalized absolute returns. A power-law dependence of the type $\beta^{-1} \propto (q-1)^{-\delta}$ is observed, with $\delta = 1.29 \pm 0.07$.

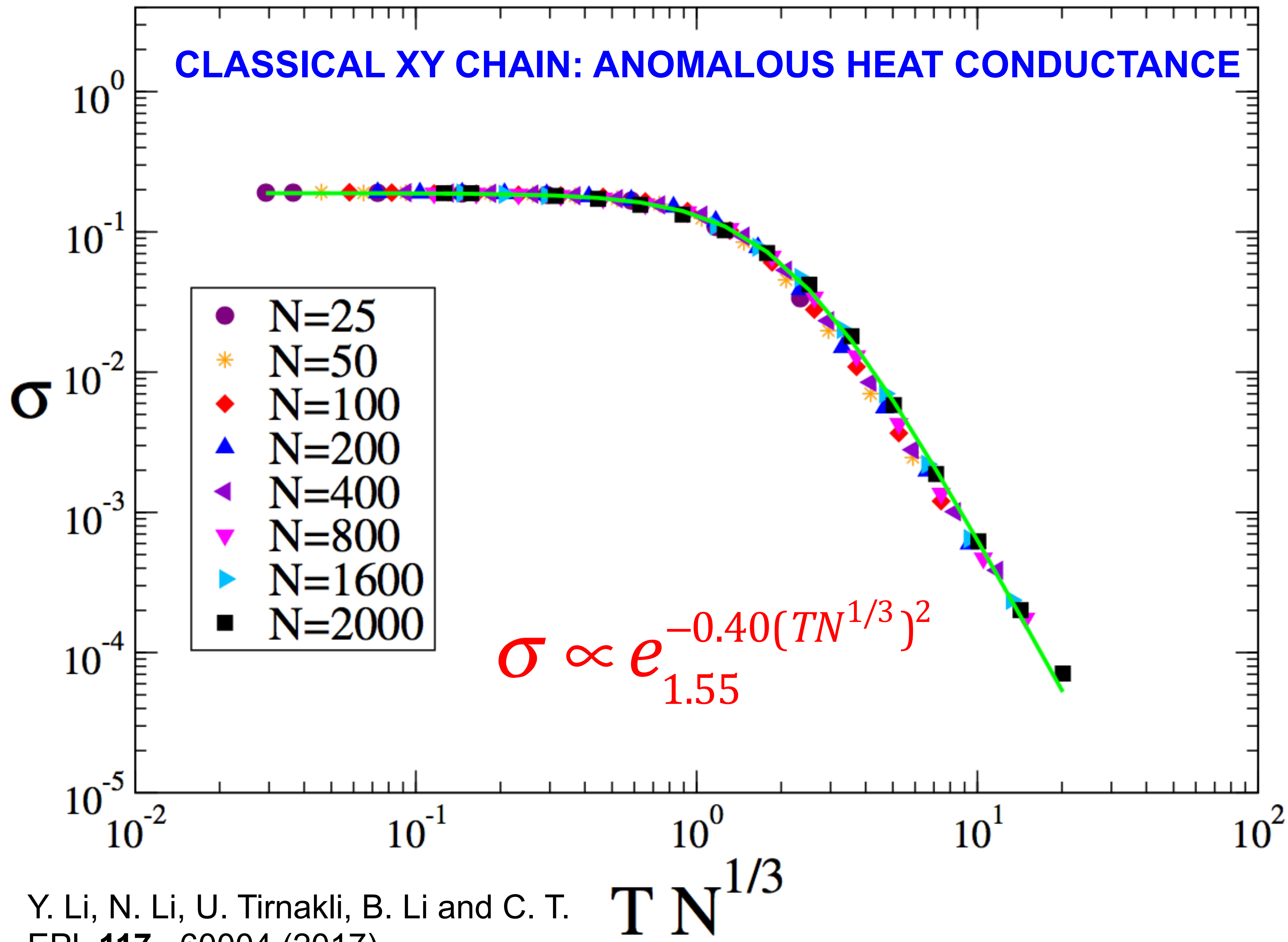
EUROPHYSICS NEWS 2015 HIGHLIGHTS

Law governing anomalous heat conduction revealed (Vol. 46 No. 5-6)



Heat conductance as the function of temperature T for different lattice size N = 50; 100; 200; 400; 800 and 1600

Y. Li, N. Li and B. Li, Temperature dependence of thermal conductivities of coupled rotator lattice and the momentum diffusion in standard map, *Eur. Phys. J. B*, **88**, 182 (2015)



Y. Li, N. Li, U. Tirnakli, B. Li and C. T.
EPL **117**, 60004 (2017)

J.W. GIBBS

Elementary Principles in Statistical Mechanics - Developed with Especial Reference to the Rational Foundation of Thermodynamics

C. Scribner's Sons, New York, 1902; Yale University Press, New Haven, (1981), page 35

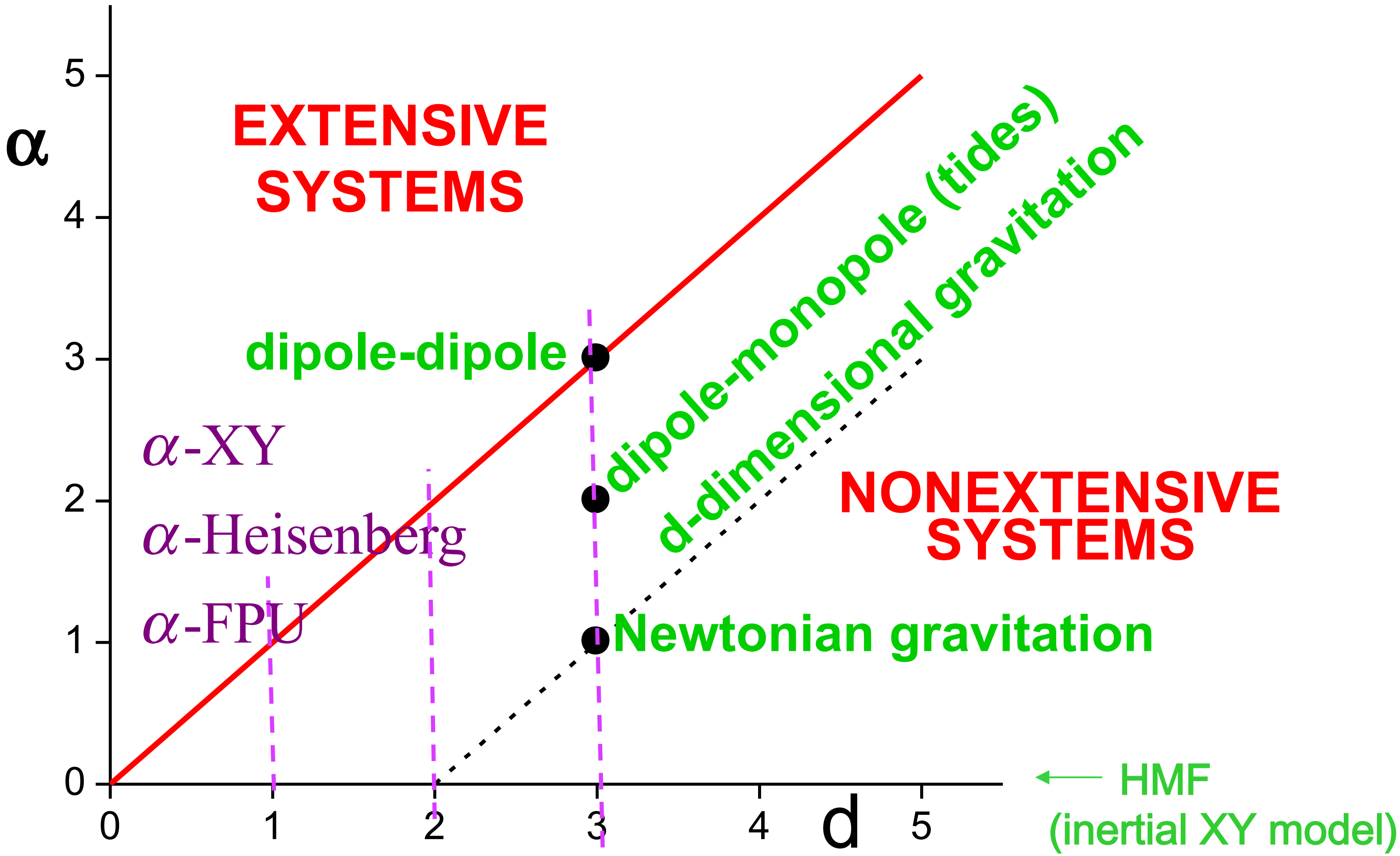
In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) [the partition function, as we call it nowadays] *to have a finite valued, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space […]. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. […]. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).*

CLASSICAL LONG-RANGE-INTERACTING MANY-BODY HAMILTONIAN SYSTEMS

$$V(r) \sim -\frac{A}{r^{\alpha}} \qquad (r \to \infty) \qquad (A > 0, \;\; \alpha \geq 0)$$

$$\textit{integrable if} \quad \alpha / d > 1 \qquad (\textit{short-ranged})$$

$$\textit{non-integrable if} \quad 0 \leq \alpha / d \leq 1 \qquad (\textit{long-ranged})$$



Breakdown of Exponential Sensitivity to Initial Conditions: Role of the Range of Interactions

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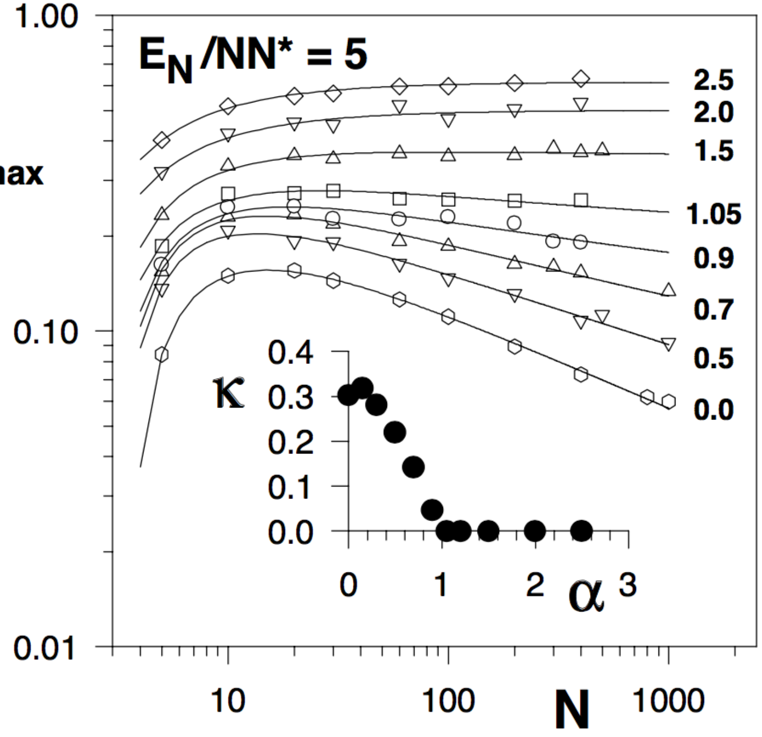
(Received 22 January 1998)

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} L_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{1-\cos(\theta_i-\theta_j)}{r_{ij}^{\alpha}}$$

$$\lambda_{\max} \propto N^{-\kappa(\alpha,d)} \equiv L^{-d\kappa(\alpha,d)}$$



A Letters Journal Exploring the Frontiers of Physics

August 2018

EPL, **123** (2018) 30003
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Validity and failure of the Boltzmann weight

L. J. L. Cirto[1], A. Rodríguez[2], F. D. Nobre[1,3] and C. Tsallis[1,3,4,5]

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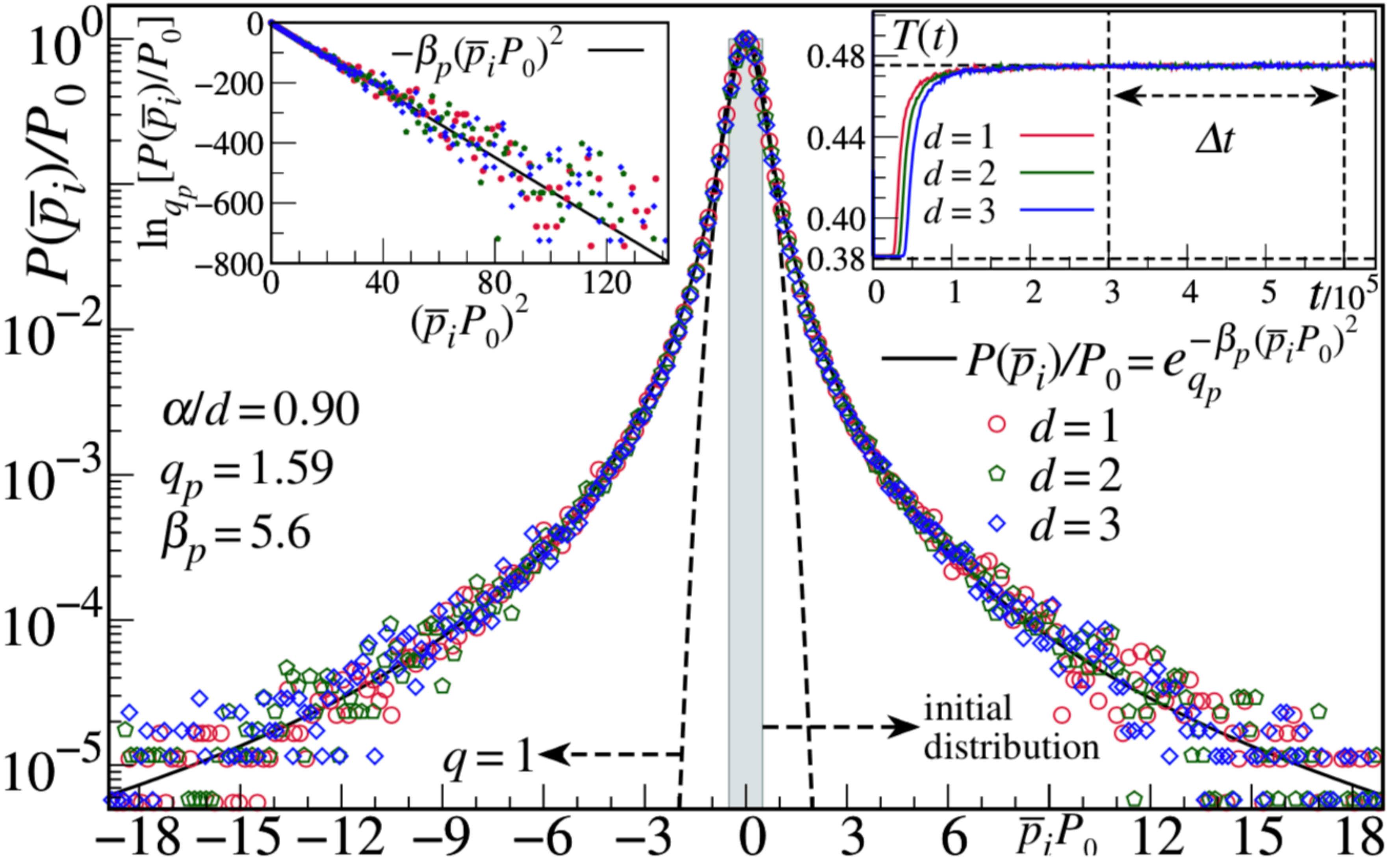
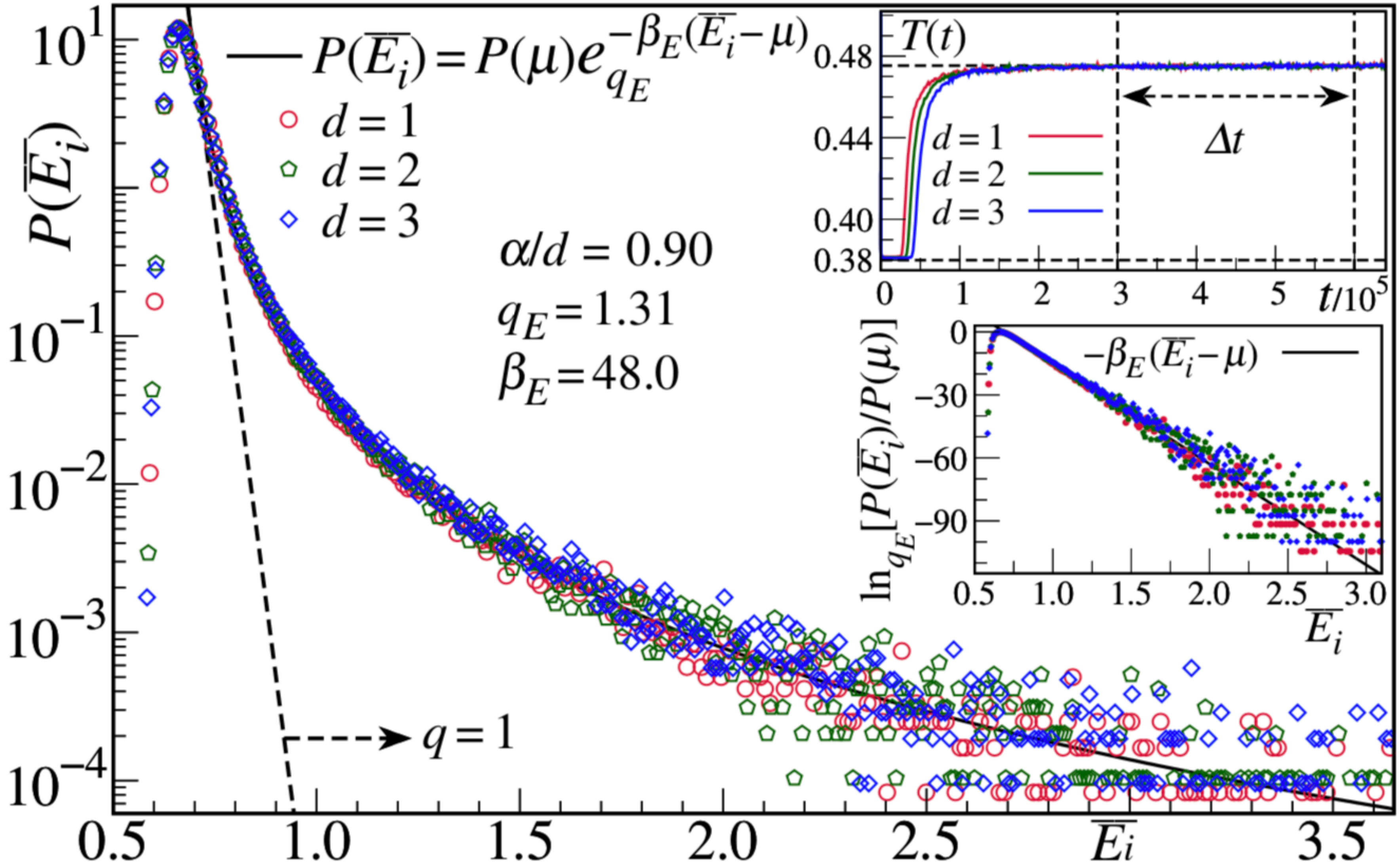
d-DIMENSIONAL XY MODEL



L.J.L Cirto, A. Rodriguez, F.D. Nobre and C.T., EPL **123**, 30003 (2018)

d-DIMENSIONAL XY MODEL



L.J.L Cirto, A. Rodriguez, F.D. Nobre and C.T., EPL **123**, 30003 (2018)

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April 2019

EPL, **126** (2019) 10005
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www.epljournal.org

New type of equilibrium distribution for a system of charges in a spherically symmetric electric field

Gabriela A. Casas[1], Fernando D. Nobre[1,2] and Evaldo M. F. Curado[1,2]

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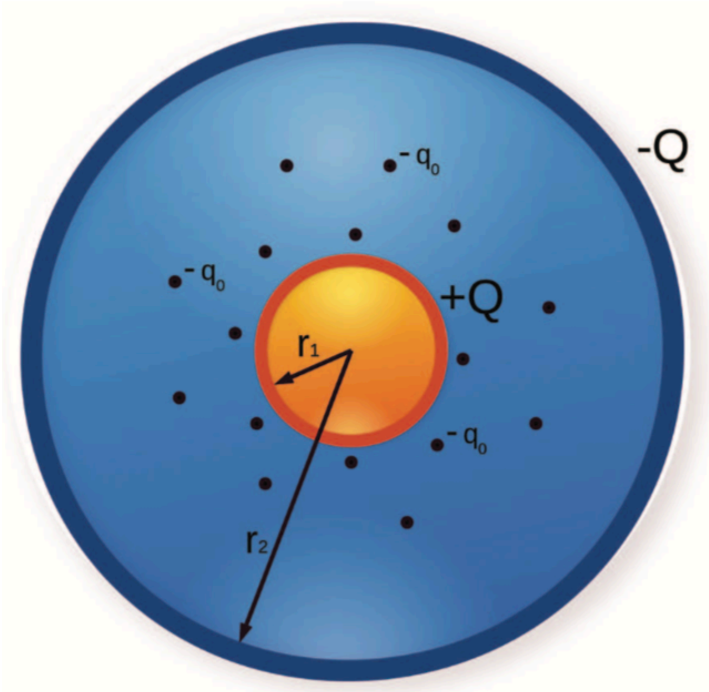
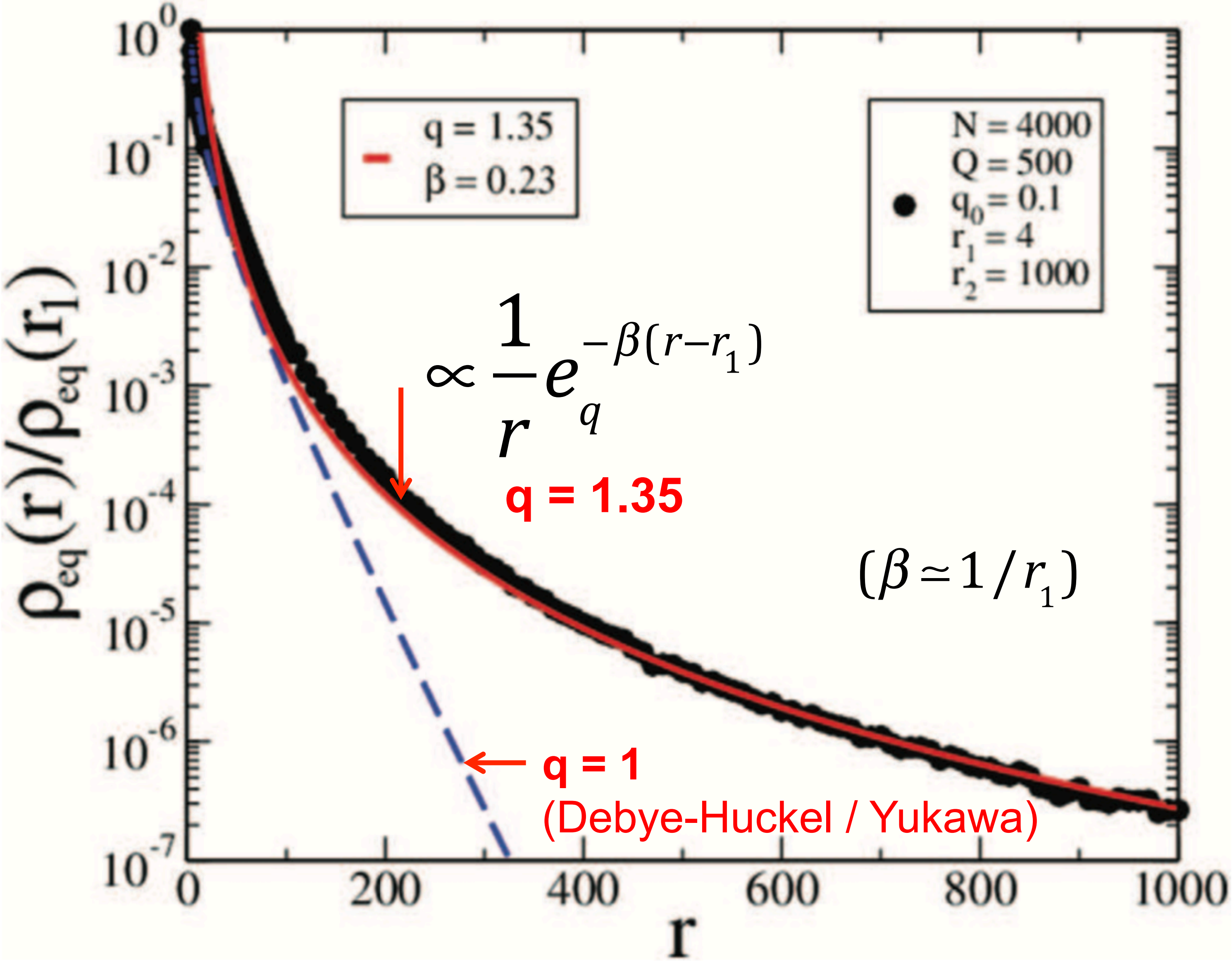


[see also P. Quarati and A. Scarfone, Astrophys. J. **666**, 1303 (2007)]



q = 1.35
β = 0.23
N = 4000
Q = 500
q0 = 0.1
r1 = 4
r2 = 1000
$\propto \frac{1}{r} e_q^{-\beta(r-r_1)}$
q = 1.35
$(\beta \simeq 1/r_1)$
q = 1
(Debye-Huckel / Yukawa)
$\rho_{eq}(r)/\rho_{eq}(r_1)$
r

Physica A 389 (2010) 3866–3869

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journal homepage: www.elsevier.com/locate/physa

Evidence for energy regularity in the Mendeleev periodic table

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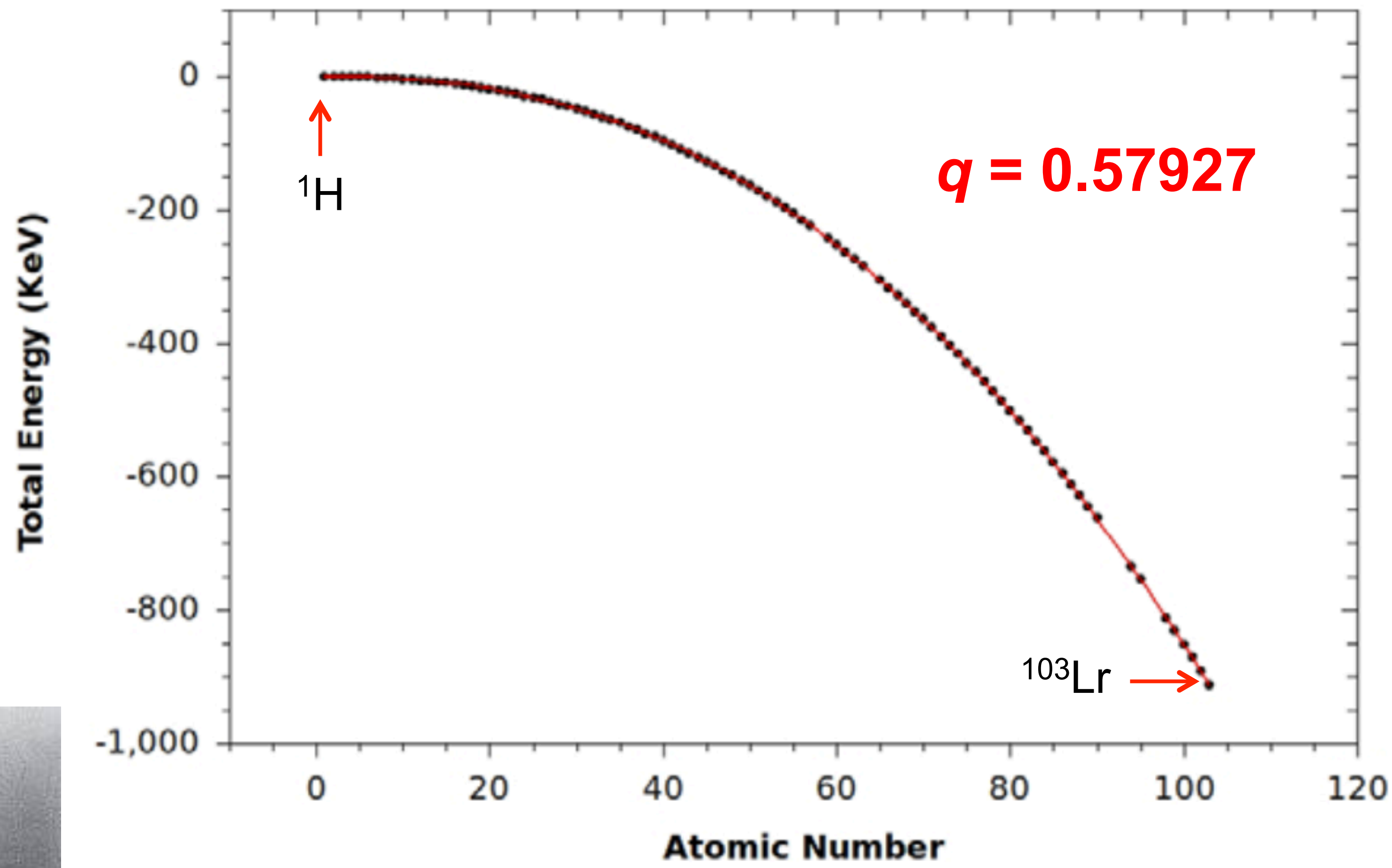
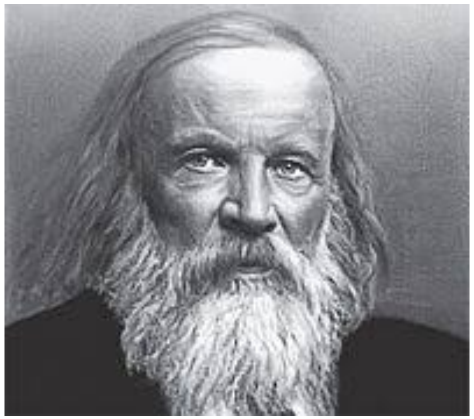
Non-extensive *q*-entropy
Atomic total energy

ABSTRACT

We show that the dependence of the total energy of the atoms on their atomic number can follow a *q*-exponential (as proposed by C. Tsallis), for practically all elements of the periodic table. The result is qualitatively explained in terms of the way the atomic configurations are arranged to minimize energy.

MENDELEEV TABLE



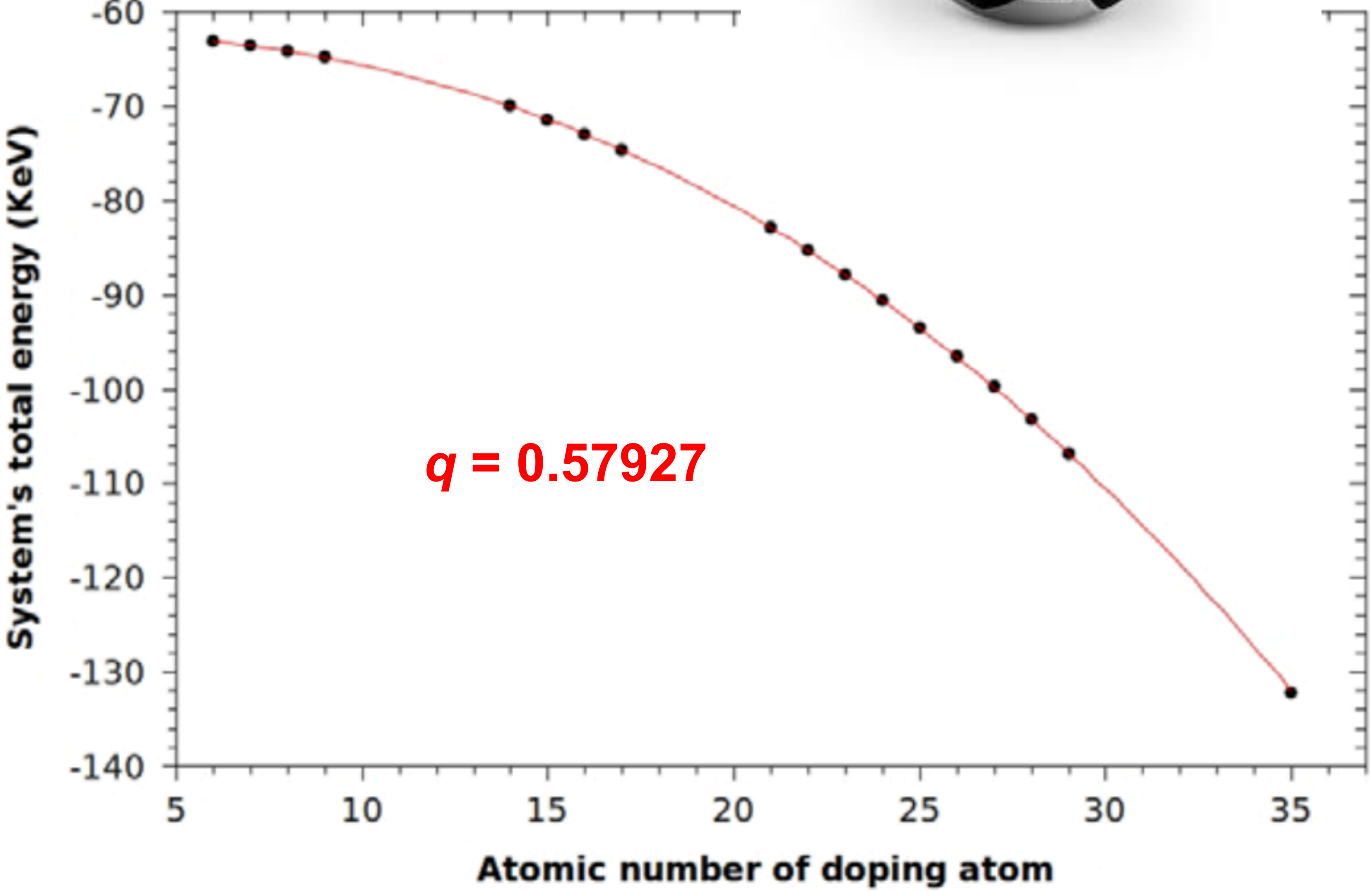


(1834-1907)

$$E = E_H \left[1 + 1.0425\,(Z-1)\right]^{2.3768}$$

(E_H = -13.6057 eV)

FULLERENES (C_{60})



Relaxation times and ergodicity properties in a realistic ionic–crystal model, and the modern form of the FPU problem

Andrea Carati* Luigi Galgani* Fabrizio Gangemi‡
Roberto Gangemi‡

March 7, 2019

arXiv:1903.02272v1 [cond-mat.stat-mech] 6 Mar 2019

The conclusion we reach is that at low temperatures ergodicity does not occur, and thus the Gibbs prescriptions are not dynamically justified, up to geological time scales.

LiF (ionic crystal)

$$H = \sum_{j,s} \frac{\mathbf{p}_{j,s}^2}{2m_s} + \sum_{j,j',s,s'} V_{s,s'}\left(\left|\mathbf{x}_{j,s} - \mathbf{x}_{j',s'}\right|\right)$$

$$V_{s,s'}(r) = a_{s,s'} e^{-b_{s,s'} r} + \frac{c_{s,s'}}{r^6} + \frac{e_s^{\text{eff}} e_{s'}^{\text{eff}}}{r}$$

(Buckingham phenomenological potential)

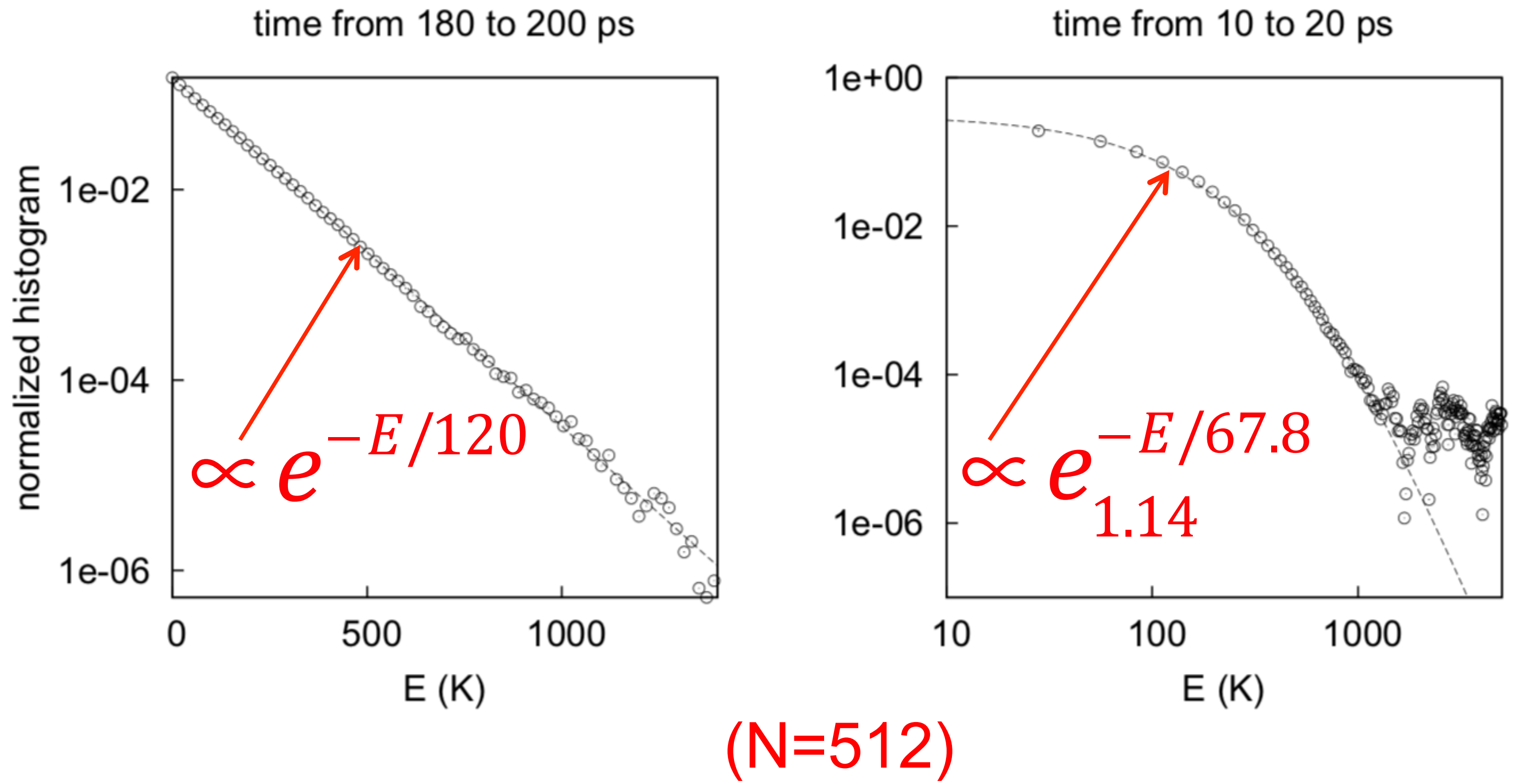


Figure 9: Maxwell–Boltzmann distribution and q–distribution. Left panel. Histogram (in semilogarithmic scale) of the energies E (circles) of the modes not initially excited, from time $t = 180$ (after their equipartition is attained) up to $t = 200$ ps. Only the 15 modes of lowest frequency were initially excited, among the total number 1536 of modes. Specific energy $\varepsilon = 120$ K, $N = 512$. Solid line is the graph of the Maxwell–Boltzmann distribution function $C \exp(-E/\varepsilon)$. Right panel. Same as left panel, in logarithmic scale, with data collected for time from 10 to 20 ps. Solid line is the graph of the Tsallis distribution function $C'(1+\beta(q-1)E)^{1/(1-q)}$ for $q = 1.14$, $\beta^{-1} = 67.8$ K.

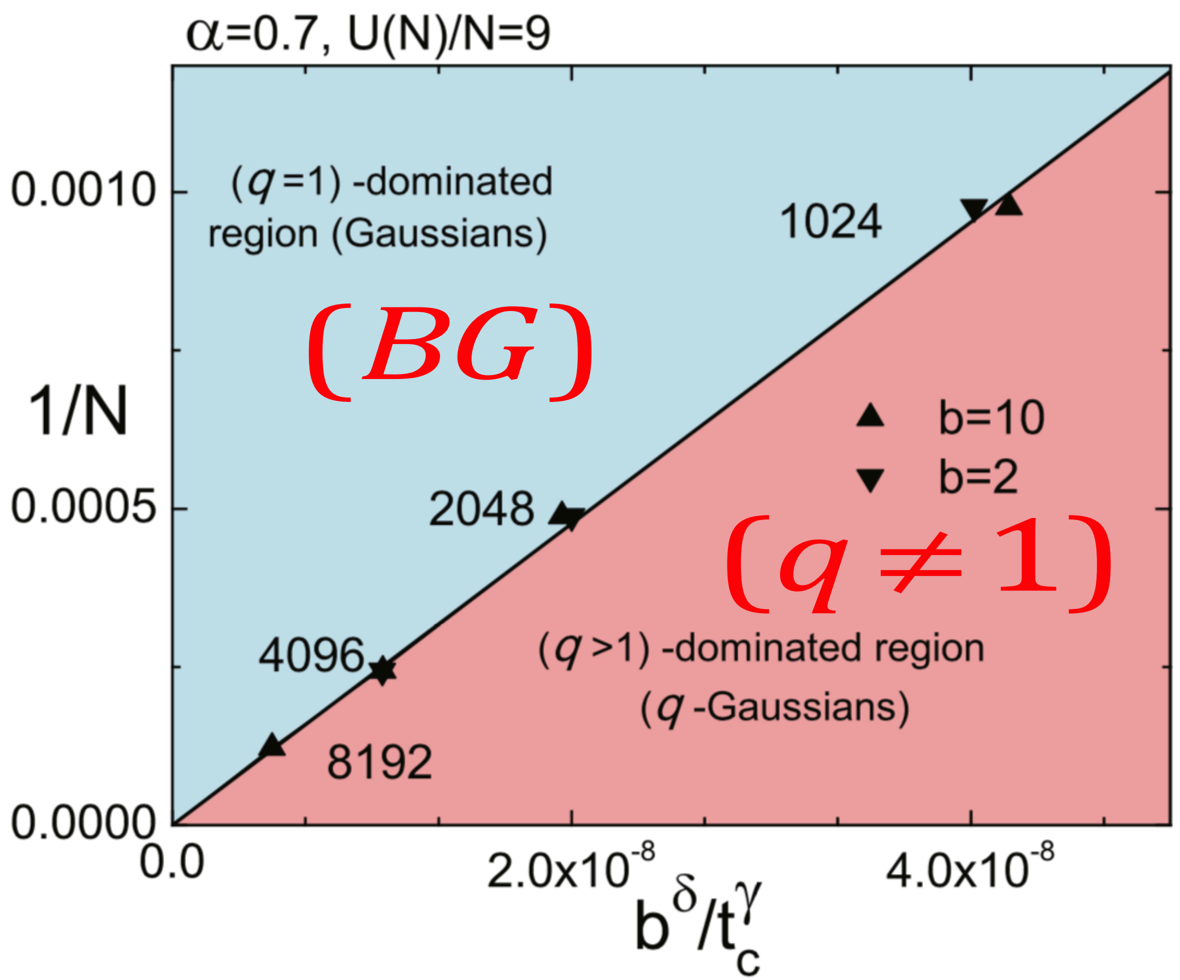


Fig. 8: (Colour on-line) A unified overview of the crossover frontier of fig. 7(b), combining the b values. The fitting straight line is $1/N = Db^{\delta}/t_c^{\gamma}$, with $D = 2.3818 \times 10^4$, $\delta = 0.27048$, and $\gamma = 1.365$.

November 2014
www.epljournal.org
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doi: 10.1209/0295-5075/108/40006

Fermi-Pasta-Ulam model with long-range interactions: Dynamics and thermostatistics

H. Christodoulidi[1], C. Tsallis[2,3] and T. Bountis[1]

www.nature.com/scientificreports

SCIENTIFIC REPORTS

OPEN

Role of dimensionality in complex networks

Samuraí Brito[1], L. R. da Silva[1,2] & Constantino Tsallis[2,3]

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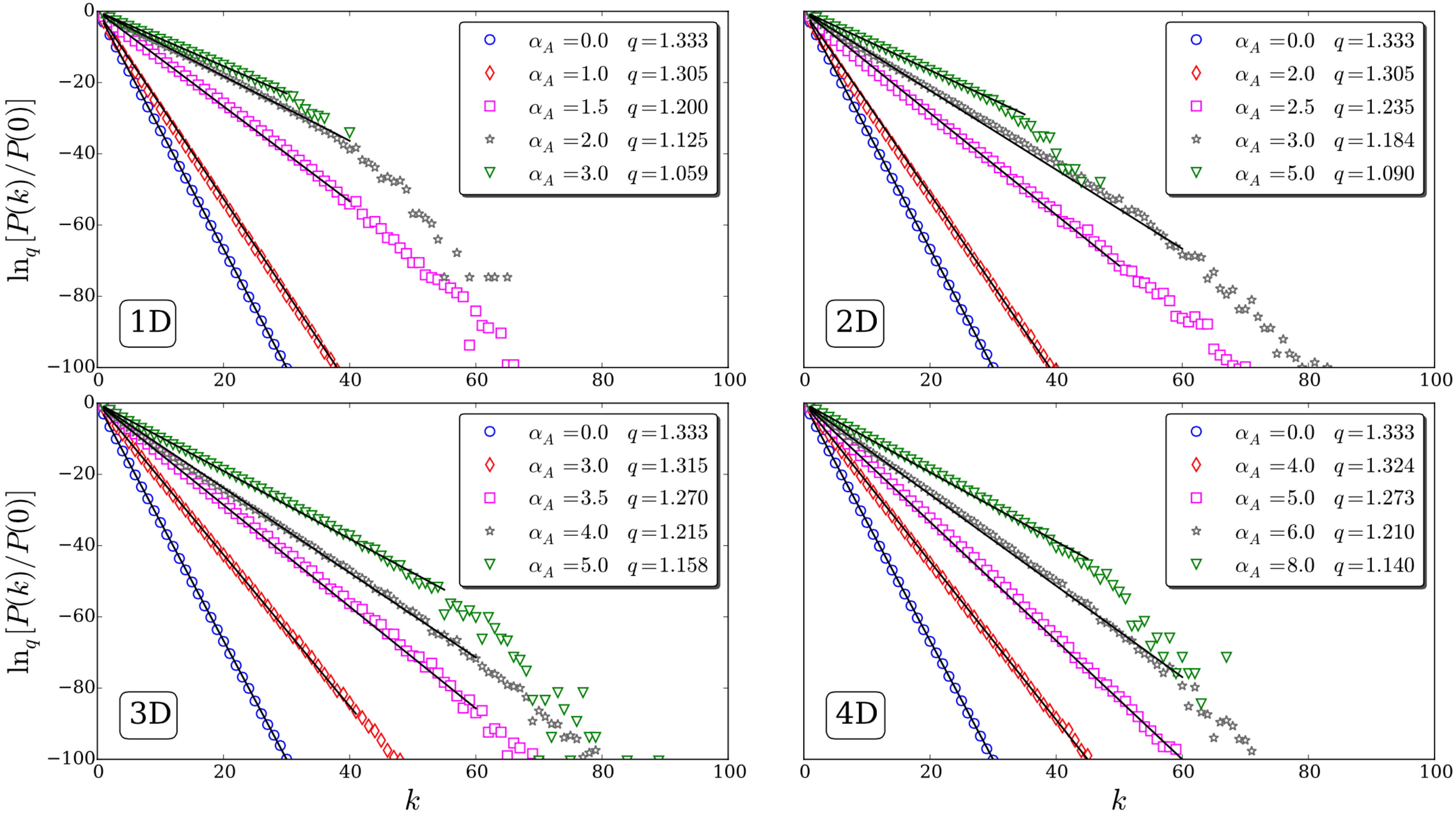
Deep connections are known to exist between scale-free networks and non-Gibbsian statistics. For example, typical degree distributions at the thermodynamical limit are of the form $P(k) \propto e_q^{-k/\kappa}$, where the q-exponential form $e_q^z \equiv [1 + (1-q)z]^{\frac{1}{1-q}}$ optimizes the nonadditive entropy S_q (which, for $q \to 1$, recovers the Boltzmann-Gibbs entropy). We introduce and study here d-dimensional geographically-located networks which grow with preferential attachment involving Euclidean distances through $r_{ij}^{-\alpha}$ ($\alpha_A \geq 0$). Revealing the connection with q-statistics, we numerically verify (for $d = 1, 2, 3$ and 4) that the q-exponential degree distributions exhibit, for both q and k, universal dependences on the ratio α_A/d. Moreover, the $q = 1$ limit is rapidly achieved by increasing α_A/d to infinity.

$$p(r)\begin{cases} = 0 & \text{if} \quad 0 \leq r < 1 \\ \propto \dfrac{1}{r^{d+\alpha_G}} & \text{otherwise} \end{cases}$$

$$(\alpha_G > 0)$$

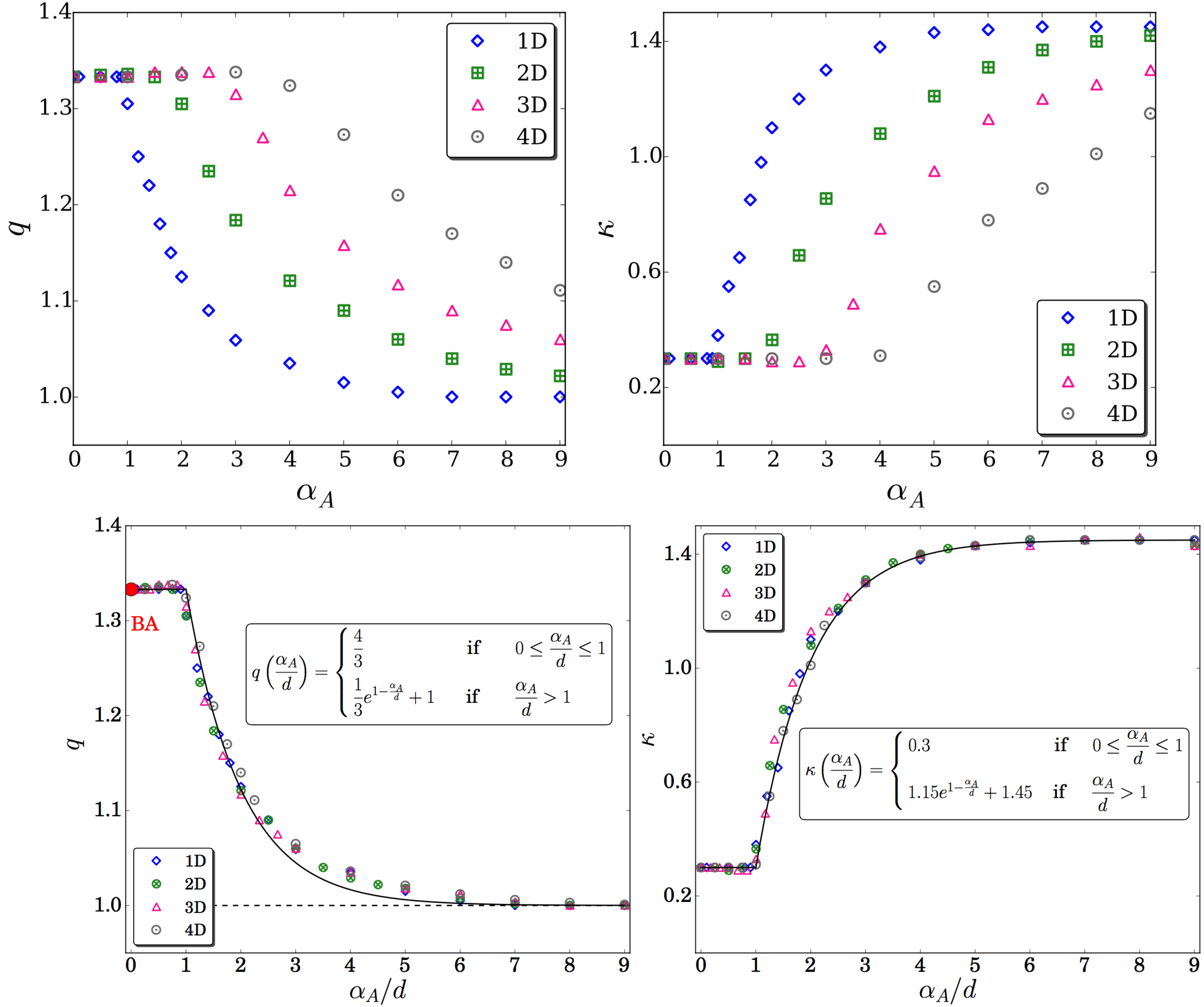
$$\Pi_{ij}(k_i) \propto \frac{k_i}{r_{ij}^{\alpha_A}}$$

$$(\alpha_A \geq 0)$$

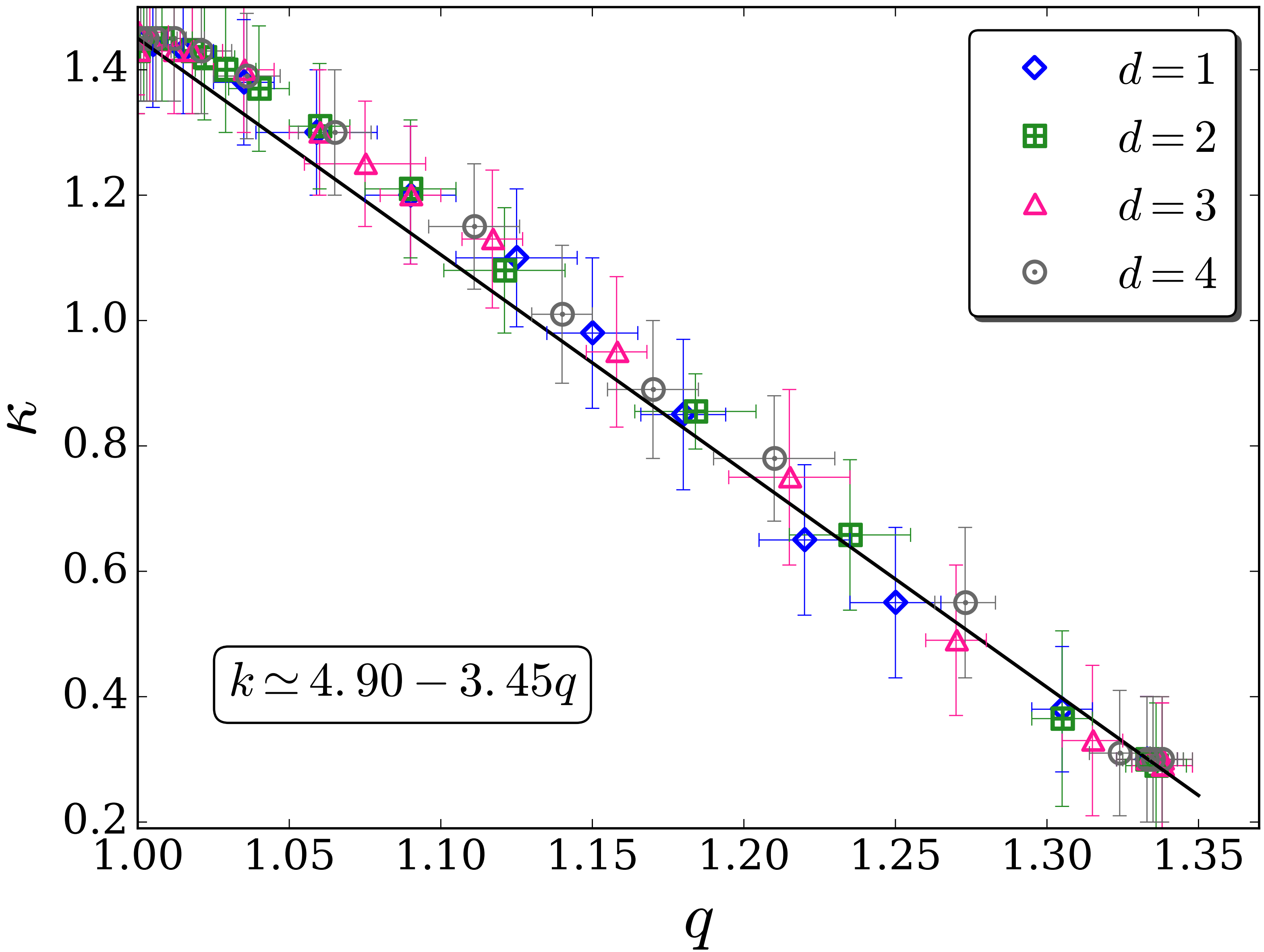


$$P(k) = P(0)\, e_q^{-k/\kappa} = \frac{P(0)}{\left[1+(q-1)k/\kappa\right]^{\frac{1}{q-1}}}$$

S.G.A. Brito, L.R. da Silva and C. T., Nature/Scientific Reports **6**, 27992 (2016)



S.G.A. Brito, L.R. da Silva and C. T.
Nature/Scientific Reports **6**, 27992 (2016)



S.G.A. Brito, L.R. da Silva and C. T., Nature/Scientific Reports **6**, 27992 (2016)

Computers and Mathematics with Applications 60 (2010) 2426–2432

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A novel automatic microcalcification detection technique using Tsallis entropy & a type II fuzzy index

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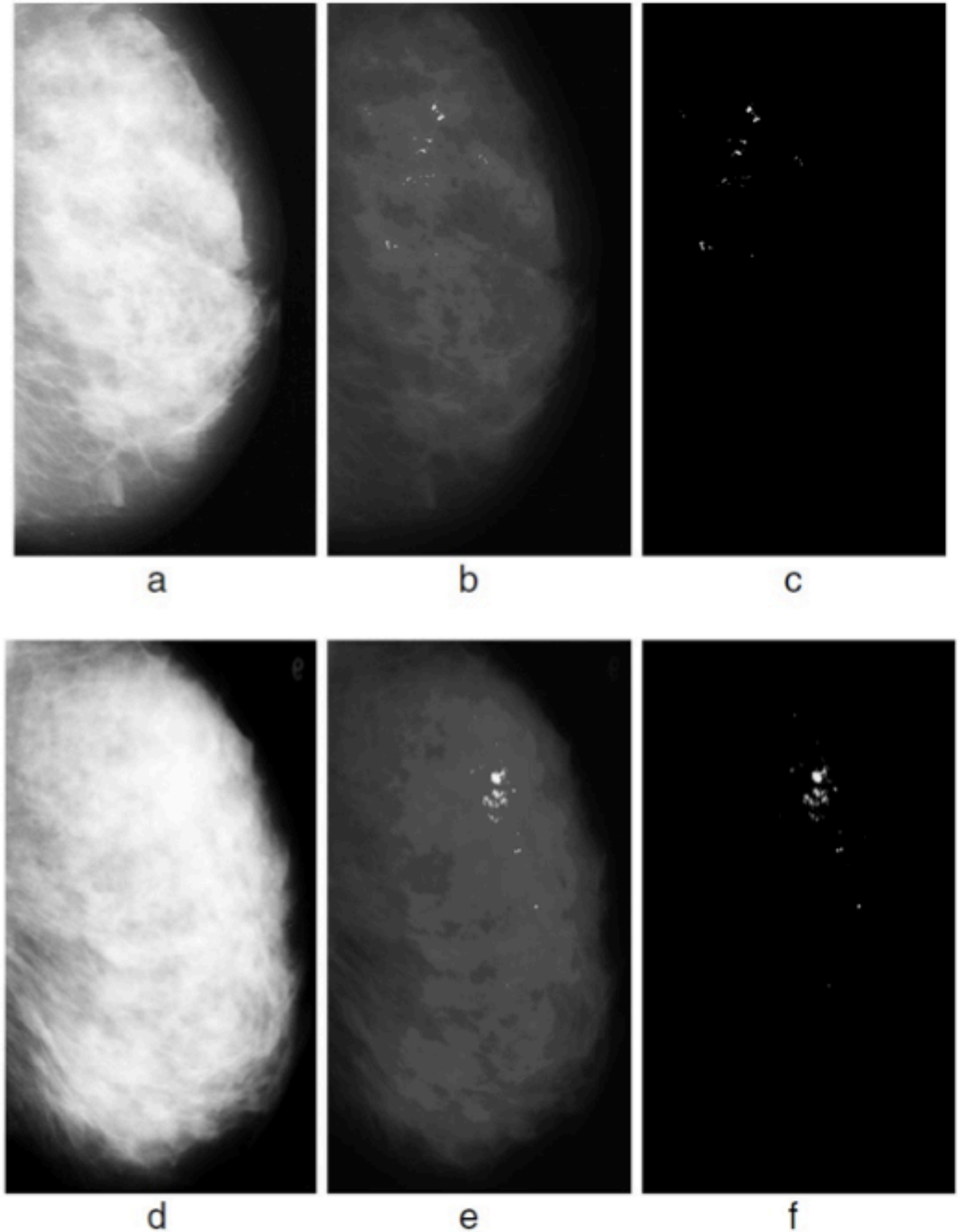
ABSTRACT

This article investigates a novel automatic microcalcification detection method using a type II fuzzy index. The thresholding is performed using the Tsallis entropy characterized by another parameter 'q', which depends on the non-extensiveness of a mammogram. In previous studies, 'q' was calculated using the histogram distribution, which can lead to erroneous results when pectoral muscles are included. In this study, we have used a type II fuzzy index to find the optimal value of 'q'. The proposed approach has been tested on several mammograms. The results suggest that the proposed Tsallis entropy approach outperforms the two-dimensional non-fuzzy approach and the conventional Shannon entropy partition approach. Moreover, our thresholding technique is completely automatic, unlike the methods of previous related works. Without Tsallis entropy enhancement, detection of microcalcifications is meager: 80.21% Tps (true positives) with 8.1 Fps (false positives), whereas upon introduction of the Tsallis entropy, the results surge to 96.55% Tps with 0.4 Fps.


a
b
c
d
e
f

Brazilian Journal of Medical and Biological Research (2010) 43: 77-84
ISSN 0100-879X

Brain tissue segmentation using q-entropy in multiple sclerosis magnetic resonance images

P.R.B. Diniz[1], L.O. Murta-Junior[3], D.G. Brum[1], D.B. de Araújo[3] and A.C. Santos[1,2]

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[3]Departamento de Física e Matemática, Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, Universidade de São Paulo, Ribeirão Preto, SP, Brasil

Abstract

The loss of brain volume has been used as a marker of tissue destruction and can be used as an index of the progression of neurodegenerative diseases, such as multiple sclerosis. In the present study, we tested a new method for tissue segmentation based on pixel intensity threshold using generalized Tsallis entropy to determine a statistical segmentation parameter for each single class of brain tissue. We compared the performance of this method using a range of different q parameters and found a different optimal q parameter for white matter, gray matter, and cerebrospinal fluid. Our results support the conclusion that the differences in structural correlations and scale invariant similarities present in each tissue class can be accessed by generalized Tsallis entropy, obtaining the intensity limits for these tissue class separations. In order to test this method, we used it for analysis of brain magnetic resonance images of 43 patients and 10 healthy controls matched for gender and age. The values found for the entropic q index were 0.2 for cerebrospinal fluid, 0.1 for white matter and 1.5 for gray matter. With this algorithm, we could detect an annual loss of 0.98% for the patients, in agreement with literature data. Thus, we can conclude that the entropy of Tsallis adds advantages to the process of automatic target segmentation of tissue classes, which had not been demonstrated previously.

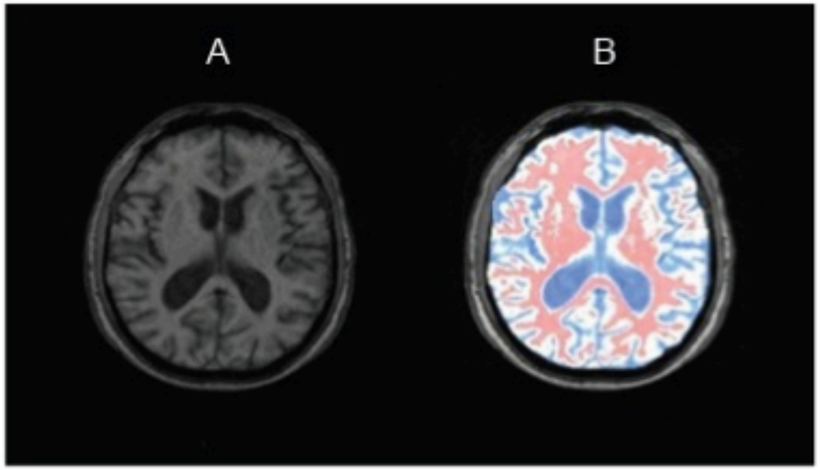

Figure 3. Maximum entropy segmentation example. *A*, Original image; *B*, image with the segmentation masks. Blue indicates cerebrospinal fluid, white indicates the gray matter, and red indicates the white matter.

The ideal q values for the segmentation of the classes are: CSF = 0.2, WM = 0.1, GM = 1.5, which have not been shown previously.

These characteristics allow its application to clinical routine.

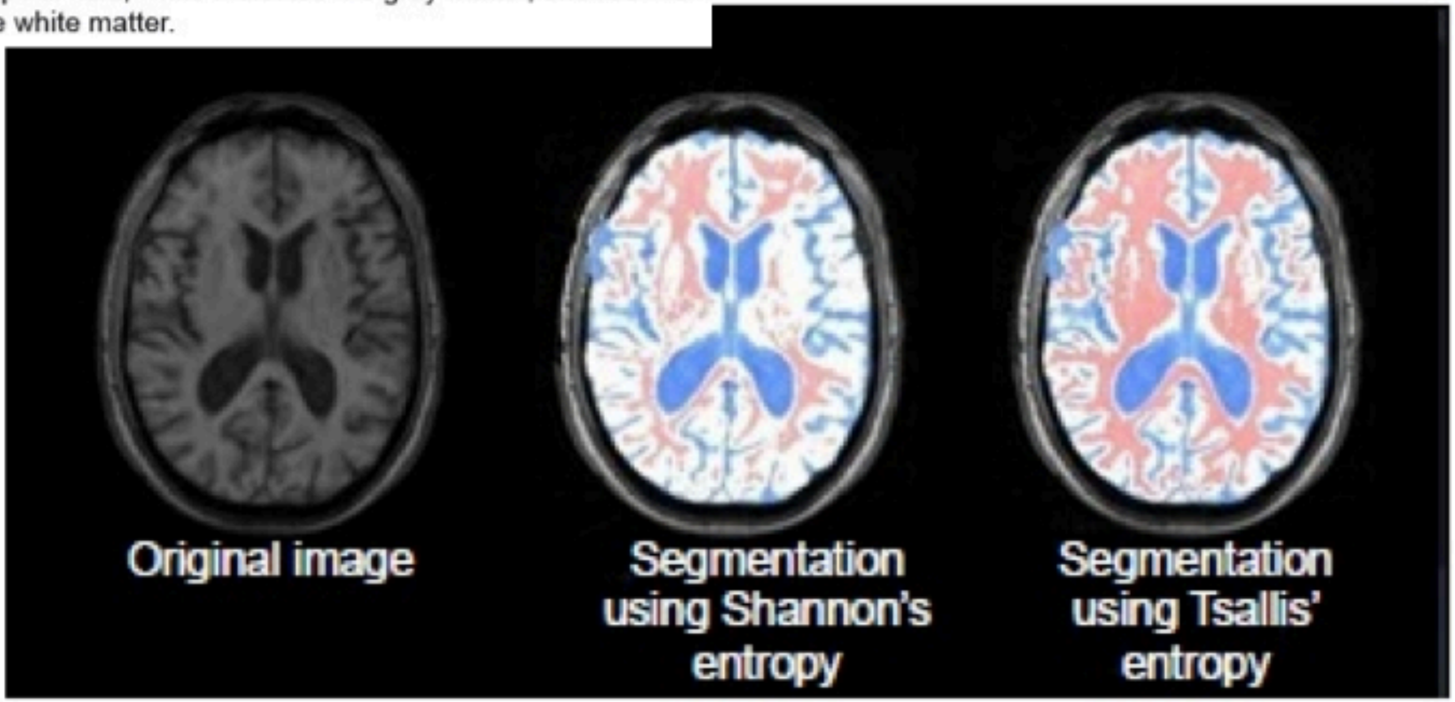

Figure 6. Segmentation using Shannon and Tsallis entropies.

PRB Diniz, LO Murta Jr., DG Brum, DB de Araujo and AC Santos
Braz. J. Med. Bio. Res. **43**, 77 (2010)

New combinational therapies for cancer using modern statistical mechanics

Jorge A. González[1]*, M. Acanda[2], Z. Akhtar[3], D. Andrews[4], J. I. Azqueta[2], E. Bass[4], A. Bellorín[5], J. Couso[8], Mónica A. García-Ñustes[9], Y. Infante[8], S. Jiménez[10], L. Lester[6], L. Maldonado[4], Juan F. Marín[9]†, L. Pineda[7], I. Rodríguez[7], C. C. Tamayo[2], D. Valdes[2], L. Vázquez[11].

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February 5, 2019

PRL **106,** 140601 (2011) PHYSICAL REVIEW LETTERS week ending 8 APRIL 2011

Nonlinear Relativistic and Quantum Equations with a Common Type of Solution

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(Received 25 October 2010; published 4 April 2011)

Generalizations of the three main equations of quantum physics, namely, the Schrödinger, Klein-Gordon, and Dirac equations, are proposed. Nonlinear terms, characterized by exponents depending on an index q, are considered in such a way that the standard linear equations are recovered in the limit $q \to 1$. Interestingly, these equations present a common, solitonlike, traveling solution, which is written in terms of the q-exponential function that naturally emerges within nonextensive statistical mechanics. In all cases, the well-known Einstein energy-momentum relation is preserved for arbitrary values of q.

See also:

R.N. Costa Filho, M.P. Almeida, G.A. Farias and J.S. Andrade, PRA **84**, 050102(R) (2011)

F.D. Nobre, M.A. Rego-Monteiro and C. T., EPL **97**, 41001 (2012)

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L.G.A. Alves, H.V. Ribeiro, M.A.F. Santos, R.S. Mendes and E. K. Lenzi, Physica A (2015)

A.R. Plastino and C. T., arxiv 1510.00415 [cond-mat.stat-mech]

A. Plastino and M.C. Rocca, Physica A (2016), in press

$q-$ generalized Schroedinger equation

(quantum non-relativistic spinless free particle)

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\left[\frac{\Phi(\vec{x},t)}{\Phi_0}\right]^{2-q} \qquad (q \in R)$$

Its exact solution is given by

$$\Phi(\vec{x},t) = \Phi_0\ e_q^{i(\vec{p}\ .\ \vec{x}-Et)/\hbar} = \Phi_0\ e_q^{i(\vec{k}\ .\ \vec{x}-\omega t)}$$

with

$$E = \frac{p^2}{2m} \quad \text{(Newtonian relation!)}$$

$$E = \hbar\omega \quad \text{(Planck relation!)}$$

$$p = \hbar k \quad \text{(de Broglie relation!)}$$

$\forall q$

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)

q-generalized Klein-Gordon equation:

(quantum relativistic spinless free particle: e.g., mesons π)

$$\nabla^2\Phi\left(\vec{x},t\right)=\frac{1}{c^2}\frac{\partial^2\Phi\left(\vec{x},t\right)}{\partial t^2}+q\frac{m^2c^2}{\hbar^2}\Phi\left(\vec{x},t\right)\left[\frac{\Phi\left(\vec{x},t\right)}{\Phi_0}\right]^{2(q-1)} \qquad (q\in R)$$

Its exact solution is given by

$$\Phi\left(\vec{x},t\right)=\Phi_0\ e_q^{i\left(\vec{p}\ .\ \vec{x}-Et\right)/\hbar}=\Phi_0\ e_q^{i\left(\vec{k}\ .\ \vec{x}-\omega\ t\right)}$$

with

$$E^2=p^2c^2+m^2c^4 \qquad (\forall q) \qquad \text{(Einstein relation!)}$$

Particular case: $m=0\ \Rightarrow\ q$-plane waves

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)

q-generalized Dirac equation:

(quantum relativistic spin $1/2$ matter and anti-matter free particles:

e.g., electron and positron)

$$i\hbar \frac{\partial\, \Phi\left(\vec{x},t\right)}{\partial t} + i\hbar c\left(\vec{\alpha}.\vec{\nabla}\right)\Phi\left(\vec{x},t\right) = \beta m c^2 A^{(q)}\left(\vec{x},t\right)\, \Phi\left(\vec{x},t\right) \qquad (q \in R)$$

with

$$\vec{\alpha} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}; \quad \beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4\times 4 \text{ matrices})$$

$$A_{i\,j}^{(q)}\left(\vec{x},t\right) \equiv \delta_{ij}\left[\frac{\Phi_j\left(\vec{x},t\right)}{a_j}\right]^{q-1} \qquad \left(A_{i\,j}^{(1)}\left(\vec{x},t\right) = \delta_{ij}\right) \qquad (4\times 4 \text{ matrix})$$

where $\{a_j\}$ are complex constants.

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)

Its exact solution is given by

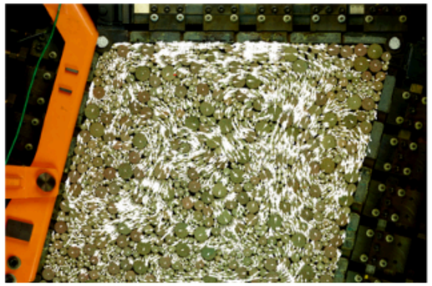
$$\Phi(\vec{x},t) \equiv \begin{pmatrix} \Phi_1(\vec{x},t) \\ \Phi_2(\vec{x},t) \\ \Phi_3(\vec{x},t) \\ \Phi_4(\vec{x},t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} e_q^{i(\vec{p}\,.\,\vec{x}-Et)/\hbar} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} e_q^{i(\vec{k}\,.\,\vec{x}-\omega\, t)}$$

with $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ being the same $\forall q$

hence

$$E^2 = p^2c^2 + m^2c^4 \quad (q \in R) \quad \text{(Einstein relation!)}$$

F.D. Nobre, M.A. Rego-Monteiro and C. T., Phys Rev Lett **106**, 140601 (2011)



EDITORS' SUGGESTION

Experimental Validation of a Nonextensive Scaling Law in Confined Granular Media

The velocity distribution of sheared granular media shows unexpected similarities with turbulent fluid flows.

Gaël Combe, Vincent Richefeu, Marta Stasiak, and Allbens P.F. Atman

Phys. Rev. Lett. **115**, 238301 (2015)

PRL **115,** 238301 (2015) PHYSICAL REVIEW LETTERS week ending 4 DECEMBER 2015

Experimental Validation of a Nonextensive Scaling Law in Confined Granular Media

Gaël Combe,* Vincent Richefeu, and Marta Stasiak
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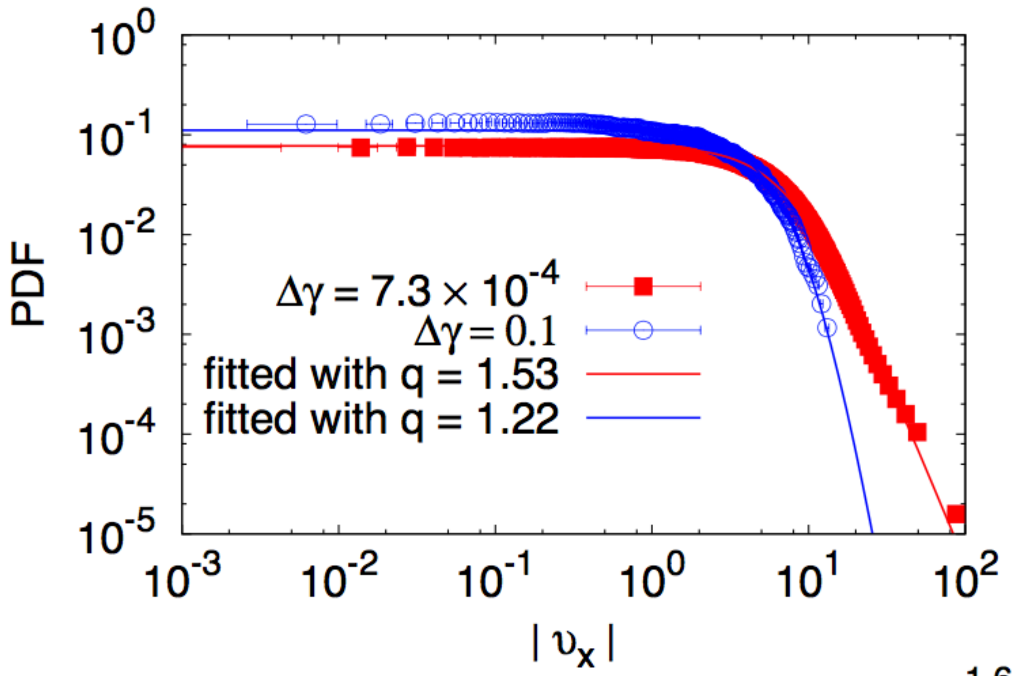
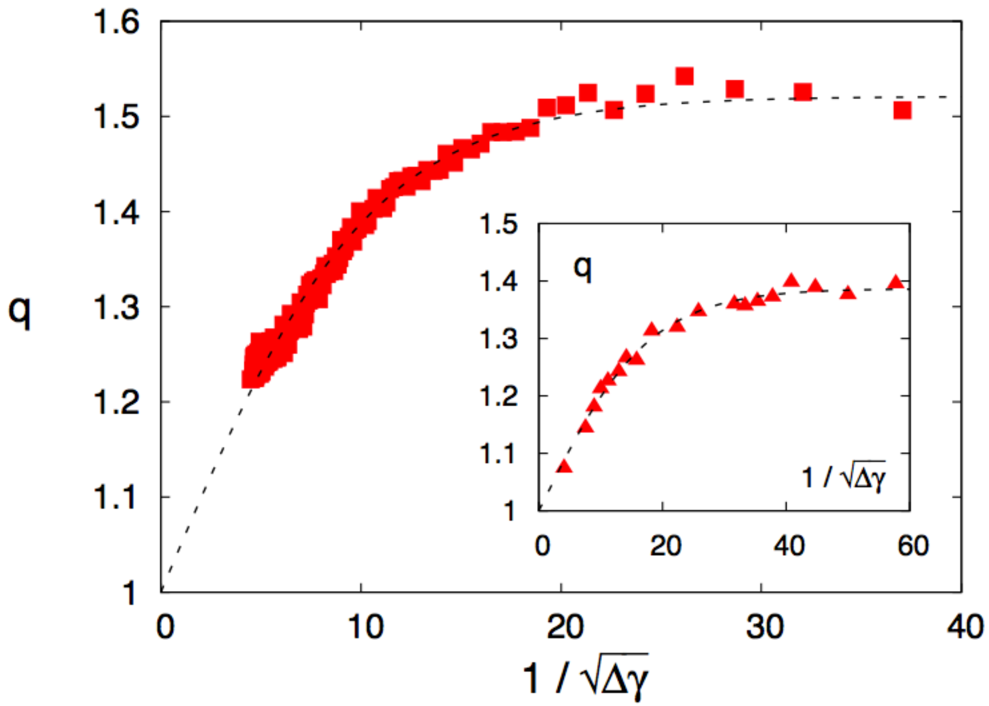
(Received 28 July 2015; published 1 December 2015)





Combe, Richefeu, Stasiak and Atman
PRL **115**, 238301 (2015)

$$\langle x^2 \rangle \propto t^{\alpha}$$

Combe, Richefeu, Stasiak and Atman
PRL **115**, 238301 (2015)

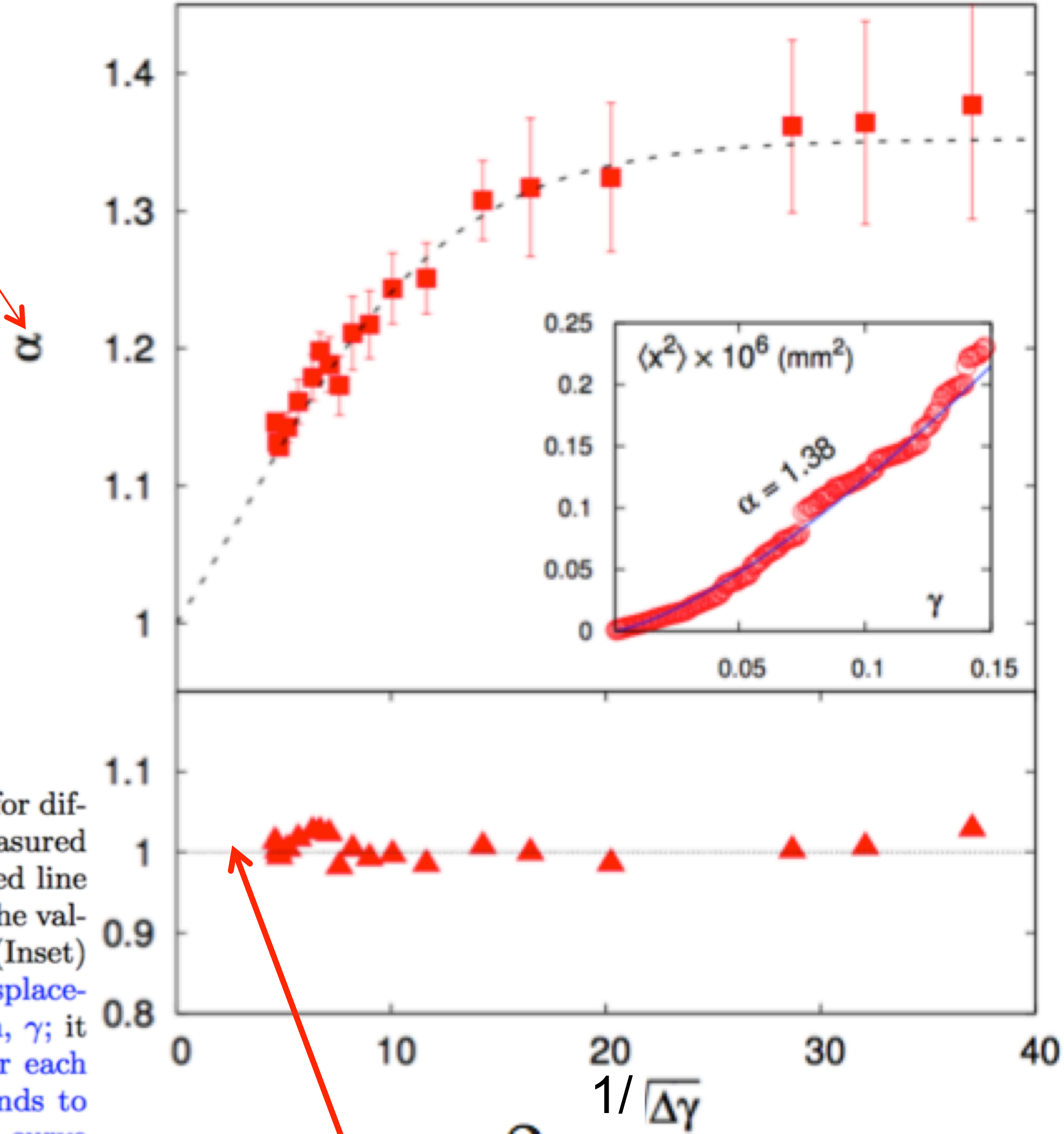


FIG. 4. Verification of the Tsallis-Bukman scaling law for different regimes of diffusion. (top) Evolution of the measured diffusion exponent α as a function of $1/\sqrt{\Delta\gamma}$ the dashed line is a direct application of the scaling law from the fit of the values shown in Fig. 3, $\alpha(1/\sqrt{\Delta\gamma}) = 2/[3 - q(1/\sqrt{\Delta\gamma})]$. (Inset) a typical diffusion curve showing the mean square displacement fluctuations, $\langle x^2 \rangle$, in function of the shear strain, γ; it allows the assessment of the diffusion exponent, α, for each strain window tested. In the case shown, it corresponds to the smallest strain window, the rightmost point in the curve at the main panel. Note that for a constant strain rate, γ is proportional to time. (Bottom) Measure of the deviation of the data relative to the scaling law prediction, as a function of $1/\sqrt{\Delta\gamma}$, showing an agreement on the order of $\pm 2\%$.

$$\alpha = \frac{2}{3-q}$$

CT and DJ Bukman, PRE **54** (1996) R2197

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SCIENTIFIC REPORTS

OPEN

Generalized statistical mechanics of cosmic rays: Application to positron-electron spectral indices

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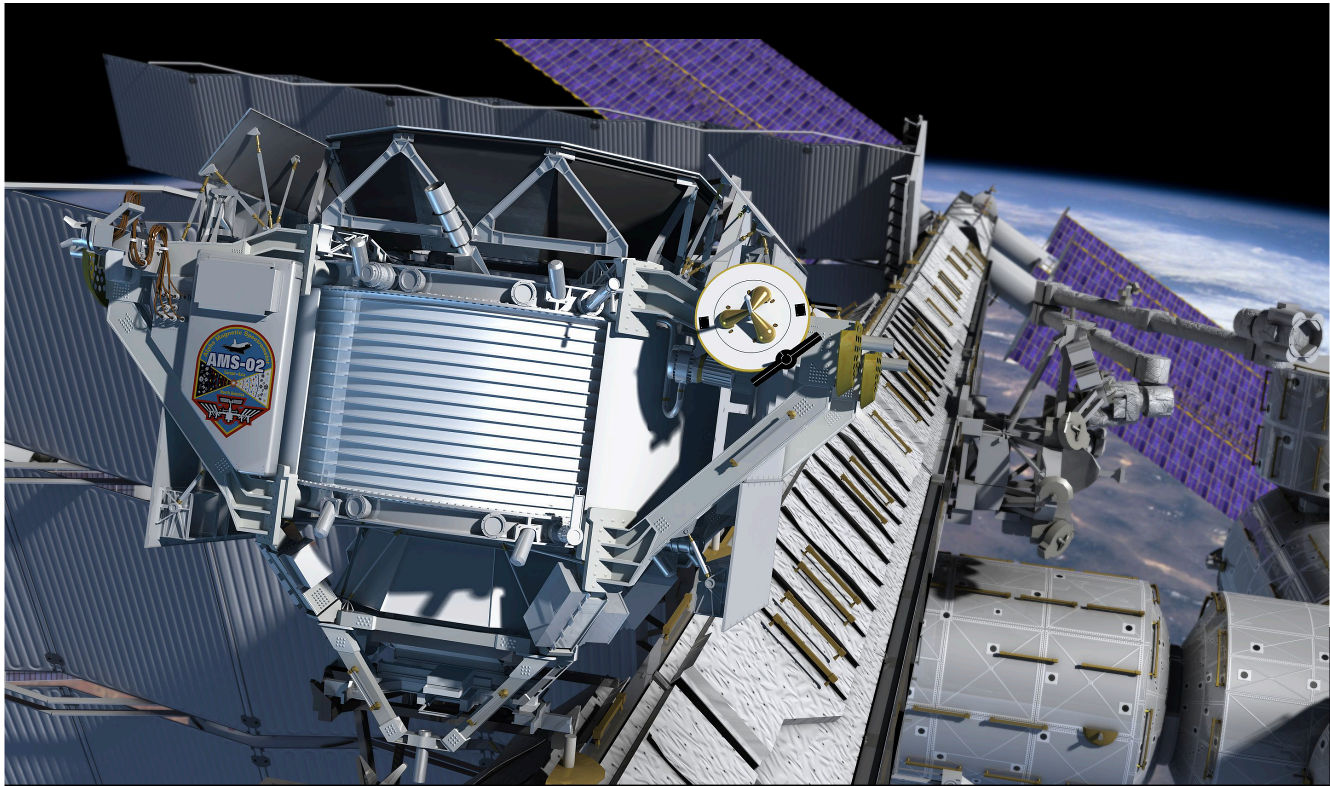
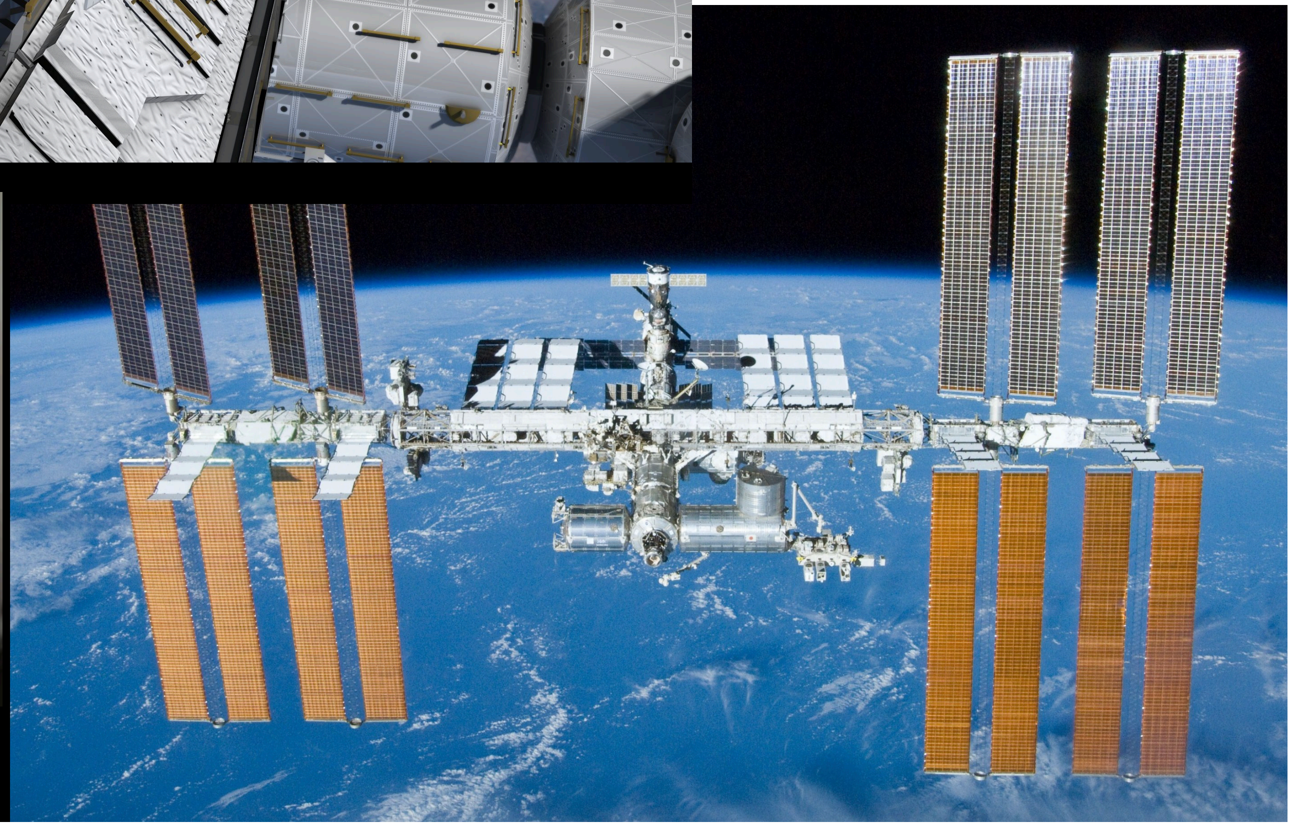
G. Cigdem Yalcin[1] & Christian Beck[2]

Cosmic ray energy spectra exhibit power law distributions over many orders of magnitude that are very well described by the predictions of q-generalized statistical mechanics, based on a q-generalized Hagedorn theory for transverse momentum spectra and hard QCD scattering processes. QCD at largest center of mass energies predicts the entropic index to be $q = \frac{13}{11}$. Here we show that the escort duality of the nonextensive thermodynamic formalism predicts an energy split of effective temperature given by $\Delta kT = \pm\frac{1}{10}kT_H \approx \pm 18$ MeV, where T_H is the Hagedorn temperature. We carefully analyse the measured data of the AMS-02 collaboration and provide evidence that the predicted temperature split is indeed observed, leading to a different energy dependence of the e^+ and e^- spectral indices. We also observe a distinguished energy scale $E^* \approx 50$ GeV where the e^+ and e^- spectral indices differ the most. Linear combinations of the escort and non-escort q-generalized canonical distributions yield excellent agreement with the measured AMS-02 data in the entire energy range.



Samuel Chao Chung Ting



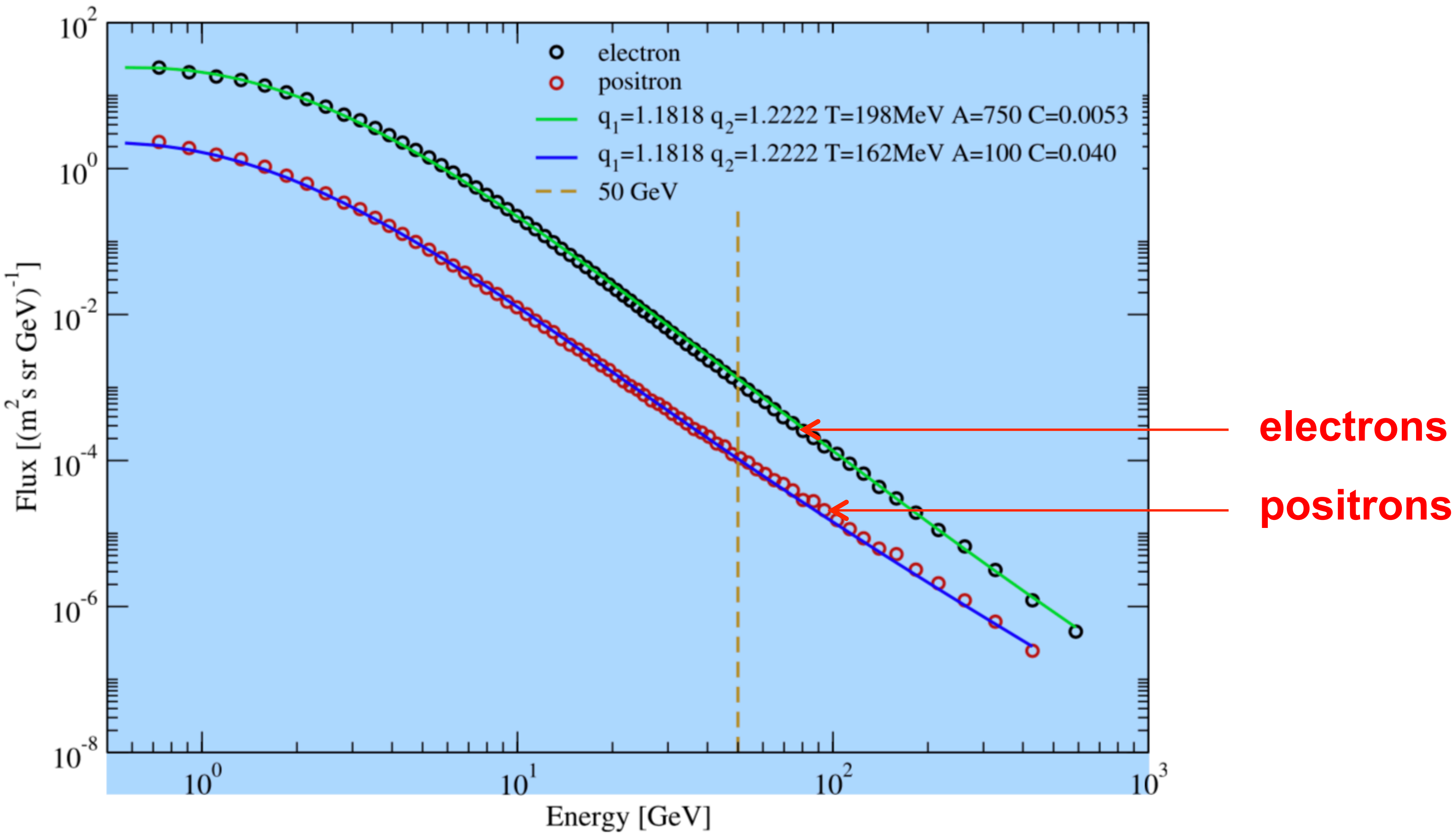


Figure 3. The measured AMS-02 data are very well fitted by linear combination of escort and n distributions (solid lines).

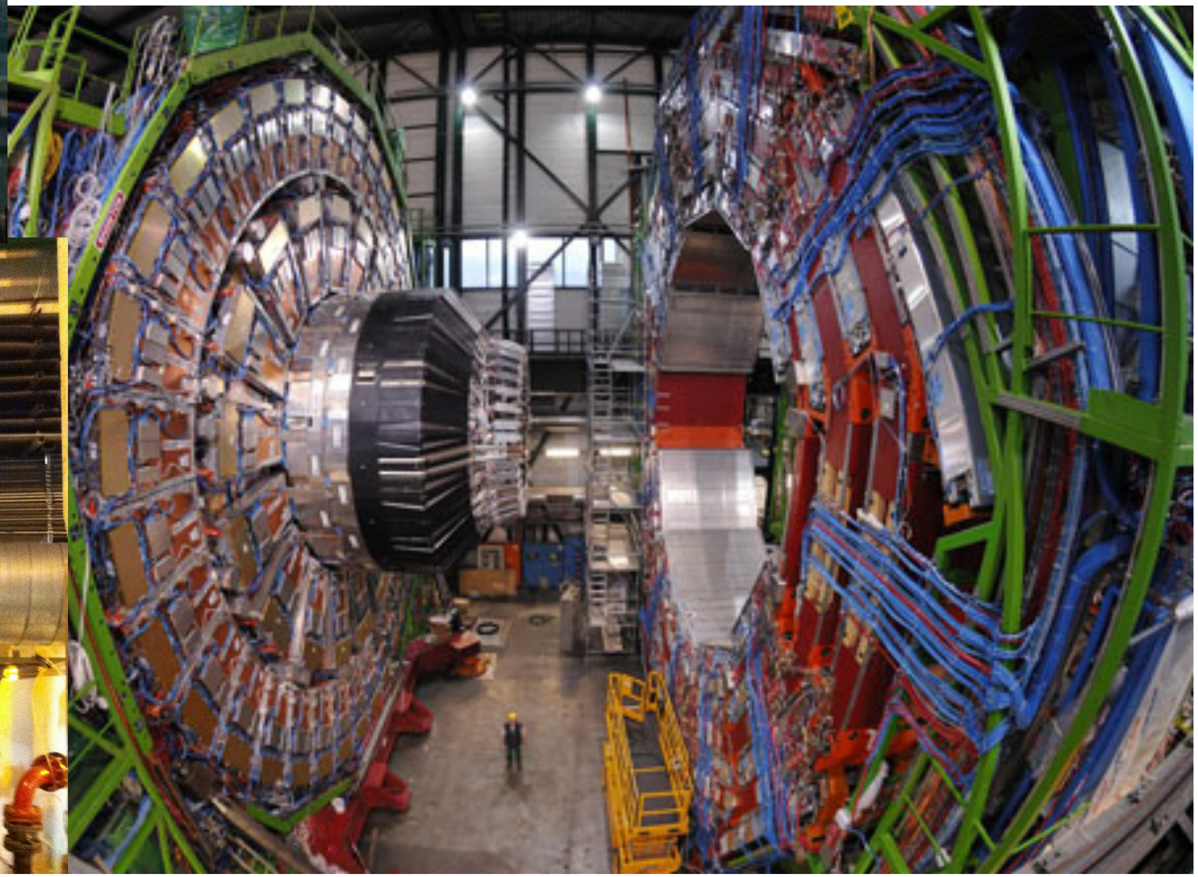
$$\text{with } q_1 = 13/11 = 1.1818...$$

$$\text{and } q_2 = \frac{1}{2-q} = 11/9 = 1.2222...$$

LHC (Large Hadron Collider)

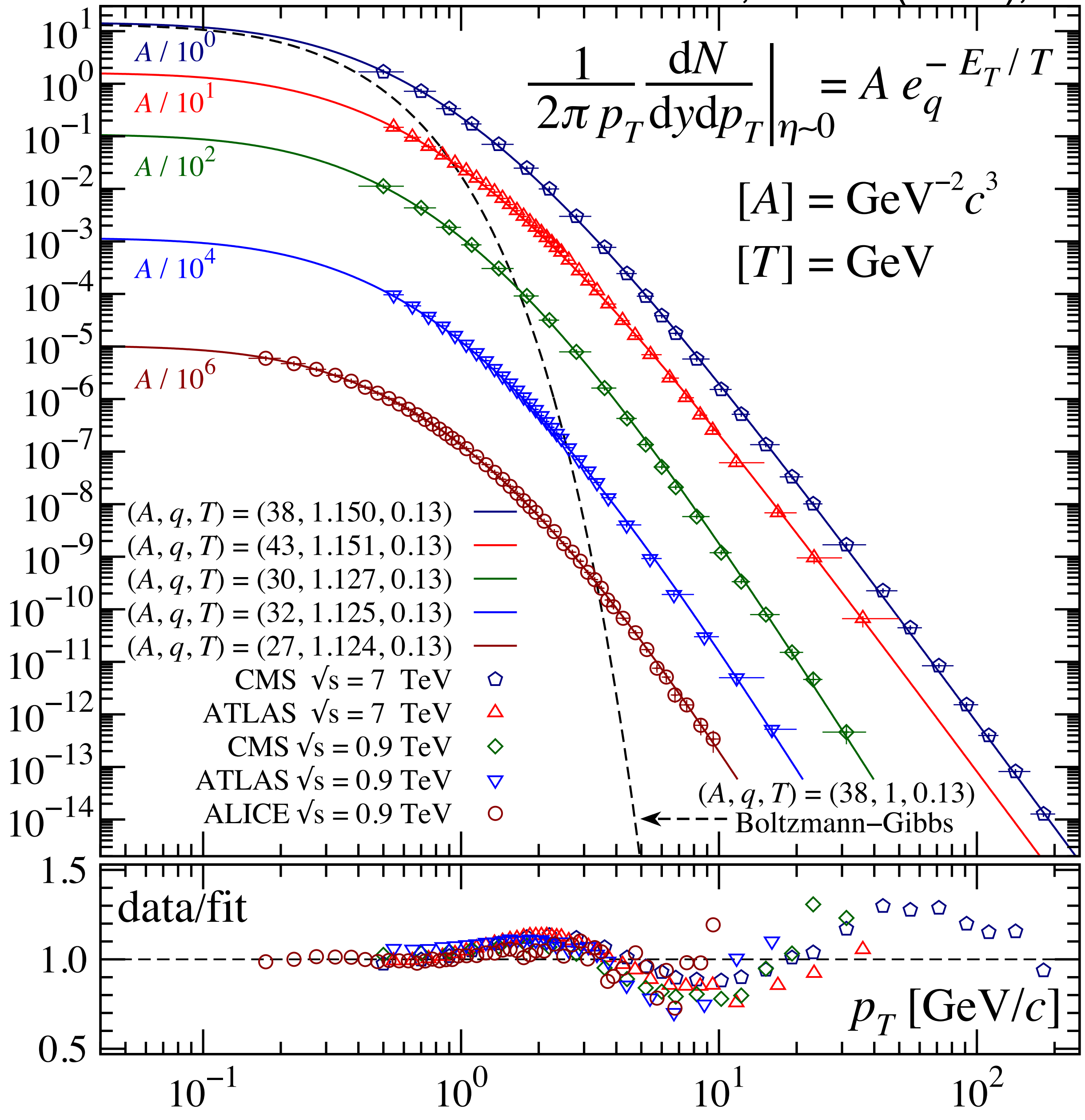
CMS, ALICE, ATLAS and LHCb detectors

~ 4000 scientists/engineers from ~ 200 institutions of ~ 50 countries

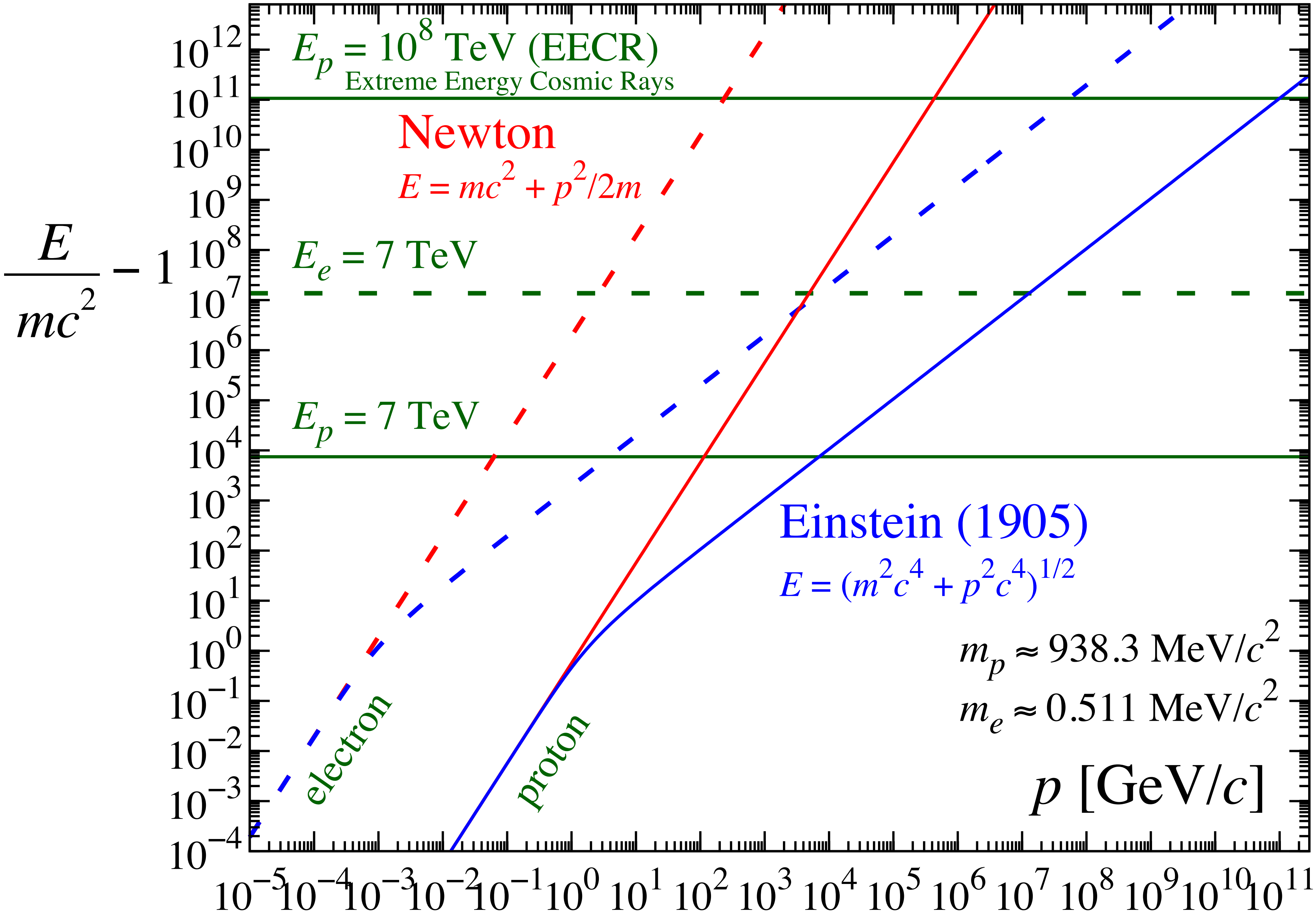


SIMPLE APPROACH: TWO-DIMENSIONAL SINGLE RELATIVISTIC FREE PARTICLE

C.Y. Wong, G. Wilk, L.J.L. Cirto and C. T.,
EPJ Web of Conferences **90**, 04002 (2015), and PRD **91**, 114027 (2015)



$$E_T = \sqrt{m^2c^4 + p_T^2c^2}$$



$E_p = 10^8$ TeV (EECR)
Extreme Energy Cosmic Rays
Newton
$E = mc^2 + p^2/2m$
$E_e = 7$ TeV
$E_p = 7$ TeV
Einstein (1905)
$E = (m^2c^4 + p^2c^4)^{1/2}$
$m_p \approx 938.3$ MeV/c^2
$m_e \approx 0.511$ MeV/c^2
electron
proton
p [GeV/c]
$\frac{E}{mc^2} - 1$

Entropy **2015**, *17*, 384-400; doi:10.3390/e17010384

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Article

Tsallis Distribution Decorated with Log-Periodic Oscillation

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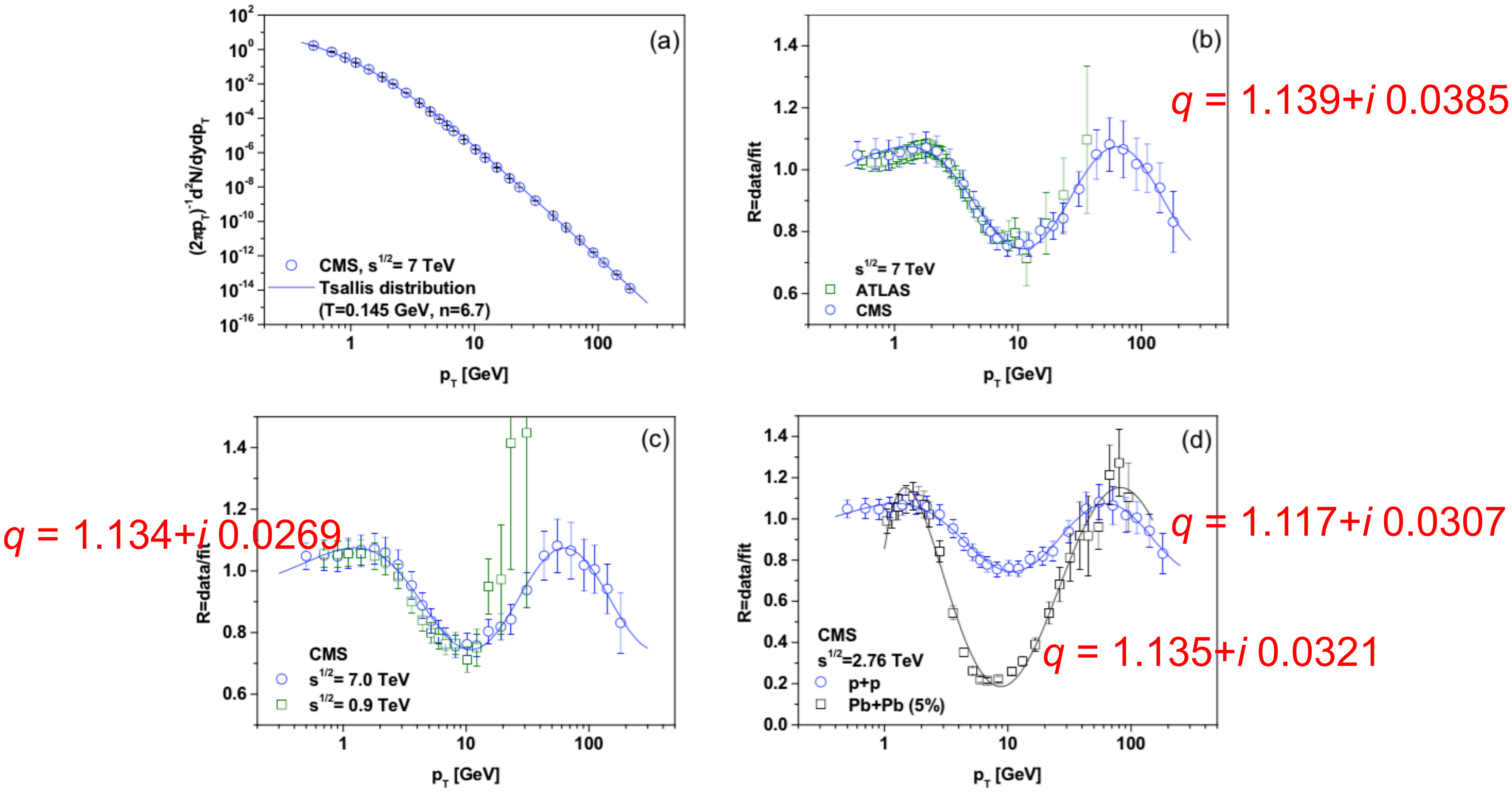

Figure 1. Examples of log-periodic oscillations. (**a**) dN/dp_T for the highest energy 7 TeV; the Tsallis behavior is evident. Only data from CMS experiment are shown [12]; others behave essentially in an identical manner. (**b**) Log-periodic oscillations showing up in different experimental data, like CMS [12] or ATLAS[15], taken at 7 TeV. (**c**) Results from CMS [12] for different energies. (**d**) Results for different systems ($p+p$ collisions compared with $Pb+Pb$ taken for 5% centrality [54]. Results from ALICE[55] are very similar. Fits for $p+p$ collision at 7, 2.76 and 0.9 TeV are performed with $q = 1.139 + i \cdot 0.0385$, $1.134 + i \cdot 0.0269$ and $1.117 + i \cdot 0.0307$, respectively. The fit for central $Pb+Pb$ collisions at 2.76 TeV is done with $q = 1.135 + i \cdot 0.0321$. See the text for more details.

Grazie!