Confluence and Convergence in Probabilistically Terminating Reduction Systems

Maja H. Kirkeby   Henning Christiansen

Computer Science, Roskilde University, Denmark

LOPSTR – 2017 – Namur, Belgium

Probabilistic Abstract Reduction Systems

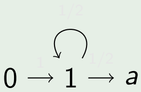
PARS cover

- probabilistic algorithms and programs
- scheduling strategies
- protocols
- . . .

Background

- Almost-sure convergence and almost-sure termination introduced for a subset of probabilistic programs by Hart et al 1983
- PARS formulated by Bournez and Kirchner 2002
- Almost-surely confluence formulated by Frühwirt et al 2002, Bournez and Kirchner 2002

Probabilistic Abstract Reduction Systems

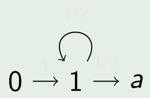


Probabilistic Abstract Reduction Systems

$$0 \longrightarrow 1 \longrightarrow a$$

Abstract Reduction System

is a pair $R = (A, \rightarrow)$ where A is countable and $\rightarrow \subseteq A \times A$.

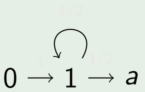


Abstract Reduction System

is a pair $R = (A, \rightarrow)$ where A is countable and $\rightarrow \subseteq A \times A$.

R is *terminating*

if there are no infinite paths



Abstract Reduction System

is a pair $R = (A, \to)$ where A is countable and $\to \subseteq A \times A$.

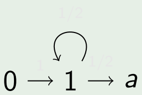
R is *terminating*

if there are no infinite paths

R is *confluent*

if for all paths $s_1 \leftarrow^* s \to^* s_2$ there is a t such that $s_1 \to^* t \leftarrow^* s_2$

Probabilistic Abstract Reduction Systems



Abstract Reduction System

is a pair $R = (A, \to)$ where A is countable and $\to \subseteq A \times A$.

R is *terminating*

if there are no infinite paths

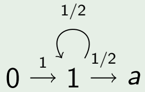
R is *confluent*

if for all paths $s_1 \leftarrow^* s \to^* s_2$ there is a t such that $s_1 \to^* t \leftarrow^* s_2$

R is *convergent*

if R is confluent and terminating

Probabilistic Abstract Reduction Systems



Abstract Reduction System

is a pair $R = (A, \rightarrow)$ where A is countable and $\rightarrow \subseteq A \times A$.

R* is *terminating

if there are no infinite paths

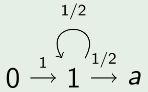
R* is *confluent

if for all paths $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a t such that $s_1 \rightarrow^* t \leftarrow^* s_2$

R* is *convergent

if R is confluent and terminating

Probabilistic Abstract Reduction Systems



R is *almost-sure terminating*

if the probability of reaching a normal form is 1.

Abstract Reduction System

is a pair $R = (A, \to)$ where A is countable and $\to \subseteq A \times A$.

R is *terminating*

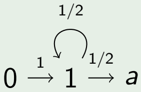
if there are no infinite paths

R is *confluent*

if for all paths $s_1 \leftarrow^* s \to^* s_2$ there is a t such that $s_1 \to^* t \leftarrow^* s_2$

R is *convergent*

if R is confluent and terminating



Abstract Reduction System

is a pair $R=(A,\to)$ where A is countable and $\to \subseteq A \times A$.

R is *terminating*

if there are no infinite paths

R is *almost-sure terminating*

if the probability of reaching a normal form is 1.

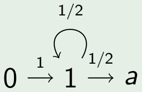
$$\sum_{i=0}^{\infty}(1/2)^i$$

R is *confluent*

if for all paths $s_1 \leftarrow^* s \to^* s_2$ there is a t such that $s_1 \to^* t \leftarrow^* s_2$

R is *convergent*

if R is confluent and terminating



R is *almost-sure terminating*

if the probability of reaching a normal form is 1.

$$\sum_{i=0}^{\infty}(1/2)^i = 1$$

Abstract Reduction System

is a pair $R=(A,\rightarrow)$ where A is countable and $\rightarrow \subseteq A \times A$.

R is *terminating*

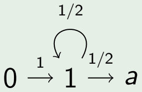
if there are no infinite paths

R is *confluent*

if for all paths $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a t such that $s_1 \rightarrow^* t \leftarrow^* s_2$

R is *convergent*

if R is confluent and terminating



R is *almost-sure terminating*

if the probability of reaching a normal form is 1.

$$\sum_{i=0}^{\infty} (1/2)^i = 1$$

R is *almost-surely convergent*

if for all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a <u>normal form</u> t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$

Abstract Reduction System

is a pair $R = (A, \rightarrow)$ where A is countable and $\rightarrow \subseteq A \times A$.

R is *terminating*

if there are no infinite paths

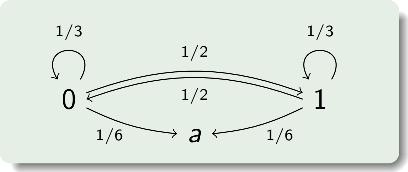
R is *confluent*

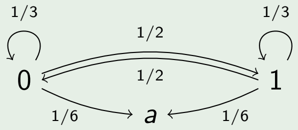
if for all paths $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a t such that $s_1 \rightarrow^* t \leftarrow^* s_2$

R is *convergent*

if R is confluent and terminating

Probabilistic Abstract Reduction Systems





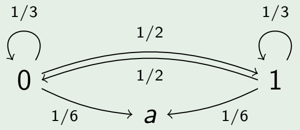
Probabilistic Abstract Reduction System

$R^P = (R, P)$ where $R = (A, \to)$ is an ARS.
For each $s \in A \setminus R_{NF}$,

$$\sum_{s \to t} P(s \to t) = 1.$$

For all s and t, $P(s \to t) > 0$ if and only if $s \to t$.

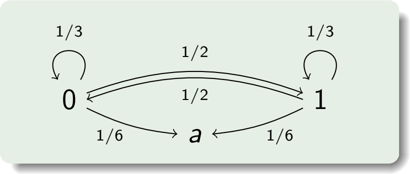
Probabilistic Abstract Reduction Systems



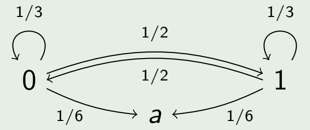
Probability of a finite path $s_0 \to s_1 \to \ldots \to s_n$ with $n \geq 0$

$$P(s_0 \to s_1 \to \ldots \to s_n) = \prod_{i=1}^{n} P(s_{i-1} \to s_i).$$

Probabilistic Abstract Reduction Systems



Probabilistic Abstract Reduction Systems



Probability of s reaching $t \in R_{NF}(s)$

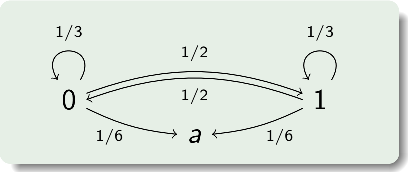
$$P(s \to^* t) = \sum_{\delta \in \Delta(s,t)} P(\delta)$$

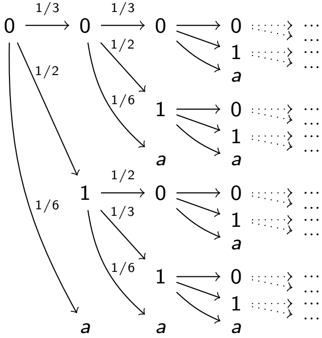
where $\Delta(s,t) = \{\delta \mid \delta = s \to \ldots \to t\}$.

Probability of *diverging* from s

$$P(s \to^\infty) = 1 - \sum_{t \in R_{NF}(s)} P(s \to^* t).$$

Probabilistic Abstract Reduction Systems



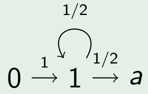


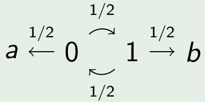
Probability of *diverging* from s

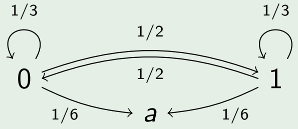
$$P(s \to^{\infty}) = 1 - \sum_{t \in R_{NF}(s)} P(s \to^{*} t).$$

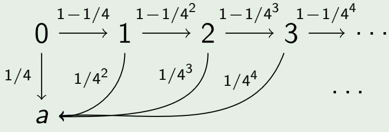
R is *almost-surely convergent*

For all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a normal form t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$.









Almost-surely convergent

R is *almost-surely convergent*

For all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a normal form t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$.

Theorem

A PARS is almost-surely terminating and confluent if and only if it is almost-surely convergent.

Almost-surely convergent

R is *almost-surely convergent*

For all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a normal form t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$.

Theorem

A PARS is almost-surely terminating and confluent if and only if it is almost-surely convergent.

Two tasks:

Almost-surely convergent

R is *almost-surely convergent*

For all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a normal form t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$.

Theorem

A PARS is almost-surely terminating and confluent if and only if it is almost-surely convergent.

Two tasks:

- *almost-sure termination*

R is *almost-surely convergent*

For all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a normal form t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$.

Theorem

A PARS is almost-surely terminating and confluent if and only if it is almost-surely convergent.

Two tasks:

- *almost-sure termination*
- *confluence*

Almost-surely convergent

R is *almost-surely convergent*

For all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a normal form t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$.

Theorem

A PARS is almost-surely terminating and confluent if and only if it is almost-surely convergent.

Two tasks:

- *almost-sure termination*:
  e.g. Lyapunov ranking functions.[1]
- *confluence*

[1] L.M.Ferrer Fioriti, H.Hermanns, *Probabilistic Termination* POPL '15.

Almost-surely convergent

R is *almost-surely convergent*

For all $s_1 \leftarrow^* s \rightarrow^* s_2$ there is a normal form t such that $s_1 \rightarrow^* t \leftarrow^* s_2$ and $P(s_1 \rightarrow^* t) = P(s_2 \rightarrow^* t) = 1$.

Theorem

A PARS is almost-surely terminating and confluent if and only if it is almost-surely convergent.

Two tasks:

- *almost-sure termination*:
  e.g. Lyapunov ranking functions.[1]
- *confluence*:
  e.g. transform to systems suited for automatic confluence analysis.[2]

---

[1] L.M.Ferrer Fioriti, H.Hermanns, *Probabilistic Termination* POPL '15.
[2] P.-L.Curien, G.Ghelli, *On Confluence for Weakly Normalizing Systems* '91.

Almost-sure termination

Definition

A measure $\mathcal{V} : A \to \mathbb{R}^+$ is a **Lyapunov ranking function** if

$$\forall s \in A, \exists \epsilon > 0: \qquad \mathcal{V}(s) \geq \sum_{s \to s'} P(s \to s') \cdot \mathcal{V}(s') + \epsilon$$

Lemma (Ferrer Fioriti, Hermanns POPL '15)

A PARS $R^P = ((A, \to), P)$ is a-s. terminating if there is a Lyapunov ranking function over A.

Almost-sure termination

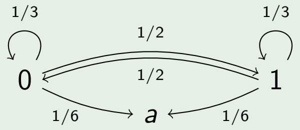
Definition

A measure $\mathcal{V} : A \to \mathbb{R}^+$ is a **Lyapunov ranking function** if

$$\forall s \in A, \exists \epsilon > 0 : \qquad \mathcal{V}(s) \geq \sum_{s \to s'} P(s \to s') \cdot \mathcal{V}(s') + \epsilon$$

Lemma (Ferrer Fioriti, Hermanns POPL '15)

A PARS $R^P = ((A, \to), P)$ is a-s. terminating if there is a Lyapunov ranking function over A.



Almost-sure termination

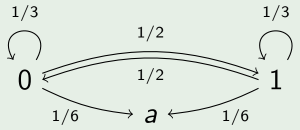
Definition

A measure $\mathcal{V} : A \to \mathbb{R}^+$ is a **Lyapunov ranking function** if

$$\forall s \in A, \exists \epsilon > 0 : \qquad \mathcal{V}(s) \geq \sum_{s \to s'} P(s \to s') \cdot \mathcal{V}(s') + \epsilon$$

Lemma (Ferrer Fioriti, Hermanns POPL '15)

A PARS $R^P = ((A, \to), P)$ is a-s. terminating if there is a Lyapunov ranking function over A.



$$\mathcal{V}(0) > 1/3\mathcal{V}(0) + 1/2\mathcal{V}(1) + 1/6\mathcal{V}(a)$$

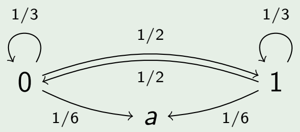
Definition

A measure $\mathcal{V} : A \to \mathbb{R}^+$ is a **Lyapunov ranking function** if

$$\forall s \in A, \exists \epsilon > 0 : \qquad \mathcal{V}(s) \geq \sum_{s \to s'} P(s \to s') \cdot \mathcal{V}(s') + \epsilon$$

Lemma (Ferrer Fioriti, Hermanns POPL '15)

A PARS $R^P = ((A, \to), P)$ is a-s. terminating if there is a Lyapunov ranking function over A.



$$\mathcal{V}(0) = 100$$
$$\mathcal{V}(1) = 100$$
$$\mathcal{V}(a) = 1$$

$$\mathcal{V}(0) > 1/3\mathcal{V}(0) + 1/2\mathcal{V}(1) + 1/6\mathcal{V}(a)$$

Almost-sure termination

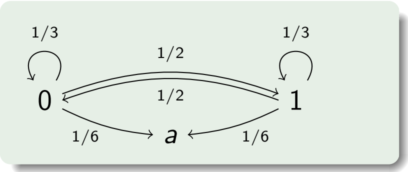
Definition

A measure $\mathcal{V} : A \to \mathbb{R}^+$ is a **Lyapunov ranking function** if

$$\forall s \in A, \exists \epsilon > 0 : \qquad \mathcal{V}(s) \geq \sum_{s \to s'} P(s \to s') \cdot \mathcal{V}(s') + \epsilon$$

Lemma (Ferrer Fioriti, Hermanns POPL '15)

A PARS $R^P = ((A, \to), P)$ is a-s. terminating if there is a Lyapunov ranking function over A.



$$\begin{aligned} \mathcal{V}(0) &= 100 \\ \mathcal{V}(1) &= 100 \\ \mathcal{V}(a) &= 1 \end{aligned}$$

$$100 = \mathcal{V}(0) > 1/3\mathcal{V}(0) + 1/2\mathcal{V}(1) + 1/6\mathcal{V}(a) = 83 + 1/2$$

Confluence

Lemma (Curien, Ghelli RTA '91)

Given two ARS $R = (A, \to_R)$ and $R' = (A, \to_{R'})$ and a mapping $G : A \to A'$, then R is confluent if the following holds.

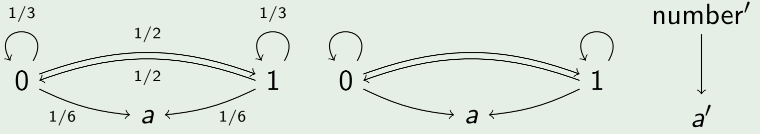
- *R' is confluent,*
- *R is normalizing,*
- *if $s \to_R t$ then $G(s) \leftrightarrow^*_{R'} G(t)$,*
- *$\forall t \in R_{NF},\ G(t) \in R'_{NF}$, and*
- *$\forall t, u \in R_{NF},\ G(t) = G(u) \Rightarrow t = u$.*

Confluence

Lemma (Curien, Ghelli RTA '91)

Given two ARS $R = (A, \to_R)$ and $R' = (A, \to_{R'})$ and a mapping $G\colon A \to A'$, then R is confluent if the following holds.

- *R' is confluent,*
- *R is normalizing,*
- *if $s \to_R t$ then $G(s) \leftrightarrow^*_{R'} G(t)$,*
- *$\forall t \in R_{NF},\ G(t) \in R'_{NF}$, and*
- *$\forall t, u \in R_{NF},\ G(t) = G(u) \Rightarrow t = u$.*

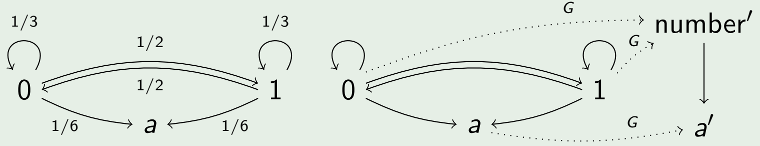


Confluence

Lemma (Curien, Ghelli RTA '91)

Given two ARS $R = (A, \to_R)$ and $R' = (A, \to_{R'})$ and a mapping $G\colon A \to A'$, then R is confluent if the following holds.

- *R' is confluent,*
- *R is normalizing,*
- *if $s \to_R t$ then $G(s) \leftrightarrow^*_{R'} G(t)$,*
- *$\forall t \in R_{NF},\ G(t) \in R'_{NF}$, and*
- *$\forall t, u \in R_{NF},\ G(t) = G(u) \Rightarrow t = u$.*

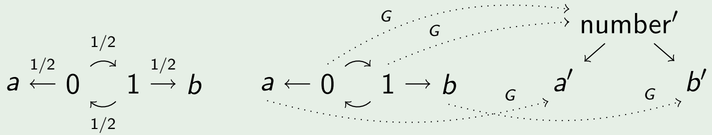


Lemma

Given two ARS $R = (A, \to_R)$ and $R' = (A, \to_{R'})$ and a mapping $G\colon A \to A'$, satisfying

- *(surjective)* $\forall s' \in A',\ \ \exists s \in A, G(s) = s'$,
- R *and* R' *are normalizing,*
- *if* $s \to_R t$ *then* $G(s) \leftrightarrow^*_{R'} G(t)$, *and*
  if $G(s) \leftrightarrow^*_{R'} G(t)$ *then* $s \leftrightarrow^*_R t$,
- $\forall t \in R_{NF},\ G(t) \in R'_{NF}$, *and* $\forall t' \in R'_{NF},\ G^{-1}(t') \subseteq R_{NF}$,
- $\forall t, u \in R_{NF},\ G(t) = G(u) \Rightarrow t = u$,

then R *is confluent iff* R' *is confluent.*



Contributions

- Self-contained definitions of PARS; proving basic properties
- Almost-sure convergence generalized to PARS
- Theorem: almost-sure convergence $\Leftrightarrow$
  almost-sure termination $\wedge$ confluence
- Unfolded methods for (dis-) proving almost-sure convergence