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Fuzziness in Financial Certainty Equivalents Maria Rosaria Simonelli Faculty of Economics, I. U. N., Via Acton, 38, 80133-Naples, Italy e-mail: Simonelli@naval.uninav.it Abstract. Our aim is to use the utility theory of von Neumann and Morgenstern and the approach of certainty equivalent in a fuzzy framework for pricing a future random amount that we call *risk*. The fuzzy theory is more adaptable to represent this problem because the financial and insurance evaluations depende on perspectives which are fuzzy known. We see that the variance-fuzziness functional (cf.[12]) can be used as an expected utility function, so we define a seller and buyer pricing functionals as a certainty equivalent.

Keywords. Utility theory, certainty equivalent, pricing theory, fuzziness.

1.Introduction In this paper, we link the theory of fuzzy numbers used in [2] and the notion of "variance - fuzziness" defined in [12] with the utility theory of von Neumann and Morgenstern, in particular with the notion of certainty equivalent (c.e.) to obtain immediate prices of a risk. We call *risk* a future random amount, for example, a future price of a stock, a price of an insurance contract, a price of a derivative contract on a security, etc..

The c.e. is a crucial tool for pricing when standard finance theory does not apply as in the case of incomplete hedging and in absence of the standard hypotheses concerning perfection and efficiency of financial markets. In these cases, the pricing of options can not be based on Black and Scholes formula [3].

Obviously, our pricing take in account of the discrepancy between the moment at which the price is paid and that at which the asset produces future fuzzy random money.

The objects of financial transactions are future scenarios which are inherently fuzzy. In this paper we give fuzzy pricing models.

Using either Lebesgue's, or the fuzzy integrals of Choquet and Sugeno and an utility function which entail variance and "fuzziness", we build different seller and buyer fuzzy pricing functionals.

These fuzzy pricing functionals distinguish between the positions of the seller and the buyer: the seller ceeds a risk and the buyer assumes it. This important distinction is used in actuarial literature e.g. Borch [4] and in finance e.g. Peccati [6, 10].

The paper is organized as follows: in section 2, we set the preliminaries, in 3, we give the pricing functionals, in 4 there is the fuzzy models and cues for further researches.

2.Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a probability space where Ω is a set representing all the eventual future prices of an asset, $\mathcal{A}$ is a σ-field representing all the random events and P is a probability defined on $\mathcal{A}$.

In this paper we take Ω equal to an interval $[a, b]$ of real positive numbers where a represents the minimum future price and b the maximum future price of the asset and $\mathcal{A}$ is equal to the σ-field of Borel $\mathcal{A} = \mathcal{B}([a, b])$.

The set of the fuzzy events corresponding to $(\Omega, \mathcal{A})$ is the set, $\mathcal{F}$, of all $\mathcal{A}$-measurable functions $f : \Omega \longrightarrow [0, 1]$. We refer to Dubois and Prade [7], Zadeh [14] and Zimmermann [15] for the theory of the fuzzy sets.

The choices of Ω and P depend on our knowledge. In general, P is linked to the observed frequencies, for example, it may be a probability which interpolates the frequencies in the past.

So, instead of P we may also consider a *fuzzy measure*, μ, i.e. a non decreasing set function $\mu : \mathcal{A} \longrightarrow [0, 1]$, continuous from below, with $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$.

The future price of an asset may be represented in two different ways:

(i) by a positive random variable X, i.e., a $\mathcal{A}$-measurable function such that $\forall \omega \in [a, b] \longrightarrow X(\omega) > 0$. We will denote with $\mathcal{X}$ the appropriate set of random variables, e.g., the space L^1 or L^2, or the set of bounded and $\mathcal{A}$-measurable functions;

(ii) by a convex, normal and upper semi continuous fuzzy number (u.s.c.f.n.) $f : \Omega \longrightarrow [0, 1]$. We denote with $\mathcal{N}$ the set of these fuzzy numbers (f.n.). 1 Given a f.n., f, and a real number $y \in]0, 1]$ we say $y - cut$ of f pitched at a level y the subset $C_y^f \subset \Omega$ defined by:

$$C_y^f = \{\omega \in \Omega : f(\omega) \geq y\}.$$

We may write the y-cut in the following way:

$$(\forall y \in [0,1]) \quad C_y^f = [f_1^f(y), f_2^f(y)],$$

where f_1^f, f_2^f are called the cut-functions of f; f_1^f is increasing, f_2^f is decreasing, $f_1^f \leq f_2^f$ and $f(\omega)$ may be written in this way:

$$f(\omega) = sup_{y\in[0,1]} y \cdot \chi_{[f_1^f(y), f_2^f(y)]}(\omega),$$

where χ_A is the characteristic function of $A \in \mathcal{A}$.

2 If $f \in \mathcal{N}$, then the cut-functions f_1^f, f_2^f are left-continuous in]0,1]. If $f \in \mathcal{N}$, then fixed $y* \in]0,1]$ we have:

$$\omega \in [f_1^f(y*), f_2^f(y*)] \Longleftrightarrow f(\omega) \geq y*; \ \forall y < y*, \ f(\omega) \geq y \Longleftrightarrow \omega \in \cap_{y<y*} C_y^f.$$

So

$$[f_1^f(y*), f_2^f(y*)] = \cap_{y<y*} C_y^f.$$

Moreover

$$\forall y < y*, f(\omega) \geq y \Longleftrightarrow \ \forall y < y*, f_1^f(y) \leq \omega \leq f_2^f(y).$$

Hence

$$[sup_{y<y*} f_1^f(y), inf_{y<y*} f_2^f(y)] = \cap_{y<y*} C_y^f.$$

As a result

$$[f_1^f(y*), f_2^f(y*)] = \cap_{y<y*} C_y^f = [sup_{y<y*} f_1^f(y), inf_{y<y*} f_2^f(y)].$$

Since f_1^f is increasing and f_2^f is decreasing, it follows:

$$lim_{y\to y*^-} f_1^f(y) = sup_{y<y*} f_1^f(y) = f_1^f(y*)$$

and

$$lim_{y\to y*^-} f_2^f(y) = inf_{y<y*} f_2^f(y) = f_2^f(y*)$$

that is f_1^f and f_2^f are left-continuous.

In the classical case and using a compound interest, at rate r, an investment of 1 will accumulate

$$m = (1+r)^t$$

at the end of a period of duration t. We call m *the compound factor.*

Symmetrically,

$$v = (1+r)^{-t}$$

is the price to pay now for having 1 at the end of the period of duration t. v is called *discount factor.*

If $r \in \mathcal{N}$ represents a fuzzy rate of interest then, the discount and the compound factors are fuzzy numbers defined by

$$(1 \oplus r)^{-t}, \quad (1 \oplus r)^t,$$

respectively, where $\oplus$ is defined with Zadeh's extension principle and t is crisp.

If s is a fuzzy future amount, then we can define the present value of s as a solution, d, of the equation

$$s = d \otimes (1 \oplus r)^t, \tag{2.1}$$

where $\otimes$ is defined with Zadeh's extension principle.

$\forall y \in [0,1]$ we will indicate with

$$f_1^s(y), f_2^s(y)$$
$$f_1^r(y), f_2^r(y)$$
$$f_1^d(y), f_2^d(y)$$

the values of the cut-functions of s, r, d respectively.

We refer to Biacino- Simonelli [2] for the theory of the fuzzy mathematics of finance. 3 If $d, r \in \mathcal{N}$ are given fuzzy positive numbers, then s defined by (2.1) is in $\mathcal{N}$ if and only if

$f_1^d(y)(1+f_1^r(y))^t$ is increasing
$f_2^d(y)(1+f_2^r(y))^t$ is decreasing and

$$f_1^d(y)(1+f_1^r(y))^t \leq f_2^d(y)(1+f_2^r(y))^t.$$

While, if $s, r \in \mathcal{N}$ are given fuzzy positive numbers, then there exists a solution $d \in \mathcal{N}$ of the equation (2.1) if and only if

$f_1^s(y)(1+f_1^r(y))^{-t}$ is increasing
$f_2^s(y)(1+f_2^r(y))^{-t}$ is decreasing and

$$f_1^s(y)(1+f_1^r(y))^{-t} \leq f_2^s(y)(1+f_2^r(y))^{-t}.$$

If we know d and r then we can calculate s passing from the equation (2.1) to the cut-functions:

$$f_1^s(y) = f_1^d(y)(1+f_1^r(y))^t; \quad f_2^s(y) = f_2^d(y)(1+f_2^r(y))^t, \tag{2.2}$$

and s is given by

$$s(\omega) = sup_{y\in[0,1]} y \cdot \chi_{[f_1^s(y), f_2^s(y)]}(\omega)$$

From the hypotheses and Proposition 2 we have that $s \in \mathcal{N}$.

While, if $s, r \in \mathcal{N}$ are given fuzzy positive numbers, then from (2.2) it follows that

$$f_1^d(y) = f_1^s(y)(1+f_1^r(y))^{-t}; \; f_2^d(y) = f_2^s(y)(1+f_2^r(y))^{-t}.$$

From these equalities, the hypotheses and Proposition 2, we have that the cut-functions $f_1^d(y), f_2^d(y)$, are left-continuous,

$$d(\omega) = sup_{y\in[0,1]} y \cdot \chi_{[f_1^d(y), f_2^d(y)]}(\omega),$$

and $d \in \mathcal{N}$.

In [11] Pratt gives the definition of *certain equivalent* (c.e.) of X, as the solution, x, of the equation

$$E[u(X)] = u(x)$$

where u denotes an agent's von Neumann and Morgenstern utility function and E denotes the expectation of the random variable $u(X)$.

Take a crisp discount factor, v, then, $y = xv$ is the *present value* (p.v.) of the c.e. x.

While, the solution x of the equation

$$E[u(vX)] = u(x)$$

is the c.e. of the p.v..

The p.v. of the c.e. and the c.e. of the p.v. represent easy ways to define two immediate prices which a person has to pay now for receiving at time t the random amount X.

If u is increasing and concave, then the second is less than the first.

We call these two immediate c.e. *explicit* because the present value is explicitly defined.

In these two definitions we do not specify whether the agent (whose utility is u) is a seller or a buyer of X. A way to specify this is defining the *seller's*, z_s, and *buyer's*, z_b, immediate c. e. implicitly defined through a compound factor by the following equations respectively:

$$E[u(z_s(1+r)^t)] = E[u(X)]$$

for the seller,

$$u(0) = E[u(X - z_b(1+r)^t)]$$

for the buyer, where we may take $u(0) = 0$ (see Peccati [6, 10]).

We call *financial* c. e. the explicit and implicit c.e. defined above.

The distinction between buyer and seller c.e. is more used in the insurance literature (for example, see [4]), than in the financial one. But, in [10, 13] we see that this distinction is used for pricing options in absence of the main standard hypotheses concerning perfection and efficiency of financial markets.

For pricing an option call (to one share) on the expiration date, t, we have to see what is its value at time t. Set Z the price on t of the underlying security and k the striking price. Then, the value on t is given by:

$$X := max\{0,\ Z - k\}.$$

We may use one of the pricing functional defined in this paper for pricing X.

3.Princing Functionals A *pricing functional,* H, is a functional, defined in $\mathcal{X}$ or in $\mathcal{F}$ and with positive values. Consider the following forms:

(i) the financial c. e. defined above;
(ii) the *fuzzily financial* c.e., that we define below;
(iii) the expected value principle: $H(X) = E[X](1 + \gamma)$, the number $\gamma > 0$ is the rate of loading;
(iv) the variance principle: $H(X) = E[X] + aVar[X]$, $a > 0$;
(v) the variance-fuzziness principle defined in [12]:

$$H(X) = E[X] + c\{Var[X] + E[X\psi(f)]\}; \tag{3.1}$$

where

$$E[X\psi(f)] = \int_\Omega X(\omega)\psi(f(\omega))dP(\omega)$$

represents *the fuzziness index on the knowledge* of the risk X, $\psi(t)$ is a norm function, for example the Shannon's norm function

$$\psi(t) := -t\log_2(t) - (1-t)\log_2(1-t),$$

and f is a fuzzy number linked to X.

We define the pricing functional (3.1) in [12] where we see in which hypotheses (3.1) is homogeneous, additive and monotonic. These are properties of no arbitrage in the insurance and reinsurance market (see Borch in [4]). But, they are probably not appropriate in the financial market if the position size directly influences risk (see the paper of Artzner- Delbaen-Eber-Heath [1]).

We want to see that the expected utility function may be a pricing functional which include variance and fuzziness: 4 Set $u(x) := c[(x - E[X])^2 + x\psi(f)]$,

maxX:=k and c be a constant such that

$$c < \frac{k}{Var[X] + E[X \cdot \psi(f(\omega))]}. \tag{3.2}$$

Then, the expected utility function is given by

$$E[u(X)] = c\{Var[X] + E[X\psi(f)]\} \qquad (3.3),$$

and is a pricing functional less than k. The maximum value of X is k, if c satisfies (3.2), then we have that $E[u(X)] < k$. Let us give the fuzzy versions of the c.e..

Using Proposition 3, we may define a fuzzy number which we call *fuzzy explicit c.e.* with the following proposition: 5 If the rate of interest r is an element of $\mathcal{N}$, then the present value of the c.e. and the c.e of the present value are elements of $\mathcal{N}$. Consider the c.e., that is, the solution, x, of the crisp equation $E[u(X)] = u(x)$. Using the Proposition 3 with s=x we have that the present value of the c.e. is defined by the following cut-functions:

$$\forall y \in [0,1],\ f_1^d(y) = x(1 + f_1^r(y))^{-t};\ f_2^d(y) = x(1 + f_2^r(y))^{-t}$$

because $s = x$ is crisp and $(\forall y \in [0,1])$, $f_1^x(y) = x = f_2^x(y)$. Obviously, $d \in \mathcal{N}$, is defined by

$$(\forall \omega \in \Omega) \quad d(\omega) = sup_{y\in[0,1]} y \cdot \chi_{[f_1^d(y), f_2^d(y)]}(\omega).$$

The c.e. of the present value is a fuzzy number, say, x_0 , whose values of the cut-functions, $\forall y \in [0,1], \forall i \in \{1,2\}$ $f_i^{x_0}(y)$ are the solutions of these equations

$$\forall y \in [0,1],\ u(f_i^{x_0}(y)) = E[u(X(1+f_i^r(y))^{-t})],$$

then the f.n. x_0 is given by

$$(\forall \omega \in \Omega) \quad x_0(\omega) = sup_{y\in[0,1]} y \cdot \chi_{[f_1^{x_0}(y), f_2^{x_0}(y)]}(\omega).$$

We may call the f.n. d and x_0 *fuzzy explicit c.e.*.

If the rate of interest r is a f.n. we can define *the fuzzy implicit c.e.* z_s (for seller) and z_b (for buyer) by means of the cut-functions of the f.n. r:

1) $z_s(\omega)$ is defined by means its cut-functions, $\forall i \in \{1,2\}, f_i^{z_s}$, whose values are the solutions of the following equations:
$$(\forall y \in [0,1]),\ E[u\{f_i^{z_s}(y)(1+f_1^r(y))^t\}] = E[u(X)];$$
2) $z_b(\omega)$ is defined by means of the solutions, $\forall i \in \{1,2\}, f_i^{z_b}(y)$, of the following equations:
$$(\forall y \in [0,1]),\ E[u\{(X - (f_i^{z_b}(y)(1+f_i^r(y))^t)\}] = u(0)$$

In both the cases we have

$$(\forall \omega \in \Omega) \quad z(\omega) = sup_{y\in[0,1]} y \cdot \chi_{[f_1^z(y), f_2^z(y)]}(\omega),$$

where z is equal to z_s in the case of the seller and z_b in that of the buyer.

We call *fuzzy financial c.e.* (f.f.c.e.) an explicit or implicit fuzzy c.e..

A f.f.c.e. is a f. n., say, f.

For having a positive real number from f, we may take the element of the support of f with the maximum degree of membership, say x_m. If x_m is not unique, then we call *fuzzily immediate financial* c.e. the mean of them. So, we have built a pricing functional.

We may define a fuzzily immediate financial c.e. with the following variance-fuzziness functional (cf.[12]):

$$\forall y \in [0,1], \quad E[u(X)] = E[X] + c\{Var[X] + E[X\psi(y)]\},\ c > 0 \tag{1}$$

because (1) is an expected utility function taking

$$\forall y \in [0,1], \quad u(x) := x + c[(x - E[X])^2 + x\psi(y)].$$

If the constant c is such that

$$\forall y \in [0,1], \quad c < \frac{k - E[X]}{Var[X] + E[X\psi(y)]},$$

then, we have that (1) is less than maxX:=k.

4.Fuzzy Models with Choquet's and Sugeno's integrals

Alternative models for the financial certainty equivalents are obtained using Choquet's and Sugeno's integrals.

Let a fuzzy measure space $(\Omega, \mathcal{A}, \mu)$ be given, where μ is an increasing set function with $\mu(\Omega) = 1, \mu(\emptyset) = 0$.

6 The *expected* utility with Choquet's integral is the functional

$$C : \mathcal{X} \longrightarrow R^+$$

defined in this way:

$$C(\mu, X) = \int_{[0,+\infty[} \mu(\omega \in \Omega : u(X(\omega)) > z) dz.$$

This may be interpreted as an expectation of $u(X)$ with respect to μ, then we can replace $E[u(X)]$ with $C(\mu, u(X))$ and define a certainty equivalent as the solution, x, of the equation

$$C(\mu, u(X)) = u(x).$$

We may consider the *expectation* also with the Sugeno's integral. Let us give the definition. 7 Let $\mathcal{F}$ the set of all $\mathcal{A}$-measurable functions, $f : \Omega \longrightarrow [0, 1]$, representing a future fuzzy amount.

The *expected* utility of f with Sugeno's integral is defined in the following way:

$$S(\mu, f) := \vee_{z \in [0,1]} \left[z \wedge (\mu\{\omega \in \Omega : u(f(\omega)) > z\}) \right].$$

The *financial certainty equivalent* is defined as before replacing the functional E with either C or S and the random variable X may be replaced by a fuzzy number.

Future researches could implicate the study of the properties of our fuzzy pricing functionals and their numerical and statistical applications also using the parallel calculus.

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