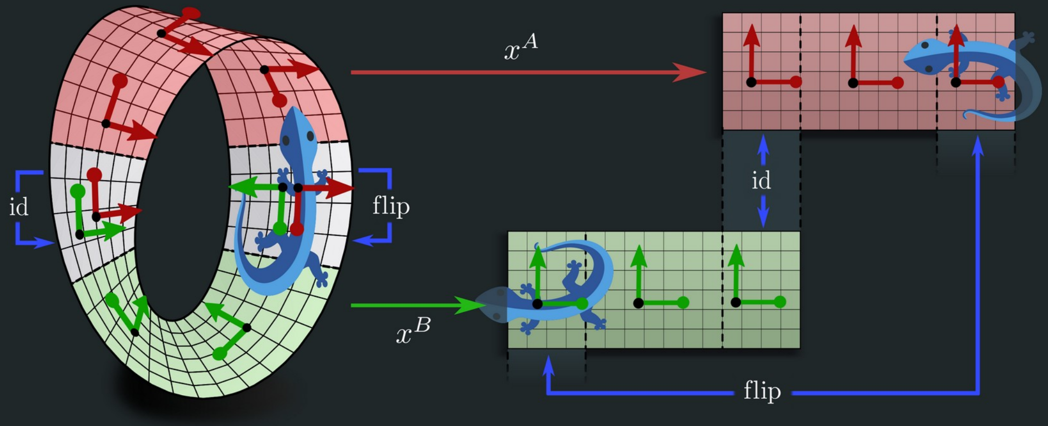
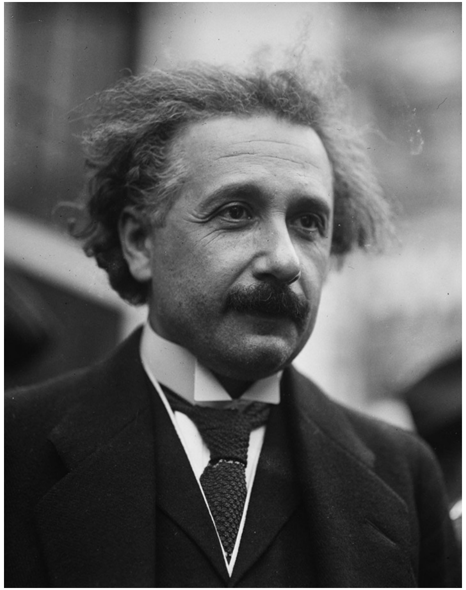
Equivariant & coordinate independent CNNs on Riemannian manifolds

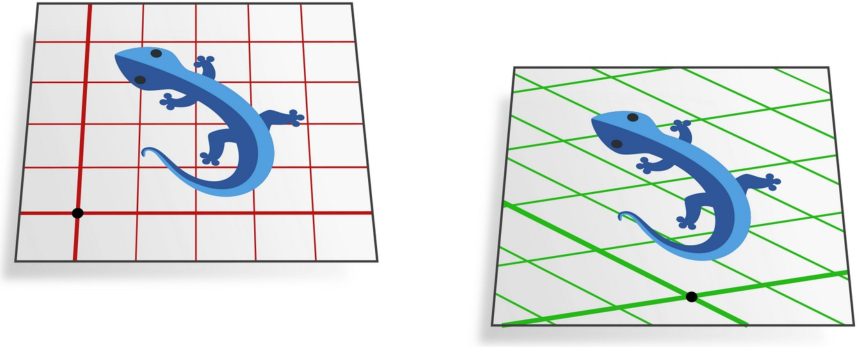
Maurice Weiler
AMLab, QUVA Lab
University of Amsterdam
@maurice_weiler



Principle of Covariance (Einstein, 1916)

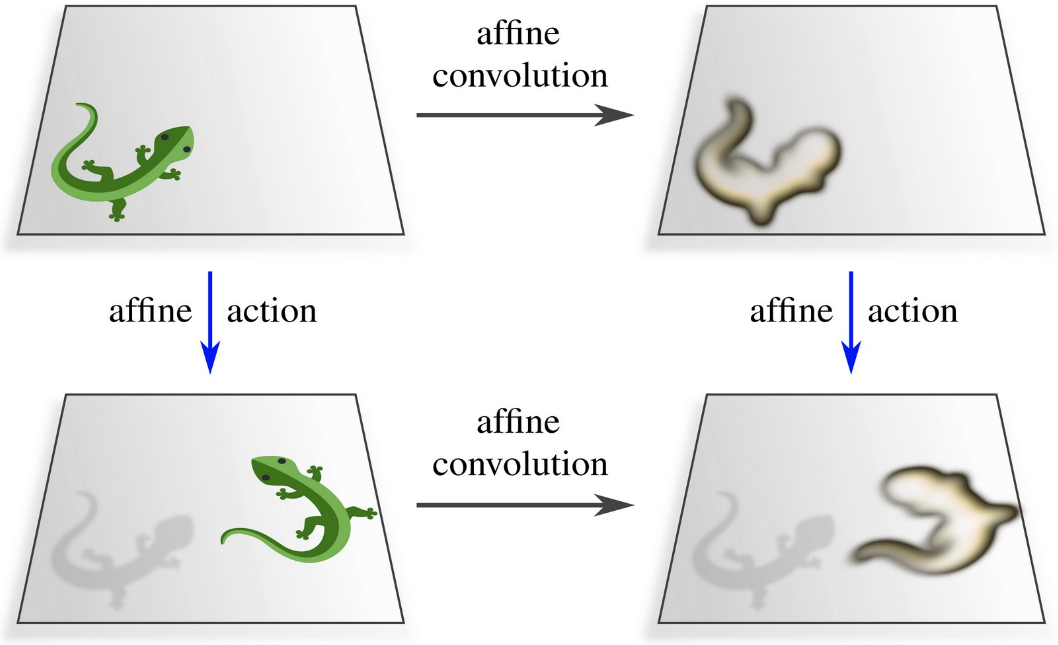


*"Universal **laws of nature** are to be expressed by equations which hold good for all systems of coordinates […] "*

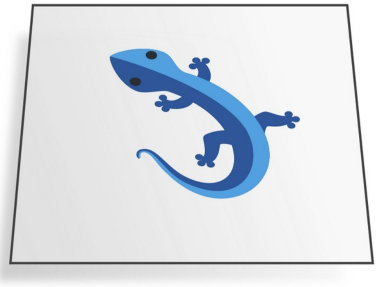


*"The **inference of CNNs** is to be expressed by equations which hold good for all systems of coordinates […] "*

Equivariant convolutions



active transformations - acting on the data itself:

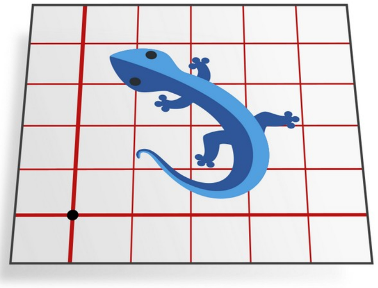


vs.

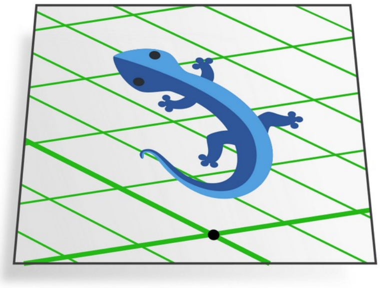
$\Rightarrow$ equivariant NNs

$\nparallel$

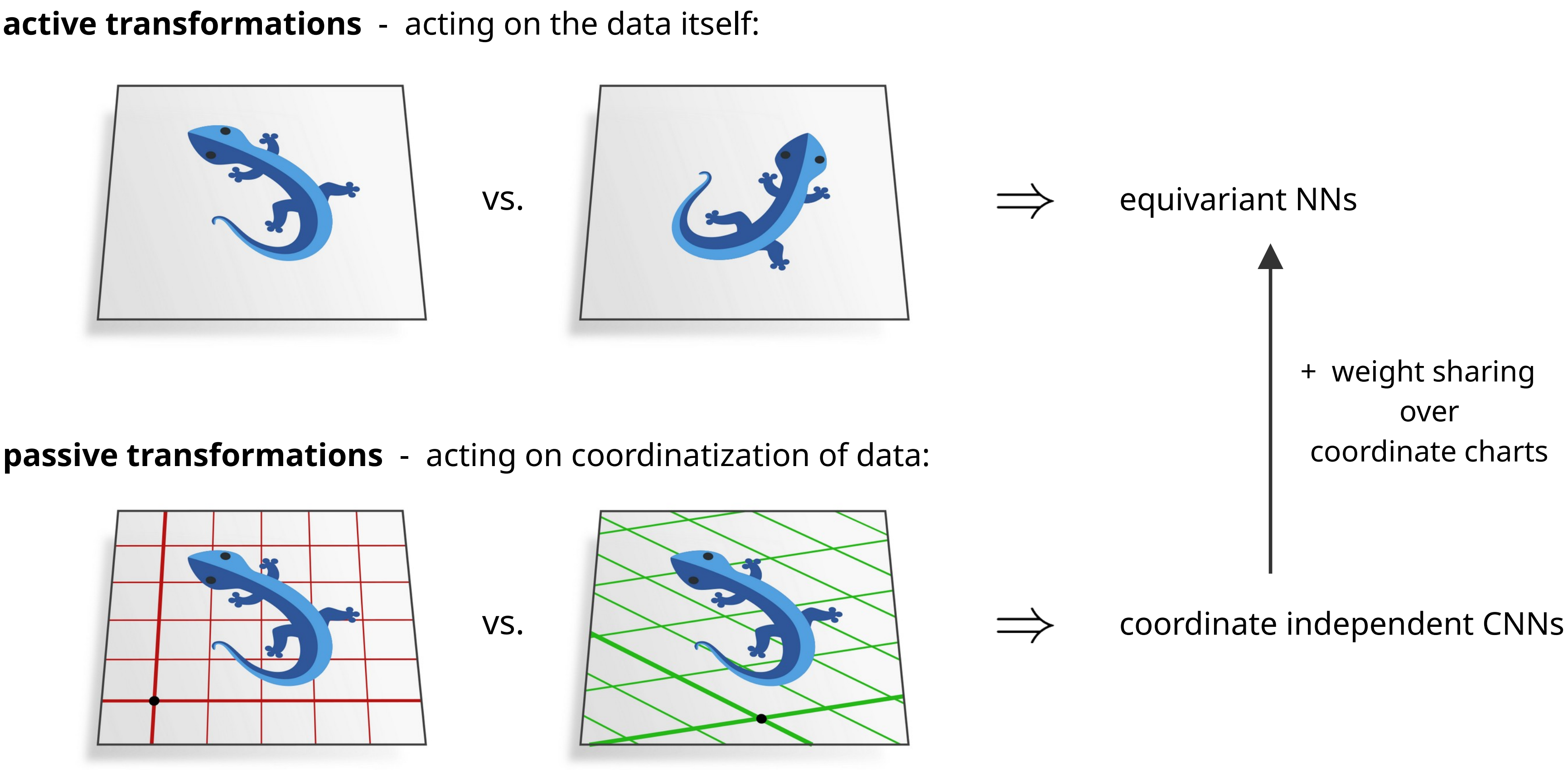
passive transformations - acting on coordinatization of data:



vs.



$\Rightarrow$ coordinate independent CNNs


active transformations - acting on the data itself:
vs.
⇒ equivariant NNs
+ weight sharing
over
coordinate charts
passive transformations - acting on coordinatization of data:
vs.
⇒ coordinate independent CNNs

Outline

Data gauges & gauge independent neural networks

Coordinate independent CNNs on Riemannian manifolds

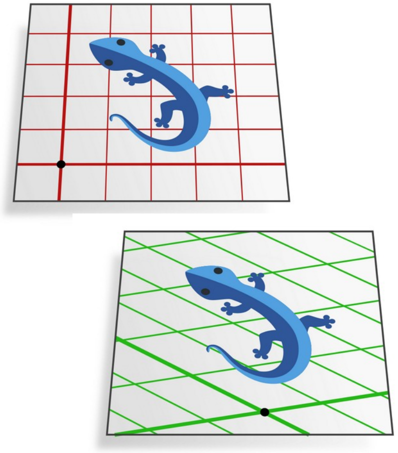
Gauge theories

physical systems or data often have *no canonical mathematical description*

a *gauge* has to be chosen *arbitrarily*

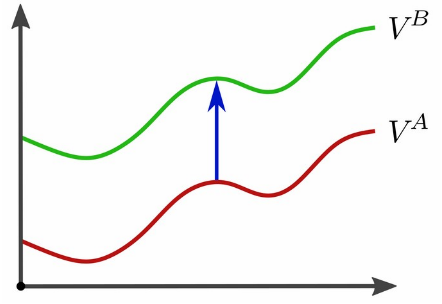
gauge theories ensure that their prediction is *equivalent for any gauge*

space / coordinates

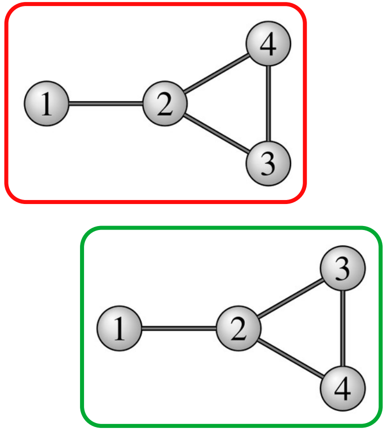


set / ordering

potential / reference potential



graph / node indexing



Weighing scale gauging

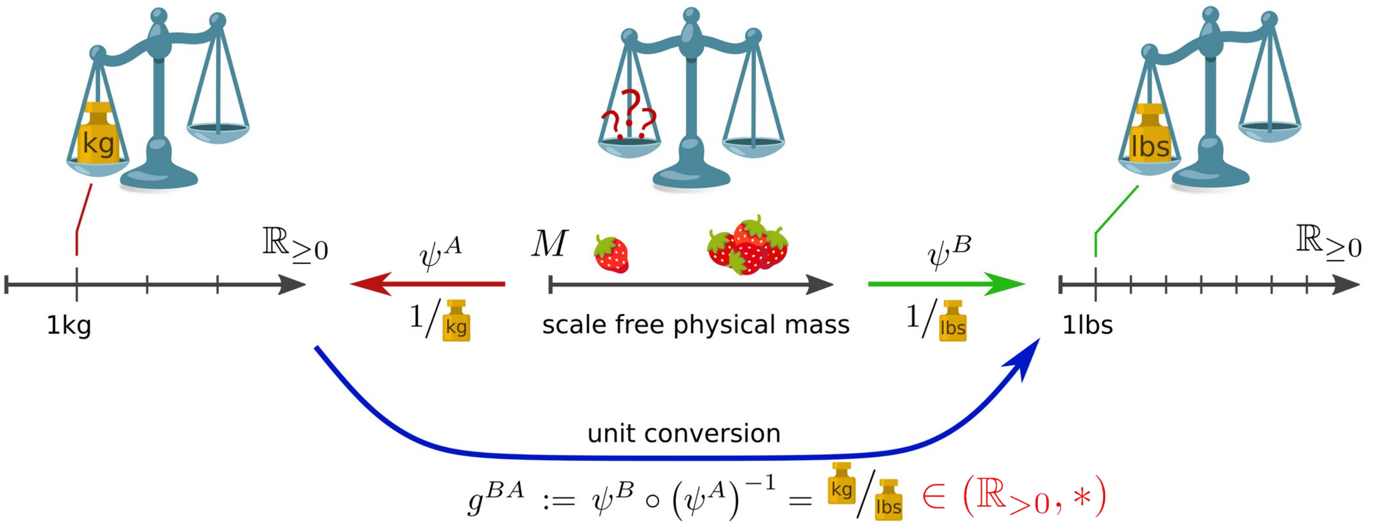
physical mass has *no canonical units*

gauging: introduce units by choosing some *reference weight*

different gauges are equally valid - within some set of structurally distinguished gauges

gauge transformations translate between these choices (= unit conversions)

determines *gauge group*



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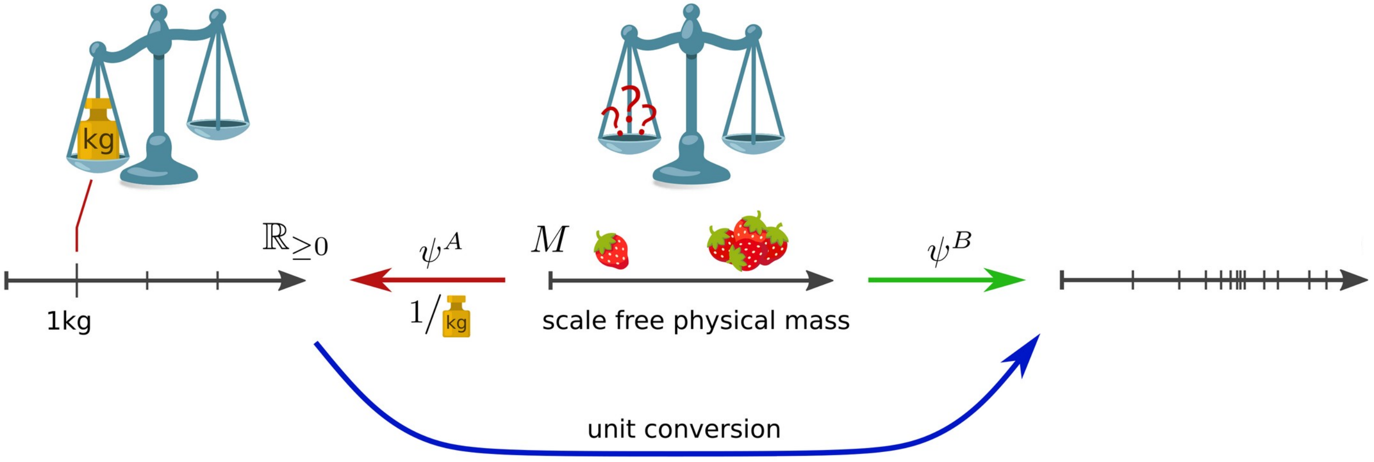


$$g^{BA} := \psi^B \circ \left(\psi^A\right)^{-1} \in \mathrm{Diff}(\mathbb{R}_{\geq 0})$$

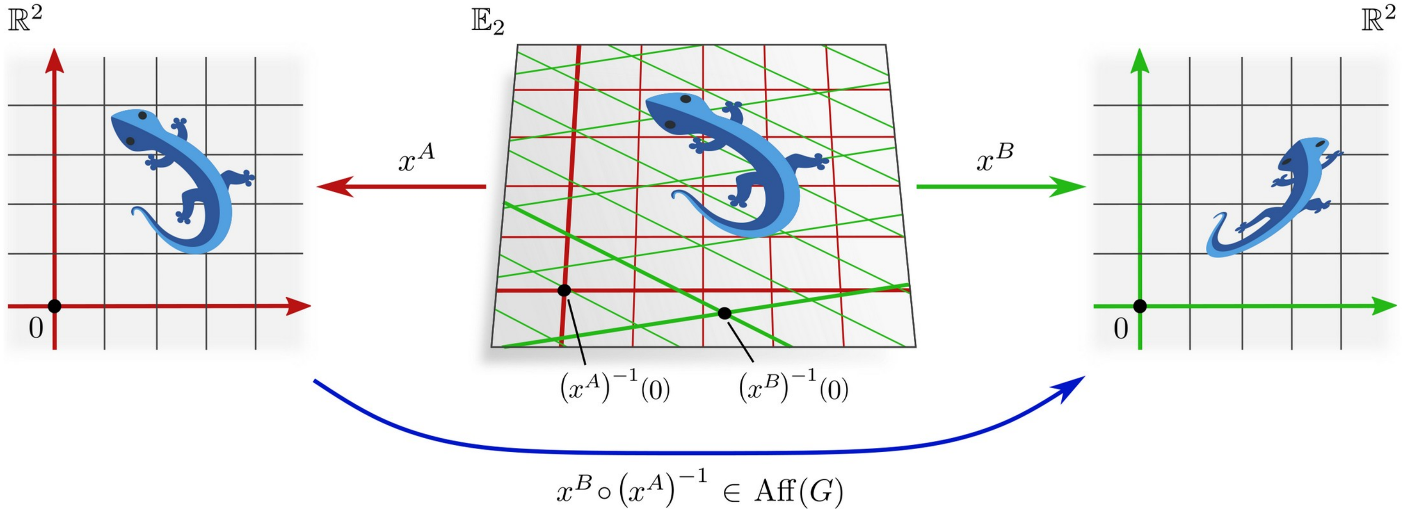
Euclidean affine space gauging

Euclidean affine spaces $\mathbb{E}_d$ are *coordinate free*

gauges = *coordinate charts* $x^A : \mathbb{E}_d \to \mathbb{R}^d$

gauge transformations = *chart transition maps* $x^B \circ \left(x^A\right)^{-1} : \mathbb{R}^d \to \mathbb{R}^d$

additional geometric structure map prefer a subset of charts and reduce the gauge group



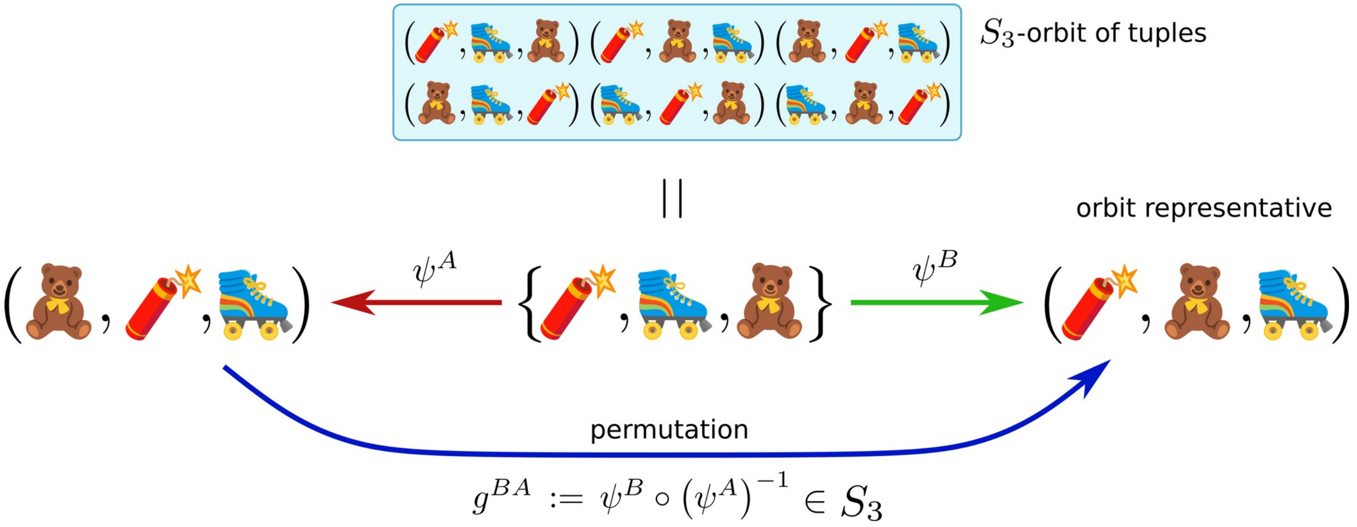
(Multi) set gauging

(multi) sets have *no canonical order*

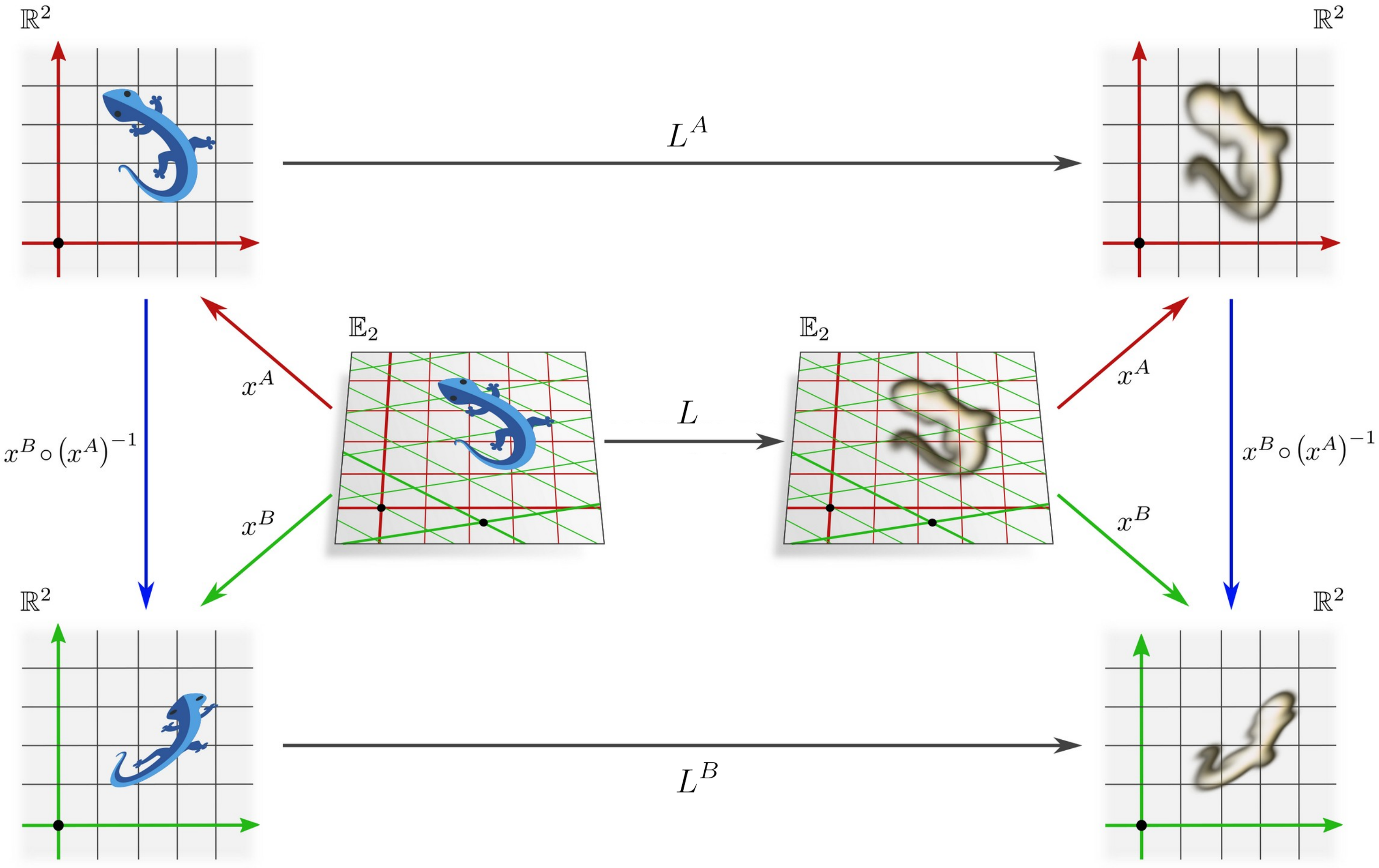
a *gauge* introduces an order of set elements, representing the set by a *tuple*

gauge transformations are permutations

the (multi) set can be viewed as an *equivalence class* (orbit) of tuples

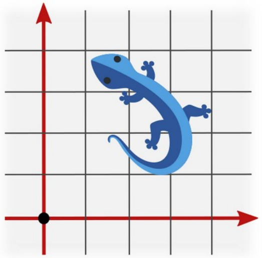


Coordinate independent NN layers



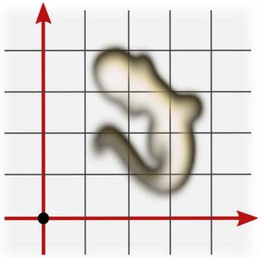
Coordinate independent NN layers + equivariance

$\mathbb{R}^2$



$\xrightarrow{\boxed{L^A}}$

$\mathbb{R}^2$



"almost" an equivariance constraint:

$$L^B \circ \left(x^B \circ (x^A)^{-1}\right) \;=\; \left(x^B \circ (x^A)^{-1}\right) \circ L^A$$

$x^B \circ (x^A)^{-1}$

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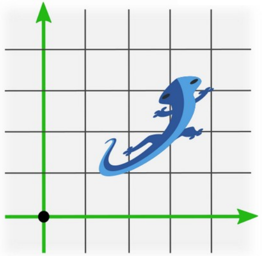
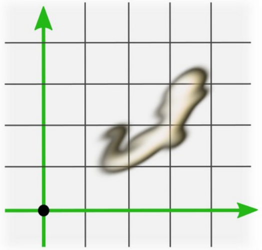
$\mathbb{R}^2$



$\xrightarrow[\boxed{L^B}]{}$

$\mathbb{R}^2$



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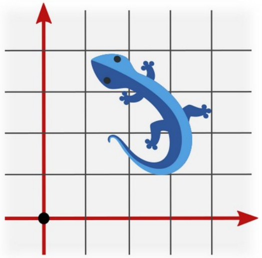
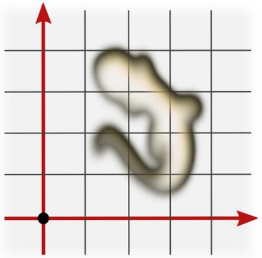


$L^A = \widetilde{L}$

$\mathbb{R}^2$



$x^B \circ (x^A)^{-1}$

“almost” an equivariance constraint:

$$L^B \circ \left(x^B \circ (x^A)^{-1}\right) \;=\; \left(x^B \circ (x^A)^{-1}\right) \circ L^A$$

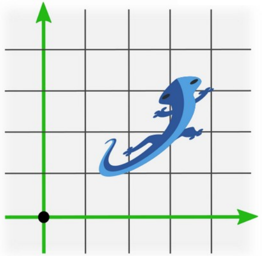
in which gauge should a *given* layer $\widetilde{L}$ be applied?

ambiguous! $\Longrightarrow$ *weight sharing over gauges*:

$$L^A = L^B = \widetilde{L}$$

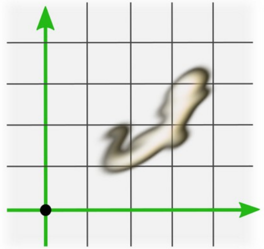
$x^B \circ (x^A)^{-1}$

$\mathbb{R}^2$



$L^B = \widetilde{L}$

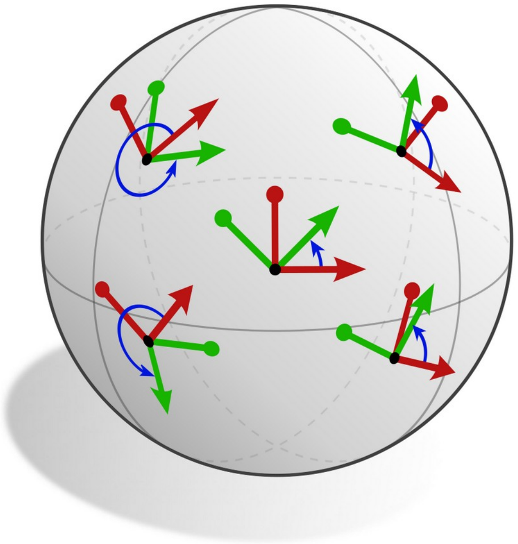
$\mathbb{R}^2$



Local reference frame independence

on manifolds: independence from *local reference frames* (observers)

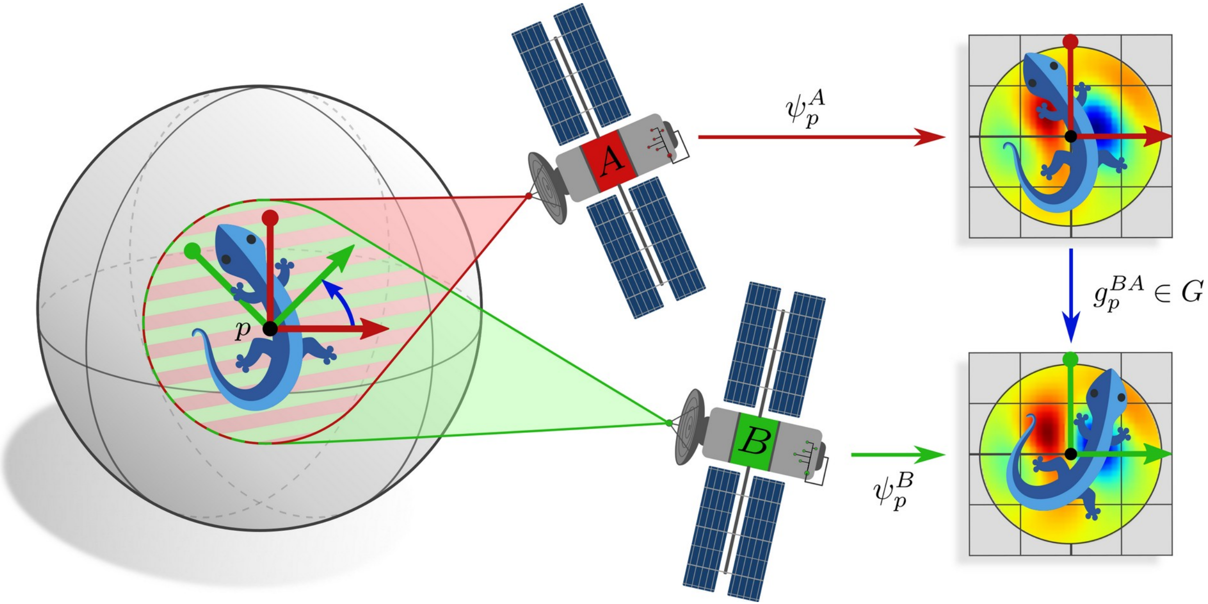
local *gauge equivariance*



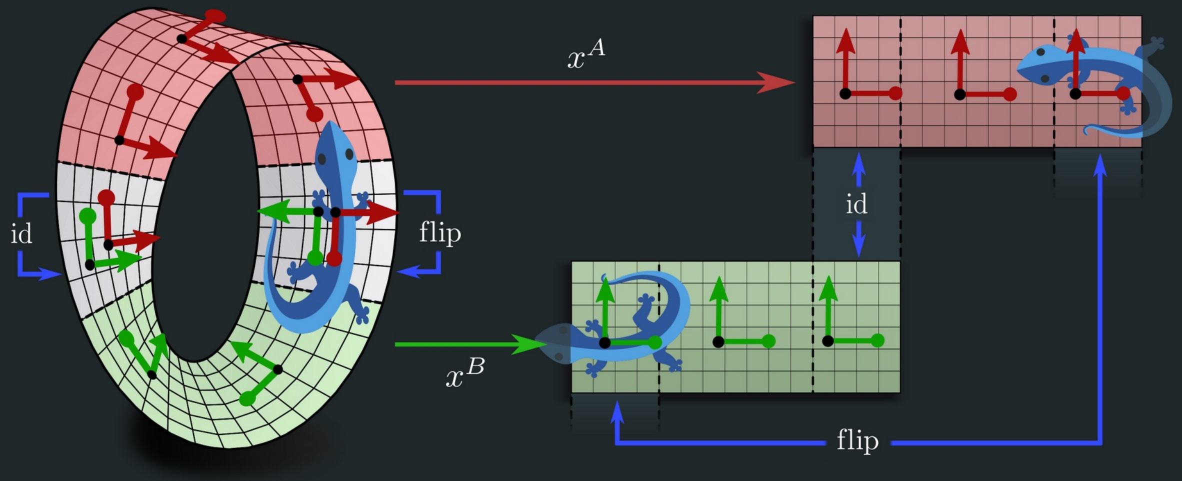
Local reference frame independence

on manifolds: independence from *local reference frames* (**observers**)

local *gauge equivariance*

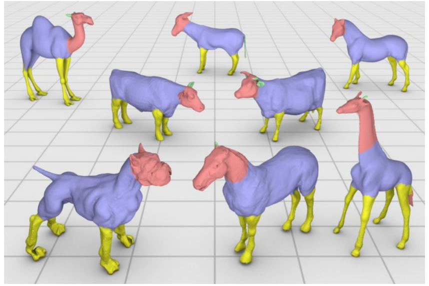


Coordinate independent CNNs on Riemannian manifolds

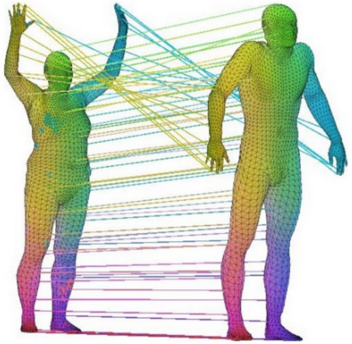


Convolutions on Riemannian manifolds

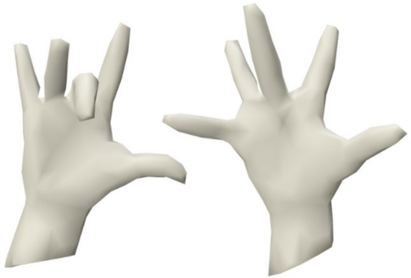
mesh segmentation



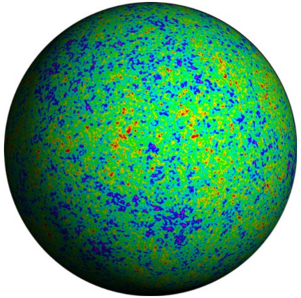
shape correspondence



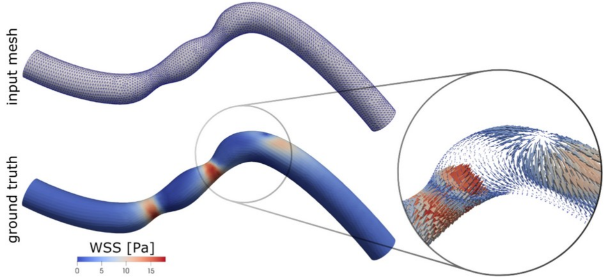
deformations (metric field)



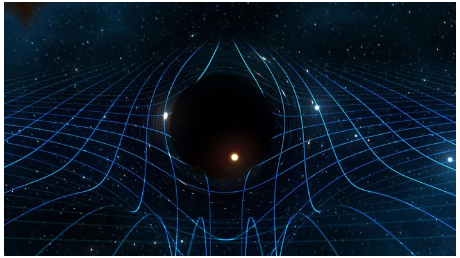
spherical CNNs



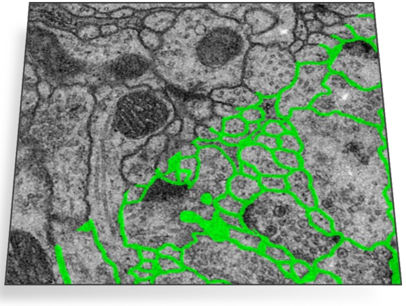
artery wall stress estimation



general relativity

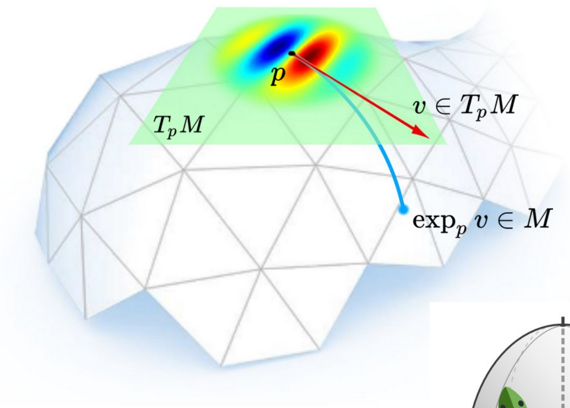


Euclidean CNNs



Design questions

Image adapted from Konakovic-Lukovic et al.



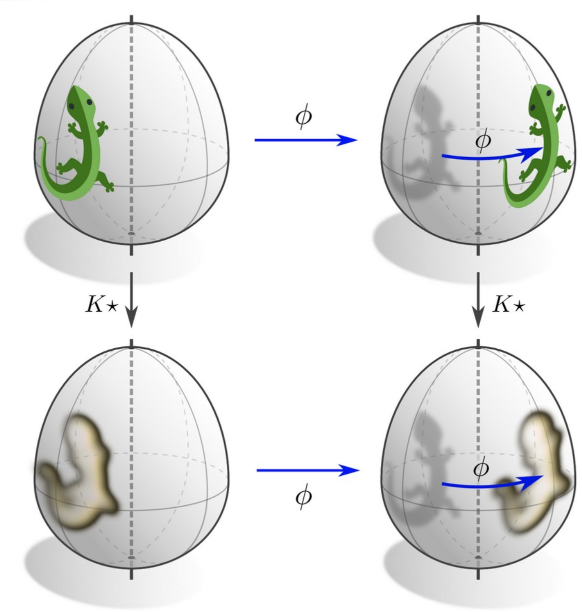
how to ...

... define *feature fields* on M ?

... define *convolution kernels* on M ?

... *share weights* over M ?

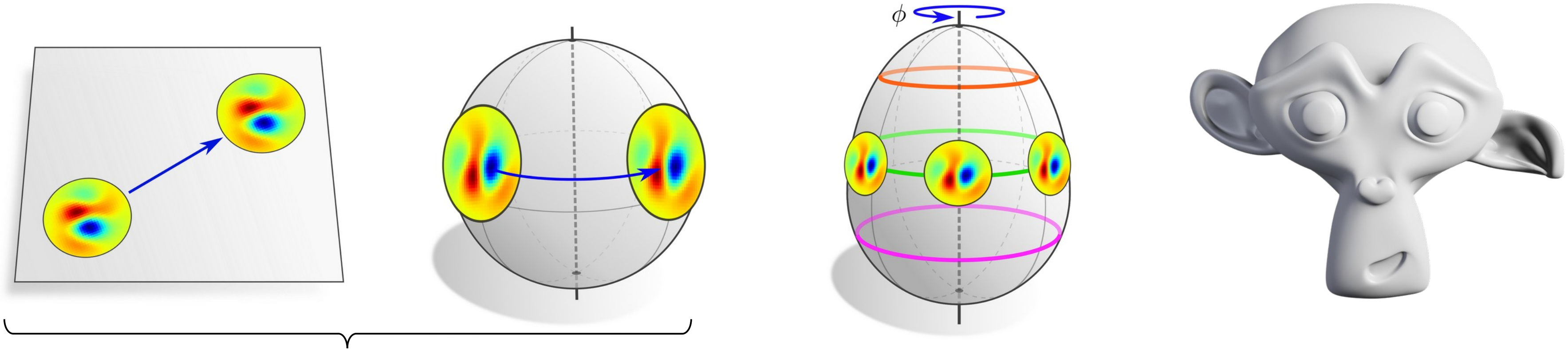
... guarantee *isometry equivariance* ?



Weight sharing - via global symmetries

weight sharing by *demanding equivariance* w.r.t. global symmetries (isometries)

↯ can only share over *symmetry orbits* (in general non-transitive)



homogeneous spaces,
transitive orbits

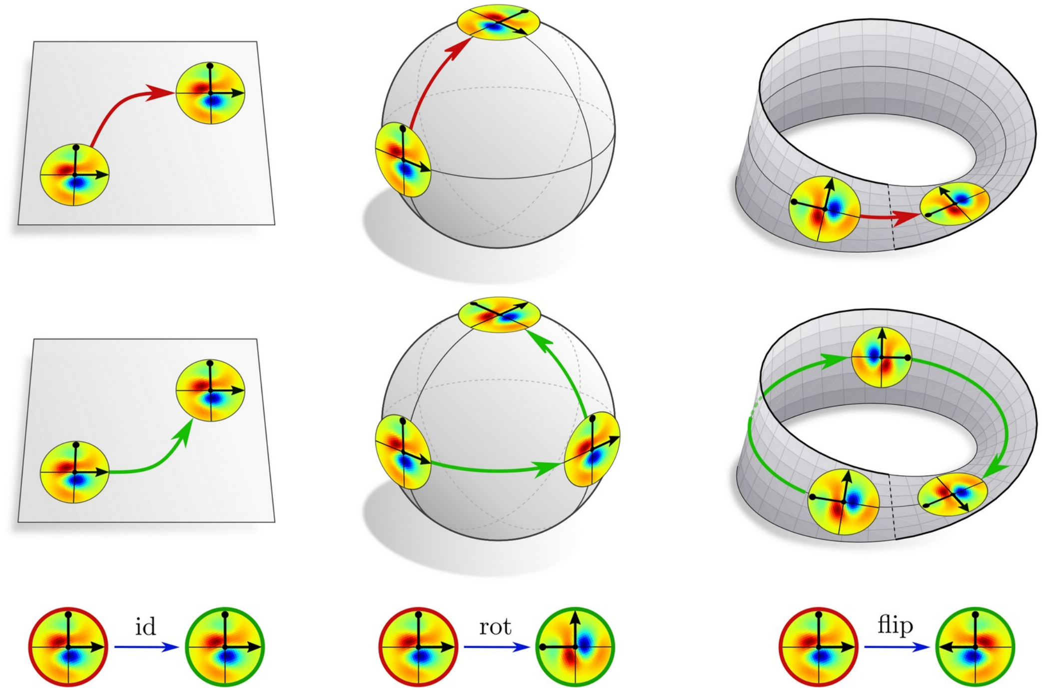
SO(2) orbits

trivial orbits

Weight sharing - via parallel transport

sharing weights by “shifting” kernel over manifold ?

⚡ parallel transport in general path dependent



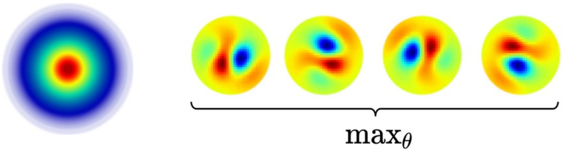
Weight sharing - approaches in the literature

the kernel alignment (“*gauge*”) on manifolds is *inherently ambiguous*!

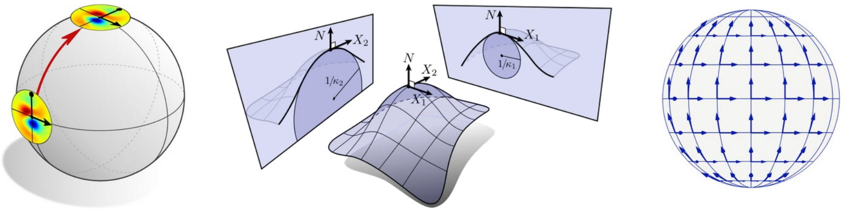
$\Leftrightarrow$ topological obstructions to the existence of G-structures

solution approaches in the literature:

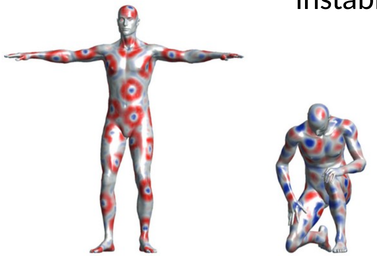
1) **gauge *invariant*** features ↯ low expressiveness



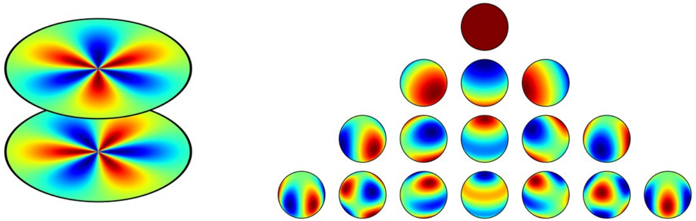
2) **heuristic** gauges ↯ instable under deformations



3) **spectral** approaches ↯ gauge independent but instable under deformations

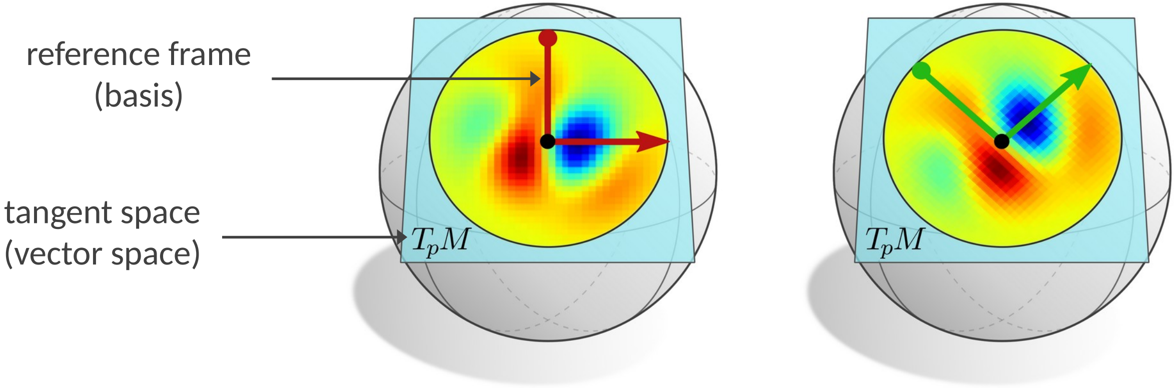


4) **gauge *equivariant*** features (ours, covers 1,2 as special cases)



Reference frames and kernel alignments

identify kernel alignment with a choice of reference frame

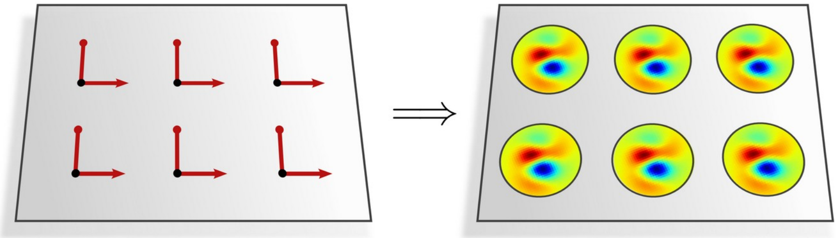


Reference frames and kernel alignments

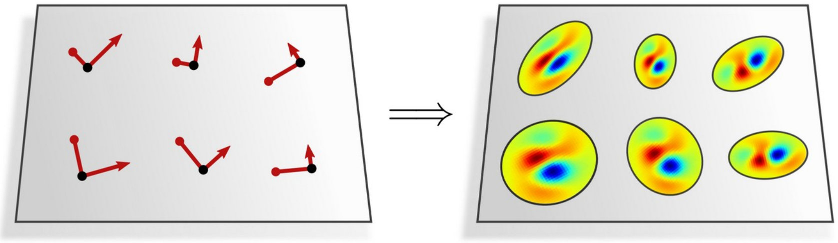
identify kernel alignment with a choice of reference frame

frame field ⟷ kernel field

standard (canonical)
frame / kernel field / CNN
on $\mathbb{R}^2$



alternative
frame / kernel field / CNN
on $\mathbb{R}^2$



Reference frames and kernel alignments

identify kernel alignment with a choice of reference frame

frame field $\longleftrightarrow$ kernel field

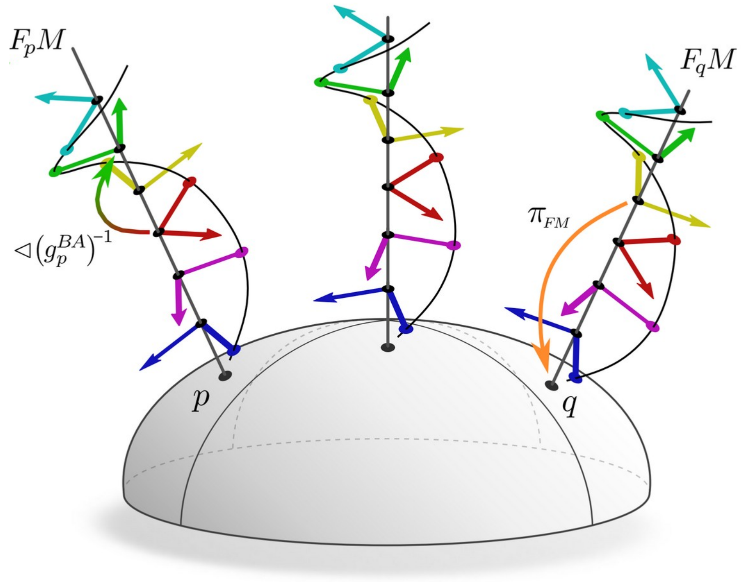
ambiguity of kernel alignments == ambiguity of reference frames

“G-structure”

G-structures

frame bundle FM = "set" of all frames (GL(d)-valued transition functions)

G-structures GM = sub-bundles of frames with $G \leq \mathrm{GL}(d)$ valued transition functions



$$\begin{tikzcd}
 & U \times G \\
\pi_{GM}^{-1}(U) \arrow[ur, "\Psi_{GM}^{B}"] \arrow[r, "\Psi_{GM}^{A}"] \arrow[d, "\pi_{GM}"'] & U \times G \arrow[u, "(\mathrm{id} \times g^{BA} \cdot)"'] \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}$$

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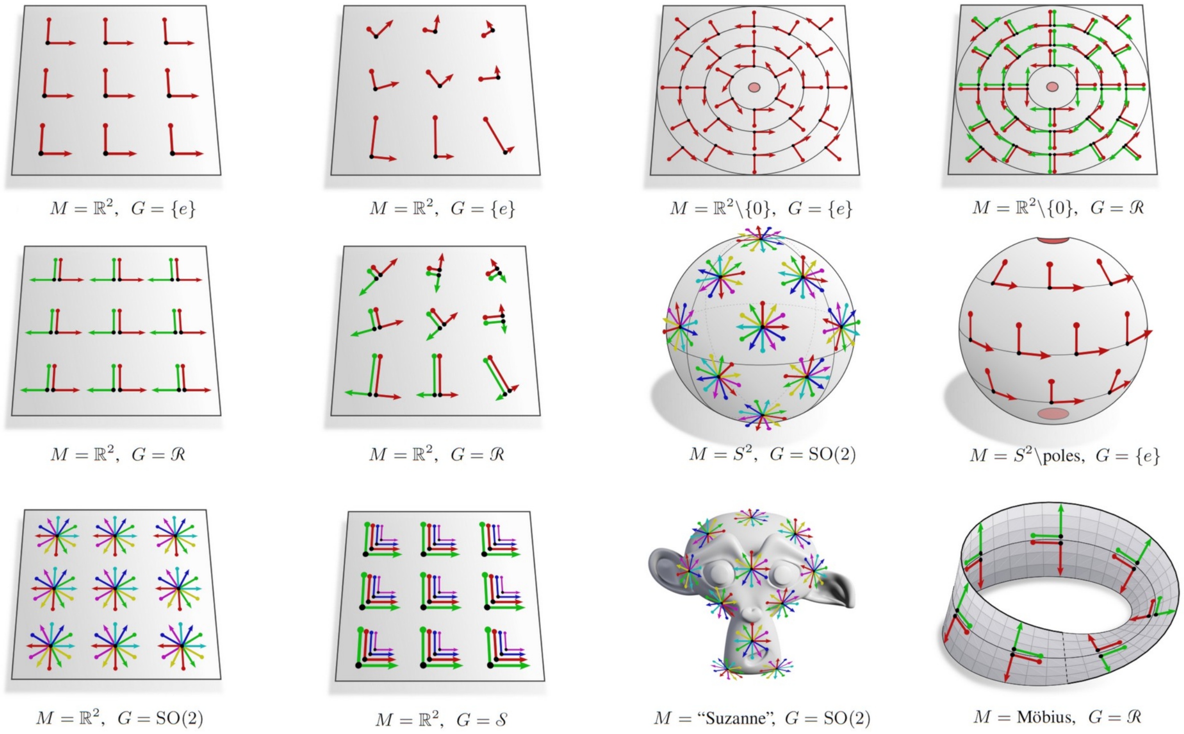


GM-coordinate independence - tangent vectors

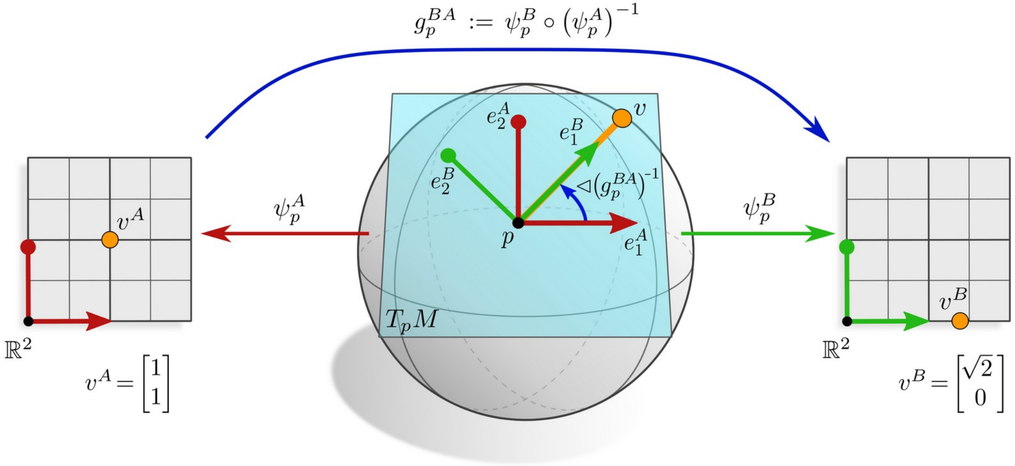
all frames of the *G*-structure are equally valid

$\Longrightarrow$ any *object* or *morphism* should be expressible relative to any frame in *GM*

example:

- tangent vectors $v \in T_pM$ are *coordinate free*
- in gauge *A*, v is expressed by *coefficients* $v^A \in \mathbb{R}^d$
- in gauge *B*, v is expressed by *coefficients* $v^B \in \mathbb{R}^d$
- *gauge trafos* $g^{BA} \in G$ relate coefficients: $v^B = g^{BA} v^A$

different coefficients, same information content!



GM-coordinate independence - feature vector fields

all frames of the *G*-structure are equally valid

$\Longrightarrow$ any *object* or *morphism* should be expressible relative to any frame in *GM*

coordinate independent *feature vectors* transform according to *group representation* ρ :

$$f^A,\, f^B \in \mathbb{R}^c \qquad\qquad f^B = \rho(g^{BA})\, f^A$$

scalar field	trivial representation	$\rho(g) = id$
tangent vector field	standard representation	$\rho(g) = g$
tensor field	tensor representation	$\rho(g) = (g^{-T})^{\otimes s} \otimes g^{\otimes r}$
irrep field	irreducible representation	
regular feature field	regular representation	

GM-coordinate independence - feature vector fields

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coordinate independent *feature vectors* transform according to *group representation* ρ :

$$f^A,\, f^B \in \mathbb{R}^c \qquad\qquad f^B = \rho(g^{BA})\, f^A$$

formally, feature vectors are elements of a *G-associated feature vector bundle* $(GM \times \mathbb{R}^c)/\sim_\rho$

$$\begin{tikzcd}
 & U \times \mathbb{R}^c \\
\pi_{\mathcal{A}}^{-1}(U) \arrow[ur, "\Psi_{\mathcal{A}}^B"] \arrow[r, "\Psi_{\mathcal{A}}^A"] \arrow[d, "\pi_{\mathcal{A}}"'] & U \times \mathbb{R}^c \arrow[u, "(\mathrm{id} \times \boxed{\rho(g^{BA})} \cdot)"'] \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U \times G \\
\pi_{GM}^{-1}(U) \arrow[ur, "\Psi_{GM}^B"] \arrow[r, "\Psi_{GM}^A"] \arrow[d, "\pi_{GM}"'] & U \times G \arrow[u, "(\mathrm{id} \times \boxed{g^{BA} \cdot})"'] \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}$$

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$$f^A,\, f^B \in \mathbb{R}^c \qquad\qquad f^B = \rho(g^{BA}) f^A$$

formally, feature vectors are elements of a *G-associated feature vector bundle* $(GM \times \mathbb{R}^c)/\sim_\rho$

feature vector fields are bundle sections == a choice of feature vector at each point $p \in M$

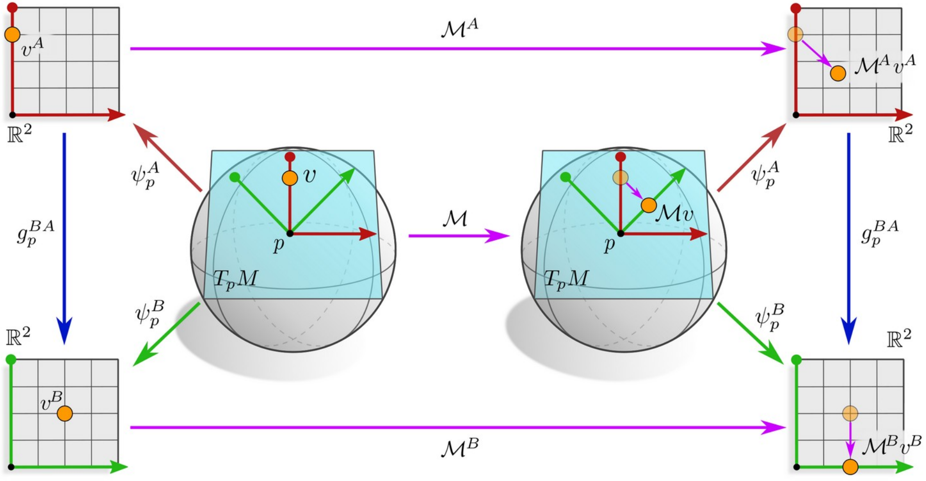
GM-coordinate independence - linear maps on T_pM

all frames of the *G*-structure are equally valid

$\Longrightarrow$ any *object* or *morphism* should be expressible relative to any frame in *GM*

example:

- linear maps $\mathcal{M} : T_pM \to T_pM$ are *coordinate free*
- in gauge *A*, $\mathcal{M}$ is expressed by *coefficients* $\mathcal{M}^A \in \mathbb{R}^{d\times d}$
- in gauge *B*, $\mathcal{M}$ is expressed by *coefficients* $\mathcal{M}^B \in \mathbb{R}^{d\times d}$
- *gauge trafos* $g^{BA} \in G$ relate coefficients: $\mathcal{M}^B = g^{BA}\mathcal{M}^A (g^{BA})^{-1}$

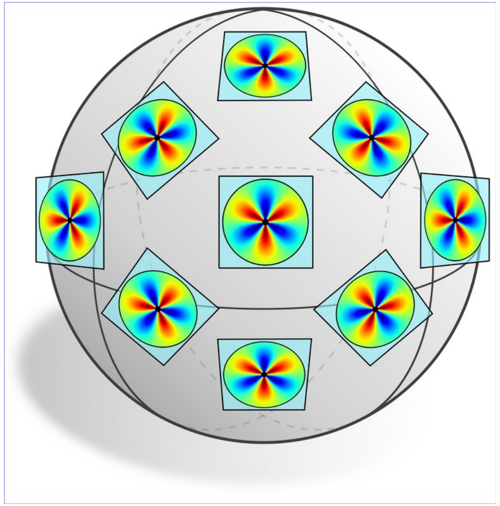


GM-convolutions

we want to design a convolution operation that is

1) mapping a ρ_{in}-field to a ρ_{out}-field
2) parameterized by a *spatially shared kernel*
3) *GM*-coordinate independent

$\Big\}$ will require G-steerability!



how to define **kernels / kernel fields**?

in coordinates, a convolution kernel is a map $K : \mathbb{R}^d \to \mathbb{R}^{c_{\text{out}} \times c_{\text{in}}}$

$\Rightarrow$ coordinate free, at $p \in M$: $\mathcal{K}_p : T_pM \to \mathrm{Hom}(\mathcal{A}_{\text{in},p}, \mathcal{A}_{\text{out},p})$

($\mathcal{A}_{\text{in/out}}$ are the feature bundles)

coordinate free, global (kernel field):

$$\begin{tikzcd} TM \arrow[rr, "\mathcal{K}"] \arrow[dr, "\pi_{TM}"'] & & \mathrm{Hom}(\mathcal{A}_{\text{in}}, \mathcal{A}_{\text{out}}) \arrow[dl, "\pi_{Hom}"] \\ & M & \end{tikzcd}$$

(allowing potentially for a different kernel at each $p \in M$, i.e. here pre-weight-sharing)

convolutional kernel fields should have the *same kernel at each position*

GM-convolutions

relation between *coordinate free kernels* $\mathcal{K}_p$ and their *coordinate expressions* $\mathcal{K}_p^A$ and $\mathcal{K}_p^B$:

$$\begin{tikzcd}
\mathbb{R}^d \arrow[rrr, "\boxed{\mathcal{K}_p^A}"] \arrow[ddd, "g_p^{BA}\cdot"'] & & & \mathbb{R}^{c_\text{out}\times c_\text{in}} \arrow[ddd, "\frac{1}{|\det g_p^{BA}|}\left(\rho_\text{in}^{-\top}\otimes\rho_\text{out}\right)(g^{BA})"] \\
& & & \\
& T_pM \arrow[uul, "\psi^A_{TM,p}"'] \arrow[ddl, "\psi^B_{TM,p}"] \arrow[r, "\mathcal{K}_p"] & \mathrm{Hom}\big(\mathcal{A}_\text{in}|_p,\mathcal{A}_\text{out}|_p\big) \arrow[uur, "\psi^A_{\text{Hom},p}"] \arrow[ddr, "\psi^B_{\text{Hom},p}"'] & \\
\mathbb{R}^d \arrow[rrr, "\boxed{\mathcal{K}_p^B}"'] & & & \mathbb{R}^{c_\text{out}\times c_\text{in}}
\end{tikzcd}$$

the coordinate expressions relate by $\boxed{\mathcal{K}^B} = \dfrac{1}{|\det g^{BA}|}\left(\rho_\text{in}^{-\top}\otimes\rho_\text{out}\right)(g^{BA}) \circ \boxed{\mathcal{K}^A} \circ (g^{BA})^{-1}$

only coordinate independence,
no equivariance constraint!

in which gauge should we share a given template kernel? Ambiguous, need to **share in all gauges *X*:** $\mathcal{K}_p^X = K$

GM-coordinate independence
spatial weight sharing $\Big\}$ requires *G*-steerability $K = \dfrac{1}{|\det g|}\left(\rho_\text{in}^{-\top}\otimes\rho_\text{out}\right)(g) \circ K \circ g^{-1}$

GM-convolutions

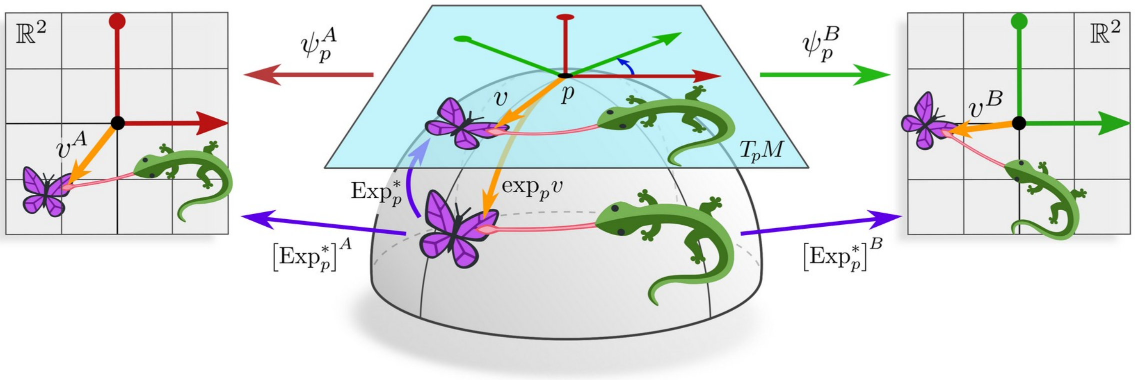
given …

… a Riemannian manifold *M*

… a *G*-structure *GM*

… a *G*-compatible connection

… a *G*-steerable convolution kernel K



the corresponding *GM*-convolution is performed by:

1) applying "*transporter pullbacks*" Exp_p^* of the feature field to the tangent spaces (a local observer's viewpoint)
2) matching this pullback with the kernel K

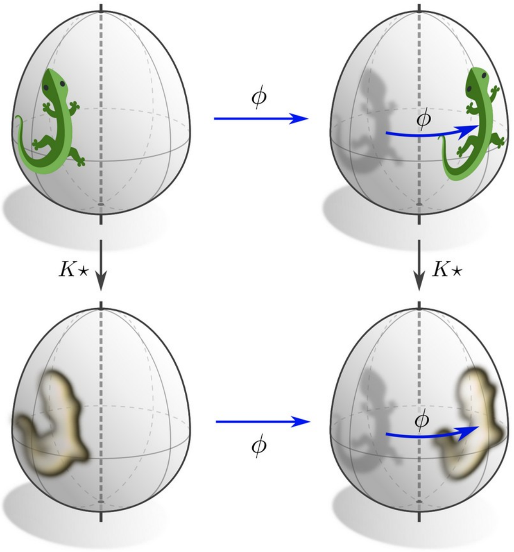
$$f_{\mathrm{out}}^A(p) \;:=\; \int_{\mathbb{R}^d} \mathrm{d}v^A\, K(v^A) \big[\, \mathrm{Exp}_p^* f_{\mathrm{in}} \big]^A (v^A)$$

the kernel's *G*-steerability ensures that the chosen gauge / kernel alignment is irrelevant

Isometry equivariance

"*kernel field transform*": similar to convolution, but not assuming weight sharing

parameterized by a *kernel field*



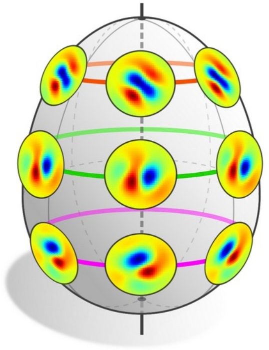
Theorem (Isometry equivariance)

Let $\mathcal{I} \leq \mathrm{Isom}(M)$, then:

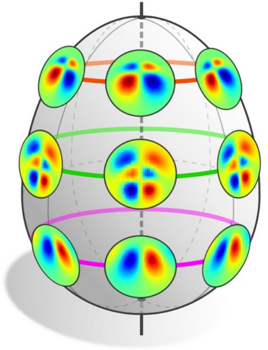
$\mathcal{I}$-*equivariant* kernel field transform

$\iff$

$\mathcal{I}$-*invariant* kernel field



SO(2)-invariant kernel field



O(2)-invariant kernel field

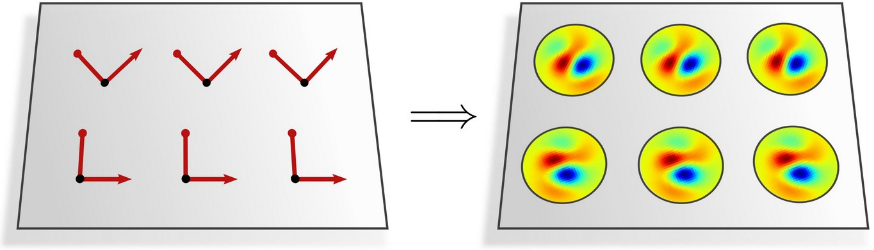
Isometry equivariance - *GM*-convolutions

Let $\mathrm{Isom}_{GM} \leq \mathrm{Isom}(M)$ be the subgroup of isometries that are symmetries of *GM*

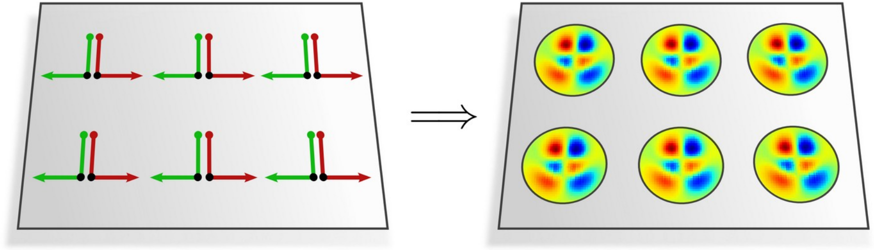
G-steerable (convolutional) kernel fields inherit this Isom_{GM}-invariance

$\implies$ *GM*-convolutions are Isom_{GM}-equivariant

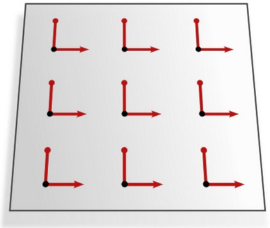
- horizontal translations



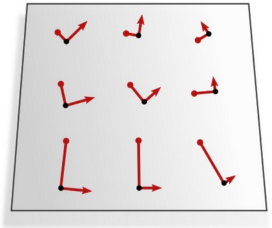
- horizontal translations
- vertical translations
- horizontal reflections



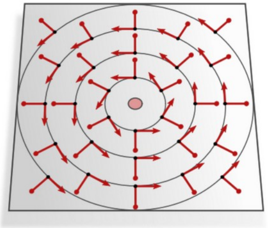
Isometry equivariance - *GM*-convolutions



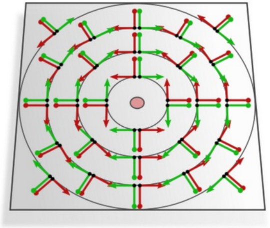
$M = \mathbb{R}^2, \; G = \{e\}$



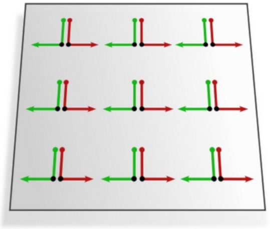
$M = \mathbb{R}^2, \; G = \{e\}$



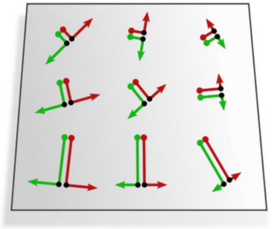
$M = \mathbb{R}^2 \backslash \{0\}, \; G = \{e\}$



$M = \mathbb{R}^2 \backslash \{0\}, \; G = \mathcal{R}$

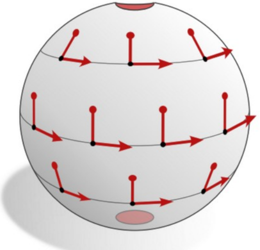


$M = \mathbb{R}^2, \; G = \mathcal{R}$

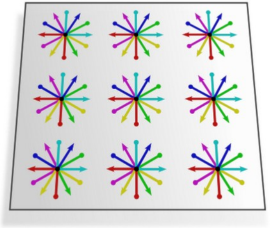


$M = \mathbb{R}^2, \; G = \mathcal{R}$

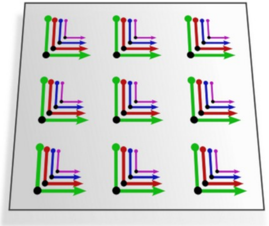
$M = S^2, \; G = \mathrm{SO}(2)$



$M = S^2 \backslash \text{poles}, \; G = \{e\}$

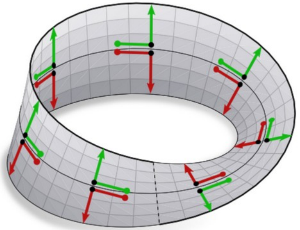


$M = \mathbb{R}^2, \; G = \mathrm{SO}(2)$



$M = \mathbb{R}^2, \; G = \mathcal{S}$

$M = \text{“Suzanne”}, \; G = \mathrm{SO}(2)$



$M = \text{Möbius}, \; G = \mathcal{R}$

Möbius CNNs - geometry

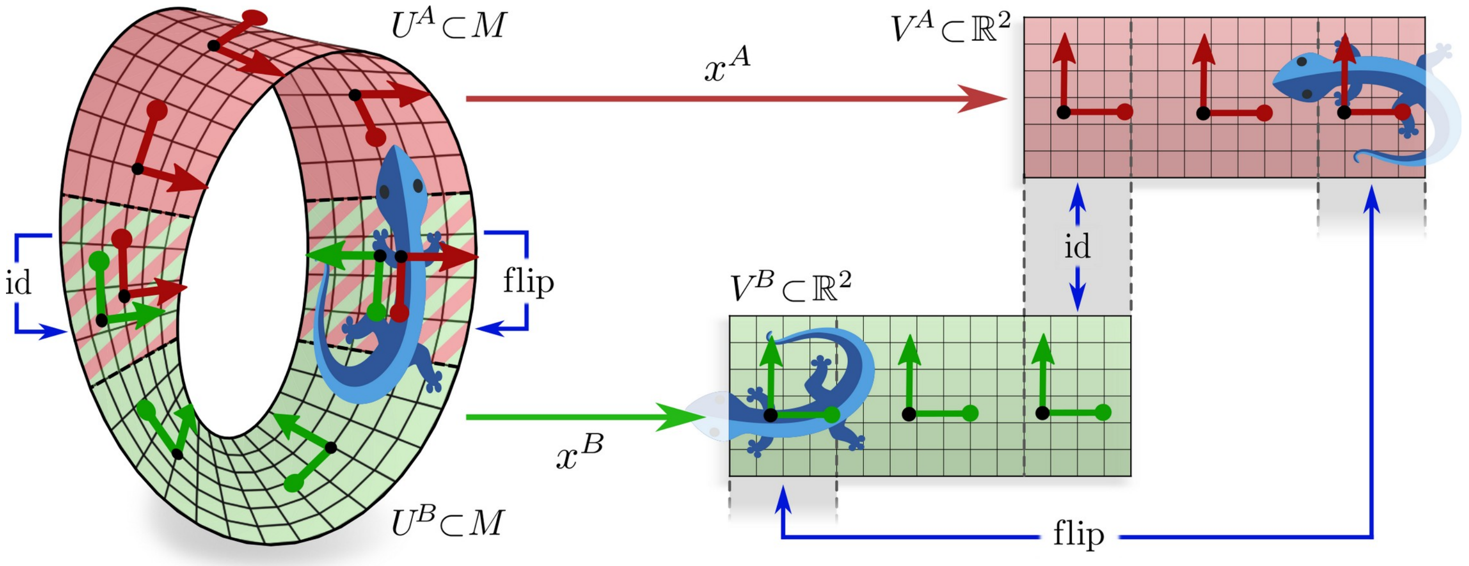
github:
https://github.com/mauriceweiler/MobiusCNNs

locally flat geometry

regular pixel grid

Levi-Civita connection has holonomy group $\mathbb{Z}_2$ (reflections)

admits $\mathbb{Z}_2$-structure



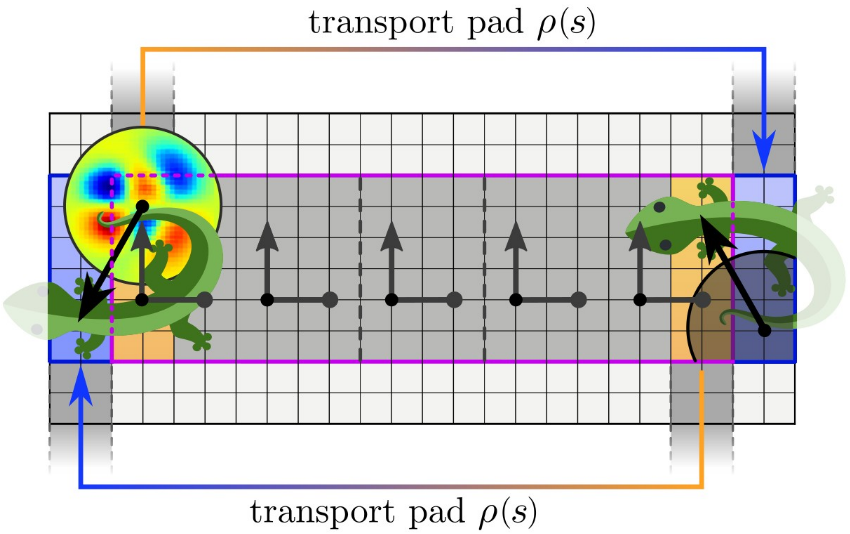
Möbius CNNs - implementation

cut and flatten the strip

parallel transport: - trivial within strip

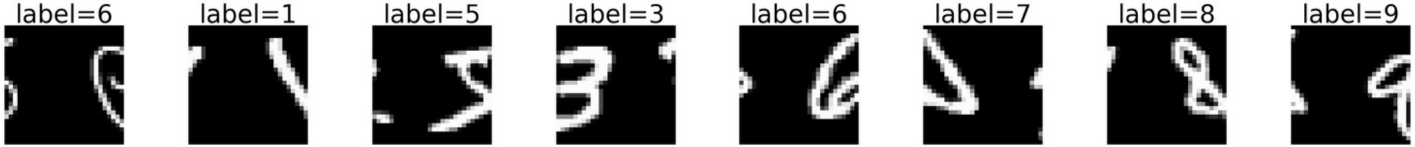
- "*reflection padding*" over the cut

after transport padding: Euclidean convolution with $\mathbb{Z}_2$-steerable kernels



Möbius CNNs - experiments

Möbius MNIST toy dataset



performance depends on field types

perfect isometry equivariance

model	field types ρ_i			params	test error (%)	
	trivial	sign-flip	regular		shifted train digits	centered train digits
CNN (channels)		—		1501 k	1.97 ± 0.11	42.99 ± 2.65
CNN (params)		—		832 k	2.08 ± 0.10	43.68 ± 2.85
gauge CNN (scalar)	✓	×	×	902 k	1.60 ± 0.10	1.60 ± 0.09
gauge CNN (sign-flip)	×	✓	×	820 k	4.27 ± 0.24	4.89 ± 0.36
gauge CNN (regular)	×	×	✓	752 k	1.24 ± 0.08	1.23 ± 0.07
gauge CNN (irreps)	✓	✓	×	752 k	1.65 ± 0.09	1.64 ± 0.12
gauge CNN (mixed)	✓	✓	✓	752 k	1.43 ± 0.09	1.42 ± 0.10

Icosahedral CNNs - geometry

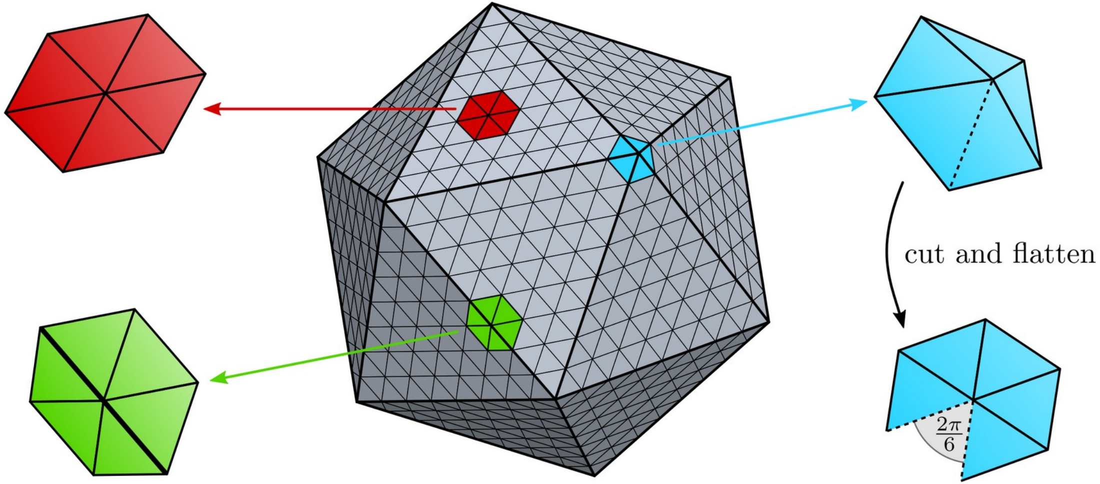
Platonic solid

locally flat approximation of the sphere

hexagonal grid

Levi-Civita connection has holonomy group $\mathbb{Z}_6$

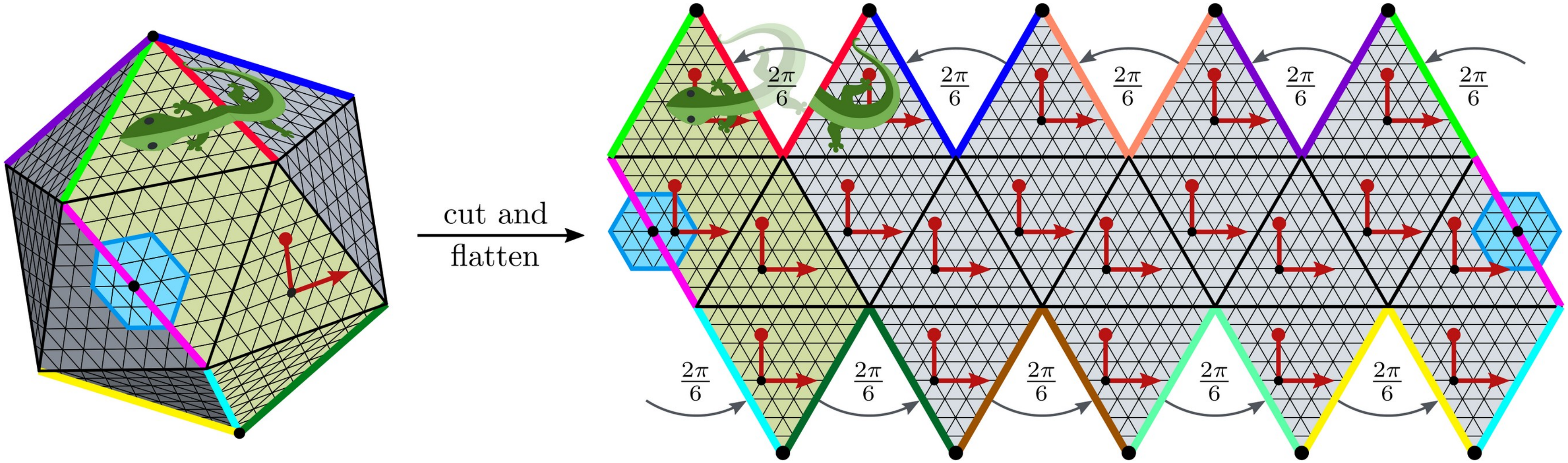
admits $\mathbb{Z}_6$-structure



Icosahedral CNNs - implementation

flatten icosahedron by cutting it at north and south pole

parallel transport non-trivial over cut edges



Icosahedral CNNs - implementation

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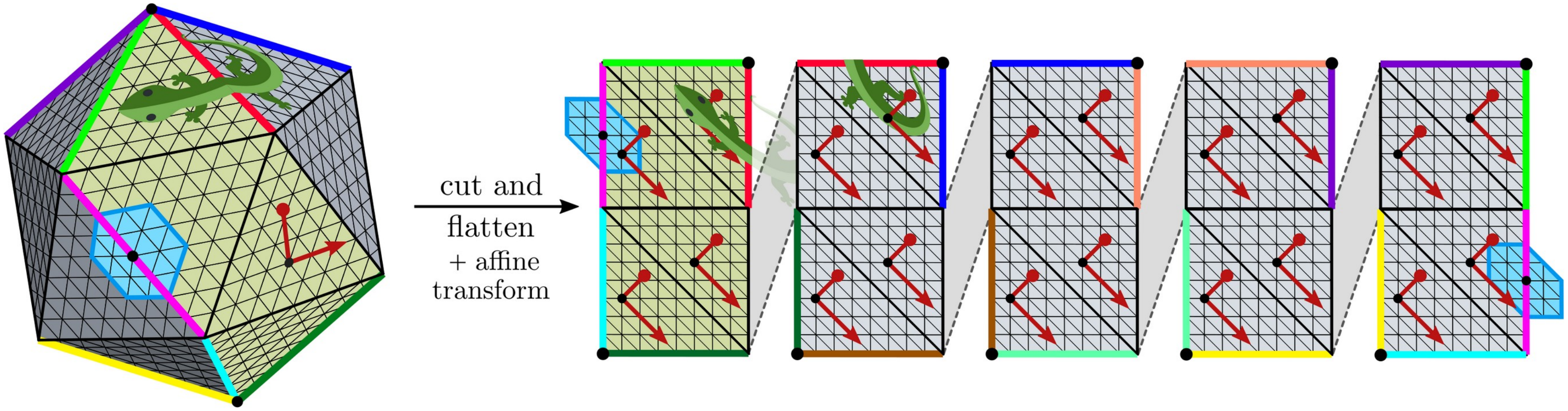


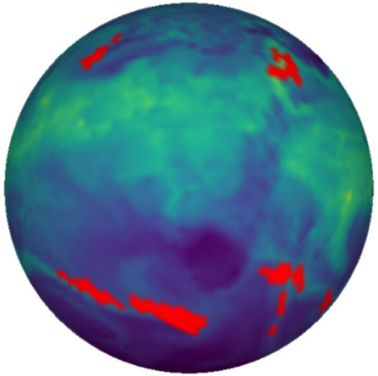
Icosahedral CNNs - experiments

perfect isometry equivariance

kernel	N/N	N/I	I/ I
non-steerable	99.42	25.41	98.67
$\mathbb{Z}_6$-steerable	$\mathbf{99.43}$	$\mathbf{99.43}$	$\mathbf{99.38}$

climate pattern segmentation

Model	BG	TC	AR	Mean	mAP
Mudigonda et al.	97	74	65	78.67	-
Jiang et al.	97	94	93	94.67	-
Ours (S2R)	97.3	97.8	97.3	97.5	0.686
Ours (R2R)	97.4	**97.9**	**97.8**	**97.7**	**0.759**



omnidirectional RGB-D image segmentation

	mAcc	mIoU
(Jiang et al., 2018)	0.547	0.383
Ours (R2R-U-Net)	**0.559**	**0.394**

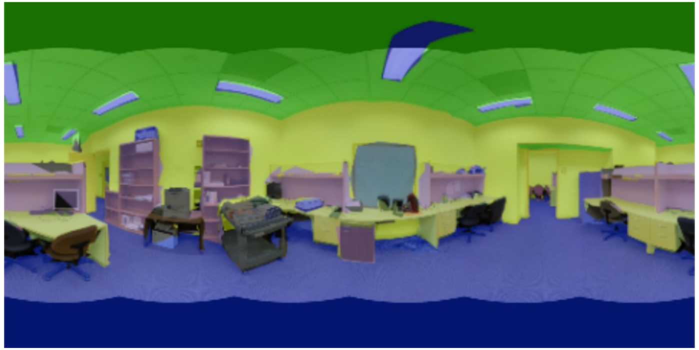


Image credits: Jiang et al., 2018

Gauge Equivariant Mesh CNNs

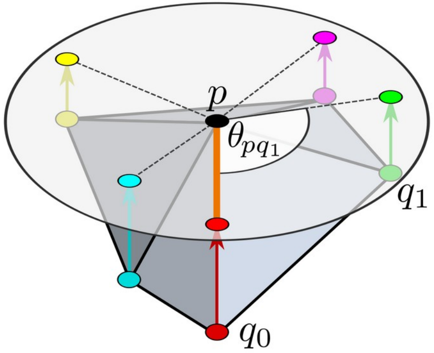
Gauge Equivariant Mesh CNNs
Anisotropic convolutions on geometric graphs

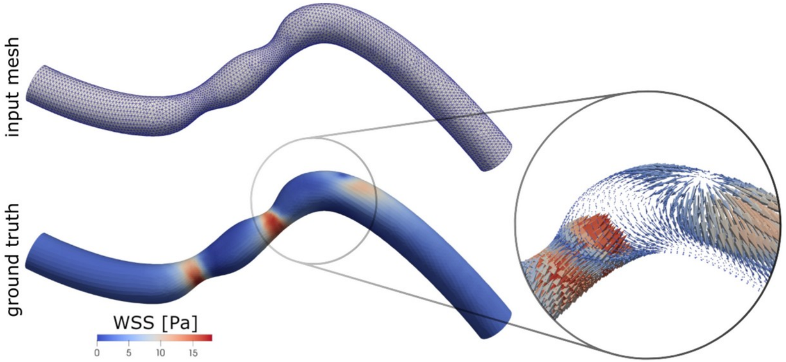
Pim de Haan [*, 1, 2, 3] **Maurice Weiler** [*, 2, 3] **Taco Cohen** [1] **Max Welling** [1, 3]

Pim de Haan

Mesh Convolutional Neural Networks for Wall Shear Stress Estimation in 3D Artery Models

Julian Suk[1], Pim de Haan[3,2], Phillip Lippe[2], Christoph Brune[1], and Jelmer M. Wolterink[1]

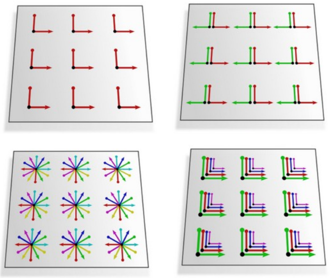




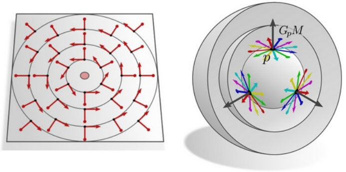
Literature review

Coordinate independent CNNs unify a wide range of related work:

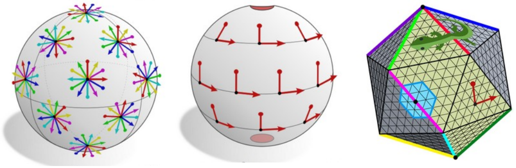
Euclidean steerable CNNs



punctured Euclidean

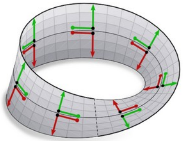


spherical / icosahedral



general 2d surfaces / meshes

Möbius



	manifold M	structure group G	global symmetry Aff_{GM} or Isom_{GM}	representation ρ	citation
1	$\mathbb{E}_d$	$\{e\}$	$\mathcal{T}_d$	trivial	[130, 253] + any conventional CNN
2	$\mathbb{E}_1$	$\mathcal{S}$	$\mathcal{T}_1 \rtimes \mathcal{S}$	regular	[186]
3		$\mathcal{R}$	$\mathcal{T}_2 \rtimes \mathcal{R}$	regular	[234]
4				irreps	[244, 234, 231]
5		SO(2)	SE(2)	regular	[51, 33, 257, 34, 236, 8, 93, 192] [234, 79, 125, 210, 232, 185, 158] [201, 7, 67, 227, 83, 159, 231, 92] [50, 206, 19, 207, 208, 164, 29, 86]
6				quotients	[34, 234]
7				regular $\xrightarrow{\text{pool}}$ trivial	[33, 143, 234]
8				regular $\xrightarrow{\text{pool}}$ vector	[144, 234]
9	$\mathbb{E}_2$			trivial	[110, 234]
10				irreps	[234]
11		O(2)	E(2)	regular	[51, 33, 93, 34, 234] [159, 79, 201]
12				quotients	[34]
13				regular $\xrightarrow{\text{pool}}$ trivial	[234]
14				induced SO(2)-irreps	[234]
15		$\mathcal{S}$	$\mathcal{T}_2 \rtimes \mathcal{S}$	regular	[243, 212, 7, 258]
16				regular $\xrightarrow{\text{pool}}$ trivial	[77]
17				irreps	[235, 224, 156, 120, 2, 6]
18		SO(3)	SE(3)	quaternion	[250]
19				regular	[67, 241, 242]
20				regular $\xrightarrow{\text{pool}}$ trivial	[3]
21				regular	[241]
22	$\mathbb{E}_3$	O(3)	E(3)	quotient O(3) / O(2)	[103]
23				irrep $\xrightarrow{\text{norm}}$ trivial	[174]
24		C_4	$\mathcal{T}_3 \rtimes \mathrm{C}_4$	regular	[219]
25		D_4	$\mathcal{T}_3 \rtimes \mathrm{D}_4$	regular	[219]
26	$\mathbb{E}_{d-1,1}$	$\mathrm{SO}(d-1,1)$	$\mathcal{T}_d \rtimes \mathrm{SO}(d-1,1)$	irreps	[205]
27	$\mathbb{E}_2 \backslash \{0\}$	$\{e\}$	SO(2)	trivial	[30, 67]
28			$\mathrm{SO}(2) \times \mathcal{S}$		[62, 67]
29	$\mathbb{E}_3 \backslash \{0\}$	O(2)	O(3)	trivial	[178]
30		$\{e\}$	$\{e\}$	trivial	[13]
31	S^2	SO(2)	SO(3)	irreps	[122, 64]
32				regular	[35, 111]
33		O(2)	O(3)	trivial	[61, 169, 245]
34	$S^2 \setminus$ poles	$\{e\}$	SO(2)	trivial	[39, 222, 254, 149, 105] [217, 218, 55, 131]
35	icosahedron	C_6	$\mathrm{I}\ (\approx \mathrm{SO}(3))$	regular	[38]
36	ico $\setminus$ poles	$\{e\}$	$\mathrm{C}_5\ (\approx \mathrm{SO}(2))$	trivial	[251, 139]
37				irreps	[238]
38	surface ($d=2$) (e.g. meshes)	SO(2)	$\mathrm{Isom}_+(M)$	regular	[173, 220, 246, 48]
39				regular $\xrightarrow{\text{pool}}$ trivial	[150, 151, 160, 220]
40		D_4	$\mathrm{Isom}_{\mathrm{D}_4 M}$	trivial	[98]
41		$\{e\}$	$\mathrm{Isom}_{\{e\}M}$	trivial	[160, 194, 106, 221, 133]
42	Möbius strip	$\mathcal{R}$	SO(2)	irreps	Section 5
43				regular	

Coordinate Independent Convolutional Networks

Isometry and Gauge Equivariant Convolutions on Riemannian Manifolds