Topology

An extremely short introduction

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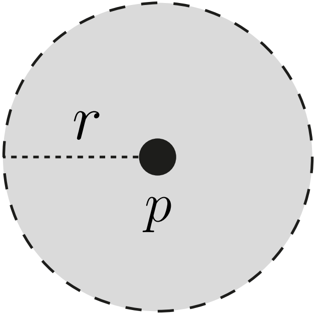
Metric Spaces

Metric spaces are sets X equipped with a distance $d(x, y)$ between the points $x, y \in X$.



The open ball of radius r around a point p.

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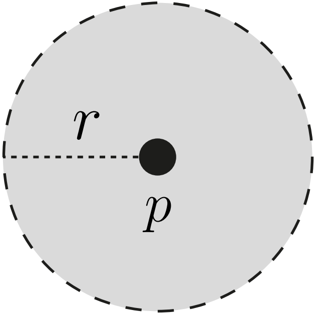

The open ball of radius r around a point p.

In these spaces we can specify what "nearby" and "close" means in a quantifiable manner.

Topological Spaces

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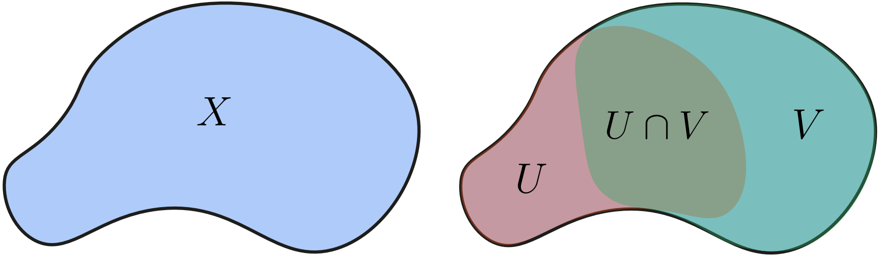
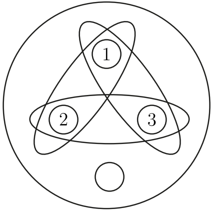


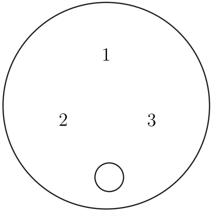
The open sets can be thought as a neighbourhood of the points they contain.

Some (non)examples of topological spaces

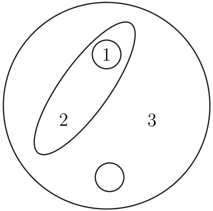
First, let us look over some topologies over the finite set $X = \{1, 2, 3\}$.



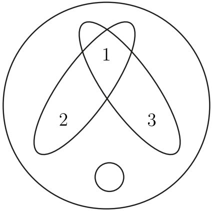
Discrete Topology



Trivial Topology

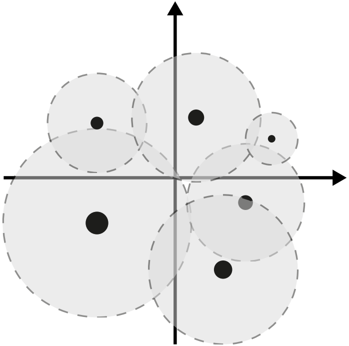


$\{\{1\}, \{1, 2\}, \{1, 2, 3\}, \emptyset\}$



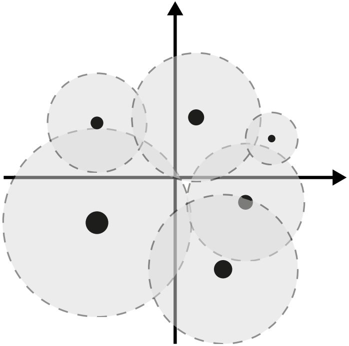
Not a topology

Euclidean topology



The Euclidean Topology is the topology given by the open sets induced by the Euclidean metric, i.e., all the open balls and their arbitrary unions.

Euclidean topology



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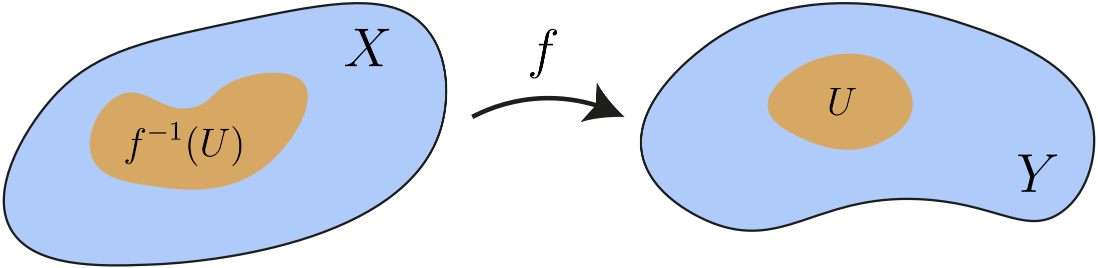
More generally, for any metric space (X, d) the open balls generate a topology on X called the *topology generated by d*.

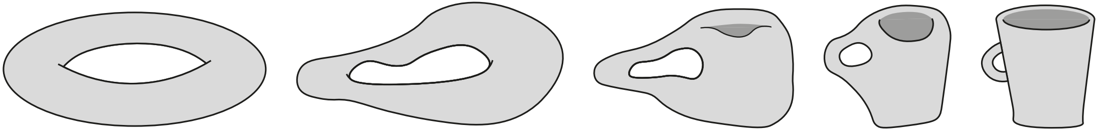
Continuity

A function $f : X \to Y$ between topological spaces is continuous if for every open subset $U \subseteq Y$ its preimage $f^{-1}(U)$ is open in X.



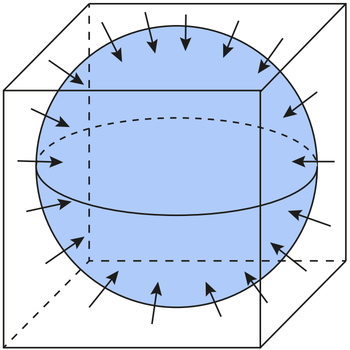
Homeomorphisms

A *homeomorphisms* $\phi : X \to Y$ between topological spaces is a bijective map such that both ϕ and ϕ^{-1} are continous. If a homeomorphism exists between topological spaces X, Y, then we say they are *homeomorphic*.



It is a classic joke that to a topologist, a mug and a donut are the same thing.

Example Homeomorphism



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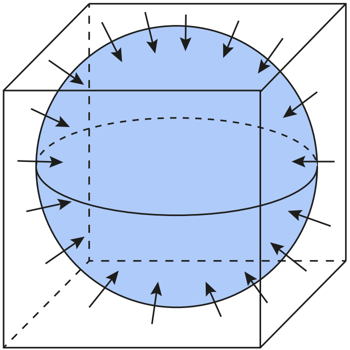

Let $C = \{(x, y, z) \mid \max(|x|, |y|, |z|) = 1\}$ be the cubical surface in the image

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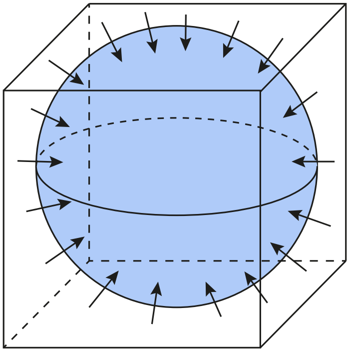

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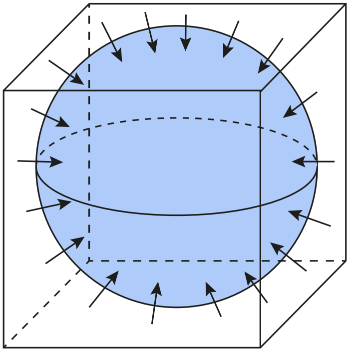

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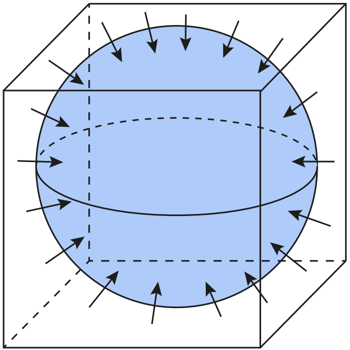

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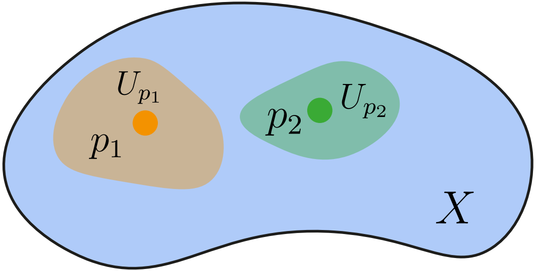
Corollary: "Smoothness" is not a topological property.

Hausdorff Spaces

The physical space respects the intuition that we can “separate” any two points. Hausdorff spaces are spaces where any two points can be “housed off”.

Felix Hausdorff
(Source: Wikipedia)



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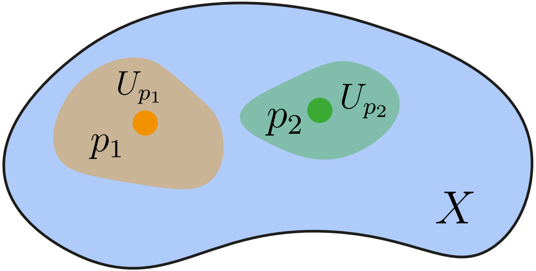
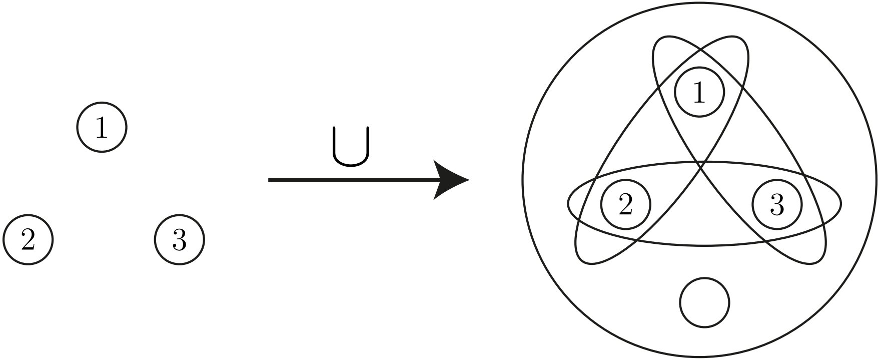


Any two points have some disjoint neighbourhoods.

A topological space X is said to be a *Hausdorff space* if given any points $p_1, p_2 \in X$, there exists neighbourhoods U_1 of p_1 and U_2 of p_2 with $U_1 \cap U_2 = \emptyset$.

Basis of a topological space

Let X be a topological space. A collection $\mathcal{B}$ of open subsets of X is called a *basis for the topology of X* if every open subset is the union of some collection of elements of $\mathcal{B}$.



The singleton subsets of X provide a basis for the discrete topology.

Adjusting the resolution: second countable spaces

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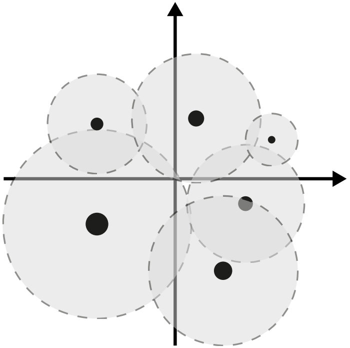

We can use the Euclidean topology in $\mathbb{R}^d$ as a reference and observe that we could select as a basis all the open balls with rational coordinates and radius.

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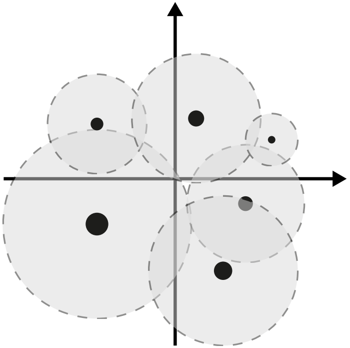

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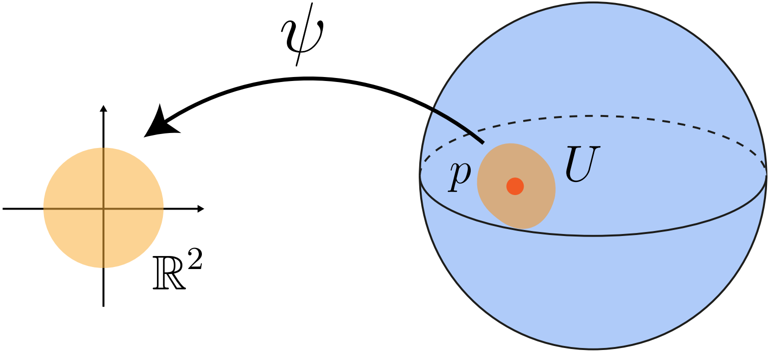
This motivates the following definition: A topological space is *second countable* if it admits a countable basis for its topology.

Manifolds

An *n*-dimensional *topological manifold* is a second countable Hausdorff space such that every point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$.



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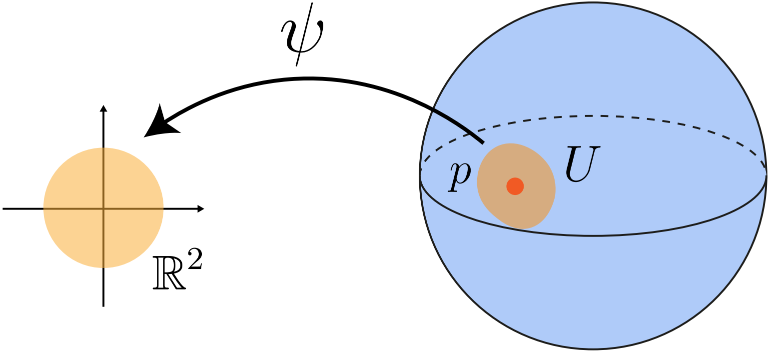
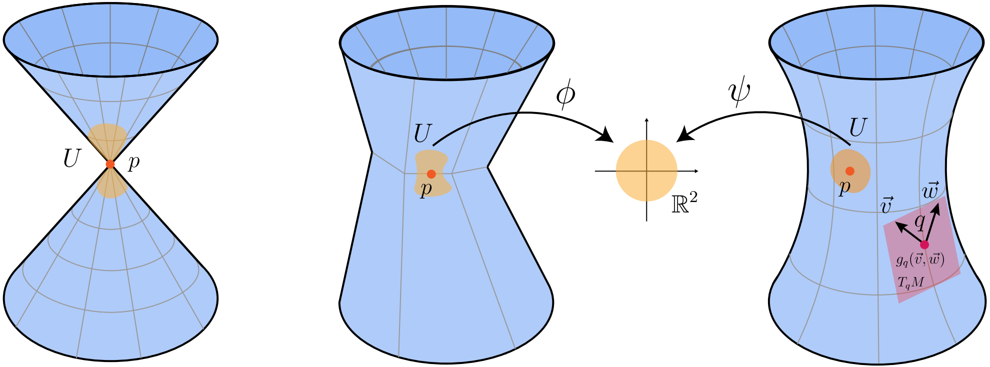


The pair (U, ψ) is called a *chart* and U is called a *coordinate domain*.

The big picture

There is a chain of structures we can equip spaces with.



The road from topological spaces to Riemannian manifolds.