Some applications of time delay systems

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Outline

1. Types and effects of time delays

2. Applications: Shower problem, Population dynamics, Social science

3. STN-GP network with three delays
Types and effects of time delays

- Types of time delays
  - Discrete time delay
  - Distributed time delay

- Effects of time delay coupling
  - Amplitude death
  - Oscillation death
  - Chimera state
  - Synchronization (isochronal, anti-phase, and splay-phase synchronous)

Figure: Janus the god of transitions
The hot shower problem

\[
\frac{dT(t)}{dt} = -\alpha (T(t-\tau) - T_d)
\]

\[
\alpha = T_d = \tau = 1
\]

\[
T = 0.5 \quad (-\tau < t \leq 0)
\]

**Delay**: time needed for hot water to come in

Population of Lemmings

- Density of lemmings (number of individuals per hectare) in the Churchill area in Canada

\[
\frac{dN}{dt} = rN \left(1 - \frac{N(t - \tau)}{K}\right)
\]

\( r = \frac{1}{0.3 \text{ years}} \), \( \tau = 0.72 \text{ years} \)

broken line: solution of the delayed logistic equation

Dynamics on stick-slip vibrations of deep hole drilling with time delay

\[
\begin{align*}
    m\ddot{z}(t) + \beta_A \dot{z}(t) + k_A z(t) &= -\zeta s a N [z(t) - z_\tau] \\
    I\ddot{\phi}(t) + \beta_T \dot{\phi}(t) + k_T \phi(t) &= -\frac{1}{2} s a^2 N [z(t) - z_\tau],
\end{align*}
\]

where \(z_\tau = z(t - t_n)\).

- \(z\) and \(\phi\) are the disturbed axial displacement and angular displacement under stable drilling of the drilling system. \(m\) mass of bit, \(I\) rotary of bit, \(\beta_A\) and \(\beta_T\) axial and torsion damping, respectively. \(\kappa_A\) and \(\kappa_T\) axial and torsion stiffness. Time delay required for the bit to rotate an angle \(2\pi/N\) to its current position.

The model of a multiparty political system is given by the following system of coupled delay differential equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= a_1 x_1 - d_1 x_1 + \frac{\beta_1 x_1^2 x_3(t - \tau)}{x_1 + x_2} + d_2 p_{21} x_2, \\
\frac{dx_2}{dt} &= a_2 x_2 - d_2 x_1 + \frac{\beta_2 x_2^2 x_3(t - \tau)}{x_1 + x_2} + d_2 p_{12} x_1, \\
\frac{dx_3}{dt} &= a_3 x_3 - d_3 x_3 - \frac{\beta_1 x_1^2 x_3}{x_1 + x_2} - \frac{\beta_2 x_2^2 x_3}{x_1 + x_2} + d_1 p_{13} x_1 + d_2 p_{23} x_2, \\
\frac{dx_4}{dt} &= \frac{\beta_1 x_1^2 x_3}{x_1 + x_2} - \frac{\beta_2 x_2^2 x_3}{x_1 + x_2} - \frac{\beta_1 x_1^2 x_3(t - \tau)}{x_1 + x_2} - \frac{\beta_2 x_2^2 x_3(t - \tau)}{x_1 + x_2}
\end{align*}
\]

- \(x_i\) the number of ruling (R), opposition(O), third party(T), non-above parties (N) \(i, i = 1, ..., 4\). \(a_i\) rates of members enter into the R, O, and T. \(d_i\) members rate of the R, O, and T entering into other parties. \(x_3(t - \tau)\) rates of T who leave the party at time \(t - \tau\) and entering into new party at time \(t\). \(P_{ij}\) are the probabilities of successful transition. \(\beta_i\) are the conversion rates.

Consider a coupled two sub-networks with time delays

\[
\begin{align*}
\dot{u}_1(t) &= -u_1(t) + a_{12}f(u_2(t - \tau)) + \alpha \int_{0}^{\infty} g(s)f(u_4(t - s))ds, \\
\dot{u}_2(t) &= -u_2(t) + a_{21}f(u_1(t - \tau)), \\
\dot{u}_3(t) &= -u_3(t) + a_{12}f(u_4(t - \tau)) + \alpha \int_{0}^{\infty} g(s)f(u_2(t - s))ds, \\
\dot{u}_4(t) &= -u_4(t) + a_{21}f(u_3(t - \tau)),
\end{align*}
\]

- \(u_i\) are voltages of neurons \(i, i = 1, \ldots, 4\).
- \(a_{12}\) and \(a_{21}\) are the strength of connections.
- \(\tau\) is discrete time delay.
- \(\alpha\) is long-rang coupling strength.

Distributed time delays between sub-networks.

A mosquito delayed mathematical model

A mathematical model to break the life cycle of mosquito

\[ \dot{x}_1(t) = bN - (\eta + \mu)x_1(t) + \rho x_4(t) \]
\[ \dot{x}_2(t) = \eta x_1(t) - (\gamma + \mu)x_2(t) \]
\[ \dot{x}_3(t) = \gamma x_2(t) - \nu x_3(t - \tau) - \mu x_3(t) \]
\[ \dot{x}_4(t) = \nu x_3(t - \tau) - (\rho + \mu)x_4(t) \]

- \( x_i \): Adult mosquitoes, Eggs, Larva, and Pupa at time \( t \), \( i = 1, \ldots, 4 \) respectively.
- \( b \) and \( \mu \): birth and death rate respectively.
- \( \eta \): rate adult mosquitoes oviposit.
- \( \gamma \): rate the eggs hatch.
- \( \nu \): rate larva develops to pupa.
- \( \rho \): rate pupa develops to adult mosquitoes.

Neuroscience

Video: ARQHIE
EXCITATORY AND INHIBITORY INTERACTIONS IN LOCALIZED POPULATIONS OF MODEL NEURONS

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ABSTRACT Coupled nonlinear differential equations are derived for the dynamics of spatially localized populations containing both excitatory and inhibitory model neurons. Phase plane methods and numerical solutions are then used to investigate population responses to various types of stimuli. The results obtained show simple and multiple hysteresis phenomena and limit cycle activity. The latter is particularly interesting since the frequency of the limit cycle oscillation is found to be a monotonic function of stimulus intensity. Finally, it is proved that the existence of limit cycle
Improved conditions for the generation of beta oscillations in the subthalamic nucleus–globus pallidus network

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Abstract

A key pathology in the development of Parkinson's disease is the occurrence of persistent beta oscillations, which are correlated with difficulty in movement initiation. We investigated the network model composed of the subthalamic nucleus (STN) and globus pallidus (GP) developed by A. Nevada Holgado et al. (2010) Journal of Neuroscience, 30, 12340–12352, who identified the conditions under which this circuit could generate beta oscillations. Our work extended their analysis by deriving improved analytic stability conditions.
STN-GP network with three delays

Consider a STN-GP model introduced by Pavlides et al. (2012),

\[
\tau_S S'(t) = F_S(-w_{GS} G(t-T_{GS}) + w_{CS} Ctx) - S(t), \\
\tau_G G'(t) = F_G(w_{SG} S(t-T_{SG}) - w_{GG} G(t-T_{GG}) - w_{XG} Str) - G(t),
\]

- \( S(t) \) and \( G(t) \) are the firing rates.
- \( T_{GS}, T_{SG}, T_{GG} \geq 0 \) are time delays.
- The synaptic weights \( w_{GS}, w_{CS}, w_{SG}, w_{GG}, \) and \( w_{XG} \) are all non-negative constants.
- \( \tau_S \) and \( \tau_G \) are the membrane time constants of the neurons.
- \( Ctx \) and \( Str \) are the constant inputs from cortex and striatum.
- \( F_S(\cdot) \) and \( F_G(\cdot) \) are the sigmoid activation function.

\[
F_S(\cdot) = \frac{M_S}{1 + \left(\frac{M_S - B_S}{B_S}\right) e^{-4(\cdot)/M_S}} \\
F_G(\cdot) = \frac{M_G}{1 + \left(\frac{M_G - B_G}{B_G}\right) e^{-4(\cdot)/M_G}}
\]
Previous analysis
- The membrane time constants are exactly the same.
- The transmission delays in the neural populations are taken to be equal.
- Nonlinear activation functions are replaced by linear functions.

Our analysis
- The membrane time constants are taken to be different.
- The three time delays in the connections between the excitatory and inhibitory populations of neurons are taken to be different.
- We consider a general nonlinear class of activation functions.

Stability analysis: single delay

Figure: (a) Stability of the non-trivial steady state, for $T_1 = 0$ and $T_2 > 0$. (b) Amplitude and (c) period of the periodic solutions.
Stability analysis: single delay

Figure: (a) Stability of the non-trivial steady state, for $T_1 > 0$ and $T_2 = 0$. (b) Amplitude and (c) period of the periodic solutions.
Stability analysis: two time delays

Figure: (a)-(d) Stability of the non-trivial steady state, for $T_1 > 0$ and $T_2 > 0$. (e) Amplitude and (f) period of the periodic solutions.
Thank you!