Some applications of time delay systems

Bootan Rahman

University of Kurdistan Hewlêr

bootan.rahman@ukh.edu.krd

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University of KURDISTAN Hewlêr

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Outline

1. Types and effects of time delays
2. Applications: Shower problem, Population dynamics, Social science
3. STN-GP network with three delays

Types and effects of time delays

- Types of time delays
  - Discrete time delay
  - Distributed time delay

Figure: Janus the god of transitions

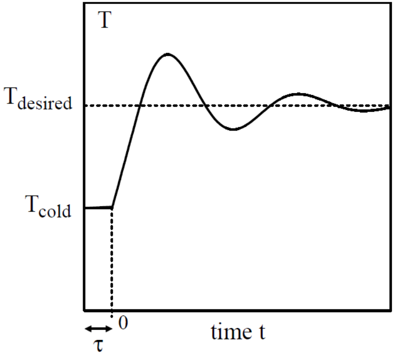
- Effects of time delay coupling
  - Amplitude death
  - Oscillation death
  - Chimera state
  - Synchronization (isochronal, anti-phase, and splay-phase synchronous)

The hot shower problem

$$\frac{dT(t)}{dt} = -a(T(t-\tau) - T_d)$$

$$a = T_d = \tau = 1$$

$$T = 0.5 \quad (-\tau < t \leq 0)$$



Delay: time needed for hot water to come in

T. Erneux, Applied delay differential equations. Vol. 3. Springer, 2009.

Population of Lemmings

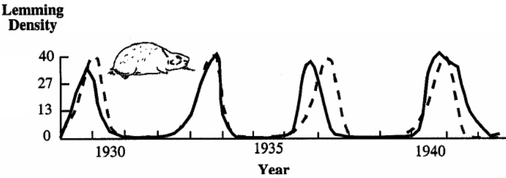
- Density of lemmings (number of individuals per hectare) in the Churchill area in Canada

$$\frac{dN}{dt} = rN\left(1 - \frac{N(t-\tau)}{K}\right)$$

r = 1/(0.3 years),

τ = 0.72 years

broken line: solution of the delayed logistic equation



T.J. Case, An Ilustrated Guide to Theoretical Ecology, Oxford University Press, Oxford, 2000.

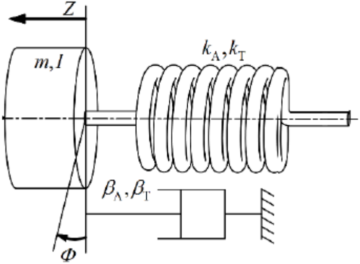
Deep Hole Drilling System

Dynamics on stick-slip vibrations of deep hole drilling with time delay

$$m\ddot{z}(t) + \beta_A \dot{z}(t) + k_A z(t) = -\zeta s a N[z(t) - z_\tau]$$

$$I\ddot{\phi}(t) + \beta_T \dot{\phi}(t) + k_T \phi(t) = -\frac{1}{2} s a^2 N[z(t) - z_\tau],$$

where $z_\tau = z(t - t_n)$.



- z and ϕ are the disturbed axial displacement and angular displacement under stable drilling of the drilling system. m mass of bit, I rotary of bit, β_A and β_T axial and torsion damping, respectively. κ_A and κ_T axial and torsion stiffness. Time delay required for the bit to rotate an angle $2\pi/N$ to its current position.

J. Huang et al. Bifurcation and stability analyses on stick-slip vibrations of deep hole drilling with state-dependent delay. Applied Sciences, 8(5), 2018.

Political system

The model of a multiparty political system is given by the following system of coupled delay differential equations:

$$
\begin{aligned}
\tfrac{dx_1}{dt} &= a_1x_1 - d_1x_1 + \frac{\beta_1 x_1^2 x_3(t-\tau)}{x_1+x_2} + d_2 p_{21} x_2, \\
\tfrac{dx_2}{dt} &= a_2x_2 - d_2x_1 + \frac{\beta_2 x_2^2 x_3(t-\tau)}{x_1+x_2} + d_2 p_{12} x_1, \\
\tfrac{dx_3}{dt} &= a_3x_3 - d_3x_3 - \frac{\beta_1 x_1^2 x_3}{x_1+x_2} - \frac{\beta_2 x_2^2 x_3}{x_1+x_2} + d_1 p_{13} x_1 + d_2 p_{23} x_2, \\
\tfrac{dx_4}{dt} &= \frac{\beta_1 x_1^2 x_3}{x_1+x_2} - \frac{\beta_2 x_2^2 x_3}{x_1+x_2} - \frac{\beta_1 x_1^2 x_3(t-\tau)}{x_1+x_2} - \frac{\beta_2 x_2^2 x_3(t-\tau)}{x_1+x_2}
\end{aligned}
$$

- x_i the number of ruling (R), opposition(O), third party(T), non-above parties (N) i, $i = 1, ..., 4$. a_i rates of members enter into the R, O, and T. d_i members rate of the R, O, and T entering into other parties. $x_3(t-\tau)$ rates of T who leave the party at time $t-\tau$ and entering into new party at time t. P_{ij} are the probabilities of successful transition. β_i are the conversion rates.

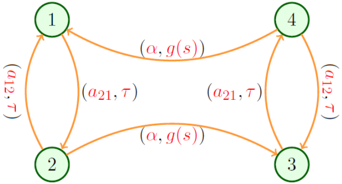
Q. J. Khan, "Hopf bifurcation in multiparty political systems with time delay in switching." Applied Mathematics Letters, 43-52, 2000.

Neural systems with discrete and distributed time delays

Consider a coupled two sub-networks with time delays

$$
\begin{aligned}
\dot{u}_1(t) &= -u_1(t) + a_{12} f(u_2(t-\tau)) + \alpha \int_0^\infty g(s) f(u_4(t-s)) ds, \\
\dot{u}_2(t) &= -u_2(t) + a_{21} f(u_1(t-\tau)), \\
\dot{u}_3(t) &= -u_3(t) + a_{12} f(u_4(t-\tau)) + \alpha \int_0^\infty g(s) f(u_2(t-s)) ds, \\
\dot{u}_4(t) &= -u_4(t) + a_{21} f(u_3(t-\tau)),
\end{aligned}
$$

- u_i are voltages of neurons i, $i = 1, ..., 4$.
- a_{12} and a_{21} are the strength of connections.
- τ is discrete time delay.
- α is long-rang coupling strength.
- Distributed time delays between sub-networks.



B. Rahman, B.K. Blyuss, and Y. N. Kyrychko. "Dynamics of neural systems with discrete and distributed time delays." SIAM Journal on Applied Dynamical Systems, 2069-2095, 2015.

A mosquito delayed mathematical model

A mathematical model to break the life cycle of mosquito

$$\begin{aligned}
\dot{x}_1(t) &= bN - (\eta + \mu)x_1(t) + \rho x_4(t) \\
\dot{x}_2(t) &= \eta x_1(t) - (\gamma + \mu)x_2(t) \\
\dot{x}_3(t) &= \gamma x_2(t) - \nu x_3(t - \tau) - \mu x_3(t) \\
\dot{x}_4(t) &= \nu x_3(t - \tau) - (\rho + \mu)x_4(t)
\end{aligned}$$

- x_i Adult mosquitoes, Eggs, Larva, and Pupa at time t i, $i = 1, ..., 4$ respectively.
- b and μ birth and death rate respectively.
- η rate adult mosquitoes oviposit.
- γ rate the eggs hatch.
- ν rate larva develops to pupa.
- ρ rate pupa develops to adult mosquitoes.

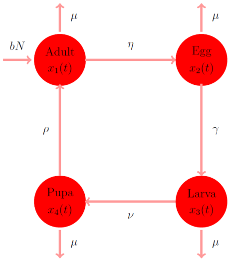


Figure 1: A flow chart of the life cycle of a mosquito

M. Yau and B. Rahman, "A Delayed Mathematical Model to break the life cycle of Anopheles Mosquito." Ratio Mathematica, 79-92, 2016.

Neuroscience

Vedio: ARQHIE

Wilson-Cowan Model

BIOPHYSICAL JOURNAL VOLUME 12 1972

EXCITATORY AND INHIBITORY INTERACTIONS IN LOCALIZED POPULATIONS OF MODEL NEURONS

HUGH R. WILSON *and* JACK D. COWAN

From the Department of Theoretical Biology, The University of Chicago, Chicago, Illinois 60637

ABSTRACT Coupled nonlinear differential equations are derived for the dynamics of spatially localized populations containing both excitatory and inhibitory model neurons. Phase plane methods and numerical solutions are then used to investigate population responses to various types of stimuli. The results obtained show simple and multiple hysteresis phenomena and limit cycle activity. The latter is particularly interesting since the frequency of the limit cycle oscillation is found to be a monotonic function of stimulus intensity. Finally, it is proved that the existence of limit cycle

Developments of Wilson-Cowan Model

EJN EUROPEAN JOURNAL OF NEUROSCIENCE

European Journal of Neuroscience, Vol. 36, pp. 2229–2239, 2012 doi:10.1111/j.1460-9568.2012.08105.x

Improved conditions for the generation of beta oscillations in the subthalamic nucleus–globus pallidus network

Alex Pavlides,[1] S. John Hogan[2] and Rafal Bogacz[3]

[1]Bristol Centre for Complexity Sciences, Department of Computer Science, University of Bristol, Bristol BS8 1UB, UK
[2]Department of Engineering Mathematics, University of Bristol, Bristol, UK
[3]Department of Computer Science, University of Bristol, Bristol, UK

Keywords: beta oscillations, globus pallidus, Parkinson's disease, subthalamic nucleus

Abstract

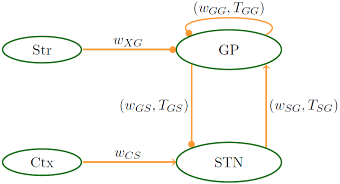
A key pathology in the development of Parkinson's disease is the occurence of persistent beta oscillations, which are correlated with difficulty in movement initiation. We investigated the network model composed of the subthalamic nucleus (STN) and globus pallidus (GP) developed by A. Nevado Holgado *et al.* [(2010) *Journal of Neuroscience*, **30**, 12340–12352], who identified the conditions under which this circuit could generate beta oscillations. Our work extended their analysis by deriving improved analytic stability conditions

STN-GP network with three delays

Consider a STN-GP model introduced by Pavlides *et al.* (2012),

$$\begin{aligned} \tau_S S'(t) &= F_S(-w_{GS} G(t - T_{GS}) + w_{CS} Ctx) - S(t), \\ \tau_G G'(t) &= F_G(w_{SG} S(t - T_{SG}) - w_{GG} G(t - T_{GG}) - w_{XG} Str) - G(t), \end{aligned}$$

- $S(t)$ and $G(t)$ are the firing rates.
- T_{GS}, T_{SG}, $T_{GG} \geq 0$ are time delays.
- The synaptic weights w_{GS}, w_{CS}, w_{SG}, w_{GG}, and w_{XG} are all non-negative constants.
- τ_S and τ_G are the membrane time constants of the neurons.
- Ctx and Str are the constant inputs from cortex and striatum.
- $F_S(\cdot)$ and $F_G(\cdot)$ are the sigmoid activation function.



$$\begin{aligned} F_S(\cdot) &= \frac{M_S}{1 + \left(\frac{M_S - B_S}{B_S}\right) e^{\frac{-4(.)}{M_S}}} \\ F_G(\cdot) &= \frac{M_G}{1 + \left(\frac{M_G - B_G}{B_G}\right) e^{\frac{-4(.)}{M_G}}} \end{aligned}$$

Previous analysis

- The membrane time constants are exactly the same.
- The transmission delays in the neural populations are taken to be equal.
- nonlinear activation functions are replaced by linear functions.

Our analysis

- The membrane time constants are taken to be different.
- The three time delays in the connections between the excitatory and inhibitory populations of neurons are taken to be different.
- We consider a general nonlinear class of activation functions.

B. Rahman, Y.N. Kyrychko, K.B. Blyuss, and J.S. Hogan, Dynamics of a subthalamic nucleus-globus palidus network with three delays, IFAC-PapersOnLine, 294-299, 2018.

Stability analysis: single delay

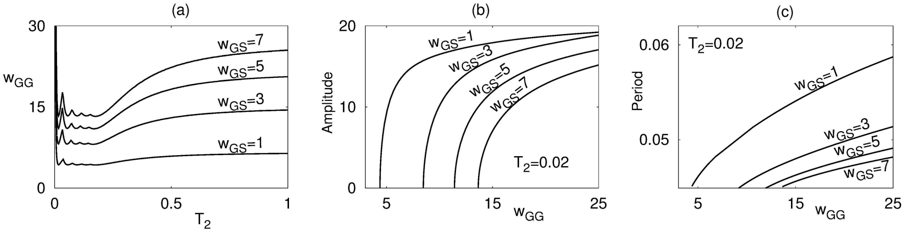


Figure: (a) Stability of the non-trivial steady state, for $T_1 = 0$ and $T_2 > 0$. (b) Amplitude and (c) period of the periodic solutions.

Stability analysis: single delay

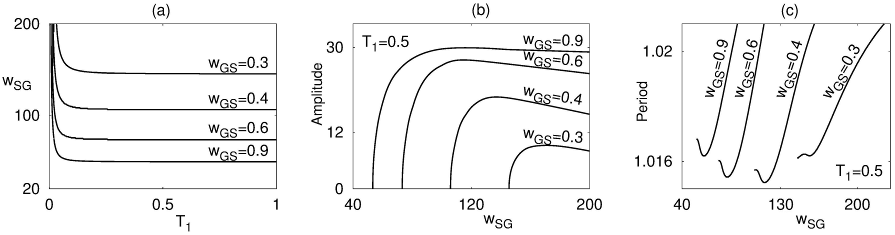


Figure: (a) Stability of the non-trivial steady state, for $T_1 > 0$ and $T_2 = 0$. (b) Amplitude and (c) period of the periodic solutions.

Stability analysis: two time delays

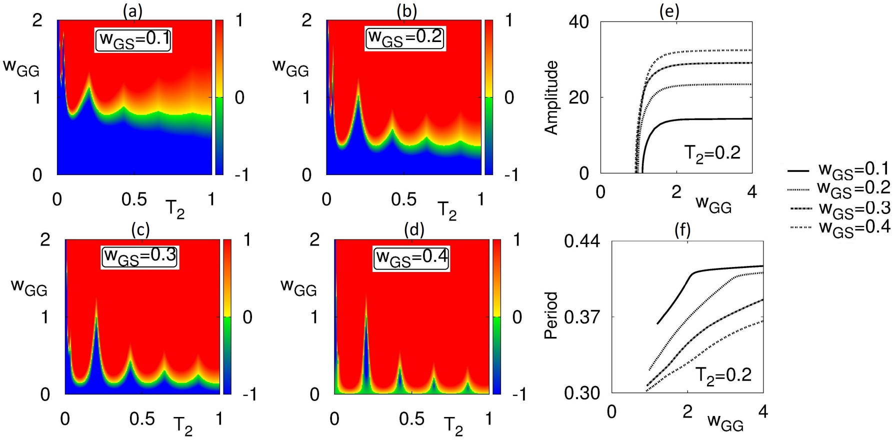
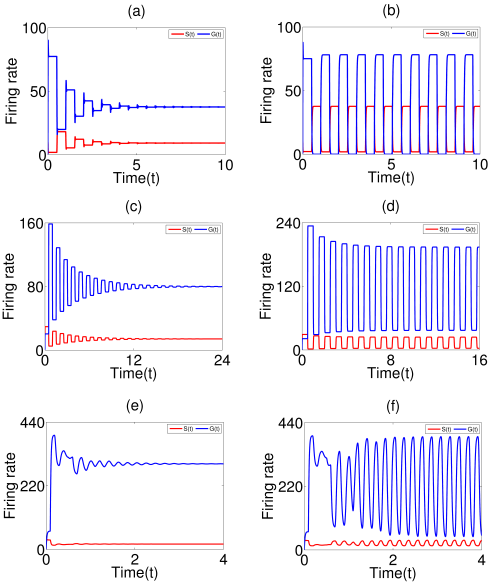


Figure: (a)-(d) Stability of the non-trivial steady state, for $T_1 > 0$ and $T_2 > 0$. (e) Amplitude and (f) period of the periodic solutions.

Numerical simulation



Thank you!