SageMath experiments in Differential and Complex Geometry

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Summary

Solve research-level problems
in Differential and Complex Geometry
by SageMath.

SageMath

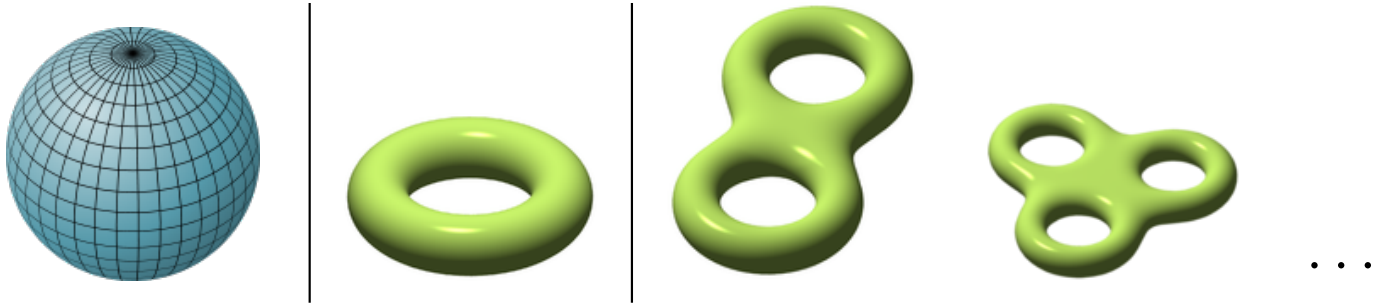
"SageMath is a free open-source mathematics software system licensed under the GPL. It builds on top of many existing open-source packages: NumPy, SciPy, matplotlib, Sympy, Maxima, GAP, FLINT, R and many more. Access their combined power through a common, Python-based language or directly via interfaces or wrappers."

SageMath, the Sage Mathematics Software System (Version 7.3), The Sage Developers, 2016, http://www.sagemath.org.

https://cloud.sagemath.com

Differential Geometry

Classification
of some class of objects up to some notion of equivalence
by means of some sort of invariants.



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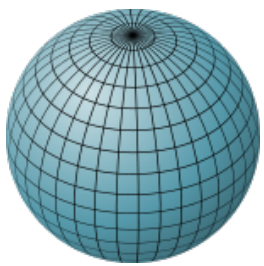
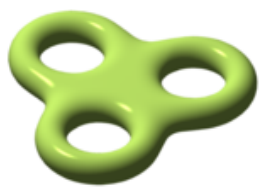

genus $= 0$


genus $= 1$


genus $= 2$


genus $= 3$

$\ldots$

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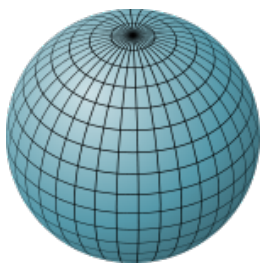
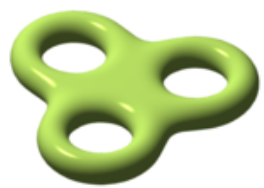


genus $= 0$
csc positive

genus $= 1$
csc zero

genus $= 2$



genus $= 3$

csc negative

$\ldots$

Two problems in Differential Geometry

We focus on a class of objects which have enough symmetries so that their geometry is reduced linearly.


D. Angella, M. G. Franzini, F. A. Rossi, Degree of non-Kählerianity for 6-dimensional nilmanifolds, *Manuscripta Math.* 148 (2015), no. 1-2, 177–211.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Two problems in Differential Geometry

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Objects	6-dim nilmanifolds
Invariants	cohomologies

Objects	4-dim Lie algebras
Invariants	locally conformal symplectic struct


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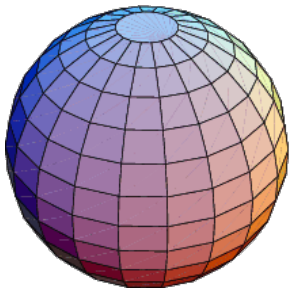
Nilmanifolds and solvmanifolds

- **manifold** = object locally modelled on $\mathbb{R}^n$, so to define a notion of smooth map



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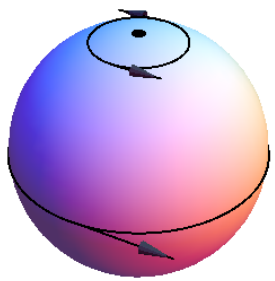
- **manifold** = object locally modelled on $\mathbb{R}^n$, so to define a notion of smooth map
- **nil/solvmanifold** = homogeneous manifold of a connected nilpotent/solvable Lie group



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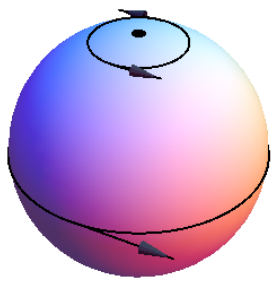

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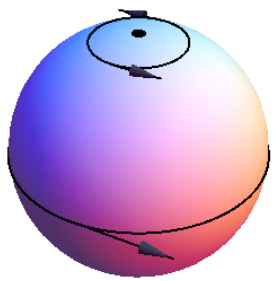
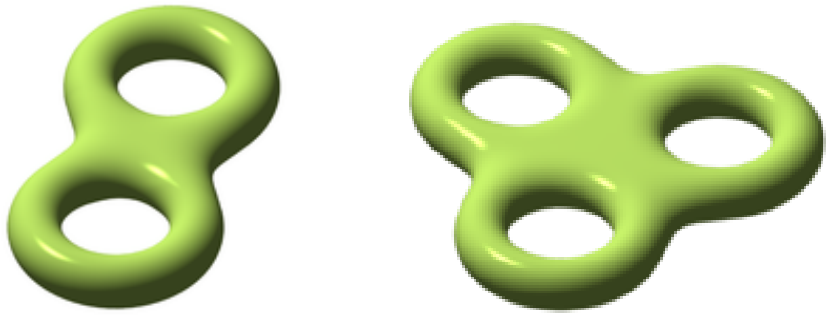


- **invariant objects** for the action of the Lie group give (partial) informations on the geometry
- they are encoded in the **Lie algebra** = vector space $\mathbb{R}^n$ with a differential $d\colon \wedge^k \mathbb{R}^n \to \wedge^{k+1}\mathbb{R}^n$ (Leibniz rule and $d \circ d = 0$)

Cohomologies



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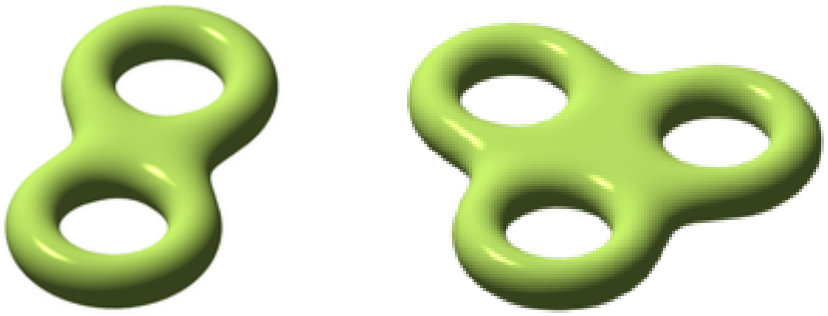

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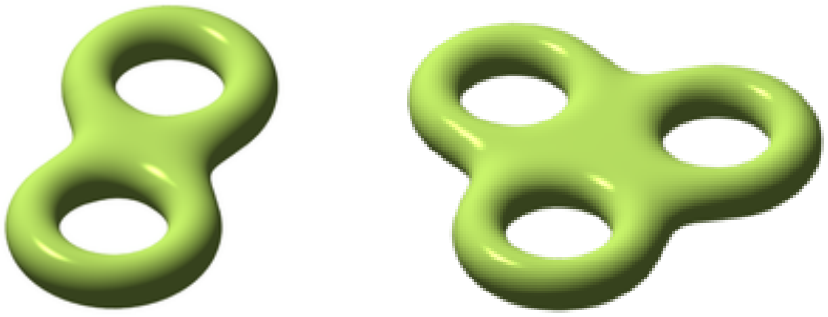

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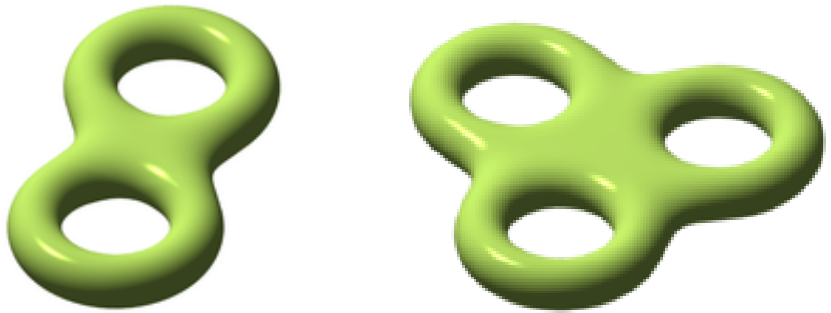
- *e.g.* **de Rham cohomology**

Cohomologies



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- for *nil/solvmanifolds*: recall that we have a differential $d\colon \wedge^k \mathbb{R}^n \to \wedge^{k+1}\mathbb{R}^n$ with $d \circ d = 0$: **Lie algebra cohomology** $\frac{\ker d}{\operatorname{im} d}$

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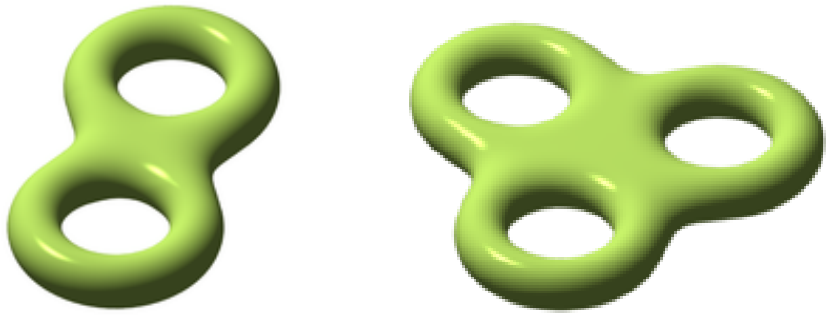

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Thm (Nomizu)

For nilmanifolds, de Rham cohomology = Lie algebra cohomology

Cohomologies with SageMath

X nilmanifold with Lie algebra

$$\mathfrak{h}_8 := (-23, 0, 0, 0, 0, 0)$$

that is, $\mathfrak{h}_8 = \mathrm{span}\{e^0, \dots, e^5\}$ with $de^0 = -e^2 \wedge e^3$, $de^1 = \cdots = de^5 = 0$
and Leibniz rule ($d(\alpha \wedge \beta) = d\alpha \wedge \beta \pm \alpha \wedge d\beta$)

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[-e2^e3, 0, 0, 0, 0, 0]
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sage: [d(b) for b in E.gens()]
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sage: all([d(d(b)) == 0 for b in E.gens()])
True
```

Cohomologies with SageMath

```
sage: cplx_dR = d.chain_complex()
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Cohomologies with SageMath

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Chain complex with at most 7 nonzero terms over Symbolic Ring
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Cohomologies with SageMath

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sage: cplx_dR.homology()
{0: Vector space of dimension 1 over Symbolic Ring,
    1: Vector space of dimension 5 over Symbolic Ring,
    2: Vector space of dimension 11 over Symbolic Ring,
    3: Vector space of dimension 14 over Symbolic Ring,
    4: Vector space of dimension 11 over Symbolic Ring,
    5: Vector space of dimension 5 over Symbolic Ring,
    6: Vector space of dimension 1 over Symbolic Ring}
```

$$b_0(X) = b_6(X) = 1, \qquad b_1(X) = b_5(X) = 5,$$
$$b_2(X) = b_4(X) = 11, \qquad b_3(X) = 14.$$

Cohomologies with SageMath

```
sage: t = cputime()
sage: E = ExteriorAlgebra(SR,'e',6)
sage: for algebra in alg_nilp_6.keys():
          d = E.coboundary(alg_nilp_6[algebra])
          H = d.chain_complex().homology()
          Poincare_poly = sum([H[j].dimension() * x^j \
              for j in range(len(H))])
          print algebra, "\n\t", Poincare_poly
sage: cputime(t)
```

Cohomologies with SageMath

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\mathfrak{g}_{6.N18}^{-1} :
        x^6 + 2*x^5 + 4*x^4 + 6*x^3 + 4*x^2 + 2*x + 1
6\mathfrak{g}_{1} :
        x^6 + 6*x^5 + 15*x^4 + 20*x^3 + 15*x^2 + 6*x + 1
\mathfrak{g}_{3.1}\oplus 3g_{1} :
        x^6 + 5*x^5 + 11*x^4 + 14*x^3 + 11*x^2 + 5*x + 1
...
0.7200000000000024
```

Cohomologies with SageMath

Introducing a complex structure:

```
sage: mat_J = matrix(6,6,[[ 0,-1, 0, 0, 0, 0], \
                          [ 1, 0, 0, 0, 0, 0], \
                          [ 0, 0, 0,-1, 0, 0], \
                          [ 0, 0, 1, 0, 0, 0], \
                          [ 0, 0, 0, 0, 0,-1], \
                          [ 0, 0, 0, 0, 1, 0]])
sage: J = E.lift_morphism(mat_J)
```

Cohomologies with SageMath

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                          [ 1, 0, 0, 0, 0, 0], \
                          [ 0, 0, 0,-1, 0, 0], \
                          [ 0, 0, 1, 0, 0, 0], \
                          [ 0, 0, 0, 0, 0,-1], \
                          [ 0, 0, 0, 0, 1, 0]])
sage: J = E.lift_morphism(mat_J)

sage: for j in range(3):
          varphi[j] = E.gens()[2*j] - I * J(E.gens()[2*j])
          print varphi[j], "|-->", J(varphi[j])
e0 - I*e1 |--> I*e0 + e1
e2 - I*e3 |--> I*e2 + e3
e4 - I*e5 |--> I*e4 + e5
```

Cohomologies with SageMath

```
sage: E.<varphi0, varphi1, varphi2, \
          barvarphi0, barvarphi1, barvarphi2> = \
          ExteriorAlgebra(SR)
sage: str_eq = {(1,4): I/2 * varphi0 + I/2 * barvarphi0}
sage: delbar = E.coboundary(str_eq)
```

Cohomologies with SageMath

```
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sage: delbar = E.coboundary(str_eq)

sage: HDol = delbar.chain_complex().homology() ; HDol
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Cohomologies with SageMath

- Ceballos-Otal-Ugarte-Villacampa classify invariant complex structures on 6-dimensional nilmanifolds in infinite families
- other cohomologies: Bott-Chern and Aeppli

```
sage: hDol = {}
sage: for j in range(len(E.gens())+1):
          V = VectorSpace(SR,len(E.basis(j)))
          Z = V.subspace(d_mat[j].transpose().kernel())
          B = Z.subspace([V([d_mat[j-1][h,k] \
              for h in range(d_mat[j-1].nrows())]) \
              for k in range(d_mat[j-1].ncols())])
          H = Z.quotient(B)
          hDol[j] = dim(H)
sage: hDol
{0: 1, 1: 3, 2: 5, 3: 6, 4: 5, 5: 3, 6: 1}
```

lcs structures

- **symplectic** = non-degenerate 2-form ω such that

$$d\omega = 0$$

- **locally conformally symplectic (lcs)** = locally, up to conformal change, of the type above:

$$d\omega = \vartheta \wedge \omega \quad \text{with} \quad d\vartheta = 0$$

lcs structures

classify lcs structure on 4-dimensional Lie algebra

$$\mathfrak{r}_4 = (03 + 13, 13 + 23, 23, 0)$$

lcs structures

classify lcs structure on 4-dimensional Lie algebra

$$\mathfrak{r}_4 = (03 + 13, 13 + 23, 23, 0)$$

```
sage: E = ExteriorAlgebra(SR, 'e', 4)
sage: str_eq = {
          (0,3) : E.gens()[0],
          (1,3) : E.gens()[0] + E.gens()[1],
          (2,3) : E.gens()[1] + E.gens()[2],
      }
sage: d = E.coboundary(str_eq)
```

lcs structures

classify lcs structure on 4-dimensional Lie algebra

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      }
sage: d = E.coboundary(str_eq)

sage: print([d(b) for b in E.gens()])
[e0^e3 + e1^e3, e1^e3 + e2^e3, e2^e3, 0]
```

lcs structures

$$d\vartheta = 0 \qquad \text{and} \qquad d\omega = \vartheta \wedge \omega$$

```
sage: thetacoeff = var(["theta%d" % j \
          for j in range(len(E.basis(1)))])
sage: theta = sum([thetacoeff[j] * E.gens()[j] \
          for j in range(len(E.basis(1)))])
```

lcs structures

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sage: d(theta)
```

$$\vartheta_0 e^0 \wedge e^3 + (\vartheta_0 + \vartheta_1) e^1 \wedge e^3 + (\vartheta_1 + \vartheta_2) e^2 \wedge e^3$$

lcs structures

$$d\vartheta = 0 \qquad \text{and} \qquad d\omega = \vartheta \wedge \omega$$

```
sage: theta = theta3 * e3
sage: Omegacoeff = var(["omega%d%d" % (i,j) \
          for i in range(len(E.basis(1))) \
          for j in range(i+1,len(E.basis(1)))])
sage: Omega = sum([Omegacoeff[j] * list(E.basis(2))[j] \
          for j in range(len(E.basis(2)))])
```

lcs structures

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sage: d(Omega) - theta * Omega
```

$$(-\omega_{01}\vartheta_3 - 2\,\omega_{01})\, e_0\wedge e_1\wedge e_3 + (-\omega_{02}\vartheta_3 - \omega_{01} - 2\,\omega_{02})\, e_0\wedge e_2\wedge e_3$$
$$+ (-\omega_{12}\vartheta_3 - \omega_{02} - 2\,\omega_{12})\, e_1\wedge e_2\wedge e_3$$

lcs structures

- if $\vartheta_3 \neq -2$, then $\omega_{01} = \omega_{02} = \omega_{12} = 0$;
  then $\Omega^2 = 0$, so Ω is degenerate
- if $\vartheta_3 = -2$, then we get the lcs structures

$$\begin{aligned} \vartheta &= -2 \cdot e^3 \\ \Omega &= \omega_{03} \cdot e^0 \wedge e^3 + \omega_{12} \cdot e^1 \wedge e^2 + \omega_{13} \cdot e^1 \wedge e^3 + \omega_{23} \cdot e^2 \wedge e^3 \end{aligned}$$

where $\omega_{23}, \omega_{13}, \omega_{12}, \omega_{03} \in \mathbb{R}$ satisfy $\omega_{12} \cdot \omega_{03} \neq 0$

lcs structures

Are they equivalent?

lcs structures

Are they equivalent?

Equivalences ψ of the Lie algebra are induced by invertible matrices

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

such that

$$\psi(d(\_)) = d(\psi(\_))$$

lcs structures

$$\begin{pmatrix} \frac{1}{\omega_{03}} & 0 & 0 & 0 \\ 0 & \frac{1}{\omega_{03}} & 0 & 0 \\ 0 & 0 & \frac{1}{\omega_{03}} & 0 \\ 0 & \frac{\omega_{01}}{\omega_{02}} & -\frac{\omega_{12}}{\omega_{02}} & 1 \end{pmatrix}$$

```
sage: mat1 = matrix(4, 4, [ ... ])
sage: morphism1 = E.lift_morphism(mat1)
```

lcs structures

$$\begin{pmatrix} \frac{1}{\omega_{03}} & 0 & 0 & 0 \\ 0 & \frac{1}{\omega_{03}} & 0 & 0 \\ 0 & 0 & \frac{1}{\omega_{03}} & 0 \\ 0 & \frac{\omega_{01}}{\omega_{02}} & -\frac{\omega_{12}}{\omega_{02}} & 1 \end{pmatrix}$$

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sage: all([morphism1(d(b)) == d(morphism1(b)) \
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True
```

lcs structures

$$\begin{pmatrix} \frac{1}{\omega_{03}} & 0 & 0 & 0 \\ 0 & \frac{1}{\omega_{03}} & 0 & 0 \\ 0 & 0 & \frac{1}{\omega_{03}} & 0 \\ 0 & \frac{\omega_{01}}{\omega_{02}} & -\frac{\omega_{12}}{\omega_{02}} & 1 \end{pmatrix}$$

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           for b in E.gens()])
True

sage: mat1.determinant()
omega03^(-3)
```

lcs structures

```
sage: theta = morphism1(theta)
sage: Omega = morphism1(Omega)
```

lcs structures

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sage: theta ; Omega
-2*e3
e0^e3 + r4/r5^2*e1^e2
```

lcs structures

```
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sage: theta ; Omega
-2*e3
e0^e3 + r4/r5^2*e1^e2
```

$$\begin{cases} \vartheta &= -2 \cdot e^3 \\ \Omega &= e^0 \wedge e^3 + \sigma \cdot e^1 \wedge e^3, \quad \text{for } \sigma \neq 0 \end{cases}$$

lcs structures

Are they different?

lcs structures

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```
sage: theta = -2 * E.gens()[3]
sage: var_Omega = var("sigma1 sigma2")
sage: Omega1 = ... + sigma1 * E.gens()[1] * E.gens()[2]
sage: Omega2 = ... + sigma2 * E.gens()[1] * E.gens()[2]
sage: var_mat = var(["a%d%d" % (i,j) for ...])
sage: mat = matrix(len(E.gens()), len(E.gens()), list(var_mat))
sage: morphism = E.lift_morphism(mat)
```

lcs structures

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sage: morphism = E.lift_morphism(mat)

sage: Anello = QQ[var_mat + var_Omega]
sage: ideal0 =[(d(morphism(b)) - morphism(d(b))).interior_product(c) \
          for c in E.basis(2) for b in E.basis(1)]
sage: ideal1 = [(morphism(theta) - theta).interior_product(c) \
          for c in E.basis(1)]
sage: ideal1 += [(morphism(Omega1) - Omega2).interior_product(c) \
          for c in E.basis(2)]
```

lcs structures

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          for c in E.basis(1)]
sage: ideal1 += [(morphism(Omega1) - Omega2).interior_product(c) \
          for c in E.basis(2)]

sage: B = Anello.ideal(ideal0 + ideal1).groebner_basis() ; B
[a21^2 + a31*sigma2 - a20, ..., sigma1 - sigma2]
```

Thanks for the attention

GeCo GeDi project:

`http://gecogedi.dimai.unifi.it`