A CRITICAL REVIEW OF TECHNIQUES FOR TERM STRUCTURE ANALYSIS

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Abstract. Fixed income markets share many features with the equity markets. However there are significant differences as well and many attempts have been done in the past to develop specific tools which describe (and possibly forecasts) the behavior of such markets.

For instance, a correct pricing of fixed income securities with fixed cache flows requires the knowledge of the *term structure* of interest rates. A number of techniques have been proposed for estimating and interpreting the term structure, yet solid theoretical foundations and a comparative assessment of the results produced by these techniques are not available.

In this paper we define the fundamental concepts with a mathematical terminology. Besides that, we report about an extensive set of experiments whose scope is to point out the strong and weak points of the most widely used approaches in this field.

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0. Introduction

Many ideas and techniques developed for the analysis of financial markets can be applied both to equities and to fixed income securities. For instance the distribution and the dynamical evolution of price variations are studied under the assumption that such variations can be described by a random process, regardless of the asset category (equity or fixed income) in exam.

However fixed income markets have many special features, thus specific concepts and tools have been introduced in the past to study (and possibly forecast) the behavior of such markets. For instance, a correct pricing of fixed income securities with fixed cash-flows requires the knowledge of the *term structure of interest rates*.

A number of "numerical recipes" have been proposed for estimating and interpreting the term structure of interest rates, yet a solid theoretical foundation of such concept and a comparative assessment of the results produced by the different techniques is not available.

In this paper we define formally the concept of absence of arbitrage opportunities and derive rigorously the existence of discount factors. Moreover the most widely used approaches in estimating the term structure are presented and their performance is checked with an extensive set of experiments.

The structure of the paper is the following: section 1 is a reminder of the terminology used in fixed income markets; section 2 defines a mathematical framework for the discount function; section 3 describes the major techniques currently in use to estimate the term structure of interest rates; section 4 presents the results of our numerical experiments; section 5 concludes the work with the future perspectives.

1. Fixed income securities

A bond is a credit instrument issued by a public institution or a corporate company on raising a debt. When the size and the timing of the payments due to the investor are fully specified in advance, the bond falls within the *fixed-income securities* category.

The main source of uncertainty for a bond markets investor is the default risk of the issuer, that is the chance that the issuer becomes unable to fulfill the commitment of paying all promised cash-flows on time. For this reason the only (true) fixed-income securities are those issued by the governments of some developed countries or by "well-known" large corporate institutions to which credit rating agencies also give their highest rating: these are the borrowers least likely to default on their debt repayments.

Likewise should not be regarded as fixed-income securities those bonds having cash-flows indexed to the rate of inflation (index-linked bonds) or those with embedded optionality giving either the issuer or the holder some discretion to redeem early or to convert to another security [1] .

The lack of any uncertainty makes fixed-income bonds a useful means for measuring market interest rates.

An investor can purchase "new" bonds in the primary market (*i.e.* directly from the issuer) or formerly issued bonds from other investors in the secondary market. Furthermore large institutional investors can sell bonds to the issuer as well, that is they can pay the price of the bond and undertake to repay the corresponding future cash-flows. Hence the bid and ask prices of a bond [2] are set continuously, as long as it remains outstanding, by transactions occurring between market participants.

In principle a bond can promise any pattern of future cash-flows. However there are two main categories.

Zero-coupon bonds, known also as discount bonds, make a single payment at a date in the future known as the maturity date. The size of this payment is called the *face value*, or par value, of the bond. The period of time up to the maturity date is the *maturity* of the bond. US Treasury bills (Treasury obligations with maturity at issue of up to twelve months) take this form.

Coupon bonds make interest payments (often referred to as coupon payments) of a given fraction of face value, called the *coupon rate*, at equally spaced dates up to and including the maturity date when the face value is also paid. The frequency at which interest payments are made varies from market to market, but generally they are made either annually or semi-annually. US Treasury notes and bonds (Treasury obligations with maturity at issue of twelve months up to ten years and above ten years, respectively) take this form. Coupon payments on US Treasury notes and bonds are made every six months.

In the following, for sake of simplicity, we will focus on US Treasury bond markets only. There are a number of advantages in this choice. First of all these markets are extremely large regardless of whether size is measured by outstanding or traded quantities, are very liquid and have uniform rules. Moreover the number of issues is quite high (at present there are about two hundreds obligations outstanding). These features make the study of such markets somewhat easier.

1.1. **The Law of One Price.** The key idea behind asset valuation in fixed-income markets is the absence of *arbitrage* opportunities, also known as the 'Law of One Price'.

[1] To give some examples, we recall the US, UK and Japanese government callable bonds, giving the Treasury the option to redeem the bond at face value at any time between two dates specified at the time of issue; or the Italian government putable bonds (CTO) giving the holder the option to sell the bond back to the Treasury at face value on pre-specified dates; or the UK government and corporate convertible bonds giving the holder the option to convert the bond into another pre-specified bond at a pre-specified ratio on one or more pre-specified dates (see *e.g.* [1])

[2] respectively the price received when selling and paid when buying that bond

The 'Law of One Price' states that two portfolios of fixed-income securities (that is two positions taken in the marketplace through buying and selling fixed-income securities) that guarantee the investor the same future cash-flows and give him the same future liabilities, must sell for the same net price. Phrased another way, a portfolio of fixed-income securities giving the investor neither the right to receive any future cash-flow nor any future liability, must have zero initial net cost. Conversely a portfolio requiring no initial net investment must give the investor some "real" liability, that is some liability whose size is larger than the net amount of money earned till then.

If any violation of these constraints on the prices of all fixed-income securities outstanding at a given time occurs, then a profitable and risk-less investment opportunity, called an arbitrage opportunity, or simply an arbitrage, arises.

Arbitrage opportunities may arise (sporadically) in financial markets but they cannot last long. In fact, as soon as an arbitrage becomes known to sufficiently many investors, the prices will be affected as they move to take advantage of such an opportunity. As a consequence, prices will change and the arbitrage will disappear. This principle can be stated as follows: in an *efficient market* [3] there are no permanent arbitrage opportunities.

The prices of all fixed-income securities (and in particular those of US government bonds) as they are quoted at any given time, cannot, then, be independent one of the other and the existence of (moderate) arbitrage opportunities in financial markets can be viewed as a relative mispricing between correlated assets. Such mispricing can be ascribed to taxation, transaction costs and commissions associated with trading that make the net price of a bond different from its quoted gross price (and, hence, make the corresponding arbitrage opportunity not really profitable), to liquidity effects and other market frictions, or to non-synchronous quotations.

2. The mathematical setting

The cash-flows that a US government bond holder is entitled to receive are completely determined by the bond face value that, unless otherwise specified, will always be supposed equal to \$100, by its coupon rate and by its maturity. For this reason an outstanding bond can be identified with a triplet (c, m, p), where $c \geq 0$ is its (semi-annual) coupon rate in percentage points, $m > 0$ is its maturity in years (computed using '30/360' convention) and $p > 0$ is its *gross price* in dollars. For a US Treasury bill $c = 0$ and $m \leq 1$; for a US Treasury note or bond $c > 0$.

We will state, now, formally the condition of absence of arbitrage opportunities and derive its consequences on the prices of US government bonds. This will lead us to the definition of the discount factors or zero-coupon bond prices. We start by giving some definitions.

Let $\mathcal{B} := [0, +\infty) \times \mathbb{R}_+ \times \mathbb{R}_+$ be the set of all bonds, actually issued by the US Treasury or not, and still outstanding at the date in exam.

Secondly, given a subset $\mathcal{S}$ of $\mathcal{B}$, denote by $T(\mathcal{S})$ the vector of the maturities of all coupon payments of all bonds belonging to $\mathcal{S}$, sorted in increasing order. This amounts to say that if $T(\mathcal{S}) = (t_1, \dots, t_N)$ for some $N \in \mathbb{N}$, then

$$\{t_1, \dots, t_N\} := \cup_{(c,m,p)\in\mathcal{S}} \{m - i/2;\ i \in \mathbb{N} \cup \{0\} \text{ and } m - i/2 > 0\}$$

[3] we recall that according to Malkiel [11] "A capital market is said to be efficient if it fully and correctly reflects all relevant information in determining security prices"

and $t_i < t_{i+1}$ for every i. Moreover for every $(c, m, p) \in \mathcal{S}$ define the *cash-flows vector* $\boldsymbol{\varphi}(c, m) = (\varphi_1(c, m), \dots, \varphi_N(c, m)) \in \mathbb{R}^N$ by

$$\varphi_i(c, m) := \begin{cases} 100 + c & \text{, if } m = t_i \\ c & \text{, if } t_i \in m - \frac{\mathbb{N}}{2} \\ 0 & \text{, otherwise} \end{cases} \qquad (i = 1, \dots, N).$$

Thus $\varphi_i(c, m)$ represents the cash-flow (in dollars) that an investor holding bond (c, m, p) is entitled to receive in t_i years' time.

Thirdly, we say that a subset $\mathcal{T}$ of $\mathcal{B}$ is a *complete coupon term structure* if and only if it contains at least a bond maturing on each coupon payment day (see *e.g.* [4]). Observe that for a complete coupon term structure $\mathcal{T}$ the set of the maturities of all coupon payments of all bonds belonging to $\mathcal{T}$ is the same as the set of the maturities of all bonds belonging to $\mathcal{T}$, that is, if $T(\mathcal{T}) = (t_1, \dots, t_N)$, then

$$\{t_1, \dots, t_N\} := \cup_{(c,m,p) \in \mathcal{T}} \{m\}.$$

Finally we represent a *portfolio* of B bonds $(c_1, m_1, p_1), \dots, (c_B, m_B, p_B)$ by a vector $(q_1, \dots, q_B) \in \mathbb{R}^B$ such that for each j, q_j represents the traded quantity of bond (c_j, m_j, p_j). By definition, $q_j > 0$ means that the investor has bought $\$100 q_j$ face value of bond j whereas $q_j < 0$ means that he has sold $\$100 |q_j|$ face value of that bond.

We are, now, in a position to state formally the condition of absence of arbitrage opportunities among the bonds belonging to a given set $\mathcal{S} \subset \mathcal{B}$, and to show which constraints on their prices such condition implies (see Theorem 2.5 below).

Definition 2.1. *Let $\mathcal{S}$ be a subset of $\mathcal{B}$ and let $T(\mathcal{S}) = (t_1, \dots, t_N)$ for some $N \in \mathbb{N}$. We say that $\mathcal{S}$ satisfies the hypothesis of Absence of Arbitrage Opportunities if and only if, given $(c_1, m_1, p_1), \dots, (c_B, m_B, p_B) \in \mathcal{S}$ ($B \in \mathbb{N}$) the following three conditions are fulfilled:*

(NA_1) *if $(q_1, \dots, q_B) \in \mathbb{R}^B$ is such that $\sum_{j=1}^B q_j \boldsymbol{\varphi}(c_j, m_j) = \mathbf{0}$, then $\sum_{j=1}^B q_j p_j = 0$;*

(NA_2) *if $(q_1, \dots, q_B) \in \mathbb{R}^B$ is such that*

$$\sum_{j=1}^{B} q_j \varphi_i(c_j, m_j) = \begin{cases} f_{\bar{\imath}} & , \quad \text{if } i = \bar{\imath} \\ 0 & , \quad \text{otherwise} \end{cases},$$

for some $f_{\bar{\imath}} > 0$, then $0 < \sum_{j=1}^B q_j p_j < f_{\bar{\imath}}$;

(NA_3) *if $(q_1, \dots, q_B) \in \mathbb{R}^B \setminus \{0\}$ is such that $\sum_{j=1}^B q_j p_j = 0$, then there exists $\bar{\imath} \in \{1, \dots, N\}$ such that $\sum_{i=1}^{\bar{\imath}} \sum_{j=1}^B q_j \varphi_i(c_j, m_j) < 0$.*

Remark 2.2. The ith component of the vector $\sum_{j=1}^B q_j \boldsymbol{\varphi}(c_j, m_j)$ represents, if it is positive, the total cash-flow that an investor holding portfolio $(q_1, \dots, q_B)$ will receive in t_i years' time; if it is negative, the total liability that he will have at that date. The quantity $\sum_{j=1}^B q_j p_j$ is the (net) price of the portfolio.

Hence, condition (NA_1) means that a portfolio of bonds giving the investor neither the right to receive any future cash-flow nor any future liability must have zero initial net cost. Conversely (NA_3) means that a non-trivial portfolio requiring no initial net investment must, sooner or later, give the investor a liability of larger size than the total net amount of money received till then.

The meaning of condition (NA_2) is self-evident and will not be discussed any further.

Definition 2.3. *Given $B + 1$ bonds $(c, m, p), (c_1, m_1, p_1), \dots, (c_B, m_B, p_B)$ in $\mathcal{B}$, a portfolio $(q_1, \dots, q_B)$ of the last B bonds such that $\sum_{j=1}^B q_j \boldsymbol{\varphi}(c_j, m_j) = \boldsymbol{\varphi}(c, m)$,* *is called a replicating portfolio of bond* (c, m, p)*. In fact it entitles an investor to receive exactly the same cash-flows as if he held bond* (c, m, p)*. If, furthermore, each bond* (c_j, m_j, p_j) *expires on a different coupon payment date of bond* (c, m, p)*, then we will refer to such a portfolio as a minimal replicating portfolio.*

Remark 2.4. Since the case $q_j \equiv 0$ is trivial, condition (NA_1) amounts to say that the price of each bond (c, m, p) in $\mathcal{S}$ equals the ones of all its replicating portfolios, whenever they exist.

Theorem 2.5. *Let* $\mathcal{T} \subset \mathcal{B}$ *be a complete coupon term structure and let* $T(\mathcal{T}) = (t_1, \ldots, t_N)$ *for some* $N \in \mathbb{N}$*. If* $\mathrm{Card}(\mathcal{T}) \geq N$ *and if condition* (NA_1) *is satisfied, then there exist* $d_1, \ldots, d_N \in \mathbb{R}$ *such that for every* $(c, m, p) \in \mathcal{T}$

$$p = \sum_{i=1}^{N} d_i \varphi_i(c, m). \tag{2.1}$$

If, furthermore, conditions (NA_2) *and* (NA_3) *are fulfilled as well, then* $1 > d_1 > d_2 > \ldots > d_N > 0$*.*

A proof of this theorem is reported in the Appendix.

Remark 2.6. In order to prove Theorem 2.5 we need to find N bonds in $\mathcal{T}$ such that the matrix whose columns are their cash-flows vectors have maximum rank N. The hypothesis that $\mathcal{T}$ forms a complete coupon term structure just helps to ensure that this choice is possible. However this is not the only case in which this is true. To realize this it is sufficient to think that if $\mathcal{S} = \{(c_1, m_1, p_1), (c_2, m_2, p_2), (c_3, m_3, p_3)\}$ with $m_1 = \frac{1}{2}$, $m_2 = m_3 = \frac{3}{2}$ and (of course) $c_2 \neq c_3$, then the matrix whose columns are the cash-flows vectors of these three bonds has maximum rank 3, even if $\mathcal{S}$ is not a complete coupon term structure.

Remark 2.7. By equation (2.1) with $c = 0$ it is apparent that each d_i represents the price of a (hypothetical) zero-coupon bond in $\mathcal{T}$ with unitary face value and maturity $m = t_i$. For this reason $d_1, \ldots, d_N$ are called the *discount factors*, or zero-coupon bond prices, corresponding to the maturities $t_1, \ldots, t_N$.

Definition 2.8. *Given* N *discount factors* $d_1, \ldots, d_N$ *corresponding to maturities* $t_1, \ldots, t_N$*, we define the spot rate (of return), or zero-coupon yield,* s_i*, by*

$$s_i := -\frac{1}{t_i} \lg d_i$$

Furthermore, given $P \in \mathbb{N}$*, if there exists* $j(i) > i$ *such that* $t_{j(i)} - t_i = \frac{P}{2}$*, we define the* P*-periods forward rate (of return)* [4] *,* f_i^P*, by*

$$f_i^P := -\frac{2}{P} \lg \frac{d_{j(i)}}{d_i}.$$

Since $\frac{1}{d_i} = \exp\left(\int_0^{t_i} s_i \, dt\right)$, then s_i represents the continuously compounded rate of return per unit time on a t_i-years investment, made today, in the bond markets. Analogously f_i^P represents the continuously compounded rate of return per unit time on a P-periods investment to be made t_i years ahead.

As it is common in financial literature, we will refer to the set of the spot rates corresponding to all maturities as the (cross-sectional) *term structure of interest rates*.

[4]as it is common in financial literature, we will refer to any six months' time interval as a period

2.1. **The method of Carleton and Cooper.** If a complete coupon term structure is available and if conditions (NA_1)-(NA_3) are (strictly) satisfied, then Theorem 2.5 tells us how to compute a complete set of decreasing discount factors (one for each coupon payment date) that summarize the whole information carried about nominal discount rates by bonds in our data set.

When small violations of condition (NA_1) are detected in the bond set $\mathcal{T}$ forming a complete coupon term structure, Carleton and Cooper [5] suggested to consider the prices of the bonds in $\mathcal{T}$ as random variables with expected values satisfying (NA_1) - (NA_3). According to their hint, equation (2.1) must be modified by adding a bond-specific error term to (say) its right-hand side:

$$p_j = \sum_{i=1}^{N} d_i \varphi_i(c_j, m_j) + \varepsilon_j, \tag{2.2}$$

where j ranges over all bonds in the set $\mathcal{T}$, the ε_j's are random variables with $E\{\varepsilon_j\} \equiv 0$, and $1 > d_1 > d_2 > \ldots > d_N > 0$.

Estimators, $\hat{d}_1, \ldots, \hat{d}_N$, of the discount factors $d_1, \ldots, d_N$, subjected to the condition $1 > \hat{d}_1 > \hat{d}_2 > \ldots > \hat{d}_N > 0$, can, then, be attained by a constrained least squares procedure with all bonds in the set $\mathcal{T}$.

We recall, however, that the least-squares estimators $\hat{d}_1, \ldots, \hat{d}_N$ of the d_i's exist and are unique provided that the regressors' matrix, $\Phi = ((\varphi_i(c_j, m_j)))$, has maximum rank N, which is certainly true if $\mathcal{T}$ is a complete coupon term structure (see Remark 2.6). If, on the contrary, the rank of the matrix Φ is less than N, then the problem of estimating the discount factors $d_1, \ldots, d_N$ by a least squares procedure is improperly posed, or ill-posed, in the sense of Hadamard (see, *e.g.*, [9]) and does not admit any solution. Hence the completeness hypothesis of the term structure of $\mathcal{T}$ (or a weaker version of its) is necessary to apply the method of Carleton and Cooper.

3. Estimation of the term structure

In general, the bond set you decide to use for your analysis does not form a complete coupon term structure. This is true, for example, for the set we used in the present paper. To realize this look at the next figure 1, where we have reported the maturity spectrum of all of 206 bonds in our sample.

Since no bond has maturity of 10 up to 15 years and the maturity of the longest (coupon) bond reaches about 30 years, the term structure of coupon bonds in our data set cannot be complete.

In 1971 McCulloch [12] introduced a new technique for studying the term structure of interest rates based on the following assumption [5] .

Hypothesis 3.1. *If the bond set $\mathcal{S} \subset \mathcal{B}$ satisfies conditions (NA_1)-(NA_3), then there exists a decreasing function d: $[0, +\infty) \to (0, 1]$ such that $d(0) = 1$, $\lim_{t\to+\infty} d(t) = 0$ and that for every bond $(c, m, p) \in \mathcal{S}$*

$$p = \sum_{i=1}^{N} d(t_i)\varphi_i(c, m).$$

Remark 3.2. Function d is called the *discount function* for the bond set $\mathcal{S}$. For each $t \geq 0$, $d(t)$ represents, then, the present value of \$1 to be received in t years' time. When $\mathcal{S} = \mathcal{T}$ forms a complete coupon term structure, the discount factors

[5]Actually McCulloch's statement of Hypothesis 3.1 differs somewhat from ours: he assumed that coupon payments arrive in a continuous stream instead of semiannually. However our formulation corresponds to the one preferred by most authors after McCulloch

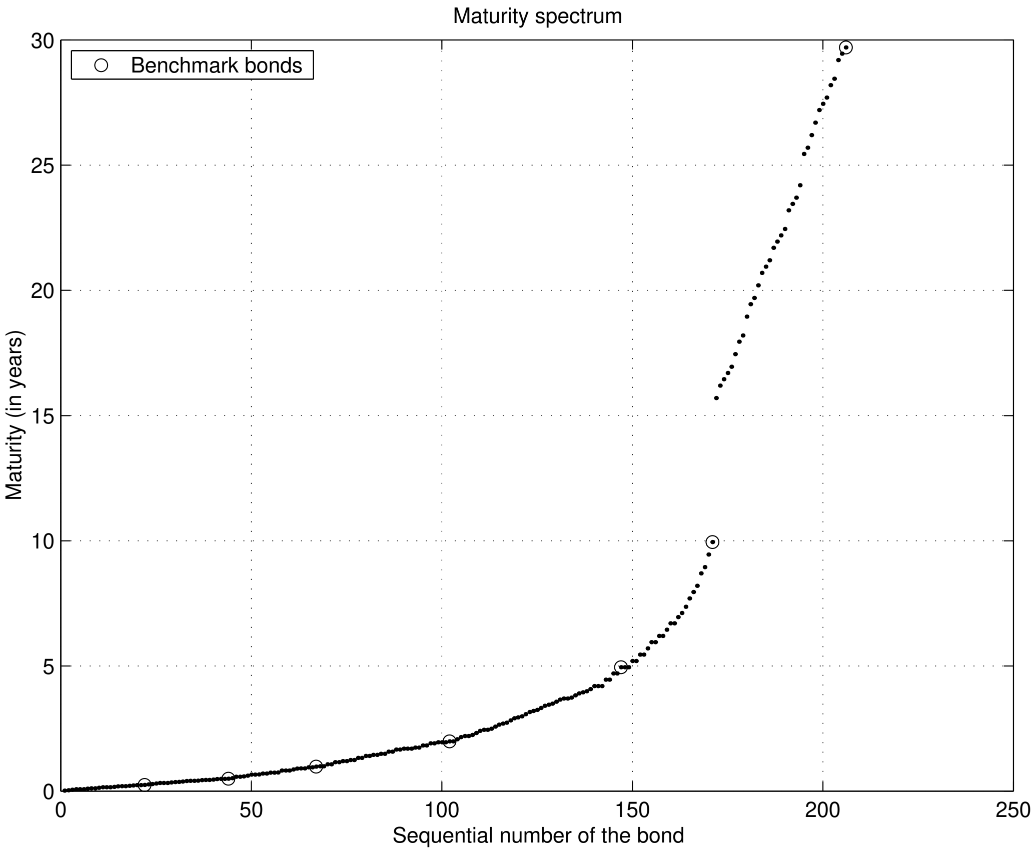


FIGURE 1.

$d_1, \dots, d_N$, whose existence is guaranteed by Theorem 2.5, are just the values that d takes on the corresponding maturities $t_1, \dots, t_N$, that is $d_i = d(t_i)$ for every i.

The definitions of spot rate and of P-periods forward rate can be easily extended, by means of the discount function d, to all maturities $t \geq 0$. In particular the spot rate function $s\colon (0, +\infty) \to \mathbb{R}_+$ is defined by

$$s(t) := -\frac{1}{t} \lg d(t) \tag{3.1}$$

and the P-periods forward rate function $f^P\colon [0, +\infty) \to \mathbb{R}_+$ by

$$f^P(t) := -\frac{2}{P} \lg \frac{d(t + P/2)}{d(t)}.$$

Furthermore, if d is differentiable, we can define the *instantaneous forward rate* (of return) function $f\colon [0, +\infty) \to \mathbb{R}_+$ by

$$f(t) := -\frac{d}{dt} \lg d(t). \tag{3.2}$$

Then $f(t)$ represents the simple net rate of return per unit time of an investment in the bond markets over an infinitesimal time interval starting t years ahead. Moreover the P-periods forward rate $f^P(t)$ equals the average of the instantaneous forward rate f over the length $P/2$ interval $[t, t + P/2]$, that is

$$f^P(t) \equiv \frac{2}{P} \int_t^{t+P/2} f(s)\, ds.$$

It is important to understand, however, that the existence of a discount function, even when conditions (NA_1)-(NA_3) are strictly satisfied, is nothing but a hypothesis that scholars and practitioners assume in order to fill the gaps in the maturity spectrum of bond samples that do not form a complete coupon term structure. It is by no means a consequence of (NA_1)-(NA_3). As we have already pointed out at the end of section 2, building a complete set of discount factors is an ill-conditioned problem when the term structure of coupon bonds in the sample is incomplete. This explains the difficulties met with by people when trying to estimate a discount function from a generic bond data set.

As to the regularity properties of the discount function, some researchers (see, *e.g.*, [7]) assert that on the ground of mere economic considerations we can only require that d be monotonic decreasing, any further restrictions being not justified. On the contrary, most authors assume also that d be twice differentiable so attaining a smooth forward rate curve. In particular Langetieg and Smoot [10] argue that a non-smooth forward rate function f should give rise to arbitrage opportunities and that, consequently, any irregularity of f should be quickly priced out in an efficient market.

If we assume Hypothesis 3.1, then the problem of estimating the term structure of interest rates from a bond data set reduces to choosing an n-parameter family of functions $d(\cdot;\boldsymbol{\alpha})\colon\ [0,+\infty)\to\mathbb{R}$ that we consider capable of capturing the main characteristics of the bond markets in exam and determining the parameter vector $\boldsymbol{\alpha}\in\mathbb{R}^n$ by minimizing the sum of squared residuals (least squares fitting procedure)

$$\min_{\boldsymbol{\alpha}} \sum_j \left| p_j - \sum_{i=1}^{N} d(t_i;\boldsymbol{\alpha})\varphi_i(c_j,m_j) \right|^2, \tag{3.3}$$

where j ranges over all bonds (c_j, m_j, p_j) in our data set. The choice of the functions $d(\cdot;\boldsymbol{\alpha})$, that at the very end is always a matter of judgment, is crucial to the quality of our fit.

The test functions $d(\cdot;\boldsymbol{\alpha})$ that are usually employed in the fitting procedure (3.3) may be roughly divided in two categories: piecewise (polynomial or exponential) functions or *spline* and functions generated by "parsimonious" models.

Spline functions are more capable of capturing genuine bends in the discount function but have the drawback of over-fitting risk; parsimonious models, on the contrary, are less flexible but do a better job in removing noise from the data. So the choice of spline functions is better for getting a "good fit" whereas parsimonious models should be privileged for sake of "smoothness".

3.1. **Spline based techniques .** The most widely used spline method, specially among practitioners, is due to McCulloch [12] (see also [13]) and is based on the use of polynomial spline functions.

We recall that, given $-\infty < K_1 < K_2 < \ldots < K_k < +\infty$ (for some $k > 2$) and $r \in \mathbb{N}\cup\{0\}$, we say that $f\colon\ [K_1,K_k]\to\mathbb{R}$ is an r-degree polynomial spline function with knot points $K_1, K_2, \ldots, K_k$, if and only if f is an r-degree polynomial in each interval $[K_i, K_{i+1}]$ and if it is continuous, along with all its derivatives up to order $r-1$ (if $r>0$), in $[K_1,K_k]$.

The reasons underlying the choice of polynomial spline functions are as follows. It is apparent that a very simple, but somehow naïve, approach to the problem of fitting a discount function to a bond data sample is to use polynomial test functions. However such functions have uniform resolving power, that is they tend to fit better at the short end of the maturity spectrum where the greatest concentration of bond maturities occurs (see figure 1), and worse at the long end. A possible way out is, of course, to increase the order of the polynomial but this can cause instability in the parameter estimates. On the other hand, practitioners often require methods endowed with a sufficiently high flexibility to have the chance to choose *a priori* how good their fit of the discount function will be in various regions of the maturity spectrum. For instance, they may desire to improve the goodness of their fit in those maturity ranges where many bonds are traded.

A fitting procedure that meets all these requirements involves the use of polynomial spline functions. By means of a convenient choice of the number and the position of the knots, such method allows to achieve the desired resolution degree along the whole maturity spectrum, even employing relatively low order polynomials (which means more stable curves).

So the choice of the number and the position of knot points affects dramatically the quality of the estimated term structure of interest rates.

McCulloch [12] suggested to choose a number k of knots equal (approximately) to the square root of bonds in the sample and to place them in such a way that $K_1 = 0$, K_k equals the maturity of the longest bond and that each subinterval $[K_i, K_{i+1}]$ contains approximately the same number of observed maturities. This choice should provide an increasing resolving power as the number of observations increases and should avoid over-fitting phenomena.

Currently, practitioners and financial analysts prefer to place knots at 0, 1, 3, 5, 7, 11 $= 10^+$ and 30 years, so splitting up the whole maturity range into the intervals $(0,1)$, $(1,3)$, $(3,5)$, $(5,7)$, $(7,11)$, and $(11,30)$. The rationale behind this choice is that most asset management companies divide their assets into groups (or categories) that always lie in one of the above intervals of the maturity spectrum. This fact is reflected also in the existence of several benchmark securities with maturity of about 1, 3, 5, 7, 10 and 30 years.

Given k knot points $K_1, \dots, K_k$, the set of all r-degree polynomial spline functions on $[K_1, K_k]$ form a linear space of dimension $k + r - 1$. As a matter of fact we have the chance of choosing arbitrarily the $r + 1$ coefficients of the polynomial in the first subinterval $[K_1, K_2]$ and one coefficient for each polynomial in the $k - 2$ remaining subintervals $[K_i, K_{i+1}]$ with $i = 2, \dots, k - 1$. The others parameters are fixed by the conditions $f^{(n)}(K_i - 0) = f^{(n)}(K_i + 0)$ for all $i \in \{2, \dots, k - 1\}$ and all $n \in \{0, \dots, r - 1\}$.

Let $\{f_0, f_1, \dots, f_{r+k-2}\}$ be a base for the linear space of r-degree polynomial spline functions with knot points $K_1, \dots, K_k$. Without loss of generality we can assume that $f_0(t) \equiv 1$ (as a constant function is a polynomial spline function) and that $f_1(0) = \dots = f_{r+k-2}(0) = 0$. Then $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{r+k-2})$ and the McCulloch model for the discount function is

$$d(t; \boldsymbol{\alpha}) = 1 + \sum_{j=1}^{r+k-2} \alpha_j f_j(t).$$

The corresponding least squares problem is therefore linear in the parameter vector $\boldsymbol{\alpha}$.

A major limitation of this method is that it does not force a monotonous decreasing behavior of the resulting discount function. As a consequence, meaningless results, like negative forward rates, may be obtained. In this respect Barzanti and Corradi [2] (see also [3] and [2]) have recently, proposed the introduction a set of linear constraints on the parameters $\alpha_1, \dots, \alpha_{r+k-2}$ that ensure the desired monotonicity of the employed spline functions.

A further criticism directed by Vasicek and Fong [19] to McCulloch's method is that most general-equilibrium models of the term structure of interest rates (see, *e.g.*, [18] and [8]) expect an exponential form for the discount function. However piecewise polynomials have a different curvature compared to an exponential function. Hence they conclude that "conventional" polynomial splines cannot provide a good fit to an exponential-like discount function.

Following the arguments of Vasicek and Fong, Langetieg and Smoot [10] and Coleman, Fisher and Ibbotson [7] proposed, respectively, to fit a polynomial spline to the spot rate function and to the instantaneous forward rate function rather than to the discount function [6]. They argued that, assuming an exponential form for the discount function, their regression procedure should give better results (*i.e.* more "realistic" from the financial viewpoint) compared to McCulloch's technique.

By (3.1) and (3.2) $d(t) \equiv \exp(-ts(t))$ and $d(t) \equiv \exp\left(-\int_0^t f(s)\,ds\right)$. Hence the method by Langetieg and Smoot amounts to take $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{r+k-2})$ (for some $r \in \mathbb{N} \cup \{0\}$ and some $k \geq 2$) and

$$d(t;\boldsymbol{\alpha}) = \exp\left(t \sum_{j=0}^{r+k-2} \alpha_j f_j(t)\right),$$

where $\{f_0, f_1, \dots, f_{r+k-2}\}$ is a base for the linear space of r-degree polynomial spline functions with knot points $K_1, \dots, K_k$.

Analogously, since the integral of an r-degree polynomial spline function is an $(r+1)$-degree polynomial spline, the method by Coleman, Fisher and Ibbotson amounts to take $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{r+k-1})$ and

$$d(t;\boldsymbol{\alpha}) = \exp\left(1 + \sum_{j=1}^{r+k-1} \alpha_j f_j(t)\right),$$

where, as above, $\{f_0, f_1, \dots, f_{r+k-1}\}$ is a base for the linear space of $(r+1)$-degree polynomial spline functions with knot points $K_1, \dots, K_k$ and with the property that $f_0(t) \equiv 1$ and that $f_1(0) = \ldots = f_{r+k-1}(0) = 0$.

The methods by Langetieg and Smoot and by Coleman, Fisher and Ibbotson share the same limitation of the method by McCulloch in that it is impossible to force a monotonous decreasing behavior of the resulting discount function by simply introducing linear constraints on the parameters to be determined. The only exception is for the method by Coleman, Fisher and Ibbotson with $r = 0$, that is when a piecewise constant functions are used to fit the instantaneous forward rate function. Hence, again, financially unacceptable results may be found.

3.2. **Parsimonious models.** It is readily apparent that, although spline based techniques are still widely used to estimate the term structure of interest rates, they present many drawbacks (as pointed out, for example, by Shea [15, 16]). As a matter of fact several choices, such as the number and the position of knots and the degree of the basic polynomials, affect dramatically the quality of fit.

An alternative approach to the estimation of the term structure of interest rates is the use of the so-called *parsimonious* models. The main idea behind this class of models is to postulate a unique functional form on the whole range of maturities for the discount function. This implies that in parsimonious models some basic properties of the discount function, such as monotonicity, we are interested in can be imposed *a priori*. Furthermore, the number of parameters to estimate is typically much smaller than that needed for a typical spline based approach.

Let us describe briefly some of the parsimonious models proposed in the literature.

[6]Actually Coleman, Fisher and Ibbotson proposed only the use of piecewise constant functions, that is of 0-degree polynomial splines. However their method can be easily extended to polynomial splines of any order.

In [6] the functional form for the discount function is proposed to be

$$d(t;\boldsymbol{\alpha}) = \exp(-\sum_{j=1}^{n} \alpha_j t^j).$$

The unknown parameters $\alpha_1, \ldots, \alpha_n$ can be estimated either by (non-linear) least square regression or by a maximum likelihood approach. The latter one is found to perform better, probably, because of the explicit consideration of the heteroskedasticity in the data. Notice that this model is a particular case of the ones by Langetieg and Smoot and by Coleman, Fisher and Ibbotson with just two knots.

A particularly relevant approach is the one proposed by Nelson and Siegel [14]. They attempted to model explicitly the implied instantaneous forward rate function f. The Expectation Hypothesis provides a heuristic motivation for their model since f is modeled as the solution of a second order differential equation. The functional form suggested for f is

$$f(t;\boldsymbol{\alpha}) = \beta_0 + \beta_1 e^{-t/\tau} + \beta_2 \frac{t}{\tau} e^{-t/\tau}, \tag{3.4}$$

where $\boldsymbol{\alpha} = (\beta_0, \beta_1, \beta_2, \tau)$. It implies the following model for the discount function:

$$d(t;\boldsymbol{\alpha}) = \exp\left(-t\left(\beta_0 + (\beta_1+\beta_2)\frac{\tau}{t}\left(1-e^{-t/\tau}\right) - \beta_2 e^{-t/\tau}\right)\right).$$

The flexibility of this model is explained by the authors by observing that the instantaneous forward rate is the linear combination of three components, $f_s(t) := e^{-t/\tau}$, $f_m(t) := \tau e^{-t/\tau}$ and $f_l(t) := 1$, modeling respectively the short-, medium-, and long-term behavior of f. The parameters β_0, β_1 and β_2 measure respectively the strength of each component, whereas τ is a time-scale factor.

In [17] an extended model for the instantaneous forward rate f was proposed to increase the flexibility of the original model by Nelson and Siegel model. It was obtained by adding a fourth term in equation (3.4), with two extra parameters, which allow the instantaneous forward rate curve for a second "hump":

$$f(t;\boldsymbol{\alpha}) = \beta_0 + \beta_1 e^{-t/\tau_1} + \beta_2 \frac{t}{\tau_1} e^{-t/\tau_1} + \beta_3 \frac{t}{\tau_2} e^{-t/\tau_2}.$$

The model proposed by a J. P. Morgan group and reported by Wiseman [20], is the so-called Exponential Model. The instantaneous forward rate function is modeled by

$$f(t;\boldsymbol{\alpha}) = \sum_{j=0}^{n} a_j \exp(-b_j t),$$

where $a_1, b_1, \ldots, a_n, b_n$ are the parameters to be estimated. It has been noticed that this model is able to capture the macro shape of the forward rate curve rather than the features of a single data set.

4. Numerical Experiments

In this section we report the results of a set of numerical experiments for estimating the term structure of interest rates from a cross-sectional data set of US government bonds.

The data set was made available by Datastream. It contains the annual coupon rate, the maturity date and the gross (or dirty or tel quel) price [7] for all (fixed-income) US Treasury bills, notes and bonds outstanding at the dates from May 31st 1999 to June 11th 1999 (two weeks).

[7] *i.e.* the sum of the quoted, or clean, price and the accrued interest (computed by the '30/360' convention)

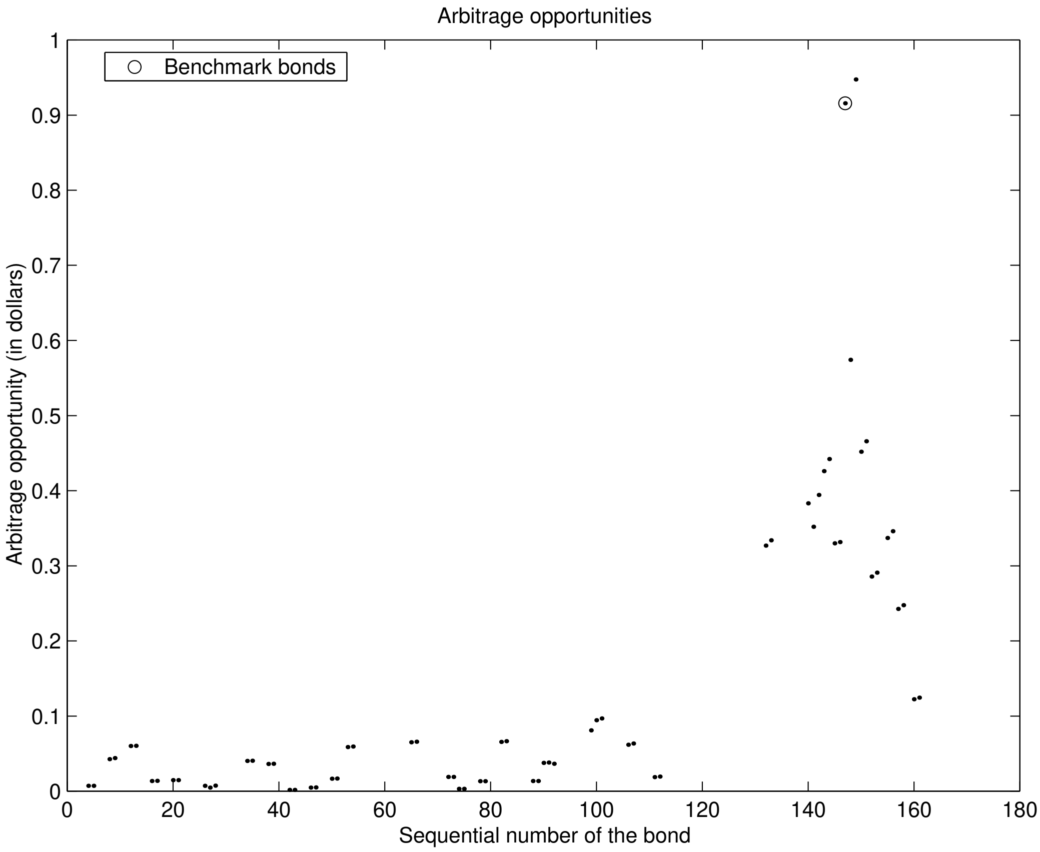


FIGURE 2.

Hereafter, for economy of presentation, since the results are very similar, only the results for June 3$^{\text{rd}}$ 1999 (when 33 bills and 173 notes and bonds were outstanding) will be presented and analyzed.

As first step, we built all minimal replicating portfolios (if any) of each available bond to double check whether condition (NA_1) is actually violated by bonds in the data set. As a "measure" of the arbitrage opportunities for a bond, we took the largest absolute value of the difference between the price of the bond and the price of all its minimal replicating portfolios. The results of this test are reported in the figure 2.

Note that just 69 bonds, out of 206, admit a (minimal) replicating portfolio, the maturity of such bonds reaching at most 7 years. This is easily seen by comparing figures 2 and 1. Moreover arbitrage opportunities amount to less than \$0.1 for bonds with maturity up to 3 years, and range between \$0.2 and \$1 for bonds with maturity within the range $3-7$ years.

To estimate the term structure of interest rates we have resorted to the approaches suggested by Carleton and Cooper [5], to the spline approaches by McCulloch [12, 13] and by Coleman, Fisher and Ibbotson [7] and to the parsimonious model by Nelson and Siegel [14] (see section 3 for a description of these methods). The technique proposed by Svensson [17] gave us the same kind of results as that proposed by Nelson and Siegel. Hence this set of experiments will not be reported in the present paper.

To apply the method of Carleton and Cooper to a bond sample, $\mathcal{S}$, whose maturity spectrum contains "gaps", it is necessary to shrink the sample itself to a suitable subset, $\mathcal{T}$, that forms a complete coupon term structure. After that we can estimate the discount factors by a constrained least squares procedure as described in section 3. In our case the subset $\mathcal{T}$ contains 152 (out of 173) US Treasury notes and bonds (the longest of which has maturity of about 7 years) and, obviously, all of 33 US Treasury bills.

For our numerical experiments, neither the covariance matrix, $\sigma_{ij} = E\{\varepsilon_i\varepsilon_j\}$ nor other information about the statistical properties of the errors ε_j's was available. For such reason we have resorted to a straight least squares procedure, instead of a weighted one, so as to obtain unbiased, even though not minimum variance, estimators of the discount factors. Our results are reported in figure 3.

Note that the discount factors approach one as their maturity tends to zero. The discount factors vary quite regularly with maturity, unlike spot rates and, even more significantly, one-period forward rates. These present irregularities for maturities of about 1 year and above 3 years, where the largest violations of condition (NA_1) occur. The plot of the residuals

$$p_j - \sum_{i=1}^{N} \hat{d}_i \varphi_i(c_j, m_j),$$

shows that, as expected, the fitting procedure performs worse where condition (NA_1) is more severely violated.

McCulloch's method has been tested on our data sample with two different sets of knot points. The first set (hereafter referred to as "knots *ala* McCulloch") is built following McCulloch's indication and consists of 14 knots placed at 0, 0.2, 0.4, 0.6, 0.9, 1.4, 1.8, 2.4, 3.4, 4.3, 6.3, 16.6, 22.8, 29.7 years. The second set is made of seven knots placed at 0, 1, 3, 5, 7, 11 and 30 years. The base we used (see section 3.1) consists of the following functions:

$$f_i(t) = t^i \qquad (i = 0, \dots, r-1),$$

$$f_i(t) = \begin{cases} 0 & , \quad \text{if } t \le K_{i-r+1}, \\ (t - K_{i-r+1})^r & , \quad \text{if } K_{i-r+1} < t \le K_{i-r+2}, \\ (t - K_{i-r+1})^r - (t - K_{i-r+2})^r & , \quad \text{if } K_{i-r+2} < t \end{cases}$$
$$(i = r, \dots, r+k-2).$$

and polynomial spline functions up to the fifth degree have been used. Our results are reported in figure 4 for the first set of knots, and in figure 5 for the second set.

These graphs show all drawbacks of McCulloch's method. First the outcome of the regression procedure depends strongly on the positions of the knots and on the degree of the polynomial spline employed. This feature is apparent in the forward rate curves (compare figure 4 and 5) whereas the fits for the discount function and the spot rate are more stable. Moreover, a polynomial spline is not, in general, a decreasing function, hence negative forward rates may occur. It is interesting to note how results for one-period forward rates obtained both by the method of McCulloch (or Coleman, Fisher and Ibbotson as we will see soon) and by the method of Carleton and Cooper share the same oscillating behaviour within 3 and 7 years. This suggests that such behaviour actually depends on the violations of the condition of absence of arbitrage opportunities that occur at those maturities (see figure 2).

To assess the criticism of Vasicek and Fong [19] to McCulloch's technique (see section 3), we assumed an exponential discount function of the form

$$d(t) \equiv e^{-0.06t}, \tag{4.1}$$

which corresponds to a spot rate function and a forward rate function identically equal to 6% per year, and generated two sets of artificial data as follows. For each bond (c_j, m_j, p_j) in the original data set, $\mathcal{S}$, we defined a fake price:

$$\hat{p}_j := \sum_{i=1}^{N} d(t_i)\varphi_i(c_j, m_j) + \varepsilon_j,$$

where $T(\mathcal{S}) = (t_1, \dots, t_N)$ and $\varepsilon_j \equiv 0$ for the first set of artificial data (named hereafter "exact" data) whereas $\varepsilon_j \sim \mathcal{N}(0,1)$ for the second set ("noisy" data).

After that McCulloch's method, with knots *ala* McCulloch, has been applied to both "exact" and "noisy" bond data so attaining an estimate, $\hat{d}$, of the discount function d defined in (4.1). The results reported in figures 6 and 7 seem to confirm Vasicek and Fong's opinion. McCulloch's method performs pretty well on the "exact" data, whereas it provides a bad fit when "noisy" data are used, even if the artificial perturbations introduced are quite small compared to the typical price of a bond ($< 1\%$ on average).

The method of Coleman, Fisher and Ibbotson has been tested with the same sets of knot points and the same base of spline functions that we used to test McCulloch's method. Polynomial spline functions of various degree have been used. For 0-degree spline functions, that is when a piecewise constant function is fitted to the instantaneous forward rate function, the least squares procedure has been constrained so as to attain a monotonic decreasing discount function. Our results are reported in figure 8 for knots *ala* McCulloch, and in figure 9 for knots placed at 0, 1, 3, 5, 7, 11 and 30 years.

These graphs show that the method of Coleman, Fisher and Ibbotson shares the same drawbacks of McCulloch's method. However the risk of getting negative forward rates seem to be reduced.

The graphs in figure 10 are produced following the approach of Nelson and Siegel. The main features of the estimates based on parsimonious models are readily seen. Although for the discount function the differences with other approaches are pretty mild, the constraints imposed on the functional form of the instantaneous forward rate function get rid of the oscillating behavior peculiar to spline based techniques.

We conclude this section by showing a table where the Root Mean Square Error (RMSE) [8] obtained in our numerical experiments is reported. This allows us to compare the quality of the fit among all employed methods (for various choices of the parameters, such as knot points or degree).

As expected the quality of the fit is higher for spline based techniques and lower for parsimonious models. Moreover it is worth noting that it is not true that increasing the order of the spline or the number of knots yields a better fit.

5. Conclusions

We have presented a critical review of the main methods for the estimation of the term structure of interest rates in fixed income markets.

It is apparent that most of the activity in this field lacks of a solid mathematical basis and there is not a single technique which is definitely *better* than others.

A number of elements contribute to form this scenario:

Bad practice: there is a general trend of re-cycling existing techniques without any critical evaluation of their real effectiveness.

Low-quality data: errors can be present in the input data (wrong prices). A large error is immediately detected whereas smaller errors may induce to take into account "fake" arbitrage opportunities.

[8] *i.e.* the square root of the arithmetic mean of squared residuals, see equation (3.3)

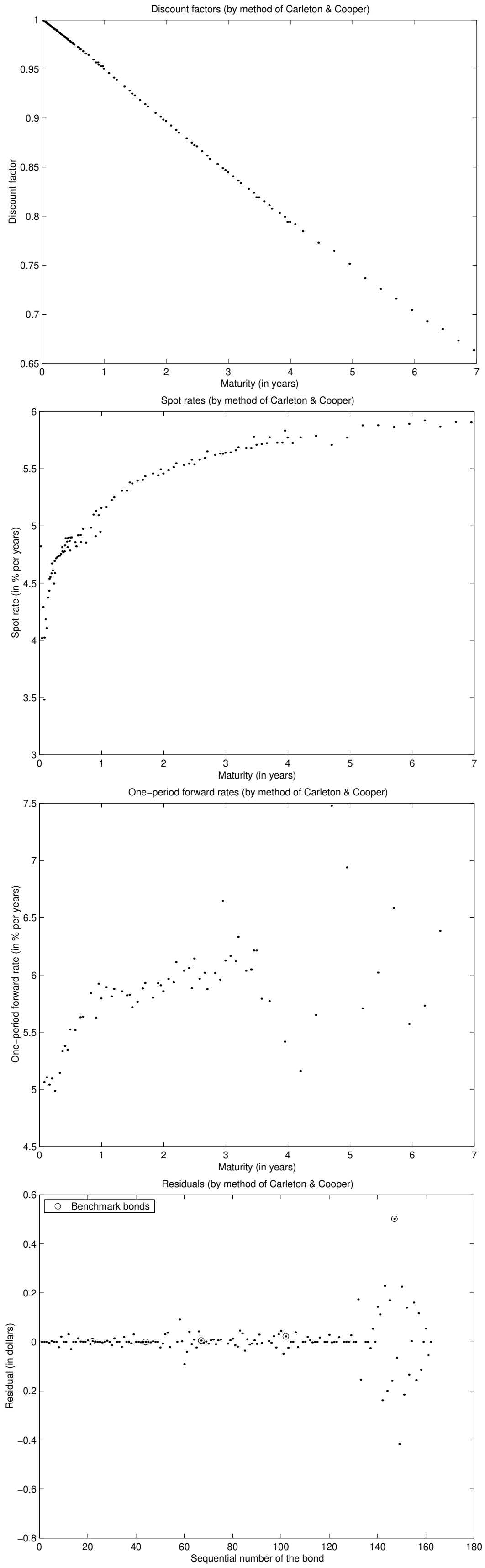

Discount factors (by method of Carleton & Cooper)
Discount factor
Maturity (in years)
Spot rates (by method of Carleton & Cooper)
Spot rate (in % per years)
Maturity (in years)
One−period forward rates (by method of Carleton & Cooper)
One−period forward rate (in % per years)
Maturity (in years)
Residuals (by method of Carleton & Cooper)
Benchmark bonds
Residual (in dollars)
Sequential number of the bond

FIGURE 3.

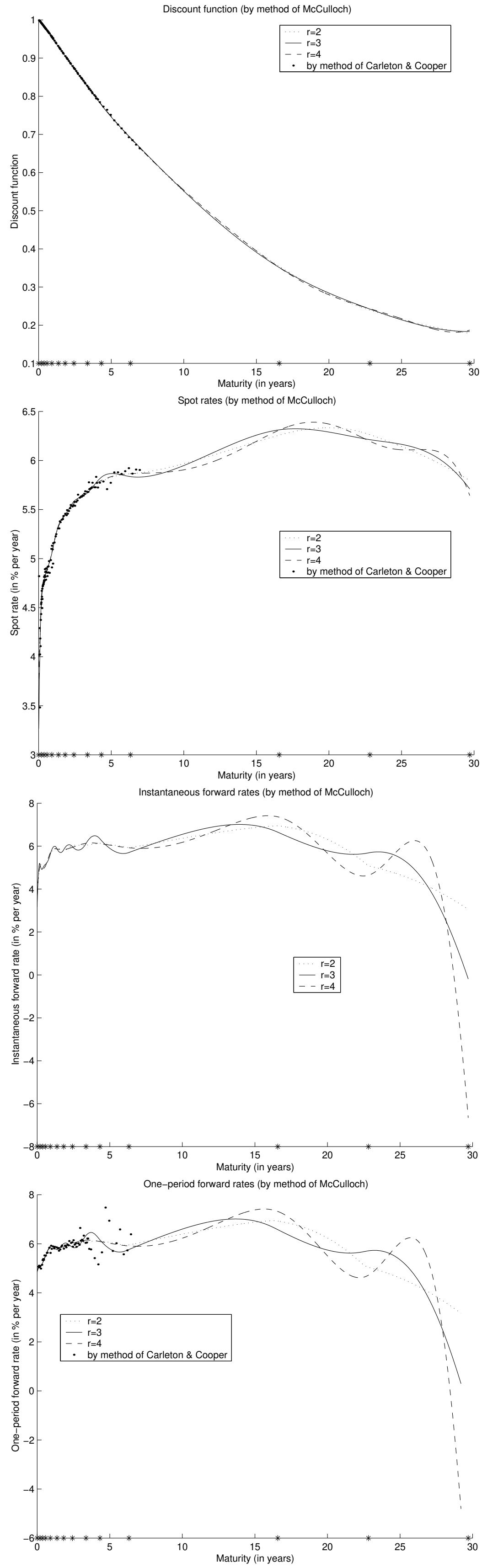

Discount function (by method of McCulloch)
r=2
r=3
r=4
by method of Carleton & Cooper
Discount function
Maturity (in years)
Spot rates (by method of McCulloch)
Spot rate (in % per year)
Instantaneous forward rates (by method of McCulloch)
Instantaneous forward rate (in % per year)
One−period forward rates (by method of McCulloch)
One−period forward rate (in % per year)

FIGURE 4.

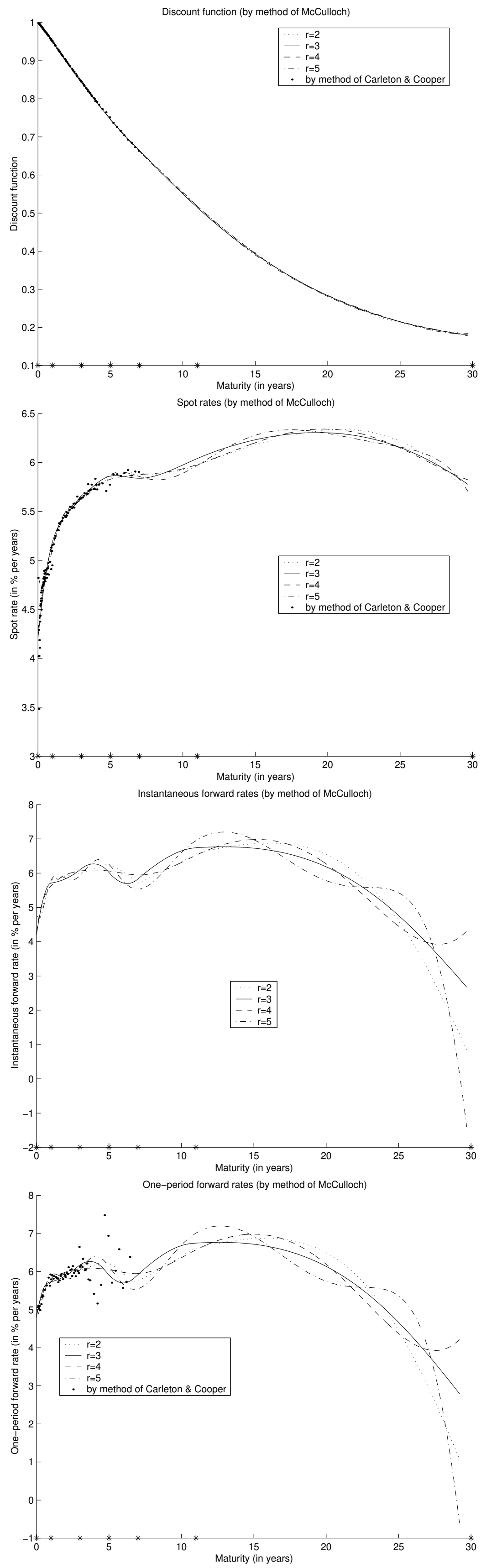

Discount function (by method of McCulloch)
Discount function
Maturity (in years)
r=2
r=3
r=4
r=5
by method of Carleton & Cooper
Spot rates (by method of McCulloch)
Spot rate (in % per years)
Maturity (in years)
r=2
r=3
r=4
r=5
by method of Carleton & Cooper
Instantaneous forward rates (by method of McCulloch)
Instantaneous forward rate (in % per years)
Maturity (in years)
r=2
r=3
r=4
r=5
One−period forward rates (by method of McCulloch)
One−period forward rate (in % per years)
Maturity (in years)
r=2
r=3
r=4
r=5
by method of Carleton & Cooper

Figure 5.

Method	Knot points	spline degree	RMSE
Carleton			.0807
McCulloch	*ala* McCulloch	2	.2428
McCulloch	*ala* McCulloch	3	.2441
McCulloch	*ala* McCulloch	4	.1896
McCulloch	0, 1, 3, 5, 7, 11, 30	2	.2493
McCulloch	0, 1, 3, 5, 7, 11, 30	3	.2358
McCulloch	0, 1, 3, 5, 7, 11, 30	4	.2389
McCulloch	0, 1, 3, 5, 7, 11, 30	5	.1846
Coleman	*ala* McCulloch	0	.3003
Coleman	*ala* McCulloch	1	.2350
Coleman	*ala* McCulloch	2	.2328
Coleman	*ala* McCulloch	3	.1926
Coleman	0, 1, 3, 5, 7, 11, 30	0	.4560
Coleman	0, 1, 3, 5, 7, 11, 30	1	.2242
Coleman	0, 1, 3, 5, 7, 11, 30	2	.2405
Coleman	0, 1, 3, 5, 7, 11, 30	3	.2282
Nelson			.5102

TABLE 1. Fit quality

Differences among the markets: fixed income markets are not homogeneous. The number of data available for a cross-section analysis of the U.S. Treasury securities is very different, for instance, from the number of outstanding Italian government bonds. The features of a market should be considered very carefully before applying any technique.

We expect to extend our work in two directions:

The development of algorithms to find the best position of the knots for spline-based interpolation methods. We consider genetic algorithms very promising for this kind of problem.

A mathematical analysis of equilibrium term structure models. This is pretty important since there is already evidence of relevant differences between the predictions of existing models and the experimental data (an example is the volatility term structure).

The importance of "intuition" in financial markets analysis can not be underestimated. However a more precise distinction among hypotheses, mathematical implications of these hypotheses and empirical observations is definitely required in this field.

6. APPENDIX

This appendix is entirely devoted to the proof of Theorem 2.5 that we write again here for our convenience.

Theorem 6.1. *Let $\mathcal{T} \subset \mathcal{B}$ be a complete coupon term structure and let $T(\mathcal{T}) = (t_1, \dots, t_N)$ for some $N \in \mathbb{N}$. If $\operatorname{Card}(\mathcal{T}) \geq N$ and if the following condition:*

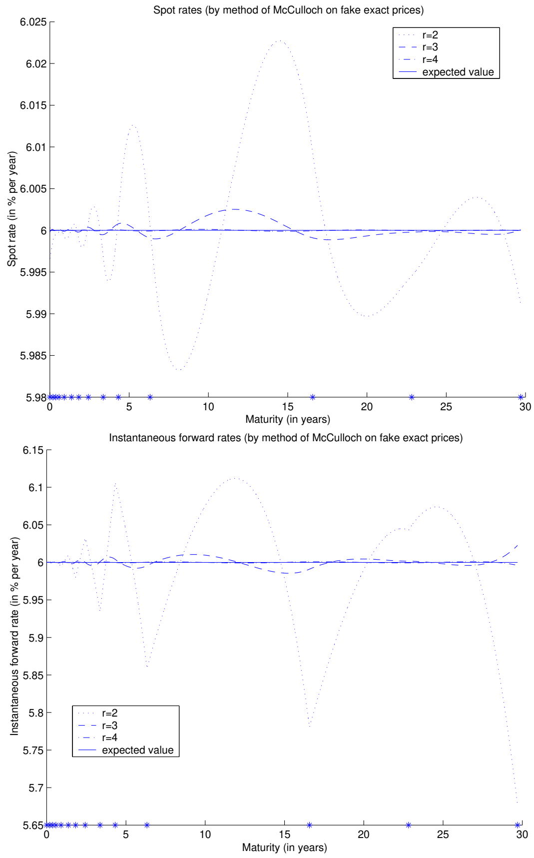


FIGURE 6.

(NA_1) *if* $(q_1, \ldots, q_B) \in \mathbb{R}^B$ *are such that* $\sum_{j=1}^{B} q_j \varphi(c_j, m_j) = 0$, *then* $\sum_{j=1}^{B} q_j p_j = 0$;

is satisfied, then there exist $d_1, \ldots, d_N \in \mathbb{R}$ *such that for every* $(c, m, p) \in \mathcal{T}$

$$p = \sum_{i=1}^{N} d_i \varphi_i(c, m). \tag{6.1}$$

If, furthermore, conditions:

(NA_2) *if* $(q_1, \ldots, q_B) \in \mathbb{R}^B$ *is such that*

$$\sum_{j=1}^{B} q_j \varphi_i(c_j, m_j) = \begin{cases} f_{\bar{\imath}} & , \quad \text{if } i = \bar{\imath} \\ 0 & , \quad \text{otherwise} \end{cases},$$

for some $f_{\bar{\imath}} > 0$, *then* $0 < \sum_{j=1}^{B} q_j p_j < f_{\bar{\imath}}$;

(NA_3) *if* $(q_1, \ldots, q_B) \in \mathbb{R}^B \backslash \{0\}$ *are such that* $\sum_{j=1}^{B} q_j p_j = 0$, *then there exists* $\bar{\imath} \in \{1, \ldots, N\}$ *such that* $\sum_{i=1}^{\bar{\imath}} \sum_{j=1}^{B} q_j \varphi_i(c_j, m_j) \geq 0$;

are fulfilled as well, then $1 > d_1 > d_2 > \ldots > d_N > 0$.

Proof. Let $(c_1, m_1, p_1), \ldots, (c_N, m_N, p_N) \in \mathcal{T}$ be such that $m_j = t_j$ for every $j \in \{1, \ldots, N\}$. The existence of N bonds like that in the set $\mathcal{T}$ is ensured by the hypotheses that $\text{Card}(\mathcal{T}) \geq N$ and that $\mathcal{T}$ forms a complete coupon term structure.

Let $\Phi = ((\Phi_{ij}))$ be an $N \times N$ matrix such that

$$\Phi_{ij} := \varphi_i(c_j, m_j) \tag{6.2}$$

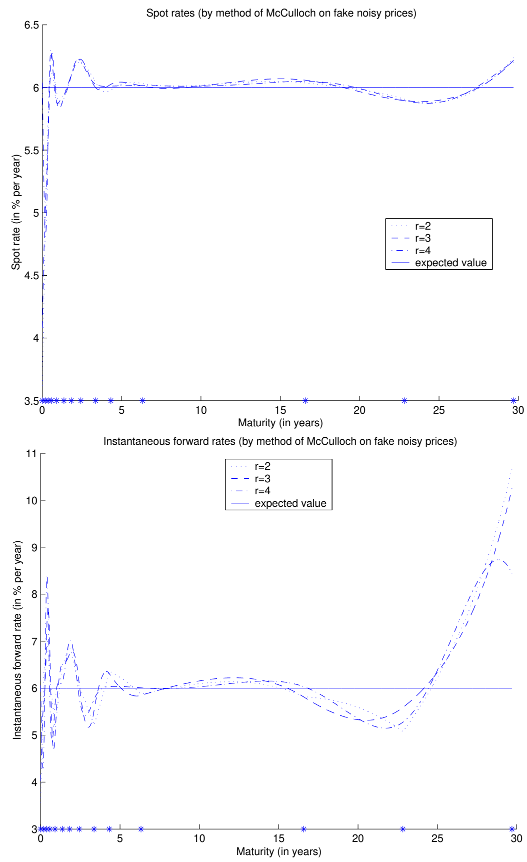


FIGURE 7.

for all $i, j \in \{1, \ldots, N\}$.

Finally let (c, m, p) be an arbitrary bond of $\mathcal{T}$.

Since Φ is an upper triangular matrix with no vanishing element on its principal diagonal, it is non-singular.

Now let $q_1, \ldots, q_N \in \mathbb{R}$ be such that

$$q_j = \sum_{i=1}^{N} (\Phi^{-1})_{ji} \varphi_i(c, m) \qquad (j = 1, \ldots, N)$$

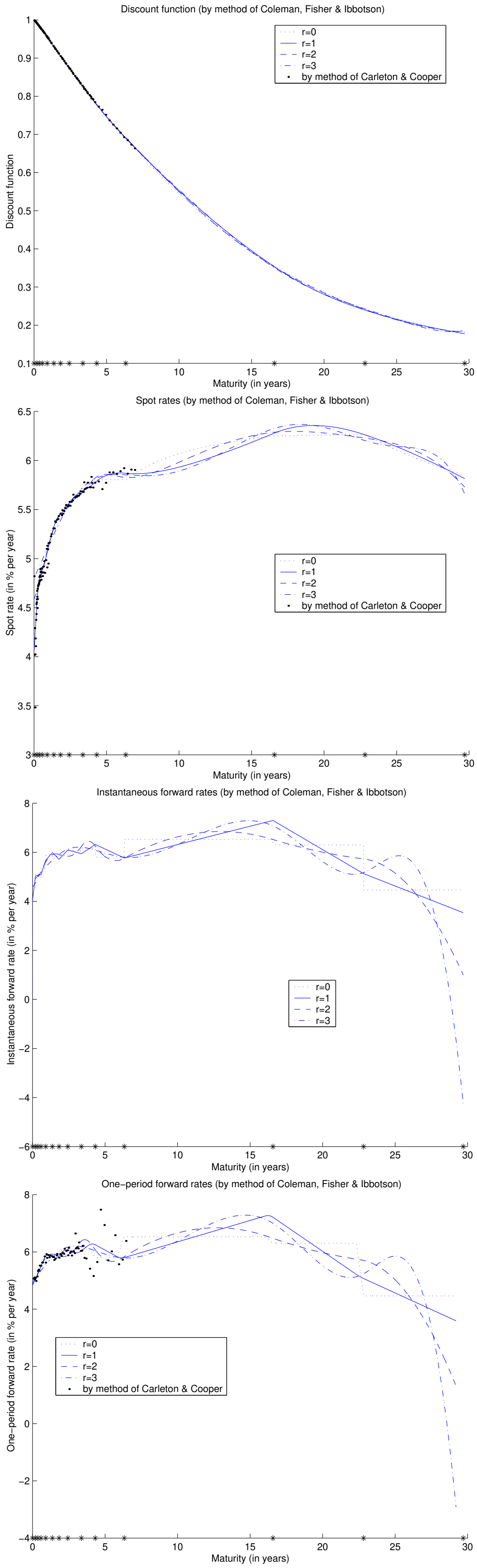
Then

$$\sum_{j=1}^{N} q_j \varphi(c_j, m_j) = \varphi(c, m).$$

By condition (NA_1), this implies equation (6.1) with

$$d_i := \sum_{j=1}^{N} p_j (\Phi^{-1})_{ji} \qquad (i = 1, \ldots, N).$$

In order to conclude the proof of the first part of the theorem, we have to show that the d_i's are independent of the choice of $(c_1, m_1, p_1), \ldots, (c_N, m_N, p_N)$ such that $m_j = t_j$ for every j if several such choices are possible. To do this, let $\bar{j} \in \{1, \ldots, N\}$ and let $(\tilde{c}_{\bar{j}}, \tilde{m}_{\bar{j}}, \tilde{p}_{\bar{j}}) \in \mathcal{T} \setminus \{(c_{\bar{j}}, m_{\bar{j}}, p_{\bar{j}})\}$ be such that $\tilde{m}_{\bar{j}} = t_{\bar{j}}$.



Discount function (by method of Coleman, Fisher & Ibbotson)
r=0
r=1
r=2
r=3
by method of Carleton & Cooper
Discount function
Maturity (in years)
Spot rates (by method of Coleman, Fisher & Ibbotson)
r=0
r=1
r=2
r=3
by method of Carleton & Cooper
Spot rate (in % per year)
Maturity (in years)
Instantaneous forward rates (by method of Coleman, Fisher & Ibbotson)
r=0
r=1
r=2
r=3
Instantaneous forward rate (in % per year)
Maturity (in years)
One−period forward rates (by method of Coleman, Fisher & Ibbotson)
r=0
r=1
r=2
r=3
by method of Carleton & Cooper
One−period forward rate (in % per year)
Maturity (in years)

FIGURE 8.

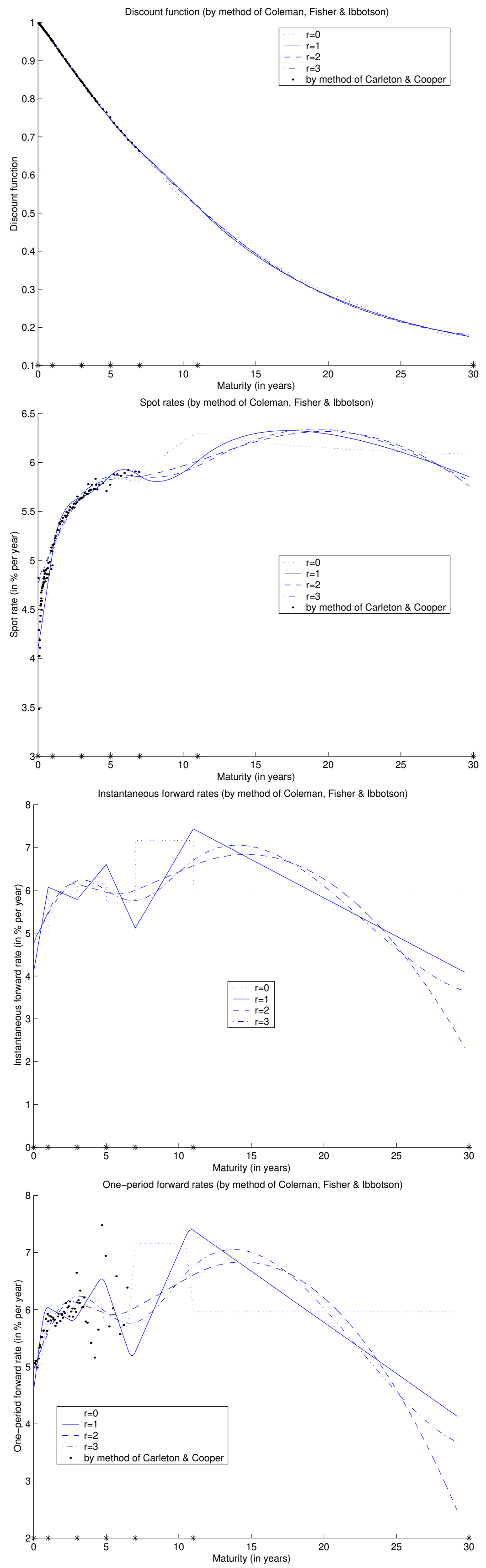
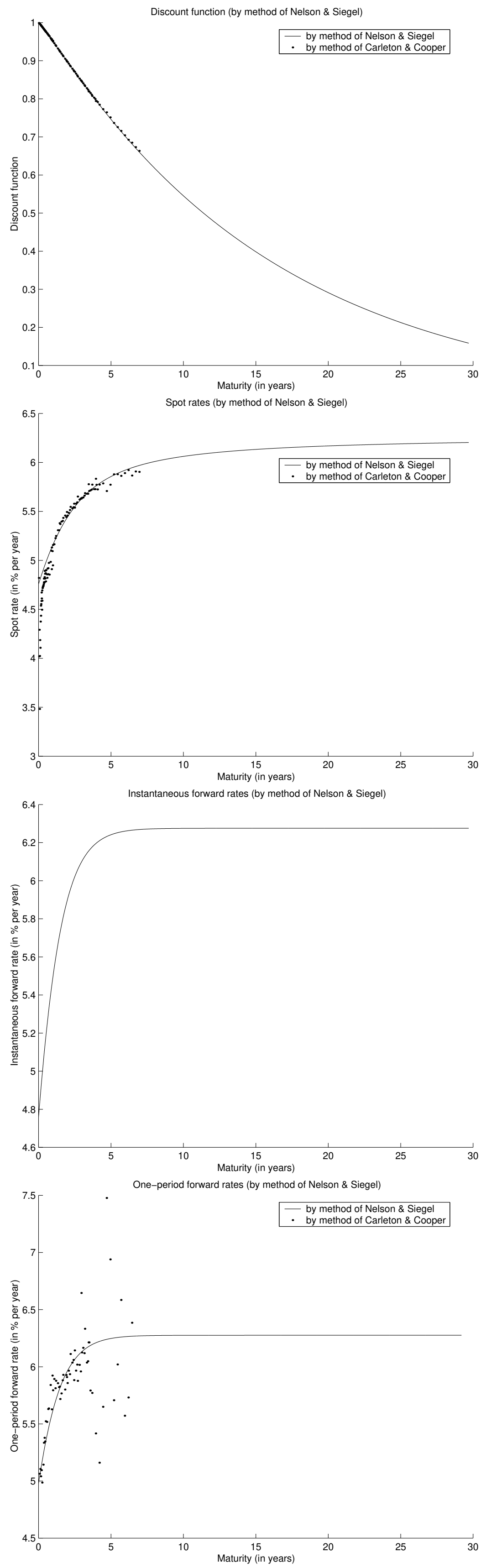

Discount function (by method of Coleman, Fisher & Ibbotson)
Discount function
Maturity (in years)
r=0
r=1
r=2
r=3
by method of Carleton & Cooper
Spot rates (by method of Coleman, Fisher & Ibbotson)
Spot rate (in % per year)
Maturity (in years)
Instantaneous forward rates (by method of Coleman, Fisher & Ibbotson)
Instantaneous forward rate (in % per year)
Maturity (in years)
One−period forward rates (by method of Coleman, Fisher & Ibbotson)
One−period forward rate (in % per year)
Maturity (in years)

FIGURE 9.



Discount function (by method of Nelson & Siegel)
by method of Nelson & Siegel
by method of Carleton & Cooper
Discount function
Maturity (in years)
Spot rates (by method of Nelson & Siegel)
by method of Nelson & Siegel
by method of Carleton & Cooper
Spot rate (in % per year)
Maturity (in years)
Instantaneous forward rates (by method of Nelson & Siegel)
Instantaneous forward rate (in % per year)
Maturity (in years)
One−period forward rates (by method of Nelson & Siegel)
by method of Nelson & Siegel
by method of Carleton & Cooper
One−period forward rate (in % per year)
Maturity (in years)

Figure 10.

Let $\tilde{\Phi} = ((\tilde{\Phi}_{ij}))$ be an $N \times N$ matrix such that

$$\tilde{\Phi}_{ij} := \begin{cases} \varphi_i(c_j, m_j) & , \quad \text{if } j \neq \bar{\jmath} \\ \varphi_i(\tilde{c}_{\bar{\jmath}}, \tilde{m}_{\bar{\jmath}}) & , \quad \text{if } j = \bar{\jmath} \end{cases} \tag{6.3}$$

for all $i, j \in \{1, \dots, N\}$, and let

$$\tilde{d}_i := \sum_{\substack{j=1 \\ j \neq \bar{\jmath}}}^{N} p_j (\tilde{\Phi}^{-1})_{ji} + \tilde{p}_{\bar{\jmath}} (\tilde{\Phi}^{-1})_{\bar{\jmath} i} \qquad (i = 1, \dots, N).$$

We want to show that $\tilde{d}_i = d_i$ for every $i \in \{1, \dots, N\}$. Since both $\bar{\jmath}$ and $(\tilde{c}_{\bar{\jmath}}, \tilde{m}_{\bar{\jmath}}, \tilde{p}_{\bar{\jmath}})$ has been arbitrarily chosen, this will conclude the proof of the first part of the theorem.

Observe that $\tilde{\Phi}_{\cdot j} = \Phi_{\cdot j}$ for all $j \neq \bar{\jmath}$ and that by equations (6.1), (6.2) and (6.3)

$$p_j = \sum_{i=1}^{N} d_i \Phi_{ij} \qquad (j = 1, \dots, N)$$

and

$$\tilde{p}_{\bar{\jmath}} = \sum_{i=1}^{N} d_i \tilde{\Phi}_{i\bar{\jmath}}.$$

Then for every $i \in \{1, \dots, N\}$

$$\begin{aligned} \tilde{d}_i &= \sum_{\substack{j=1 \\ j \neq \bar{\jmath}}}^{N} \left[(\tilde{\Phi}^{-1})_{ji} \sum_{h=1}^{N} d_h \Phi_{hj} \right] + (\tilde{\Phi}^{-1})_{\bar{\jmath} i} \left(\sum_{h=1}^{N} d_h \tilde{\Phi}_{h\bar{\jmath}} \right) \\ &= \sum_{h=1}^{N} d_h \left[\sum_{j=1}^{N} \tilde{\Phi}_{hj} (\tilde{\Phi}^{-1})_{ji} \right] = d_i. \end{aligned}$$

In order to prove the second part of the theorem, let $(c_1, m_1, p_1), \dots, (c_N, m_N, p_N) \in \mathcal{T}$ be such that $m_j = t_j$ for every $j \in \{1, \dots, N\}$ and let $(q_1, \dots, q_N) \in \mathbb{R}^N$ be a portfolio of these bonds such that

$$\sum_{j=1}^{N} q_j \varphi_i(c_j, m_j) = \begin{cases} f_{\bar{\imath}} & , \quad \text{if } i = \bar{\imath} \\ f_{\bar{\imath}+1} & , \quad \text{if } i = \bar{\imath} + 1 \\ 0 & , \quad \text{otherwise} \end{cases},$$

for some $\bar{\imath} \in \{1, \dots, N\}$ and some $f_{\bar{\imath}}, f_{\bar{\imath}+1} \in \mathbb{R}$. By (6.1), the price of this portfolio is

$$\sum_{j=1}^{N} q_j p_j = d_{\bar{\imath}} f_{\bar{\imath}} + d_{\bar{\imath}+1} f_{\bar{\imath}+1}.$$

If $f_{\bar{\imath}} > 0$ and $f_{\bar{\imath}+1} = 0$, then by condition (NA_2), $0 < d_{\bar{\imath}} f_{\bar{\imath}} < f_{\bar{\imath}}$, which implies that $0 < d_{\bar{\imath}} < 1$ and, since $\bar{\imath}$ has been chosen arbitrarily, that

$$0 < d_i < 1 \qquad (i = 1, \dots, N).$$

Now suppose, by contradiction, that $d_{\bar{\imath}} \leq d_{\bar{\imath}+1}$ and let $f_{\bar{\imath}}, f_{\bar{\imath}+1}$ be such that

$$d_{\bar{\imath}} f_{\bar{\imath}} + d_{\bar{\imath}+1} f_{\bar{\imath}+1} = \sum_{j=1}^{N} q_j p_j = 0.$$

Since $d_{\bar{\imath}}, d_{\bar{\imath}+1} > 0$, we can assume, without loss of generality, that $f_{\bar{\imath}} \geq 0$ and that $f_{\bar{\imath}} + f_{\bar{\imath}+1} \geq 0$. However condition (NA_3) implies that either $f_{\bar{\imath}} < 0$ or $f_{\bar{\imath}} + f_{\bar{\imath}+1} < 0$. This is the wanted contradiction. $\square$

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