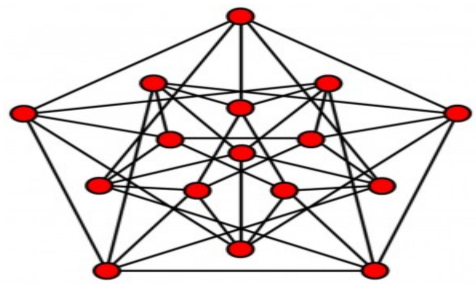
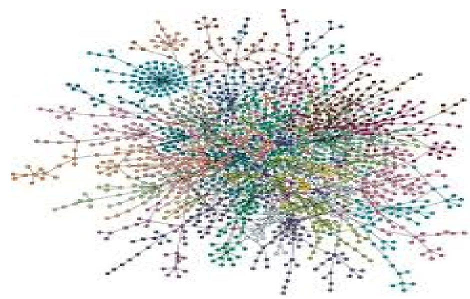
# Pescara, Italy, July 2019

# DIGRAPHS I
## Mathematical Background: Perron-Frobenius, Jordan Normal Form, Cauchy-Binet, Jacobi's Formula

**Based on various sources.**

J. J. P. Veerman,
Math/Stat, Portland State Univ., Portland, OR 97201, USA.
email: veerman@pdx.edu

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SUMMARY:

* This is a review of four theorems from linear algebra that are important for the development of the algebraic theory of directed graphs. These theorems are the Perron-Frobenius theorem, the Cauchy-Binet formula, the Jordan Normal Form, and Jacobi's Formula.

**OUTLINE:**

The headings of this talk are color-coded as follows:

**Graph Theory Definitions**

**Perron-Frobenius**

**Jordan Normal Form**

**Cauchy-Binet**

**Jacobi's Formula**

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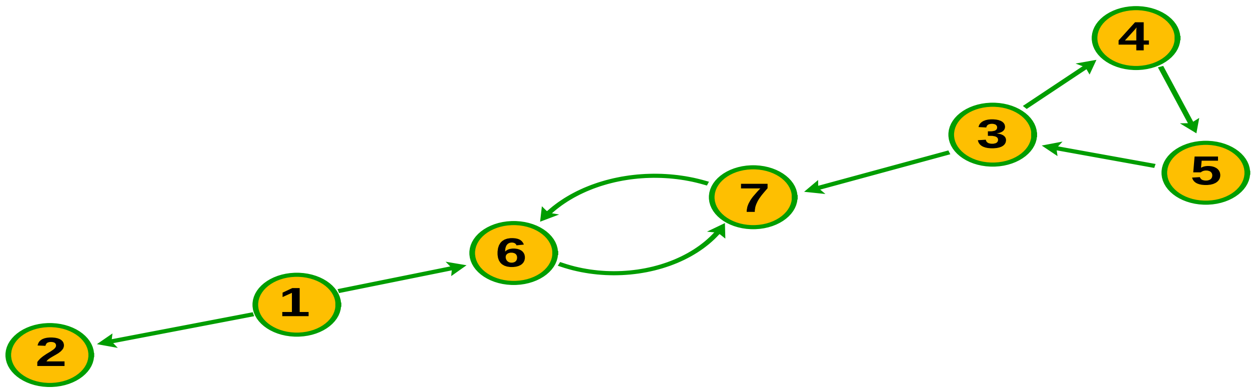
# ELEMENTARY GRAPH THEORY

# Definitions: Digraphs

**Definition:** A directed graph (or **digraph**) is a set $V = \{1, \cdots n\}$ of **vertices** together with set of ordered pairs $E \subseteq V \times V$ (the **edges**).



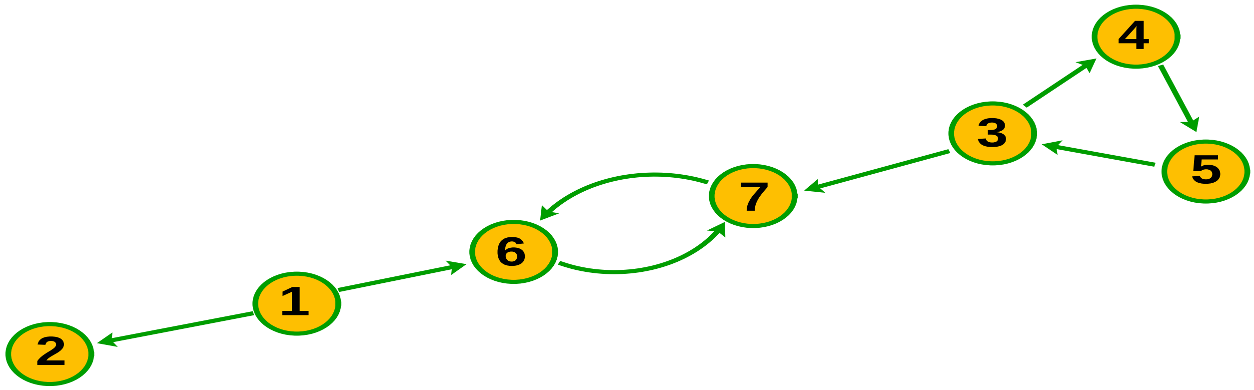

A directed edge $j \to i$, also written as $ji$.
A directed path from $j$ to $i$ is written as $j \rightsquigarrow i$.

**Digraphs are everywhere:** models of the internet [6], social networks [7], food webs [11], epidemics [10], chemical reaction networks [12], databases [5], communication networks [4], and networks of autonomous agents in control theory [8], to name but a few.

**A BIG topic:** Much of mathematics can be translated into graph theory (discretization, triangulation, etc). In addition, many topics in graph theory that do not translate back to *continuous* mathematics.

# Definitions: Connectedness of digraphs

Undirected graphs are connected or not. But...




**Definition:**

* A digraph $G$ is **strongly connected** if for every ordered pair of vertices $(i, j)$, there is a path $i \rightsquigarrow j$. **SCC !**

* A digraph $G$ is **unilaterally connected** if for every ordered pair of vertices $(i, j)$, there is a path $i \rightsquigarrow j$ or a path $j \rightsquigarrow i$.

* A digraph $G$ is **weakly connected** if the **underlying UNdirected graph** is connected.

* A digraph $G$ is **not connected:** if it is not weakly connected.

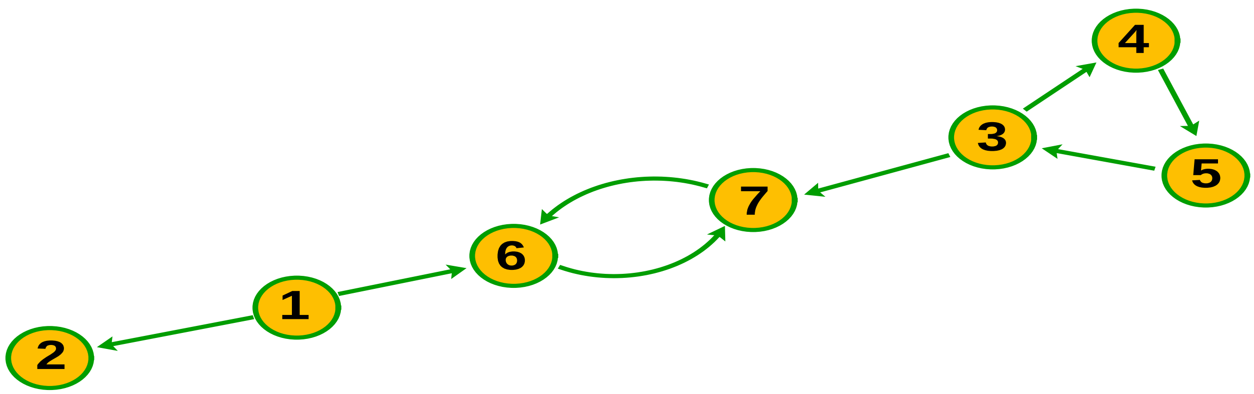
**Definition:**

**Multilaterally connected**: **weakly connected** but not **unilaterally connected**.

# The Adjacency Matrix

**Definition:** The **combinatorial adjacency matrix** $Q$ of the graph $G$ is the matrix whose entry $Q_{ij} = 1$ if there is an edge $ji$ and equals 0 otherwise.

**Interpretation:** We think of $Q_{ij} = 1$ as information going from $j$ to $i$. Or: $i$ "sees" $j$. In the graph below, both 2 and 6 "see" 1. So $Q_{21} = Q_{61} = 1$.




$$Q = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

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# THE PERRON FROBENIUS THEOREM

# Non-Negative Matrices

**Definition:** A non-negative matrix $Q$ is **irreducible** if for every $i$, $j$, there is a $k$ such that $(Q^k)_{ij} > 0$.

**OR:** for all $i$, $j$, there is path from $j$ to $i$: $j \rightsquigarrow i$.

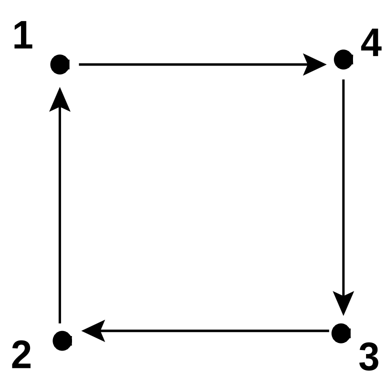
**Definition:** A non-negative matrix $Q$ is **primitive** if there is a $k$ such that for every $i$, $j$, we have $(Q^k)_{ij} > 0$.

**OR:** $\exists\ k$ such that for all $i$, $j$, there is $j \rightsquigarrow i$ of length $k$.

$Q$ is adjacency matrix of graph $G$. Both imply that $G$ is SCC.

**Irreducible** but **not primitive**: any cyclic permutation.

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



# Perron-Frobenius

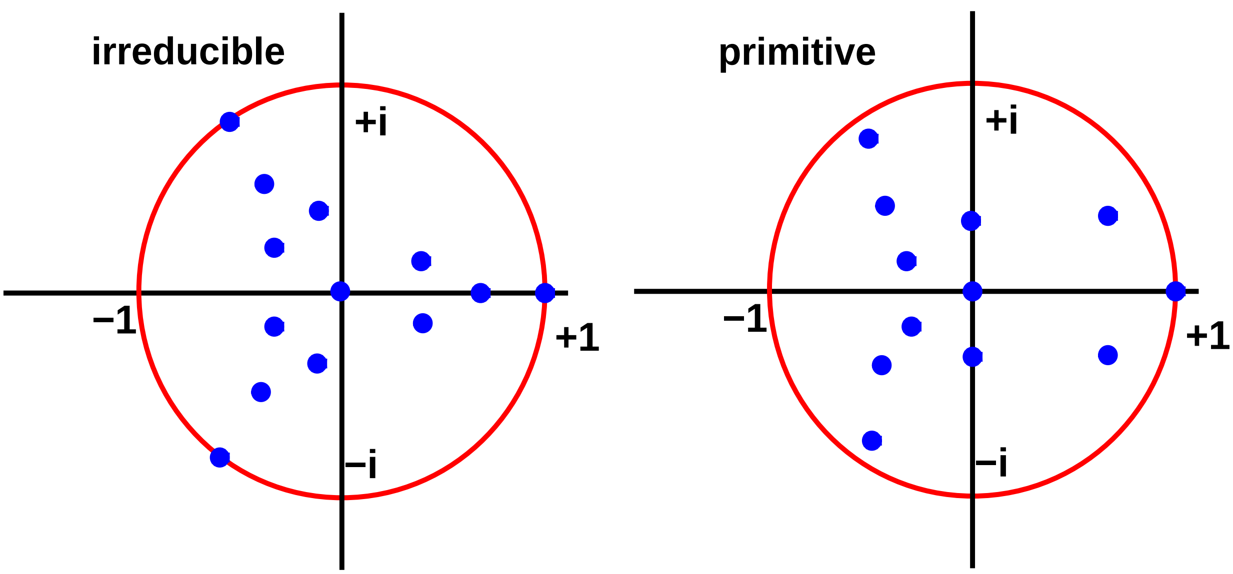
The single most important theorem in algebraic graph theory!!
Gives leading eigenpair of many important matrices.
1st order description of dynamical processes on graphs.
More details in [1] and [13].

**Theorem 1A: Let $A \geq 0$ be irreducible. Then:**
**(a) Its spectral radius $\rho(A)$ is a simple eval of $A$.**
**(b) Its associated evec is the only strictly positive evec.**

Thus its largest eval is simple, real, and positive. But there may be other evals of the same modulus.

**Theorem 1B: Let $A \geq 0$ be primitive. Then also:**
**All other evals have modulus strictly smaller than $\rho(A)$.**

(Note 3-fold rotational symmetry in irreducible case.)



# Irreducible Has Period $p$

In the irreducible case, the matrix $A$ has a **period $p > 1$**.
That is: after permutation of vertices, $A$ is **block cyclic**.
Example: $p = 3$:

$$A = \begin{pmatrix} 0 & A_1 & 0 \\ 0 & 0 & A_2 \\ A_3 & 0 & 0 \end{pmatrix}$$

In this **cyclic block form**, the $A_i$ are **rectangular**!

**Exercise 1:** Show that

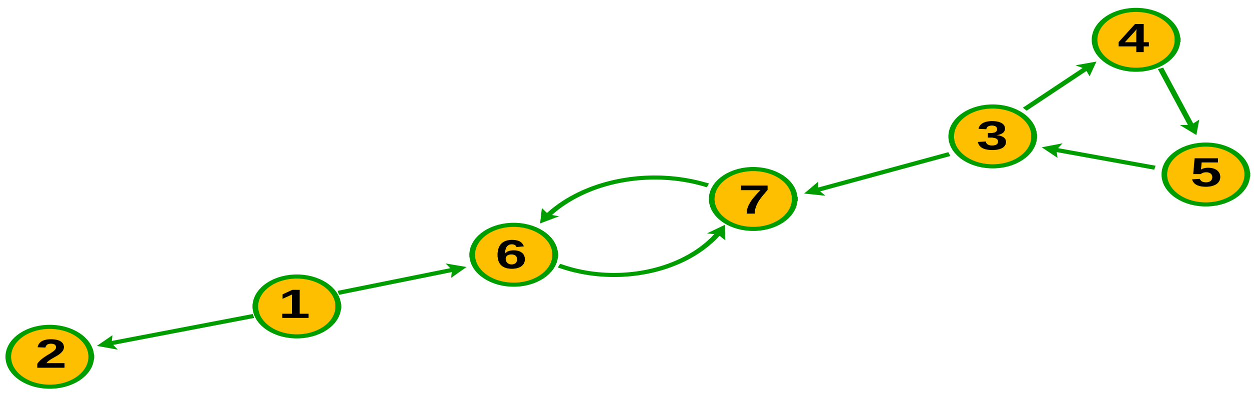
$$A^3 = \begin{pmatrix} A_1A_2A_3 & 0 & 0 \\ 0 & A_2A_3A_1 & \\ 0 & 0 & A_3A_1A_2 \end{pmatrix}$$

Now, the diagonal blocks are primitive.

By **Cauchy-Binet** (later):
each diagonal block $D$ of $A^3$ has same non-zero spectrum.
Suppose non-zero spectrum $D$ is: $\{\lambda_i\}_{i=1}^{s}$.

The non-zero spectrum of $A$ consists of **all 3rd roots** of these.

# Example




$$\sum_{i=1}^{7} A^i = \left(\begin{array}{c|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 0 & 0 \\ \hline 4 & 0 & 5 & 3 & 4 & 3 & 4 \\ 3 & 0 & 7 & 4 & 5 & 14 & 3 \end{array}\right)$$

So, $Q$ is block-triangular and thus *not* irreducible. But: The two non-trivial blocks are **irreducible but *not* primitive**. Notice the grouping of the evals.

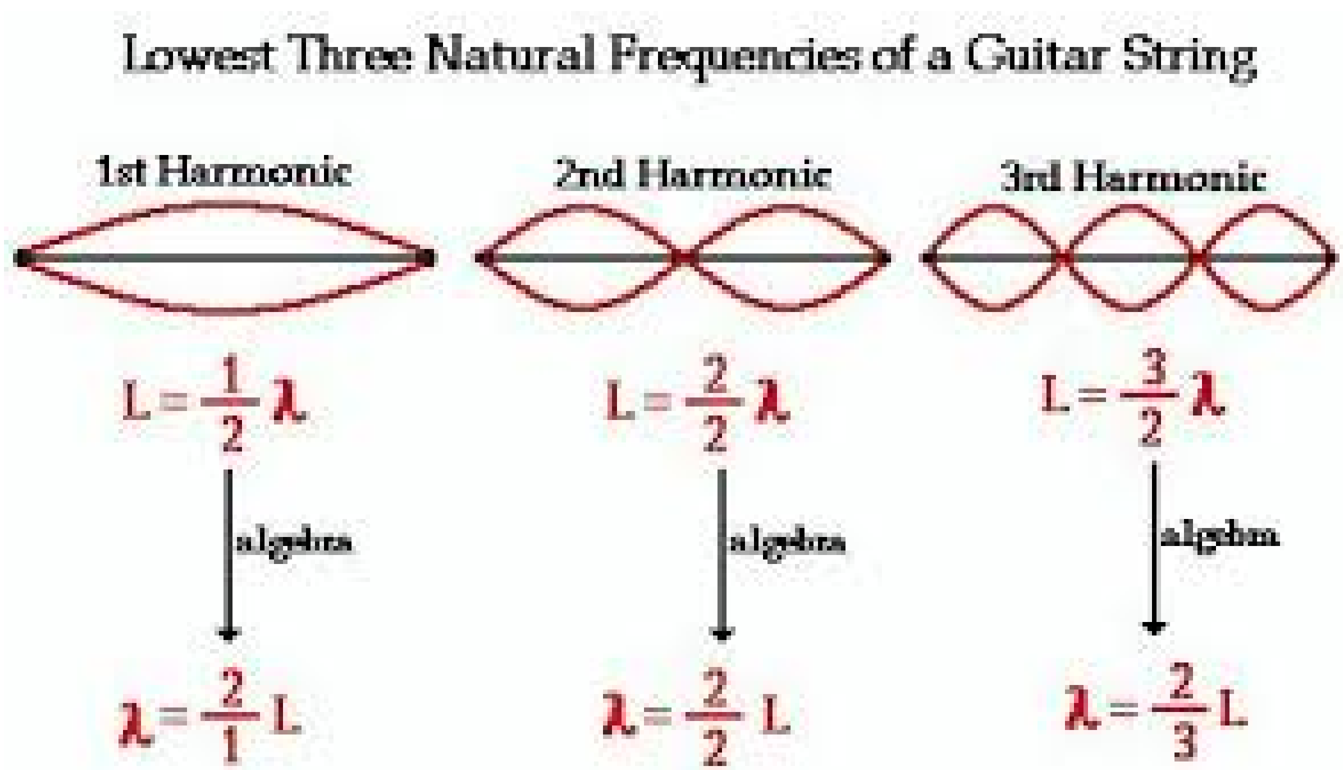
The spectrum is $\{0, 0, \quad 1, e^{2\pi i/3}, e^{-2\pi i/3}, \quad 1, -1\}$.

# Other Eigenvectors

**Theorem 1C: Let $A$ be irreducible. Any other evec but the leading <u>cannot</u> be real and non-negative.**

This is clear if the eigenvalue is non-real. So only needs proof for real evecs.

This is the beginning of the study of **Nodal Domains**.
A classical problem in analysis (since Courant): count the number of nodal domains of e.fns to the Laplace operator. See Figure.




For undirected graphs there are many results. But for digraphs very little is known. (After all, evecs may not be real!)

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# J O R D A N   N O R M A L   F O R M

# Spectral Theorem

From now: A is $n \times n$ matrix with real or complex coeff's:
**real symmetric $\subset$ self-adjoint $\subset$ normal.**
($A$ is normal if $A^*A = AA^*$.)

**Theorem 2 (spectral): $A$ has orthonormal basis of evecs $\{v_i\}_{i=1}^n$ iff $A$ normal.**

**These evals are real, if $A$ is self-adjoint.**

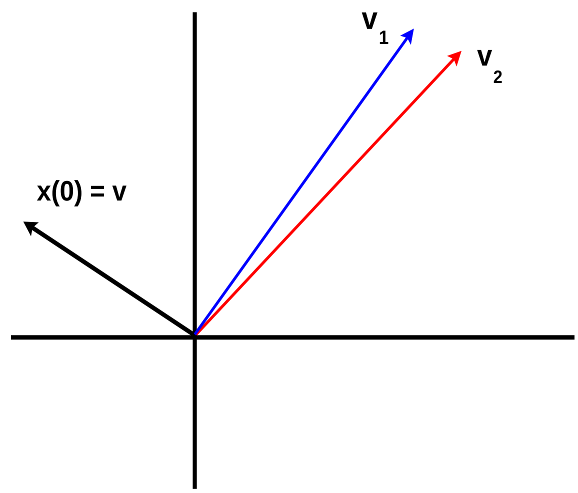
Computations simplify (e.g. quantum mechanics and statistical physics):
Let $A$ a (normal) matrix with e.pairs $\{\lambda_i, v_i\}$.
Suppose $\dot{x} = Ax$ with initial condition $x(0) = x_0$. Then:

$$\boldsymbol{x(t) = \sum_i (v_i, x_0) e^{\lambda_i t} v_i}$$

where $(.,.)$ is real or Hermitian inner product.
$(v_i, x_0)$ is the orthogonal projection of $x_0$ onto $v_i$.

**Exercise 2: The matrix norm $\|A\| \equiv \sup_x \{Ax \,|\, |x| = 1\}$ equals norm of its largest eval if $A$ is normal.**
(*Hints: a) Show* $\sum (v_i, x)^2 = 1$*; b) Show that* $Ax = \sum \lambda_i (v_i, x)$*; c) Show that* $(Ax, Ax)$ *is a weighted mean of* $\lambda_i^2$*.*)

# Life in a Non-normal Universe

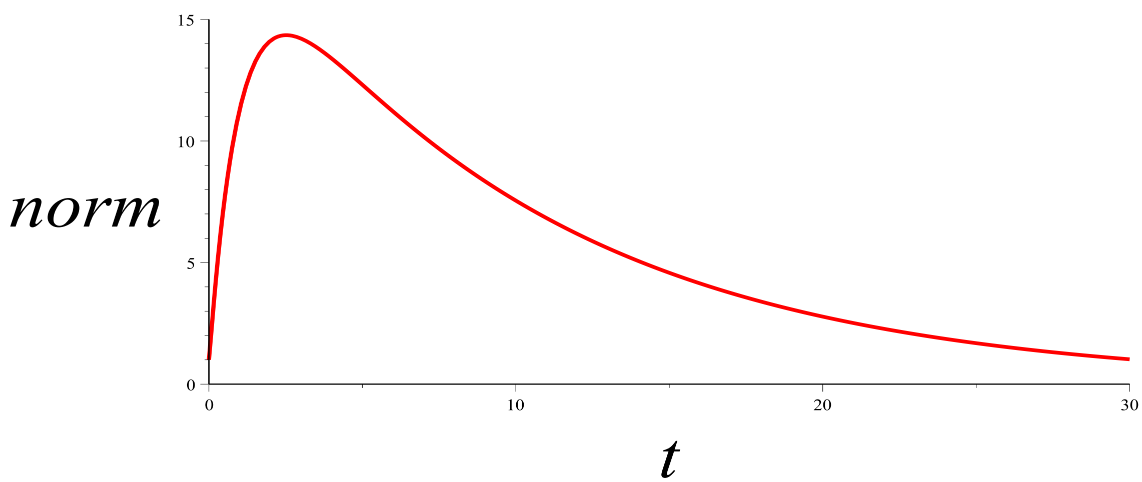



Let $\dot{x} = Ax$. Sps evecs $v_1$ and $v_2$ nearly parallel.

$$x(t) = A_1 e^{\lambda_1 t} v_1 + A_2 e^{\lambda_2 t} v_2$$

Example: $\lambda_i = \{-0.1, -1.0\}$ and init. condn $x(0)$ as indicated.

Large **transient**! Stable system may initially "look" unstable. Below we plot $|x(t)|$.




**Exercise 3:** Define a 2-dim. system of ODE plus initial condition that exhibits this type of behavior.

# Case I: $n$ Eigenvectors

Let $A$ be $n \times n$ matrix.
In general, it may have real and/or complex epairs.

Evals are the solutions $\{\lambda_i\}_{i=1}^{k}$ (with $k \leq n$) of

$$\det(A - \lambda I) = 0$$

**Case I:** $n$ **linearly independent** evecs $\{v_i\}_{i=1}^{n}$.
Given $\lambda_i$, then $\{v_i\}$ is the solution of

$$(A - \lambda_i I)v = 0$$

Let $H$ the matrix whose $i$th column equals $v_i$. Then $A$ is **diagonalizable**, or:

$$D = H^{-1} A H$$

with $D$ diagonal with $D_{ii} = \lambda_i$ (real if $A$ is self-adjoint).

**Application:** Suppose $\dot{x} = Ax$ with init. cond. $x_0$. Then:

$$x(t) = \sum_i \alpha_i e^{-\lambda_i t} v_i$$

But the $\alpha_i$ are less simple to calculate. Set $t = 0$, you get:

$$H\alpha = x_0$$

# Case II: Less than $n$ Eigenvectors

Let $A$ be $n \times n$ matrix.

**Case II:** less than $n$ **linearly independent** evecs $\{v_i\}_{i=1}^n$.

This happens when for some $i$, $\lambda_i$ is a root of order $\underline{k}$ of

$$\det(\boldsymbol{A} - \boldsymbol{\lambda} \boldsymbol{I}) = \mathbf{0}$$

but

$$(\boldsymbol{A} - \boldsymbol{\lambda}_i \boldsymbol{I})\boldsymbol{v} = \mathbf{0}$$

has less than $k$ linearly independent solutions for $v$.

**Definition:** The **algebraic multiplicity** of an eigenvalue $\lambda_i$ of $A$ is the **order** of the root $\lambda_i$ of $\det(A - \lambda I)$.
The **geometric multiplicity** of $\lambda_i$ is the **number** of linearly independent evecs associated with $\lambda_i$.

In this case $A$ is not diagonalizable but **block diagonalizable**. There is matrix $H$ so that

$$\boldsymbol{J} = \boldsymbol{H}^{-1} \boldsymbol{A} \boldsymbol{H}$$

**Exercise 4:** $J$ has diagonal **Jordan blocks** (or JB), all of the form:

$$B_i = \begin{pmatrix} \lambda_i & 1 & 0 & .. \\ 0 & \lambda_i & 1 & .. \\ .. & .. & .. & 1 \\ .. & .. & 0 & \lambda_i \end{pmatrix}$$

# Case II: Not Enough LI Eigenvectors

Find all evals $\lambda$ satisfying

$$\det(A - \lambda I) = 0$$

For each eval $\lambda_i$, find its evecs:

$$(A - \lambda_i I)v = 0$$

These vectors span the **eigenspace** of $\lambda_i$.
For simplicity: assume there is only one: $v_i$.

If geom mult$(\lambda_i)$ < alg mult$(\lambda_i)$:
Start with evec $v_i$.
Find vector $w_{i1}$ such that

$$(A - \lambda_i I)w_{i1} = v_i$$

Find $w_{i2}$ such that

$$(A - \lambda_i I)w_{i2} = w_{i1}$$

Etc. The $v_i$ together with $w_{ij}$ are **generalized eigenvectors**. They span the **generalized eigenspace** of $\lambda_i$.

Thus there are exactly $n$ **linearly independent** generalized eigenvectors $v_i$.

# Case II: Construction of the Matrix $H$

$H$ is the matrix whose columns are:

$$\{\mathbf{v_1}, \mathbf{w_{11}}, \cdot\cdot\, \mathbf{w_{1n_1}}, \mathbf{v_2}, \mathbf{w_{21}}, \cdot\cdot\, \mathbf{w_{2n_2}}, \cdot\cdot, \mathbf{v_k}, \mathbf{w_{k1}}, \cdot\cdot\, \mathbf{w_{kn_k}}\}$$

equals $v_i$. Then

$$J = H^{-1}AH$$

and $J$ has non-trivial Jordan blocks.

**Example:** If 1st block has dim $\geq 3$ (or $n_1 \geq 2$):

$$\lambda_1 e_1 \xleftarrow{H^{-1}} \lambda_1 v_1 \xleftarrow{A} v_1 \xleftarrow{H} e_1$$

$$\lambda_1 e_2 + e_1 \xleftarrow{H^{-1}} \lambda_1 w_{11} + v_1 \xleftarrow{A} w_{11} \xleftarrow{H} e_2$$

$$\lambda_1 e_3 + e_2 \xleftarrow{H^{-1}} \lambda_1 w_{12} + w_{11} \xleftarrow{A} w_{12} \xleftarrow{H} e_3$$

**Definition:** Thus $J$ becomes:

$$\begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & \cdots \\ 0 & \lambda_1 & 1 & \cdots & \cdots \\ 0 & 0 & \lambda_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

This is called **Jordan normal form**.

# $\dot{x} = Ax$, General Case

**Exercise 1:** Let $I$ be the identity and

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad J = \lambda I + N = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

a) Compute $e^{Jt}$ via the usual expansion.
(*Hint:* $e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.)
b) Use a) to give solutions of $\dot{x} = Jx$, where $x(0) = (a_1, a_2)^T$.
(*Hint:* $e^{\lambda t} \begin{pmatrix} a_1 + a_2 t \\ a_2 \end{pmatrix}$.)

The expansion of $e^{Jt}$ in the exercise

$$e^{Jt} = I + Jt + \frac{J^2 t^2}{2} + \frac{J^3 t^3}{3!} + \cdots$$

simplifies because $J = \lambda I + N$ and $N^2 = 0$.

**Back to the general problem $\dot{x} = Ax$, $x(0) = x_0$.**
**Step 1:** Write init. cond as sum of **gener. evecs.**

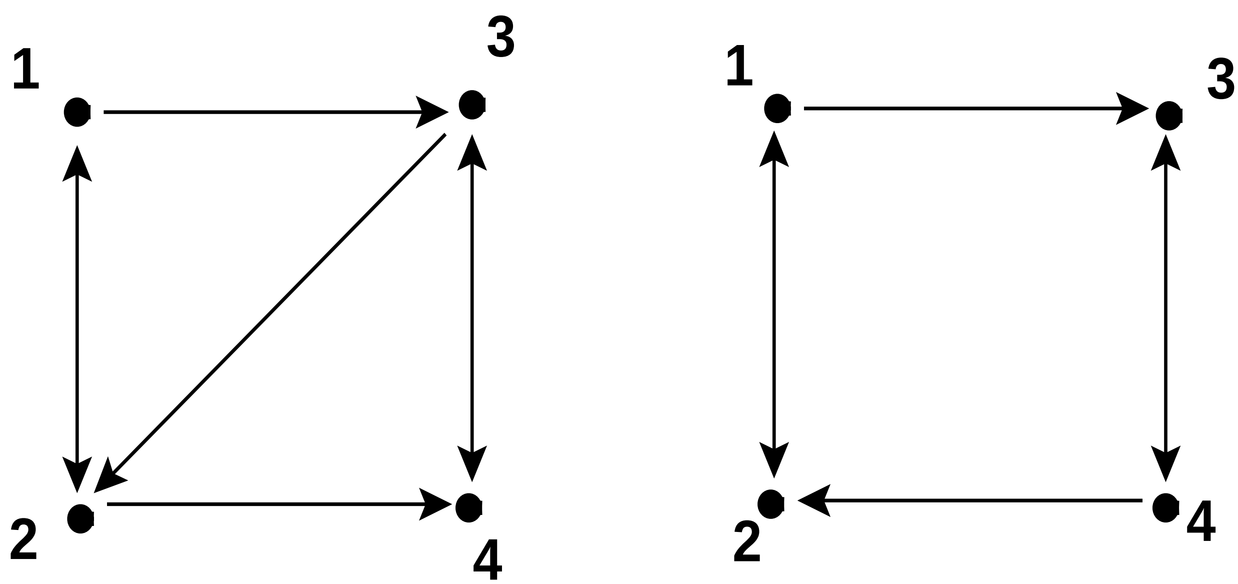
$$x_0 = \sum \alpha_i v_i \qquad \text{where} \qquad H\alpha = x_0$$

**Step 2:** Suppose $x_0 = \alpha_{12} w_{12}$. Then

$$x(t) = \alpha_{12}\, e^{\lambda t} \left( \frac{t^2}{2} v_1 + t\, w_{11} + w_{12} \right)$$

**Step 3:** Sum those contributions.

# Examples




Two digraphs. The first has adjacency matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

with spectrum $\{1.68, -1.03 \pm 0.74i, 0.37\}$ (approximately). The second has adjacency matrix

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with spectrum $\{0^{(2)}, \pm\sqrt{2}\}$. The eigenvalue 0 has an associated 2-dimensional Jordan block.

# Additional Exercises

**Exercise 2:** Show that the matrix

$$\begin{pmatrix} a-b & c \\ -cd & a+b \end{pmatrix}$$

has a non-trivial Jordan block (JB) if $b^2 = c^2 d$ and $c \neq 0$ and $d \neq 0$.

**Exercise 3:** So you may think JB's are rare (co-dimension one). But symmetries can change that. Show that
a) Newton's equation $\ddot{x} = 0$ gives rise to a JB.
b) That JB explains why two bodies without forcing separate linearly in time (Newton's first law).

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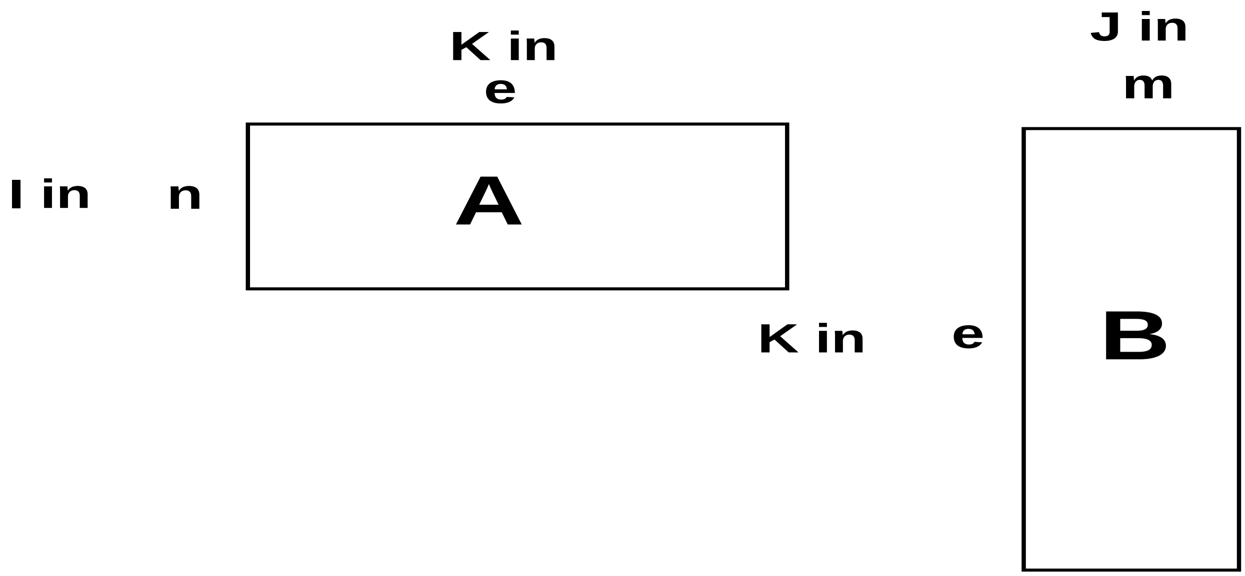
# THE CAUCHY - BINET FORMULA

# Generalized Cauchy-Binet

$A$ is a $n \times e$ matrix and $B$ is a $e \times m$ matrix.




**Notation:** $k \leq n, m \leq e$. (See figure). Let $I \subseteq \{1, \cdots n\}$, $J \subseteq \{1, \cdots m\}$, and $K \subseteq \{1, \cdots e\}$. All subsets have the same cardinality $k$.

**Definition:** The matrix consisting of the entries of $A$ in $I \times K$ is called a **minor** of $A$. **Principal minor** if $I = K$. It is denoted by $A[I, K]$.

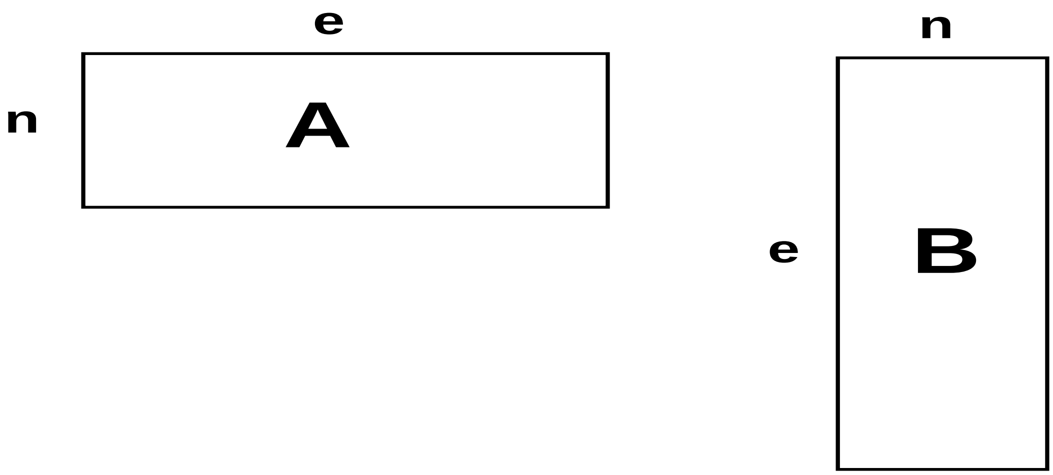
**Theorem 3 (generalized Cauchy-Binet):**

$$\det((AB)[I, J]) = \sum_K \det(A[I, K]) \det(B[K, J])$$

**where the sum is over all** $K \subseteq \{1, \cdots e\}$ **with** $|K| = k$**.**

# Corollaries

$A$ and $B$ as depicted, where $n \leq e$. Now $I = J = \{1, \cdots n\}$




**Corollary (Cauchy-Binet):** We have

$$\det(AB) = \sum \det(A[J,K])\det(B[K,J])$$

where the sum is over all $K \subseteq \{1, \cdots e\}$ with $|K| = n$.

**If $X$ is $n \times n$, by standard matrix computation**

$$\det(X + z\,Id) = \cdots + z^{n-k} \sum_{|K|=k} \det X[K,K] + \cdots$$

**By generalized C-B, we also have for $k \leq n$:**

$$\sum_{|K|=k} \sum_{|L|=k} \det A[K,L] \det B[L,K]$$

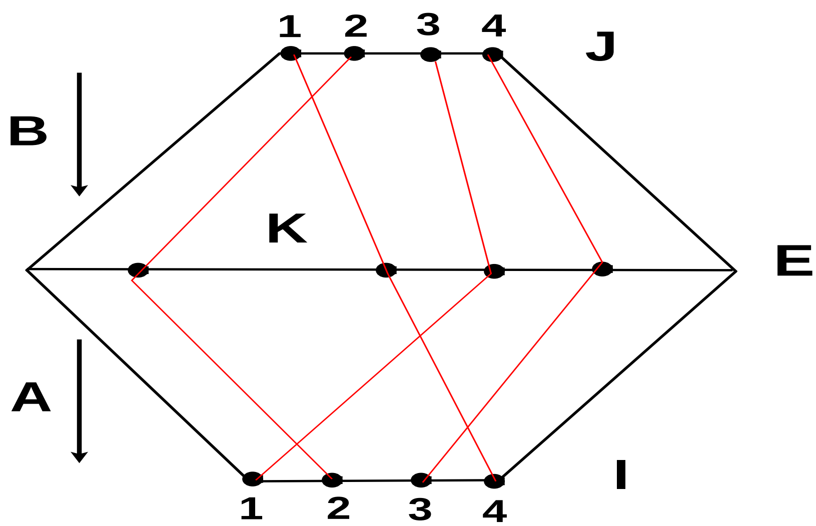
**equals** $\sum_{|K|=k} \det(AB)[K,K]$ **and** $\sum_{|L|=k} \det(BA)[L,L]$.

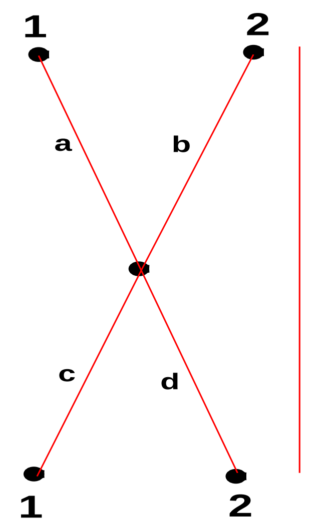
**Corollary:** We have

$$\det(BA + z\,Id) = z^{e-n} \det(AB + z\,Id)$$

# Sketch of Proof of Cauchy-Binet

Inspired by Gessel-Viennot [9].




$I = J = \{1, \cdots n\}$ and $E = \{1, \cdots e\}$ with $n \leq e$. $(n = 4.)$

$$\begin{aligned} \det AB &= \sum_\sigma \operatorname{sgn}\sigma \prod_i (AB)_{i\sigma(i)} \\ &= \sum_\sigma \operatorname{sgn}\sigma \prod_{i\in I} \sum_{\ell \in E} A_{i\ell} B_{\ell\sigma(i)} \end{aligned}$$

**Crossing paths give canceling contributions.**

For the crossing as pictured (right figure):

$$(AB)_{11}(AB)_{22}(AB)_{3\sigma(3)} \cdots = (AB)_{12}(AB)_{21}(AB)_{3\sigma(3)} \cdots$$

All other terms equal. But $\sigma$ changes by 1 transpos.: $1 \leftrightarrow 2$.

**Thus:**

$$\prod_{i\in I} \sum_{\ell\in E} A_{i\ell} B_{\ell\sigma(i)} =$$

$$\left(\sum_{\ell_1} A_{1\ell_1} B_{\ell_1\sigma(1)}\right)\left(\sum_{\ell_2} A_{2\ell_2} B_{\ell_2\sigma(2)}\right) \cdots \left(\sum_{\ell_n} A_{n\ell_n} B_{\ell_n\sigma(n)}\right)$$

**Where now $\bar{\ell} \equiv (\ell_1, \cdots \ell_n)$ runs over ALL $n$-tuples of distinct members of $E$. (No two $\ell_i$'s are the same.)**

# Sketch of Proof Continued

**Idea: Regroup the sums so that for any given $K \subset E$ of size $n$, the permutations of $K$ are grouped together. Then:**

$$\prod_{i \in I} \sum_{\ell \in E} A_{i\ell} B_{\ell\sigma(i)} =$$
$$\sum_{K,|K|=n} \left(\sum_{\ell_1} A_{1\ell_1} B_{\ell_1\sigma(1)}\right) \cdots \left(\sum_{\ell_n} A_{n\ell_n} B_{\ell_n\sigma(n)}\right)$$

For each $K$, the $\ell_i$ run over the permutations of $K$.

**For fixed** $K$, choose permutations $\rho$ and $\tau$:

$$I \xrightarrow{\tau} K \xrightarrow{\rho} J \text{ so that } \rho(\tau) = \sigma \ :$$

$$\cdots = \sum_{|K|=4} \prod_i \sum_\tau A_{i\tau(i)} B_{\tau(i)\rho(\tau(i))}$$

We obtain:

$$\det AB = \sum_{|K|=4} \sum_\sigma \operatorname{sgn}\sigma \prod_i \sum_\tau A_{i\tau(i)} B_{\tau(i)\rho(\tau(i))}$$

**For fixed** $K$, this is determ. of product of **square** matrices:

$$\cdots = \sum_{|K|=4} \det(A[I,K]) \det(B[K,J])$$

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# J A C O B I ' S

# F O R M U L A

# The Formula and Its Corollaries

$A$ a square matrix, **adj**$(A)$ its **adjugate**:

$$A \; \mathrm{adj}(A) = \mathrm{adj}(A) \; A = \det(A) \; I$$

Suppose $A$ depends (differentiably) on a parameter $t$.

**Theorem 4:** $\dfrac{d}{dt}\det(A) = \mathrm{Tr}\left(\mathrm{adj}(A)\,\dfrac{dA}{dt}\right).$

**We give some common corollaries as easy exercises.**

Replace $\dfrac{dA}{dt}$ by $B$ whose only non-zero entry is $B_{k\ell} = 1$:

**Exercise 4:** Show $\dfrac{d}{dA_{k\ell}}\det(A) = (\mathrm{adj}(A))_{\ell k}$.

Instead, replace $A$ by $e^{Bt}$ and so $\mathrm{adj}(A)$ by $e^{-Bt}\det\left(e^{Bt}\right)$:

**Exercise 5:** Show $\dfrac{d}{dt}\det(e^{tB}) = \mathrm{Tr}(B)\,\det(e^{tB})$.

The latter gives an ODE. Solve it:

**Exercise 6:** Show the latter implies: $\det(e^{tB}) = e^{\mathrm{Tr}(Bt)}$.

# Sketch of Proof

$B$ has evals $\lambda_i$ (with mult.). Then $I + \epsilon B$ has evals $1 + \epsilon\lambda_i$:

$$\det(I + \epsilon B) = \prod_i (1 + \epsilon\lambda_i)$$

Thus

$$\lim_{\epsilon\to 0} \frac{\det(I + \epsilon B) - \det(I)}{\epsilon} = \sum_i \lambda_i = \mathrm{Tr}(B)$$

For an invertible $A$:

$$\lim_{\epsilon\to 0} \frac{\det(A + \epsilon B) - \det(A)}{\epsilon} =$$
$$\lim_{\epsilon\to 0} \frac{\det(A)\left[\det(I + \epsilon A^{-1} B) - \det(I)\right]}{\epsilon} =$$
$$\det(A)\mathrm{Tr}(A^{-1}B)$$

Extend to non-invertible: replace $\det(A)\, A^{-1}$ by $\mathrm{adj}(A)$:

$$\cdots = \mathrm{Tr}\left(\mathrm{adj}(A)\, B\right)$$

... And replace $B$ by $\frac{dA}{dt}$:

$$\cdots = \mathrm{Tr}\left(\mathrm{adj}(A)\, \frac{dA}{dt}\right)$$

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