Symmetries, computers, and periodic orbits for the *n*-body problem

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Abstract

Periodic orbits play a central role in the n-body problem. In the attempt of understanding them, in the sense of computing their existence, qualitative and quantitative properties, and classifying such orbits and symmetries, computers have been extensively used in many ways since decades. I will focus on some very special symmetric orbits, which occur as symmetric critical points of the gravitational Lagrangean action functional. The exploration of the realm where such critical points live, i.e. the loop space of the n-point configuration space, raised computational, epistemological and mathematical questions that needed to be addressed and that I have found interesting. The aim of the talk is to explain how such questions and issues were (more or less naively) considered in the development of a software package that combined symbolic algebra, numerical and scientific libraries, human interaction and visualization.

1. The beginning

- Geometry and computing. A very old story.
- As a story I will tell, it will be partial and partially fictional.
- Henry Poincaré.
- Born in 1854, PhD in 1879, soon after mining engineer and lecturer.
- 1879-1881: double *annus mirabilis*.
- 1885: Lecturer at Paris University; 1886: professor.
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- May 1888: Poincaré (anonymous) submission entitled *Sur le problème des trois corps et les équations de la dynamique* (epigraph: *Numquam præscriptos transibunt sidera fines*).
- The award was given to Poincaré, since "It is the deep and original work of a mathematical genius whose position is among the greatest mathematicians of the century. The most important and difficult questions, like the stability of the world system, are treated using methods which open a new era in celestial mechanics".
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- ➔ But there was a problem.
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- ➔ He was actually right. There was a mistake.
- ➔ Poincaré had to get back all the printed issues of the journal to be destroyed, to submit a new corrected memoir (he did it in June 1890 — 270 pages long) and to pay for the new printing (the cost was more than the 2500 crowns of the prize).
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4. Consequences

- *Méthodes nouvelles de la mécanique céleste* (1892–1899).
- Motivated by the study of nonlinear ordinary differential equations and the three-body problem, between 1892 and 1901 he published the six memoirs on *Analysis situs*, where basically topology and algebraic topology were created.
- Two Poincaré conjectures, both based on a first wrong statement: the (now Perelman's Theorem) uniqueness of the topology of 3-spheres among simply-connected closed 3-manifolds, and the density of periodic orbits in the phase space for the (restricted) 3-body problem.

> *If a particular solution of the restricted problem is given, one can always find a periodic solution (with a period which might be very long) such that the difference between these two solutions is as small as desired for any given length of time.*

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5. King Oscar's prize

Given a system of arbitrarily many mass points that attract each other according to Newton's laws, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

This problem, whose solution would considerably extend our understanding of the solar system, seems capable of solution using analytic methods now at our disposal; we can at least suppose as much, since Lejeune Dirichlet communicated shortly before his death to a geometer of his acquaintance [Leopold Kronecker] that he had discovered a method for integrating the differential equations of Mechanics, and that by applying this method, he had succeeded in demonstrating the stability of our planetary system in an absolutely rigorous manner. Unfortunately, we know nothing about this method, except that the theory of small oscillations would appear to have served as his point of departure for this discovery. We can nevertheless suppose, almost with certainty, that

5. King Oscar's prize (cont.)

this method was based not on long and complicated calculations, but on the development of a fundamental and simple idea that one could reasonably hope to recover through persevering and penetrating research. In the event that this problem remains unsolved at the close of the contest, the prize may also be awarded for a work in which some other problem of Mechanics is treated as indicated and solved completely.

6. Long story

- Long after Aristarchus of Samos (310–230 BCE), the heliocentric planetary model was formulated by Nicolaus Copernicus (1473–1543) in a note in 1513, and finally published with mathematical details in *De revolutionibus orbium coelestium* (1543).
- Founding his speculations on years of astronomical data collected by Tycho Brahe (1546–1601), Kepler (1571–1630) discovered the laws governing the motion of planets around the sun, now called Kepler's three laws of planetary motion.
- After a few years, in 1632 Galileo Galilei (1564–1642) improved astronomical observations with telescope, and published the *Dialogo sopra i massimi sistemi*.

6. Long story (cont.)

- One year after Galileo died, Isaac Newton (1642–1727) was born. He is the one who found the reason of Kepler's laws, namely the law of universal gravitation. Newton's *Philosophiæ Naturalis Principia Mathematica* was published in 1687.
- With Laws of Dynamics and Universal Gravitation, the problem can simply be stated, in modern words, as a second-order differential Newton equation:

$$\frac{d^2\boldsymbol{q}}{dt^2} = \nabla U(\boldsymbol{q}),$$

where $\boldsymbol{q}(t)$ is the *configuration* at time $t \in \mathbb{R}$, and U is the gravitational *potential* force function

$$U(\boldsymbol{q}) = \sum_{i<j} \frac{m_i m_j}{|\boldsymbol{q}_i - \boldsymbol{q}_j|},$$

6. Long story (cont.)

where m_i are the masses (in a unit such that the gravitational constant is 1) adn $\boldsymbol{q}_i$ are the positions of the point masses in the euclidean space $\mathbb{R}^d$ ($d = 2, 3$).

- Newton solved the two-body problem in the first book of *Principia* (propositions 1-17, 57-60). The conical nature of Kepler orbits can also be derived by purely geometrical means To predict the position positions of planets one has to use an approximation of solutions of Kepler equation and its generalizations. Then, in propositions 65-66, Newton describes some qualitative features of the three-body problem, and speculated that and exact solution “exceeds, if I am not mistaken, the force of any human mind”.

6. (Long story)

After Newton, Johann Bernoulli (1667–1748) and Leonhard Euler (1707–1783) studied Newton's equation for some simplified problems: they could integrate the the one-center and two fixed-centers problem, which can be seen as an intermediate (integrable) approximation of the restricted three-body problem. In 1762, Euler considered the circular *restricted three-body problem*, which is related to the two-centers problem: consider Earth as rotation on a circle around the Sun, and consider the Moon as a negligible-mass body orbiting around the Earth, in rotating coordinates frame, and studied the collinear problem for generic masses (and found Euler central solutions).

6. (Long story)

Joseph-Louis Lagrange (born Giuseppe Lodovico Lagrangia, 1736–1813) expanded and generalized the results of Euler, and, with much more impact, later founded the analytical approach to mechanics, now called Lagrangean mechanics, published in subsequence editions the *Mécanique analytique* (1811,1815). In short, solutions Newton equations are local minimizers (critical points) of the Lagrangean action functional

$$\mathcal{A}[\boldsymbol{q}] = \int_{t_0}^{t_1} \frac{1}{2} \sum_j m_j |\frac{\boldsymbol{q}_j}{dt}|^2 + U(\boldsymbol{q})$$

defined on a suitable class of trajectories $\boldsymbol{q}(t)$. Lagrange found some particular periodic orbits (homographic central configurations for the (non-restricted) three-body problem, now termed Lagrange configurations) in his *Essai sur le problème des trois corps* (1772); also, he introduced the concepts of *stability* (1776) and *potential* (1773).

7. Before Poincaré

- ➔ Changes of variables, and search for integrals and reductions of the degrees of freedom.
- ➔ Jacobi (1804–1851) and Hamilton (1805–1865) : Hamilton-Jacobi formalism (with Poisson and Lagrange brackets and canonical transformations).
- ➔ The Jacobi integral for the three-dimensional restricted three-body problem was published in 1836.
- ➔ Delaunay (1816–1872) treatise on lunar theory, in 1860 and 1867. The main procedure was to expand the Hamiltonian as Fourier series with respect to position coordinates and apply suitable canonical transformations.
- ➔ After 57 iterations and 20 years of calculations, Delaunay could accurately predict the orbit of the Moon up to 1 arc second.

7. Before Poincaré (cont.)

- The approach via series seemed promising: in 1874 Simon Newcomb proved that the three-body problem can be formally solved by infinite series of purely periodic terms;
- in 1883, Lindstedt again showed that such a series existed, in Lagrange coordinates.
  - The first problem is: a formal series might not converge.
  - The second problem is: a convergent series might converge so slowly to be practically useless.

7. (Before Poincaré)

- But, actually, what is *exactly* the problem?
- And, given the problem, what does it mean to *solve* it?
- Newton equations in which space? Sobolev space H^1? C^1? C^∞. C^ω?
- And, when an equation is "solved"? Contructively giving the solution? Weierstrass mentioned "a method for integrating the differential equations of Mechanics". Why integration and not computing? Aren't they the same thing?
- Singularies and collisions.

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8. Integrable systems

- ➔ As with the integrability of Kepler problem, the first line of attack had been the one of “integrating” the equations, that is to find as many first integrals as necessary to express the solutions in terms of arbitrary constants. This approach, which is the starting point of the theory of integrable systems, did not work well.
- ➔ Bruns (1848-1919) showed that the series solutions of Lagrange can be divergent for the three-body problem (1884), and in 1887 he proved that there are no first integrals as algebraic (beyond those coming from known symmetries: the six of the centre of gravity, the three of angular momentum and the energy/Hamiltonian) functions in the phase space (positions and velocities of the bodies).
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9. After Poincaré

The ideas developed in Poincaré's *Les Méthodes nouvelles de la mécanique celeste* (1892-1899) contained seeds of innovation in many fields:

- ➔ *global* approach to dynamical system
- ➔ *qualitative*
- ➔ Poincaré-Birkhoff recurrence theorem,
- ➔ or the analogy introduced by Poincaré (see also Jacques Hadamard, E.T. Whittaker, G.D. Birkhoff, J. Moser) of periodic orbits as closed geodesics,
- ➔ the topological approach to stability
- ➔ and periodic orbits as fixed points of Poincaré section
- ➔ (and the existence of infinitely many periodic orbits in the restricted circular three-body problem, the *Last Geometric Theorem* of Poincaré proved).

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- ➔ Moulton
- ➔ G.D. Birkhoff
- ➔ Chazy
- ➔ Sundmann
- ➔ Kolmogorov
- ➔ Arnold
- ➔ Moser ($\Longrightarrow$ KAM)

With a shift in computing techniques and the need of mathematical tools for space missions and astrodynamics, the problem is still very alive also from the mathematical point of view.

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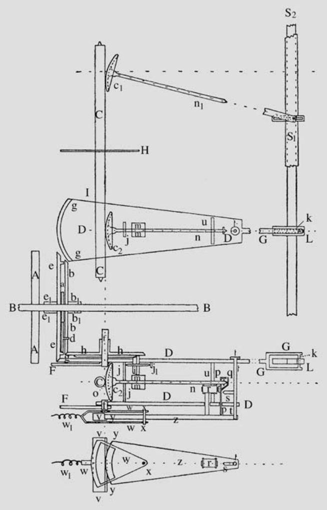

Figure: Sundman Contraption: the (never built) *perturbograph*

1913: Karl Sundman. Solutions by series in terms of $t^{1/3}$ for the full three-body problem. Regularization of binary collisions, but not for triple collisions.

1991: A generalization of Sundman's result to n-body was found by Quidong Wang.

Compare with:

- ➔ Solve polynomial equations (Galois theory and Abel-Ruffini theorem).
- ➔ Compute digits of π.

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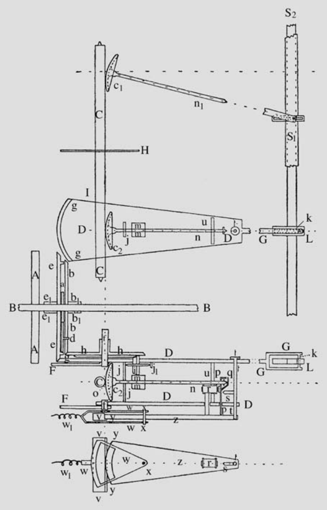

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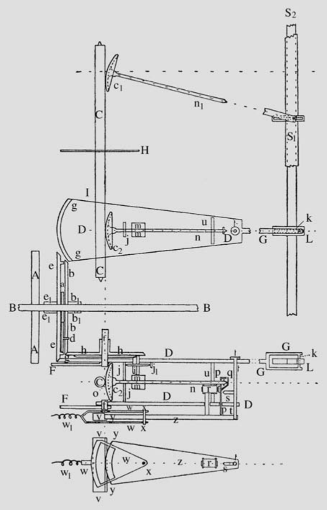

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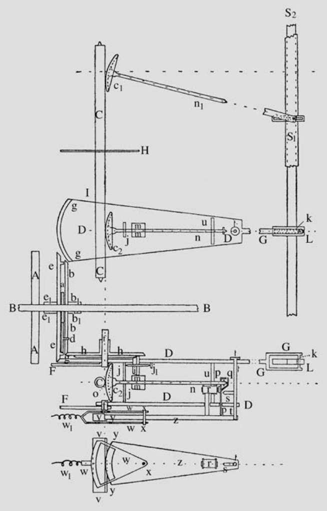

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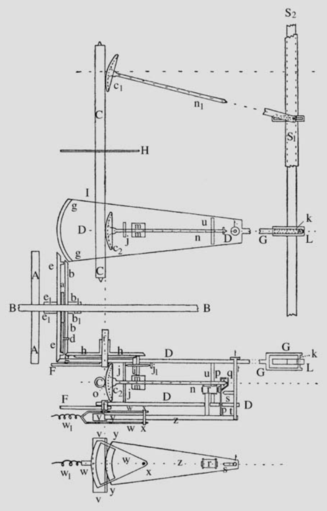

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(4) Chaotic dynamics / complexity
(5) Computability
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- Last geometric theorem (Poincaré-Birkhoff Theorem: every area-preserving, orientation-preserving homeomorphism of an annulus that rotates the two boundaries in opposite directions has at least two fixed points) $\implies$ in the PCR3BP periodic orbits are infinite.
- But, in the general problem, *proven* to exist: Euler and Lagrange orbits.
- And rotating central configurations.
- Chenciner-Montgomery remarkable figure-eight.
- Find: Symmetric Lagrangean minimizers
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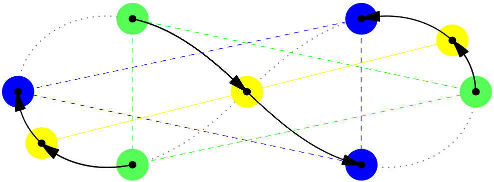
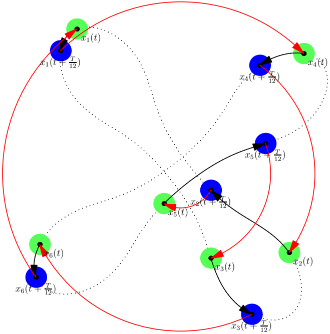
15. Symmetry and choreographies



Chenciner–Montgomery Eight Choreography [▷]



Two symmetric 3-choregraphies [▷]

1. Poincaré, topology and the n-body problem
2. Periodic orbits, symmetries, geometry and Lagrangean minimizers
3. **Qualitative features, analysis, modeling and computing**
4. Explorations and crawlers: symmetry groups, loop spaces, critical points and interactive distributed computing
5. Human interaction: visualization, CLI and interfaces, 3D manipulation and remote computations
6. Conclusions

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- Computing (periodic) orbits: simulations, ODE, qualitative features. Restricted 3BP and perturbations: Copenhagen (Stromgren) mechanical, Karl Sundman (perturbographe), electronic Hénon (Nice), Broucke, Szebehely, Bruno, Carles Simó, Stuchi, Alessandra Celletti, Luigi Chierchia (Computer-assisted proofs), Krakow School, Hans Koch).
- What to expect on an orbit? Detecting *chaos* and *instability*.
- Existence.
- Symmetry group.
- Approximation.
- Stability.
- Global flow.
- Try to put together a stratified singular infinitely dimensional Morse theory (computationally first).

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- A template numerical optimization search with symbolic data ($O(d)$, Σ_n, ...)
- Collisions: what to do about near-colliding trajectories? Strong-force trick? Smoothing? Regularizations like Sundman or Levi-Civita or McGehee?
- Coercivity: what to do of minima or critical points at infinity?
- *Visibility* of critical points: when the finite-dimensional approximations are close to real solutions?
- Closure: when the infinite-dimensional critical point can be approximate by finite-dimensional approximations?
- Ingredients: Sobolev spaces, geometry and topology, calculus of variations, numerical analysis and scientific computing, computer algebra.

  $\Longrightarrow$ A mixture of GAP, F95 and scientific libraries (IMSL, gsl, slatec, minuit, minpack, ...). Glued with paper clips, python and duct tape. Kind of a minor sage-math?

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- Ingredients: Sobolev spaces, geometry and topology, calculus of variations, numerical analysis and scientific computing, computer algebra.

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20. Preparing initial data

- ➔ The basic module: given initial data (random or given), a level of approximation (number of Fourier coefficients and intermediate steps for integral approximations of the potential), find the closest local minimum, or the closest critical point (with modified conjugate gradients, or Newton-Powell, or other standard schemes). Output the periodic orbit.
- ➔ Then: reshape and repeat, or change some parameters and use continuation methods.
- ➔ Thousands and thousands of periodic orbits found (as expected), with many symmetry groups.
- ➔ Next step: Crawling in the space of all groups. Explore the set of all possible symmetry groups, and classify them (according to features of the symmetric configuration space).
- ➔ Features of a group: representation theory and permutations. Again: GAP and some wrapping scripts.

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21. Tools

- ➔ Visualization: geomview, OpenGL, gnuplot, pdf+eps. Graphical interface, manipulate the camera, the object, point and click.
- ➔ About the minimization, a CLI with mini-language and python interactive shell. Remote 3D manipulation: sends commands to geomview via OOGL.
- ➔ Mobile manipulator/visualizator + remote connection (to a server or a cluster). It works with good open networks.
- ➔ Remote interacively usage of a cluster. MPI (OpenMPI), pytthon and ssh, pyRPC, objectify initial data, symmetry groups and periodic orbits.
- ➔ Calculate attributes: norm of gradient, Floquet multipliers, shooting, multi-shooting, stability, ...
- ➔ Then, use the cluster for frame rendering and video encoding. Fun part.

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22. Example session

```
RequirePackage("symorb");
dim:=3;

phi:=(Sqrt(5)-1)/2;

mat1:=[[phi/2,(1+phi)/2,1/2],[(1+phi)/2,-1/2,phi/2],
  [1/2,phi/2,-(1+phi)/2]];;
mat2:=[[0,1,0],[0,0,1],[1,0,0]];;
K:=GroupWithGenerators([mat1,mat2]);;
hom:=ActionHomomorphism(K,K,OnRight);
s1:=Image(hom,mat1);
s2:=Image(hom,mat2);
matrot:=[ [ 0, -1, 0 ], [ 1, 0, 0 ], [ 0, 0, -1 ] ];
a:=[[-1,0,0],[0,-1,0],[0,0,-1]];
mat3:=mat1*mat2*a;

GG:=GroupWithGenerators([mat1,mat2,mat3]);
```

22. Example session (cont.)

```
nhom:=ActionHomomorphism(GG,K,OnPoints);
Image(nhom,mat1);
Image(nhom,mat2);
rotS:=Image(nhom,mat3);

NOB:=Size(K);
kert:=GroupWithGenerators([ Tuple([mat1,s1 ]), Tuple([mat2,s2]) ]
rotV:=mat3;

LSG:=LagSymmetryGroup(0,NOB,kert, rotV,rotS,rotV,rotS);
MakeMinorbSymFile("icosa-luminy",LSG);
```

22. Example session (cont.)

```
ferrario@lkl01 ~ $ minpath
minpath -- beginning at Tue Sep 10 12:09:55 CEST 2013

symfiles:

    fourlag.sym
    [...]
    icosa-luminy.sym

MinorbShell > x=minpath()

 1)  fourlag.sym
 [...]
 19)  icosa-luminy.sym
 x)  eXit

  ...Select a Number: > 19
 You have selected file: icosa-luminy.sym
```

22. Example session (cont.)

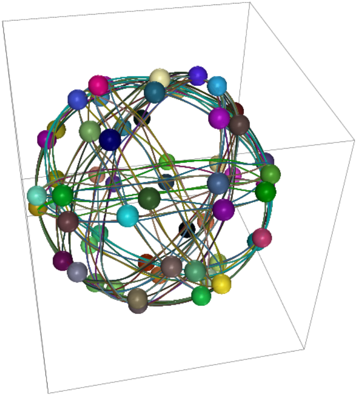
```
MinorbShell > res=remjob(x,30,"new();relax(202)")
remjob called with nsol= 30
beginning the job...
tmpsSfPIk                       100%  293KB 292.8KB/s 292.8KB/s   00:00
:: about to exec the following...:
/home/ferrario/local/symorb/py/par/parminpath --solutions=30
--output=/home/ferrario/.symorb/objsfile_1001745813788O7812.objs
--load=/home/ferrario/.symorb/obj_1001745813788O7812.obj Pypar (v
initialised MPI OK with 1 processors
parminpath: we were called with args
['/home/ferrario/local/symorb/py/par/parminpath', '--solutions=3
'--output=/home/ferrario/.symorb/objsfile_1001745813788O7812.obj
'--load=/home/ferrario/.symorb/obj_1001745813788O7812.obj']
now starting parminpath on 31 nodes...
Pypar (version 2.1.5) initialised MPI OK with 31 processors
parminpath: we were called with args ['/home/ferrario/local/bin/pa
'--parallel',
'--output=/home/ferrario/.symorb/objsfile_1001745813788O7812.obj
```

22. Example session (cont.)

```
'--solutions=30', '--load=/home/ferrario/.symorb/obj_10017458137
...
[skip]
...
 # relaxing...
 # using IMSL DUMIDH
 # Unconstrained Minimization with finite-Difference Hessian
 # using NONLINEAR DNEQNJ
 # Newton-Powell Analytic Jacobian
 # writing out...
 # done...
 # dTOL= 1.491668146240041E-154
 ==> action: 1820.4218; howsol: 2.0671e-12
received from node 28: <minpath object; NOB=60, dim=3, steps=24>
[numCompleted= 20/30 -- numFailed=0]
```

22. Example session (cont.)

OUTPUT:



(Icosahedral 60-body with 10-adic hip-hop rotation: res-lum00.data)

1. Poincaré, topology and the n-body problem
2. Periodic orbits, symmetries, geometry and Lagrangean minimizers
3. Qualitative features, analysis, modeling and computing
4. Explorations and crawlers: symmetry groups, loop spaces, critical points and interactive distributed computing
5. Human interaction: visualization, CLI and interfaces, 3D manipulation and remote computations
6. **Conclusions**

23. Conclusions

- Naive gluing together different programming paradigms, languages, fields and libraries from var contexts (symbolic algebra, computer algebra systems, AI, visualization, ...).
- Novelty of approach is granted. At the same time: almost nobody will fully understand or appreciate it, and not much funding (cf. W. Stein).
- Why has there been a partial stigma on computational pure mathematics? Why is that that if it is computational, then it has to be applied to some real-world problem?
- What does it mean to assist computationally a qualitative analysis? Just computer-assisted proofs? Computer-aided proofs? Or, topological semantic data analysis? What does it mean to analyze semantic data?
- And, epistemologically: what does it mean to let computer help us understand? What does it mean to understand?
- MCQ-XeLaTeX (OMR and test lazy grading). `http://www.matapp.unimib.it/~ferrario/var/mcqxelatex.html`

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The end

https://github.com/dlfer/symorb

«Thank you !»