Mathematical Models and Methods in Applied Sciences

THE EUROPEAN OPTIONS HEDGE "ANY" CLAIM PERFECTLY IN A POISSON-GAUSSIAN MODEL FOR THE INDEX-LINKED BOND MARKET

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We want to price and hedge inflation-linked bonds. We assume that the spot nominal intensity rate $\tilde{r}(t)$ and the general index $\tilde{p}(t)$ of the prices of consumption goods are state variables for such a sector of the market. We assume for the state variables a jump-diffusion model (an extension of the bivariate, continuous time model of Cox-Ingersoll-Ross): it is driven by two Brownian Motions and n independent Poisson Processes.
It is known the theoretical result asserting that it is possible to complete the market model if we can dispose of n+2 nonredundant assets. For a practical purpose, here we show that one nominal zero-coupon-bond and n+1 European Call Options subscribed on the index $\tilde{p}$ and having n+1 different strike prices are nonredundant assets in this model and they allow us to hedge any square integrable claim we sell perfectly.

1. Introduction

The goal of this paper is to tackle the problems of pricing and hedging inflation-linked bonds when we guess that inflation can suddenly jump as a result of exceptional happenings.
We assume that, for the prices of such assets, are state variables the spot nominal "riskless" intensity rate $\tilde{r}(t)$ and the general index $\tilde{p}(t)$ of the prices of consumption goods, whose variations give inflation. We model the evolution of the first variable by a diffusive mean-reverting and square root dynamics. Instead, for the second one we guess a geometric jump-diffusion Brownian Motion having n sources of Poissonian jump.
As a consequence of the *Second Fundamental Theorem* of the Asset Pricing Theory, which has been rigorously formalized by Bjork, Kabanov and Runggaldier for a wide class of jump-diffusion models for the bond market ([BKR]), we are guaranteed that, because we have n+2 independent sources of randomness, it has to be possible to *complete* the market by the use of n+2 *nonredundant* assets. Our question is: which assets are surely nonredundant in our framework?
We adopt the *Absence of Arbitrage* approach, so we define the price of a claim as the initial budget needed to construct a strategy replicating it. Our goal is to estabilish how concretely to choose good assets to hedge the claim. On the light of Proposition(4.5) and Section(5.2) in [BKR], the problem consists of finding assets for which the volatilities of the prices form a non-singular matrix. The result is that it is possible to perfectly hedge *any* inflation-linked square integrable claim using the following risky assets: one nominal zero coupon bond and n+1 European Call Options subscribed on the jumping index $\tilde{p}$ and having different strike prices.

The *first Fundamental meta-theorem* of the Asset Pricing Theory ([Bj]), states that the *Absence of Arbitrages* in a model driven by n+2 independent sources of

randomness substantially means that there exists an *Equivalent Martingale Measure* (EMM) for n+2 nonredundant risky assets. Such an EMM is a Probability measure Q, equivalent to the "objective" law P, under which each asset, belonging to a specified set of n+2 assets, is such that its discounted price is a martingale. Any change of equivalent measure leaves the price volatilies untouched. Because we focus our attention on the problem of the completion, motivated by such meta-theorem, we have chosen (in order to make the hedgings) an appropriate class of assets, called *primary* assets, and we have worked directly under a law which makes martingales their discounted prices. But to reach our result we need to start from an infinite class of primary assets.
We apply to our model the general theory developed in [BKR]. In particular we compose portfolios *a priori* containing an infinite number of assets. In the end, however, the hedging strategies will come out such that at any time t only n+2 risky assets appear in the hedging portfolio.

Let us explain which motivation led us to study the present framework.
The mean-reverting square-root dynamics for the "riskless" interest rate was considered for the first time by Cox, Ingersoll and Ross ([CIR]) and it seemed to fit quite accurately the historical data available for the Italian Bond Market ([Mo], [Ce]). In ([CIR]) the prices came out in a *general economic equilibrium* framework, which was considered for the *real* world, where financial agents can only exchange one consumption good. After precised assumptions on the state variables, C.I.R. endogenously reached the dynamics for the real spot rate $\tilde{x}(t)$ over certain periods. Then they introduced the money as an external, and not correlated, factor: the process *level of prices* was only needed for a change of unit of measure of the prices and was assumed to be a Geometric Brownian Motion.

Instead Moriconi took directly $\tilde{r}$ and $\tilde{p}$ as the state variables in the *nominal* world: a process having square-root and mean-reverting dynamics never becomes negative, and this is not consistent with the historical "observed" paths of the real spot interest rate $\tilde{x}$. Moreover he adopted the *Absence of Arbitrages* approach: in the nominal world the *General Equilibrium* approach does not seem justified. $\tilde{r}$ was assumed to be a mean reverting and square root process, and $\tilde{p}$ a Geometric Brownian Motion.

The present work adopts the same approach as Moriconi, but $\tilde{p}$ is assumed to be a jumping Geometric Brownian Motion: by this choice we wanted to gain an insight into the role played by inflation while linking the nominal and real spot rates. But in fact the presence of jump processes in the market model creates troubles about how, concretely, to complete it.

The jump component in our model is simply given by an n-variate Poissonian process: inflation jumps only by deterministic amplitudes: this allows us, given any square integrable claim, to exhibit explicitely, for each t, a finite set of assets (whose choice can depend on t) trading which we hedge the claim perfecly. When the jump amplitudes can randomly assume infinite values, the replication with a finite number of assets is not guaranteed ([BKR]). Here follows a survey of the works which have tackled the problem of the hedging within jump-diffusion models.

Merton ([Mer]) worked with a stock which jumps through a coumpounded Poissonian process: to price an option, he decided to construct hedging portfolios based only on the option and the jumping underlying stock, but he had to assume that the risk induced by the jumps was diversifiable.

Endogenously to an Equilibrium model, Naik-Lee reached a Merton-like dynamics for the stock. They quantified the amount which remains not hedged if, to replicate the payoff of a European Call Option on the jumping stock, one uses the hedging strategy adopted in the classical Black-Scholes model: such amount is not negligeable and they decided to perfectly hedge it by the use of the so called *jump-financing security* and of the *jump-variance* security.

The payoff of a European Option depends only on the final state $\tilde{S}_T$ of the underlying stock, so it is not path dependent: actually, within many models, by European Options it is possible to hedge any claim, whose payoff could also depend on the whole path of $\tilde{S}$.

Rubinstein [Ru], Dupire [D], Derman-Kani [DK] studied the discrete framework: from the observed prices of a finite set of Call Options on the same stock $\tilde{S}$, but having different strike prices (for Dupire and Derman-Kani also different maturities) they *estimate* the risk-neutral probabilities that $\tilde{S}_T$ touches each state, and then they solve the whole binomial tree describing the risk-neutral and arbitrage-free evolution of $\tilde{S}$. Then they explicitely evaluate other European and American options substcribed on $\tilde{S}$, and their respective hedging parameters.
We are here interested to know how much information we gain by the European Call Options in our continuous framework.

Bajeux-Besnainou and Rochet ([BR], 1996) show that when the state variable is a stock following a Geometric Brownian Motion with stochastic diffusive volatility it is still sufficient to use the stock and a European Call Option for the perfect hedgings and we can avoid to use more complicated exotic derivatives.

Duffie-Huang ([DH], 1985), Harrison-Pliska ([HP], 1985) and Bjork-Kabanov-Runggaldier ([BKR], 1997) showed that in a continuous time state variable model, where the range of values assumed by each variable is the whole $\mathbb{R}_+$ (and it could seem not feasable to replicate any claim with a finite number of assets) what really determines the possibility of hedging using only a finite number of securities is the number of "dimensions" of uncertainty, the *martingale multiplicity*, of the space of the square integrable martingales.
In fact B.K.R. give an operational tool to check that a finite family of Zero Coupon Bonds complete a jump-diffusion model for the forward rates of interest: the nonsingularity of their volatilities matrix. Such a result becomes the Heat-Jarrow-Morton ([HJM]) result when we restrict ourselves to a diffusion model, and becomes the Shirakawa result ([S]) when the sources of randomness are multidimensional Poisson and Gaussian processes.

Jarrow-Madan ([JM], 1995) deal with a jump-diffusion model both for the forward rates and for a family of assets: the jumps are driven by an m-variate Point Process. Under proper assumptions they show the existence and uniqueness of an EMM, which gives the completeness of the model. One of the assumptions is exactly that the matrix of the volatilities of a selected finite set of securities (bonds and assets) is non-singular.

2. The Model

Let us consider a frictionless and default-free market.

Definition. The default-free nominal *unitary Zero Coupon Bond*, (abbreviated ZCB), issued at t and maturing at s, is a contract guaranteeing, at the instant t, to pay, at the future epoch s, the amount of 1\$: we will denote its price at t by $\tilde{v}(t;s)$ (this is a random variable, but we leave out the dependenc on the "state of the world" ω).

Definition. The *nominal spot rate*, or nominal instantaneous intensity of interest, is

$$\tilde{r}(t) \doteq lim_{s\to t} - \frac{log\ \tilde{v}(t;s)}{s-t}, \tag{2.1}$$

the interest rate guaranteed by a bond contract of infinitesimal time to maturity (i.e. which lives from t to t+dt).

Let us consider the general index of the prices of consumption goods, and let us denote by $(\tilde{p}(t))_{t\geq 0}$ the process describing it.
Let us define the realized *inflation* rate during a time interval [t,s] as

$$\frac{\tilde{p}(s)-\tilde{p}(t)}{\tilde{p}(t)},$$

and the respective *intensity of inflation* as

$$\frac{\tilde{p}(s)-\tilde{p}(t)}{(s-t)\tilde{p}(t)}.$$

Our goal is to price and hedge inflation-linked bonds. So in our framework we assume that the nominal spot rate of interest $\tilde{r}(t)$ and the general index $\tilde{p}(t)$ are state variables, and we assume that their evolutions are:

$$\begin{gathered} d\tilde{r}(t) = k(\theta - \tilde{r}(t))dt + \sigma_r\sqrt{\tilde{r}(t)}dW^r(t), \\ \tfrac{d\tilde{p}(t)}{\tilde{p}(t-)} = (\tilde{r}(t)-\tilde{x}(t))dt + \sigma_p dW^p(t) + \textstyle\sum_{i=1..n}\delta_i[dN^i(t)-\nu_i dt]. \end{gathered} \tag{2.2}$$

on an filitered probability space $(\Omega, \{F_t\}_{t\in[0,T]}, Q)$ where $(W^r(t), W^p(t))$ is a Brownian Motion and $(N^1,...,N^n)$ a simple n-dimensional Poissonian Process. Each component N^i has respective intensity ν_i.
$(W^r(t), W^p(t), N^1,...,N^n)$ are stochastically independent ([RY], [Te]). Thus let us restrict the filtration F to the canonical one, $F^{W,N}$, and rename it F. All the parameters in the model are constant numbers which satisfy $k,\theta > 0, \sigma_r > 0; 2k\theta > \sigma_r^2$ (this is a sufficient condition ensuring that $\forall r_0 > 0,\ r_t > 0\ a.s.\forall t$, [LL]); $\sigma_p > 0$, $\delta_j > -1$ for each j=1,..,n (this ensures that $\forall p_0 > 0$, we have $p_t > 0\ a.s.\forall t$). Each δ_j represents the relative (algebraic) increment of the general level of prices at an instant of jump.

The stochastic process $\tilde{x}$ in the drift term of $\tilde{p}$ represents the spot *real* interest rate (see the Appendix), and we make the following fundamental

Assumption 1: $(\tilde{r}(t)-\tilde{x}(t))$ is a positive constant for each (t,ω).

Our plan. In order to price and hedge *claims* which are contingent at a future instant of time T we will utilize the *risk-neutrality* approach. The first step in this analysis is to impose enough additional structure to ensure that Q is a *Martingale Measure* for a proper set of assets: the *primary assets.* Our second step is to demonstrate that the economy is *complete*, which allows us to replicate any claim by dynamic *self-financing* trading strategies in the primary traded assets. In particular we give to *any* claim a unique *price* determined by the cost of the replicating strategy.

Definition. The *Bank Account* is the stochastic process $\tilde{B}(t)$ defined by

$$\begin{cases} d\tilde{B}(t) = \tilde{r}(t)\tilde{B}(t)dt \\ \tilde{B}(0) \equiv 1 \end{cases}.$$

We can interpret it as the value of an account we can keep into our Bank and which allows us to have, on the initially invested amount $\tilde{B}(0) = 1$, a continuously compounded interest at instantaneous intensity rate $\tilde{r}(t)$: at s we have cumulated $\tilde{B}(s) = e^{\int_0^s \tilde{r}(u)du}$ dollars. The Bank Account is called the "riskless" asset.

Definition. Given a price process $(\tilde{V}(t))_{t\geq 0}$, the respective *discounted* price process is $\tilde{V}_t^d \doteq \frac{\tilde{V}(t)}{\tilde{B}(t)}$.

The process $\tilde{p}$ can be interpreted (see the Appendix) as the price process of a traded stock having instantaneous rate of return $\tilde{r}$ and which pays continuously dividends at instantaneous rate $\tilde{x}(t)$.
Model (**??**) says that when we discount $\tilde{p}$ by the *Bank Account*, and at net of the dividends, we reach a martingale.

Definition. The *European Call Option* subscribed on the index $\tilde{p}$, issued at t, maturing at T, having strike price K, is the security guaranteeing to give the payoff $(\tilde{p}_T - K)_+$ at T: we will denote its (stochastic) price at t by $\tilde{C}(t;K)$.

Let us assume that the following set of assets is traded in this economy: the unitary zero-coupon bond with maturity T, the Bank Account $\tilde{B}$ and for any $K \geq 0$ the European Call Option subscribed on the index $\tilde{p}$ with maturity T and strike price K.

Definition. We will call such assets *primary assets*, because their prices are assumed already to exist (they are traded).
Starting from that we want to find a price for any other claim.

Definition. A *Claim Contingent at T* is a non negative random variable $\tilde{H} \in L^2(\Omega, F_T, Q)$.
The discounted value of a T-contingent claim $\tilde{H}$ is $\tilde{H}^d = \frac{\tilde{H}}{\tilde{B}(T)}$.

Because $\tilde{r}$ and $\tilde{p}$ are assumed to be state variables, the price processes of the primary assets have to be stochastic only via a deterministic dependence on $\tilde{r}(t)$ and $\tilde{p}(t)$ in the following way: $\tilde{v}(t;T) = v(t,\tilde{r}(t),\tilde{p}(t),T)$, $\tilde{C}(t;K) = C(t,\tilde{r}(t),\tilde{p}(t),K)$, with $v(t,r,p,T)$ and $C(t,r,p,K)$ being deterministic functions of the three real variables (t,r,p) and of the respective parameters T or K.

Assumptions 2. Q is a probability law such that
- the discounted price process $\frac{\tilde{v}(t;T)}{\tilde{B}(t)}$, of the nominal unitary ZBC of maturity T is a Q-martingale;
- the discounted price process $\frac{\tilde{C}(t;K)}{\tilde{B}(t)}$, of a Call Option on the index p, and maturity T is a Q-martingale, for any strike price K.

Definition. An *Equivalent Martingale Measure* (EMM) for a specified set of assets is a law, equivalent to Q, such that the discounted prices of each one of such assets is a martingale.

Remark. The model is driven by a finite number of sources of randomness but here we need to require that our Q is an EMM for a set of infinite assets. This happens because, even if at last the replicating strategies will be composed at each instant t by a finite number of primary assets, the proper set of assets will depend on t. And we need that each asset we use for the hedging discountes to a martingale.

3. The Outcome for the Prices of the Primary Assets

By assumptions 1 and 2 and thanks to the stochastic independence of $(\tilde{r}(t))_t$ and $(\tilde{p}(t))_t$, we explicitely reach the following price functions for the primary assets.

• For the **unitary nominal zero-coupon-bond** with maturity T:

$$v(t,r,p,T) = E^Q[e^{-\int_t^T \tilde{r}(u)du} | \tilde{r}(t) \equiv r, \tilde{p}(t) \equiv p] = A(T-t)e^{-r \cdot F(T-t)}$$

where A, F are the well known CIR deterministic functions of the time to maturity $\tau = T-t$, other than of the parameters of the model for r ([CIR, p. 393] with $\lambda = 0$, or [Mo, p.17] with $\pi = 0$):

$$F(\tau) = \frac{2 \cdot [e^{\gamma\tau} - 1]}{(k+\gamma)[e^{\gamma\tau} - 1] + 2\gamma}, \tag{3.3}$$

$$\gamma = \sqrt{k^2 + 2\sigma_r^2},$$

$$A(\tau) = \left[\frac{2\gamma \cdot e^{(k+\gamma)\tau/2}}{(k+\gamma)[e^{\gamma\tau} - 1] + 2\gamma}\right]^{\frac{2k\theta}{\sigma_r^2}}.$$

Let us remark that as the state variables are stochastically independent processes, the price function $v(t,r,p,T)$ of the ZCB turns out to depend only on (t,r) and not on p: we rename it $v(t,r,T)$ (or $v(t,r)$ when T will be fixed once forever); moreover the process $\tilde{v}(t;T) = v(t,\tilde{r}(t),T)$ has continuous paths.

Let us fix the future date T at which the claims we want to replicate are contingent.

• For the European **Call Option on p** with maturity T and strike price K we reach that:

$$C(t,r,p,K) = E^Q[(\tilde{p}(T) - K)_+ e^{-\int_t^T \tilde{r}(u)du} | \tilde{r}(t) \equiv r, \tilde{p}(t) \equiv p] =$$

$$= v(t,r,T)e^{\rho\tau} \sum_{m_1=0}^{+\infty} ... \sum_{m_n=0}^{+\infty} Q_{m_1} \cdots Q_{m_n} \cdot$$

$$\cdot F_{BS}(p(1+\delta_1)^{m_1} \cdots (1+\delta_n)^{m_n}, K, t),$$

with

$$Q_{m_i} = Q[N^i(T) - N^i(t) = m_i] = e^{-\tau\nu_i} \cdot \frac{(\tau\nu_i)^{m_i}}{m_i!}, i = 1,...,n$$

$$\tau = T - t;$$

$$\rho \doteq \tilde{r}(t) - \tilde{x}(t) - \sum_{i=1..n} \delta_i \nu_i,$$

$$F_{BS}(S,K,t) = SN[d_1(S,K,t)] - Ke^{-\rho\tau}N[d_2(S,K,t)],$$

$$d_i(S,K.t) = \left[log\left(\frac{S}{K}\right) + \left(\rho \pm \frac{\sigma_p^2}{2}\right)\tau\right]\frac{1}{\sigma_p\sqrt{\tau}}.$$

F_{BS} is the well known Black-Scholes formula for the t-price of a Call Option subscribed on a stock $\left(\tilde{S}(t)\right)_t$ having diffusion dynamics.

Notations. To reach our result we will fix the two positive variables t and r, and all the parameters characterizing the price function of the Call on the index $\tilde{p}$, except for the strike price K: we will be regarding $C(t,r,p,K)$ only as a function of p and K which vary into $[0,+\infty[$.
We will denote by $C_p(p,K)$ the partial derivative of the Call price function $C(t,r,p,K)$ with respect to the variable p and by
$\Delta^j C(p,K)$ the quantity $\doteq C(p\cdot(1+\delta_j),K)-C(p,K)$.

Properties (we remark only the properties we will need in the continuation).

- $v(t,r,T)$, $C(t,r,p,K)$ are real functions belonging to the following classes of continuously differenciable functions: for each T $v(t,r,T)\in\mathbb{C}^{1,2}$ and for each K $C(t,r,p,K)\in\mathbb{C}^{1,2,2}$. *This remark will be necessary to allow us to use the Ito differentiation rule in order to find the volatility coefficients of the prices (see below).

- The two functions $(p,K)\mapsto p\cdot C_p(p,K),(p,K)\mapsto\Delta^j C(p,K),j=1,...,n$, are postively homogeneous: $g(\alpha\cdot p,\alpha\cdot K)=\alpha\cdot g(p,K)$, for each function $g(p,K)=p\cdot C_p(p,K)$, $g(p,K)=\Delta^j C(p,K)$.

- The Call price function $C(t,r,p,K)$ is also a $\mathbb{C}^{n+1}$ function in the variable p.
Actually for each fixed strike K, $C(p,K)$ is a series of real functions of p, say $C(p,K)=\sum_{m_1=0}^{+\infty}\cdots\sum_{m_n=0}^{+\infty}f_{m_1,...,m_n}(p,K)$; let us continuously extend each function $f_{m_1,...,m_n}(p,K)$ with the value 0 at p=0: the partial derivatives $\frac{\partial^h f_{m_1,...,m_n}}{\partial p^h}$, h=1...n+1, of these extended functions converge uniformly with respect to p on $\mathbb{R}_+$.

- For each h=1..n, $lim_{p\to 0}\frac{\partial^h C}{\partial p^h}(p)=0$.

- There exists a sequence $(p_k)_{k\in\mathbb{N}}$ converging to zero such that $\frac{\partial^{n+1}C}{\partial p^{n+1}}(p_k,K)\neq 0$ for each k.
Consequently, for any $h\leq n+1$, there exists a sequence $(p_\ell)_{\ell\in\mathbb{N}}$, depending on h, which converges to zero and such that $\frac{\partial^h C}{\partial p^h}(p_\ell,K)\neq 0$.

- The Call price function satisfies, for all t, r and K,

$$\frac{C(t,r,p\cdot\alpha,K)}{p}\to_{p\to\infty}\alpha\cdot v(t,r)\cdot e^{(\tilde{r}(t)-\tilde{x}(t))\tau},\ \forall\text{ constant }\alpha$$

and

$$C_p(t,r,p,K)\to_{p\to\infty}v(t,r)\cdot e^{(\tilde{r}(t)-\tilde{x}(t))\tau},$$

where $v(t,r)$ is the price function of the ZCB having the same maturity T (which we have already fixed) as the Call Option.

*To check that, fixed K, $C(t,r,p,K)\in\mathbb{C}^{1,2,2}$ we have computed the series of the proper partial derivatives and we have seen that they converge uniformly.

3.1. *Stochastic evolutions of the ZCB and of the Calls on p: volatility coefficients*

By the Ito differentiation rule we compute the differentials $d\tilde{v}(t;T)$ and $d\tilde{C}(t;K)$; so we explicitely have all the coefficients of volatility for each primary asset. Such coefficients are all deterministic functions of (t,r,p) after evalued at $(t,\tilde{r}(t),\tilde{p}(t-))$:

$$d\tilde{v}(t;T) = \tilde{r}(t)\tilde{v}(t;T)dt + \tilde{\Sigma}_r(t-;T)dW^r(t)$$

$$d\tilde{C}(t;K) = \tilde{r}(t)\tilde{C}(t-;K)dt + \tilde{\Sigma}_r(t-;K)dW^r(t) + \tilde{\Sigma}_p(t-;K)dW^p(t)+ \tag{3.4}$$

$$+\textstyle\sum_{j=1}^n \Delta^j\tilde{C}(t-;K)[dN^j(t) - \nu_j dt].$$

Definition. The process $\frac{\tilde{\Sigma}_r(t-;T)}{\tilde{v}(t-;T)}$ is called *volatility* of the ZCB price with respect to the source of risk W^r;
the processes $\frac{\tilde{\Sigma}_r(t-;K)}{\tilde{C}(t-;K)}$, $\frac{\tilde{\Sigma}_p(t-;K)}{\tilde{C}(t-;K)}$, $\frac{\Delta^j\tilde{C}(t-;K)}{\tilde{C}(t-;K)}$ are called *volatilities* of the Call Option price with respect to each source of risk W^r,W^p,N^j.

After computations, we reach that: for $\tilde{v}(t;T)$

$$\tilde{\Sigma}_r(t-;T) = -F(T-t)\tilde{v}(t;T)\sigma_r\sqrt{\tilde{r}(t)} = \tilde{\Sigma}_r(t;T):$$

let us point out that $F(0)=0$ and: $F(T-t)=0 \Leftrightarrow T-t=0$; so the t-expected value of the instantaneous return of any zero coupon bond is $\tilde{r}(t)$.
About the volatilities of the Call price $\tilde{C}(t;K)$ we have

$$\tilde{\Sigma}_r(t-;K) = (\sigma_r\sqrt{r}\frac{\partial C}{\partial r})(t,\tilde{r}(t),\tilde{p}(t-)) =$$

$$= \sigma_r\sqrt{\tilde{r}(t)}\left\{\tilde{C}(t-;K)\left[\tau - F(\tau)\right] + \tilde{v}(t;T)K\tau\cdot\right.$$

$$\left.\cdot\sum_{m_1,\dots,m_n} Q_{m_1}\cdots Q_{m_n} N[d_2(\tilde{p}(t-)\cdot(1+\delta_1)^{m_1}\cdots(1+\delta_n)^{m_n},K,t)]\right\};$$

$$\tilde{\Sigma}_p(t-;K) = (\sigma_p p\frac{\partial C}{\partial p})(t,\tilde{r}(t),\tilde{p}(t-)) =$$

$$= \sigma_p\tilde{p}(t-)\ \tilde{v}(t;T)e^{\rho\tau}\cdot\sum_{m_1,\dots,m_n} Q_{m_1}\cdots Q_{m_n}(1+\delta_1)^{m_1}\cdots(1+\delta_n)^{m_n}\cdot$$

$$\cdot N[d_1(\tilde{p}(t-)\cdot(1+\delta_1)^{m_1}\cdots(1+\delta_n)^{m_n},K,t)] =$$

$$= v(t,\tilde{r}(t))\cdot\sigma_p\tilde{p}(t-)e^{\rho\tau}G_p(t,\tilde{p}(t-),K),$$

where

$$G(t,p,K) \doteq \frac{C(t,r,p,K)}{v(t,r)e^{\rho\tau}}, \tag{3.5}$$

and so $G_p(t,p,K) \doteq \frac{\partial G(t,p,K)}{\partial p}$, $\Delta^j G(t,p,K) = G(t,p(1+\delta_j),K) - G(t,p,K)$,
$\Delta^j \tilde{G}(t-;K) = G(t,\tilde{p}(t-)\cdot(1+\delta_j),K) - G(t,\tilde{p}(t-),K)$;

and

$$\Delta^j \tilde{C}(t-;K) \doteq \tilde{v}(t;T)e^{\rho\tau} \cdot \sum_{m_1,\ldots,m_n} Q_{m_1}\cdots Q_{m_n} \cdot$$
$$\cdot \{F_{BS}(\tilde{p}(t-)\cdot(1+\delta_1)^{m_1}\cdots(1+\delta_n)^{m_n}(1+\delta_j),K,t) +$$
$$-F_{BS}(\tilde{p}(t-)\cdot(1+\delta_1)^{m_1}\cdots(1+\delta_n)^{m_n},K,t)\} =$$
$$= \tilde{v}(t;T)e^{\rho\tau}[G(t,\tilde{p}(t-)\cdot(1+\delta_j),K) - G(t,\tilde{p}(t-),K)] = \tilde{v}(t;T)e^{\rho\tau}\Delta^j\tilde{G}(t-;K),$$
$$j = 1\ldots n.$$

Definition. The zero coupon bond *indexed to the level of prices* $\tilde{p}$, or *indexed bond*, or *inflation linked bond*, is the bond which at t guarantees exactly the cash $\tilde{p}_T$ for the future date T. We will denote its price process $(\tilde{I}(t;T))_{t\geq 0}$.
We can regard the zero coupon bond indexed to the level of prices as a particular case of a Call Option on the index $\tilde{p}$: actually the indexed bond has the same payoff $\tilde{p}_T$ as the Call having zero strike price. Therefore such a bond has price process

$$\tilde{I}(t;T) = \tilde{C}(t;0) = \tilde{v}(t;T)e^{(\tilde{r}(t)-\tilde{x}(t))\tau}\tilde{p}(t) \tag{3.6}$$

and volatilities given by:

$$(\tau - F(\tau))\sigma_r\sqrt{\tilde{r}(t)}, \text{ with respect to } dW^r,$$
$$\sigma_p \text{ with respect to } dW^p,$$
$$\delta_j \text{ with respect to } dN^j.$$

4. Evaluating Claims and Completing the Market Model

Definition. A *Trading strategy* into the primary assets is a set
$\tilde{\mathbf{U}} = (\tilde{U}^{(\tilde{B})}(s), \tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s,dK))_{s\in[0,T[}$ where:

- the components $\tilde{U}^{(\tilde{B})}(s)$, $\tilde{U}^{(\tilde{v})}(s)$ are predictable processes s.t.:
$\int_0^T |\tilde{U}^{(\tilde{B})}(s)\tilde{r}(t)(s)|ds < \infty$ a.s.
$\int_0^T |\tilde{U}^{(\tilde{v})}(s)\frac{\tilde{r}(s)\tilde{v}(s;T)}{\tilde{B}(s)}|ds < \infty$ a.s.;
$E[\int_0^T |\tilde{U}^{(\tilde{v})}(s)\frac{\tilde{\Sigma}_r(s-;T)}{\tilde{B}(s)}|^2 ds] < \infty$;

they represent respectively the number of units of the Bank Account $\tilde{B}(s)$ and of the ZCB $\tilde{v}(s;T)$ we decide to put into our portfolio at s;

- for each set A $\subset [0,+\infty[$ the stochastic process $\tilde{U}^{(\tilde{C})}(s,\omega,A)$ is predictable;
for each (s,ω), $\tilde{U}^{(\tilde{C})}(s,\omega,dK)$ is a signed finite Borel measure on the set of the admissible strike prices $[0,+\infty[$, and such that
$\int_0^T \int_{\mathbb{R}_+} |\tilde{U}^{(\tilde{C})}(t,dK)\frac{\tilde{r}(t)\tilde{C}(t-;K)}{\tilde{B}(t)}|dt < \infty$, a.s.
$E[\int_0^T \int_{\mathbb{R}_+} |\tilde{U}^{(\tilde{C})}(t,dK)\frac{\tilde{\Sigma}_r(t-;K)}{\tilde{B}(t)}|^2 dt] < \infty$,

$E[\int_0^T \int_{\mathbb{R}_+} |\tilde{U}^{(\tilde{C})}(t,dK)\frac{\tilde{\Sigma}_p(t-;K)}{\tilde{B}(t)}|^2 dt] < \infty$,
$E[\int_0^T \int_{\mathbb{R}_+} |\tilde{U}^{(\tilde{C})}(t,dK)\frac{\Delta^j \tilde{C}(t-;K)}{\tilde{B}(t)}|dt] < \infty, j = 1..n$.

The (stochastic) quantity $\tilde{U}^{(\tilde{C})}(s,\omega,dK)$ represents the "number" of units of Call Options with strike prices in the interval $[K, K+dK]$ we hold into the portfolio at time s.

Definition. The *value process* of a portfolio composed accordingly to a strategy $\left(\tilde{U}^{(\tilde{B})}(s), \tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s,dK)\right)_{s\in[0,T[}$ is

$$\tilde{V}(t) = \tilde{U}^{(\tilde{B})}(t)\tilde{B}(t) + \tilde{U}^{(\tilde{v})}(t)\tilde{v}(t;T) + \int_{K\in\mathbb{R}_+} \tilde{C}(t;K)\tilde{U}^{(\tilde{C})}(t,dK), \quad t \in [0,T[:$$

$\tilde{V}(0)$ is the initial cost of the strategy, and $\tilde{V}(t)$ is the cost of adopting such a strategy if we start from t.

Definition. A trading strategy $\left(\tilde{U}^{(\tilde{B})}(s), \tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s,dK)\right)$ is *Self Financing* (SF) if the corresponding portfolio value evolves accordingly to

$$d\tilde{V}(t) = \tilde{U}^{(\tilde{B})}(t)d\tilde{B}(t) + \tilde{U}^{(\tilde{v})}(t)d\tilde{v}(t;T) + \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t,dK)d\tilde{C}(t;K), \qquad (4.7)$$

id est

$$\tilde{V}(s) = \tilde{V}(0) + \int_0^s \tilde{U}^{(\tilde{B})}(t)d\tilde{B}(t) + \int_0^s \tilde{U}^{(\tilde{v})}(t)d\tilde{v}(t;T) + \int_0^s \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t,dK)d\tilde{C}(t;K).$$

Definition. A Claim $\tilde{H}$ is *attainable* (or *hedgeable*) if there exists a SF trading strategy into the primary assets whose portfolio value is s.t. $\tilde{V}_T = \tilde{H}$ a.s. We say that such a strategy *replicates* the claim .

Remark. The integrability conditions requested to a trading strategy are needed to make well defined the stochastic integrals appearing into the discounted value process $\tilde{V}^d$ of a SF strategy:

$$\frac{d\tilde{V}^d(t)}{\tilde{V}^d(t-)} = \frac{d\tilde{V}(t)}{\tilde{V}(t-)} - \tilde{r}(t)dt,$$

and so

$$d\tilde{V}^d(t) = \tilde{V}^d(t-)\frac{d\tilde{V}(t)}{\tilde{V}(t-)} - \tilde{V}^d(t-)\tilde{r}(t)dt = \frac{d\tilde{V}(t)}{\tilde{B}(t)} - \frac{\tilde{V}(t)}{\tilde{B}(t)}\tilde{r}(t)dt =$$

$$= \left[\tilde{U}^{(\tilde{B})}(t) - \frac{\tilde{V}(t)}{\tilde{B}(t)}\right]\tilde{r}(t)dt + \frac{\tilde{U}^{(\tilde{v})}(t)}{\tilde{B}(t)}d\tilde{v}(t;T) + \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t,dK)\frac{d\tilde{C}(t;K)}{\tilde{B}(t)}.$$

Moreover the last two differentials are such that the respective stochastic integrals are martingales.

Remark. The SF trading strategy *rolling over* just maturing ZCB indexed to the general level of prices perfectly replicates the very process $(\tilde{p}(t))_t$ (see the Appendix). So it can be interpreted as the price process of an asset actually traded in the market.

Proposition 1 *([BKR], Lemma 3.5) The trading strategy into the primary assets* $\left(\tilde{U}^{(\tilde{B})}(s), \tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s, dK)\right)$ *is SF (i.e. satisfies (**??**)) if and only if:*
the trading strategy $\left(\tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s, dK)\right)$ *into the discounted assets* $\tilde{v}^d(t;T))$, $\tilde{C}^d(t;K)$ *is SF, i.e. satisfies*

$$d\tilde{V}^d(t) = \tilde{U}^{(\tilde{v})}(t) d\tilde{v}^d(t;T) + \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK) d\tilde{C}^d(t;K),$$

and $\tilde{U}^{(\tilde{B})}(t) \equiv \frac{\tilde{V}(t)}{\tilde{B}(t)}$. □

Definition. The *t-price* of an hedgeable claim $\tilde{H}$ is the budget $\tilde{\Pi}(t)$ which allows us, starting from t, to determine a Self-Financing trading strategy $\left(\tilde{\mathbf{U}}(s)\right)_{s\in[t,T[}$ into the primary assets which replicates $\tilde{H}$. Id est $\tilde{\Pi}(t)$ is the cost at t of the replicating strategy $\tilde{\mathbf{U}}$.
The replication consists into selling the claim at t at price $\tilde{\Pi}(t)$ and using such a cash to adopt the hedging strategy.
Such a definition is well posed: two SF trading strategies having the same value at a future instant T have necessarily the same value at each $t < T$, because (by the integrability conditions imposed to a trading strategy) its discounted portfolio value $\tilde{V}^d$ is a Q-martingale and so at each t its value can only be the expected value $E^Q[\tilde{H}^d|F_t]$.

Definition. An *Arbitrage Opportunity* among a set of assets is a SF strategy into such assets whose portfolio value satisfies

$$\tilde{V}(0) =_{a.s.} 0, \ \tilde{V}(T) \geq_{a.s.} 0 \text{ and } P\{\tilde{V}(T) > 0\} > 0.$$

Remark. The discouted price of each primary asset is a martingale, then (by the integrability conditions requested to a trading strategy set) there are no arbitrage opportunities among the primary assets.

Definition. The model is *Complete* if any claim $\tilde{H} \in L^2(\Omega, Q, F_T)$ is hedgeable.

Definition. A set of $n+2$ assets is a *t-Market Basis* if the $(n+2)\times(n+2)$ matrix of their volatility coefficients, at t, is non-singular a.s.

The market basis has to contain exactly the same number of assets as the number of independent sources of randomness driving the model.

General Result (corollary of Proposition 4.5 in [BKR]: for the proof see the Appendix).
If for any $t < T$ *there exists a t-market basis, then the market model is complete.*

The proof shows how, given any contingent claim $\tilde{H}$, to construct a SF strategy which buys and sells for any $t < T$ only the assets of the t-market basis. The discounted portfolio value process $\left(\tilde{V}(t)^d\right)_{t\in[0,T]}$ of such a strategy is a Q-martingale, closed at T by the random variable $\tilde{H}^d$.

So, if for any instant a t-market basis exists, the discounted price process $\left(\tilde{V}(t)^d\right)_{t\in[0,T]}$ of *any* claim $\tilde{H}$ is a Q-martingale.

As a consequence, the t-price of any claim $\tilde{H}$ is $\Pi(t) = E^Q[\tilde{H} \cdot e^{-\int_t^T \tilde{r}(u)du}|F_t]$.

Remark. When we replicate a claim $\tilde{H}$ using the market basis, it could happen that at different instants t and s we use different basic assets because the t-basis could be different from the s-basis. Or, if they would coincide, the number of shares of each asset could be different for t and for s.
So we can perfectly hedge a claim $\tilde{H}$ by the strategy we found in the previous proof only as long as we have not transaction costs for the trading of the primary assets.

Incidentally we remark that the price $\tilde{\Pi}(t)$ actually turns out to be a function of kind $\tilde{V}(t) = V(t, \tilde{r}(t), \tilde{p}(t))$, where $V(t, r, p)$ is deterministic because of the Markovian setting. Moreover $V(t, r, p)$ is uniquely determined as solution of the partial difference-differential equation called the *Fundamental Evaluation Equation* for this model, once we assign the proper boundary conditions ([Te]).
Such an equation describes the fact that, because $\tilde{V}^d$ is a martingale, its drift coefficient is identically zero.

Our goal is now to exhibit for each t a t-market basis for this model: it will be consituted by primary assets, so: by the use of primary assets we complete the market model. This is the central result of the paper.

Theorem 2 *Whatever n is, for each $t < T$, there exist $K_2, .., K_{n+1}$ (depending on t) such that the set composed by the following n+2 assets*

- *the unitary ZCB $\tilde{v}(t;T)$,*
- *the index-linked bond $\tilde{I}(t;T)$,*
- *the n Call Options $\tilde{C}(t;K_2), ..., \tilde{C}(t;K_{n+1})$ on the index p having the respective strike prices K_2 ... K_{n+1} but the same maturity T*

is a t-market basis for the model.

In order to prove the theorem we need some preliminary facts.

Definition. Given N functions $f_1(x), ..., f_N(x)$, with $f_i : \mathbb{R} \mapsto \mathbb{R}$, they are said to be *linearly dependent* if there exist N constant numbers (i.e. not depending on x) $a_1, ..., a_N$, not all zeros, s.t.

$$\sum_{i=1}^{N} a_i f_i(x) \equiv_x 0$$

i.e. we can entirely express one of the f_i using the other N-1 functions.
$f_1(x), ..., f_N(x)$, are said *linearly independent* if they are not linearly dependent.

Remark. N functions $f_1(x), ..., f_N(x)$ are *linearly independent* if, given N constant numbers $b_1, ..., b_N$ such that $\sum_{i=1}^{N} b_i f_i(x) \equiv_x 0$, we are obliged to conclude that $b_1 = ... = b_N = 0$. $\square$

Lemma 3 : link between non singular matrix of functions and linear independence among such functions.
Given n+1 real functions $g_1(p,K), g_2(p,K), ..., g_{n+1}(p,K),\ g_i : \mathbb{R}^2_+ \mapsto \mathbb{R}$ the following holds:
if, fixed any p, the functions $g_j(p,K)$ are linearly independent w.r.t. K and $g_1(p,0) \neq 0$ then there exist n numbers $K_2, ..., K_{n+1}$ such that the matrix

$$\begin{matrix} g_1(p,0) & g_2(p,0) & g_3(p,0) & ... & g_{n+1}(p,0) \\ g_1(p,K_2) & g_2(p,K_2) & g_3(p,K_2) & ... & g_{n+1}(p,K_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_1(p,K_{n+1}) & g_2(p,K_{n+1}) & g_3(p,K_{n+1}) & ... & g_{n+1}(p,K_{n+1}) \end{matrix}$$

is non-singular.
Proof. p is arbitrary but fixed. We remark only K as variable of the g_js. We proceed by induction on the number N of functions.
N=1. g_1 is linearly independent because g_1 is not identically zero: the 1×1 matrix $g_1(0)$ has determinant different from zero.
N=2. $g_1(0)$ and $g_2(0)$ are both different from zero by assumption.
Let us compute

$$M(K) \doteq det \begin{bmatrix} g_1(0) & g_2(0) \\ g_1(K) & g_2(K) \end{bmatrix} =$$
$$= g_1(0)g_2(K) - g_1(K)g_2(0) :$$

this is a linear combination of $g_1(K), g_2(K)$ with coefficients (in particular) not all zeros, and $g_1(K), g_2(K)$ are linearly independent; so the combination cannot be constantly zero, i.e.

$$\exists K_2 :\ M(K_2) \neq 0.$$

The second step is also concluded.
Now the step $N-1 \Rightarrow N$, for any $N \leq n+1$. We can find (by induction) $K_1 = 0, K_2, ..., K_{N-1}$ s.t.

$$\begin{matrix} g_1(0) & g_2(0) & ... & g_{N-1}(0) \\ g_1(K_2) & g_2(K_2) & ... & g_{N-1}(K_2) \\ \vdots & \vdots & \vdots & \vdots \\ g_1(K_{N-1}) & g_2(K_{N-1}) & ... & g_{N-1}(K_{N-1}) \end{matrix}$$

is non singular.
Let us compute

$$M(K) \doteq det \begin{bmatrix} g_1(0) & g_2(0) & g_3(0) & ... & g_N(0) \\ g_1(K_2) & g_2(K_2) & g_3(K_2) & ... & g_N(K_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_1(K_{N-1}) & g_2(K_{N-1}) & g_3(K_{N-1}) & ... & g_N(K_{N-1}) \\ g_1(K) & g_2(K) & g_3(K) & ... & g_N(K) \end{bmatrix} :$$

set $M^{\hat{j}}_{\hat{N}}$ the determinant of the minor obtained by eliminating the row N and the column j, and develope along last line. We have

$$M(K) = (-1)^{N+1} g_1(K) M^{\hat{1}}_{\hat{N}} + (-1)^{N+2} g_2(K) M^{\hat{2}}_{\hat{N}} + ...$$

$$+(-1)^{2N} g_N(K) M_{\hat{N}}^{\hat{N}} =$$
$$= a_1 g_1(K) + a_2 g_2(K) + ... + a_N g_N(K).$$

This is a linear combination of the functions g_is with coefficients a_is not all zeros, because at least $M_{\hat{N}}^{\hat{N}}$ is different from zero.
By the linear independence of the g_is, we know that M(K) cannot be identically zero, so there exists K_N s.t. $M(K_N) \neq 0$ □

Lemma 4 *:* positive homogeneity of functions and change of variable.
Let

$$f_j(p, K), j = 1, ..., N$$

be real functions, defined for the two strictly po sitive variables p, K, which are positively homogeneous.
Then: they are linearly independent w.r.t. K, for each fixed value of p, if and only if they are linearly independent w.r.t. p, for each fixed value of K.
Proof. †Let us suppose that for a fixed value of p the functions

$$f_j, j = 1, ..., N$$

are linearly independent as functions of K. Id est: if for some group of N constant numbers $b_1, ..., b_N$ (which are constant w.r.t. K, but can depend on p), we have $\sum_{i=1}^N b_i f_i(p, K) \equiv_K 0$ then $b_1 = ... = b_N = 0$.
Now we go to verify that if we fix an arbitrary K and regard the f_is as functions only of p, and for some group of N numbers $c_1, ..., c_N$ which are constant w.r.t. p we had

$$\sum_{i=1}^N c_i f_i(p, K) \equiv_p 0$$

then we necessarily would obtain that $c_1 = ... = c_N = 0$ (this is the thesis of the lemma).
So: let us fix K and take $c_1, ..., c_N$ s.t.

$$\sum_{i=1}^N c_i f_i(p, K) \equiv_p 0;$$

we know that, for each $p > 0$, $f_i(p, K) = p f_i(1, \frac{K}{p})$, for each i, so we have

$$p \sum_{i=1}^N c_i f_i(1, \frac{K}{p}) \equiv_p 0,$$

thus, for each $p > 0$,

$$\sum_{i=1}^N c_i f_i(1, \frac{K}{p}) \equiv_p 0. \tag{4.8}$$

Now: given $\bar{x} > 0$, the value $\bar{p} = \frac{K}{\bar{x}} > 0$ ($\bar{x} = \frac{K}{\bar{p}}$, K is fixed) is such that

$$\sum_{i=1}^N c_i f_i(1, \bar{x}) = \sum_{i=1}^N c_i f_i(1, \frac{K}{\bar{p}})$$

†For the proof I thank Massimiliano Giuli, University of Florence

and this is equal to zero because (**??**) holds for each positive value of p.
Thus, because for each $x > 0$ $\exists p > 0 : x = \frac{K}{p}$, we can say that

$$\forall x > 0 \quad \sum_{i=1}^{N} c_i f_i(1, x) = 0.$$

But we have the linear independence of the functions f_i w.r.t. the second variable (for each value assumed by the first variable, in particular the value 1), therefore $c_1 = ... = c_N = 0$, which is what we wanted to prove.
The vice-versa is analogous. □

Proposition 5 : another property of the Call price function.
Let C(p) be a real function which is continuously differentiable $n+1$ *times on* $]0, +\infty[$ *and s.t.:*

- *for each h=1..n,* $lim_{p\to 0}\frac{\partial^h C}{\partial p^h}(p) = 0$, *but there exists a sequence* $\mathcal{D} \doteq \{p_\ell, \ell \in \mathbb{N}\}$ *such that* $p_\ell \neq 0$ $\forall \ell$, $lim_{\ell\to\infty} p_\ell = 0$ *and any derivative* $\frac{\partial^h C}{\partial p^h}$ *is different from zero on* $\mathcal{D}$,
- $\exists \beta \neq 0 : \forall \alpha \in \mathbb{R}$

$$\frac{C(p \cdot \alpha)}{p} \to_{p\to\infty} \alpha\beta,$$

$$C_p(p) \to_{p\to\infty} \beta.$$

Then: if $\delta_1, .., \delta_n$ *are different constant numbers, then the* $n + 1$ *functions* $p \cdot C_p(p), (\Delta^j C)(p), j = 1, ..., n$ *are linearly independent.*

Proof. Let us set $g_1(p) \doteq pC_p(p), g_j(p) \doteq (\Delta^j C)(p), j = 1, ..., n$.
Let us suppose that there exist n+1 numbers a_j, not depending on p, s.t.

$$\sum_{i=0}^{n} a_j g_j(p) \equiv_p 0. \tag{4.9}$$

For each $p > 0$ we have

$$0 \equiv_p \sum_{i=0}^{n} a_j \frac{g_j(p)}{p} = a_0 C_p(p) + \sum_{j=1}^{n} a_j \frac{(\Delta^j C)(p)}{p}. \tag{4.10}$$

Taking $p \to +\infty$ we have that

$$\frac{\Delta^j C(p)}{p} = \frac{C(p(1+\delta_j)) - C(p)}{p} \to_{p\to\infty} \delta_j\beta,$$

and

$$C_p(p) \to_{p\to\infty} \beta,$$

so the constant numbers a_i have to satisfy

$$a_0 + \sum_{j=1}^{n} a_j \delta_j = 0.$$

This means that relation (**??**) coincides with

$$0 \equiv_p - \left(\sum_{j=1}^{n} a_j\delta_j\right) pC_p(p)+\sum_{j=1}^{n} a_j(\Delta^j C)(p) = \sum_{j=1}^{n} a_j[(\Delta^j C)(p)-\delta_j pC_p(p)]. \quad (4.11)$$

Using now the Taylor expansion up to the second derivative, for each j=1...n, we have

$$lim_{p\to 0, p\in\mathcal{D}} \frac{\Delta^j C(p) - \delta_j pC_p(p)}{p^2 C_{pp}(p)} = \frac{\delta_j^2}{2}$$

and so

$$lim_{p\to 0, p\in\mathcal{D}} \sum_{j=1}^{n} a_j \left(\frac{\Delta^j C(p) - \delta_j pC_p(p)}{p^2 C_{pp}(p)}\right) = \frac{\sum_j a_j\delta_j^2}{2}. \quad (4.12)$$

But now, from (**??**), we have
$\sum_{j=1}^n a_j[\Delta^j C - \delta_j pC_p] \equiv_p 0$ and so the left member of (**??**) is zero, and therefore the constants a_j have to satisfy also

$$\sum_{j=1}^{n} a_j\delta_j^2 = 0.$$

Analogously, by induction we are going to check that $\sum_j a_j\delta_j^h = 0$ for each $h = 3, ..., n+1$:
let us know that $\sum_j a_j\delta_j^2 = ... = \sum_j a_j\delta_j^{h-1} = 0$.
For each h=1..n, by the Taylor expansion up to the h-th derivative,

$$lim_{p\to 0, p\in\mathcal{D}} \frac{\Delta^j C - \delta_j pC_p - ... - \frac{\delta_j^{h-1}p^{h-1}}{(h-1)!}\frac{\partial^{h-1}C}{\partial p^{h-1}}}{p^h\frac{\partial^h C}{\partial p^{h-1}}} = \frac{\delta_j^h}{h!}$$

and so

$$\frac{\sum_j a_j\delta_j^h}{h!} = lim_{p\to 0, p\in\mathcal{D}} \frac{\sum_j a_j\left(\Delta^j C - \delta_j pC_p - ... - \frac{\delta_j^{h-1}p^{h-1}}{(h-1)!}\frac{\partial^{h-1}C}{\partial p^h}\right)}{p^h\frac{\partial^h C}{\partial p^h}} =$$

$$= lim_{p\to 0, p\in\mathcal{D}} \frac{\sum_j a_j(\Delta^j C - \delta_j pC_p)}{p^h\frac{\partial^h C}{\partial p^h}} - lim_{p\to 0, p\in\mathcal{D}} \frac{p^2\cdot\frac{\partial^2 C}{\partial p^2}}{2p^h\frac{\partial^h C}{\partial p^h}} \sum_j a_j\delta_j^2 - ...$$

$$-lim_{p\to 0, p\in\mathcal{D}} \frac{\frac{\partial^{h-1}C}{\partial p^{h-1}}}{p(h-1)!\frac{\partial^h C}{\partial p^h}} \sum_j a_j\delta_j^{h-1}$$

Now: the functions g_js satisfy (**??**), and the inductive hypothesis says us that $\sum_j a_j\delta_j^2$ =...= $\sum_j a_j\delta_j^{h-1} = 0$: so the left member of last equality is zero, and we also have necessarily that $\sum_j a_j\delta_j^h = 0$.

Summing up: if the functions $g_1, ..., g_n$ satisfy (**??**), then the n+1 constants a_j are uniquely determined by the following conditions:

$$a_0 = -\sum_{j=1}^{n} a_j \delta_j$$

$$\sum_{j=1}^{n} a_j \delta_j^2 = 0$$

$$\vdots$$

$$\sum_{j=1}^{n} a_j \delta_j^{n+1} = 0$$

Last n equations give a linear system, into the unknowns $a_1, ..., a_n$, whose coefficients matrix reduces to one of the kind of Vandermonde, which is non singular as soon as the δ_js are different each other. This implies that the unique solution to the system is $a_1 = ... = a_n = 0$ and so the unique case in which $0 \equiv_p \sum_{j=0}^{n} a_j g_j(p)$ is when all the constants a_j are zero.
I.e. the functions g_js are linearly independent. □

Application of proposition ??: *proof of the Theorem.*

The price volatility coefficients of the set of assets constituted by the unitary ZCB $\tilde{v}(t;T)$, the index-linked bond $\tilde{I}(t;T)$ and n Call Options with strike prices $K_2, ..., K_{n+1}$ form the following $(n+2) \times (n+2)$ matrix (we frame it with a row and a column explaining the entries):

$$\begin{array}{cccccc}
 & dW^r & dW^p & dN^1 & \dots & dN^n \\
\frac{d\tilde{v}(t;T)}{\tilde{v}(t;T)} & -F(\tau)\sigma_r\sqrt{\tilde{r}(t)} & 0 & 0 & \dots & 0 \\
\frac{d\tilde{I}(t;T)}{\tilde{I}(t-;T)} & (\tau - F(\tau))\sigma_r\sqrt{\tilde{r}(t)} & \sigma_p & \delta_1 & \dots & \delta_n \\
\frac{d\tilde{C}(t;K_2)}{\tilde{C}(t-;K_2)} & \frac{\tilde{\Sigma}_r(t-;K_2)}{\tilde{C}(t-;K_2)} & \frac{\tilde{\Sigma}_p(t-;K_2)}{\tilde{C}(t-;K_2)} & \frac{\Delta^1\tilde{C}(t-;K_2)}{\tilde{C}(t-;K_2)} & \dots & \frac{\Delta^n\tilde{C}(t-;K_2)}{\tilde{C}(t-;K_2)} \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
\frac{d\tilde{C}(t;K_{n+1})}{\tilde{C}(t-;K_{n+1})} & \frac{\tilde{\Sigma}_r(t-;K_{n+1})}{\tilde{C}(t-;K_{n+1})} & \frac{\tilde{\Sigma}_p(t-;K_{n+1})}{\tilde{C}(t-;K_{n+1})} & \frac{\Delta^1\tilde{C}(t-;K_{n+1})}{\tilde{C}(t-;K_{n+1})} & \dots & \frac{\Delta^n\tilde{C}(t-;K_{n+1})}{\tilde{C}(t-;K_{n+1})}
\end{array}$$

Let us remember that, fixed t and $T, K_2, ..., K_{n+1}$, each entry is a deterministic function $f_{ij}(r,p)$ of the stochastic pair $(\tilde{r}(t), \tilde{p}(t-))$.
Our stochastic matrix is non singular at t a.s. if and only if the matrix of the deterministic functions $f_{ij}(r,p)$ is non singular for each choice of the two numbers $r > 0, p > 0$: in fact the random variables $\tilde{r}(t)$, $\tilde{p}(t)$ are stochastically independent and the support of each law $(\tilde{r}(t))\,(Q)$, $(\tilde{p}(t-))\,(Q)$ is the whole $\mathbb{R}_+$, therefore $\forall (r,p), P\{(\tilde{r}(t), \tilde{p}(t-)) \in]r-\varepsilon_1, r+\varepsilon_1[\times]p-\varepsilon_2, p+\varepsilon_2[\} > 0$, and $P\{\tilde{r}(t) > 0, \tilde{p}(t-) > 0\} = 1$.

$\tau = 0$ is the only value for which $F(\tau) = 0$: therefore, fixed $t < T$, $r > 0$, $p > 0$, the matrix of the deterministic functions $f_{ij}(r,p)$ is non singular if and only if such is the sub-matrix of the last n+1 rows and n+1 columns:

$$\begin{array}{cccc}
\sigma_p & \delta_1 & \dots & \delta_n \\
\frac{\sigma_p p \cdot C_p(p,K_2)}{C(p,K_2)} & \frac{\Delta^1 C(p,K_2)}{C(p,K_2)} & \dots & \frac{\Delta^n C(p,K_2)}{C(p,K_2)} \\
\vdots & \vdots & \dots & \vdots \\
\frac{\sigma_p p \cdot C_p(p,K_{n+1})}{C(p,K_{n+1})}) & \frac{\Delta^1 C(p,K_{n+1})}{C(p,K_{n+1})} & \dots & \frac{\Delta^n C(p,K_{n+1})}{C(p,K_{n+1})}
\end{array}.$$

• Let us keep fixed $t < T$, $r > 0$, $p > 0$. After manipulations[‡]we reach that, our matrix is non singular if and only if the following $(n+1)\times(n+1)$ matrix is non singular (we leave out t and r):

$$M(p;0,K_2,...,K_{n+1}) \doteq$$

$$\begin{bmatrix} p\cdot exp[\sum_i \delta_i\nu_i\tau] & \delta_1 p\cdot exp[\sum_i \delta_i\nu_i\tau] & ... & \delta_n p\cdot exp[\sum_i \delta_i\nu_i\tau] \\ p\cdot C_p(p,K_2) & \Delta^1 C(p,K_2) & ... & \Delta^n C(p,K_2) \\ \vdots & \vdots & ... & \vdots \\ p\cdot C_p(p,K_{n+1}) & \Delta^1 C(p,K_{n+1}) & ... & \Delta^n C(p,K_{n+1}) \end{bmatrix}.$$

• Because $p\cdot C_p(p,0) = p\cdot exp[\sum_i \delta_i\nu_i\tau]$ and $\Delta^j C(p,0) = \delta_j p\cdot exp[\sum_i \delta_i\nu_i\tau]$ for each j=1,...,n, $M(p;0,K_2,...,K_{n+1})$ is a matrix of kind

$$\begin{matrix} g_1(p,0) & g_2(p,0) & g_3(p,0) & ... & g_{n+1}(p,0) \\ g_1(p,K_2) & g_2(p,K_2) & g_3(p,K_2) & ... & g_{n+1}(p,K_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_1(p,K_{n+1}) & g_2(p,K_{n+1}) & g_3(p,K_{n+1}) & ... & g_{n+1}(p,K_{n+1}) \end{matrix}$$

with

$$g_1(p,K) \doteq pC_p(p,K), \qquad g_{j+1}(p,K) \doteq \Delta^j C(p,K), j=1,...,n. \tag{4.13}$$

Applying Proposition **??** to the Call price function $C(p,K)$ regarded as function of p, and we conclude that the functions

$$g_1(p,K) \doteq pC_p(p,K), \qquad g_{j+1}(p,K) \doteq \Delta^j C(p,K), j=1,...,n$$

are p-linearly independent for any fixed value of K.
Therefore, by Lemma **??**, for any fixed value of p they turn out to be K-linearly independent, and so, by Lemma **??**, it is guaranteed the existence of n different strike prices $K_2,...,K_{n+1}$ s.t. the matrix $M(p;0,K_2,...,K_{n+1})$ is non singular. And this happens for any $t < T$ and for any realization p of the general index of prices at t. □

Remark. The K_js whose existence is guaranteed by the Theorem do not depend on r, but only on t, and p.
In fact we can divide $M(p;0,K_2,...,K_{n+1})$ by $v(t,r)\cdot e^{\rho\tau}$: any entry of $\frac{M(p;0,K_2,...,K_{n+1})}{v(t,r)\cdot e^{\rho\tau}}$ is determined only by the function $G(t,p,K)$, given in (**??**), which does not depend on r.

Remark. When n=1, the prices volatilities matrix is

$$\begin{array}{cccc} & dW^r & dW^p & dN \\ \frac{d\tilde{v}(t;T)}{\tilde{v}(t;T)} & -F(\tau)\sigma_r\sqrt{\tilde{r}(t)} & 0 & 0 \\ \frac{d\tilde{I}(t;T)}{\tilde{I}(t-;T)} & (\tau - F(\tau))\sigma_r\sqrt{\tilde{r}(t)} & \sigma_p & \delta \\ \frac{d\tilde{C}(t;K_2)}{\tilde{C}(t-;K_2)} & \frac{\tilde{\Sigma}_r(t-;K_2)}{\tilde{C}(t-;K_2)} & \frac{\tilde{\Sigma}_p(t-;K_2)}{\tilde{C}(t-;K_2)} & \frac{\Delta^1\tilde{C}(t-;K_2)}{\tilde{C}(t-;K_2)} \end{array}, \tag{4.14}$$

[‡]1. In the selected sub-matrix we divide the whole first column by σ_p,
2. multiply row i by $C(p,K_i)$ for each i=2,...,n+1, and
3. multiply the first row by $pe^{\sum_i \delta_i\nu_i\tau}$.

so (once fixed arbitrarily and left out $t < T$ and $r > 0$ and after manipulations) we reduce to deal with

$$M(p;K) \doteq \left[\begin{array}{cc} 1 & \delta \\ pC_p(p,K) & C(p(1+\delta),K) - C(p,K) \end{array} \right]. \tag{4.15}$$

This matrix has positive determinant for each $p > 0$ and each $K > 0$, because for any fixed positive K the function $C(p,K)$ is convex on $p > 0$, and so the quantity $C(p(1+\delta),K) - C(p,K) - \delta p \cdot C_p(p,K)$ is always strictly positive.
This means that when n=1 the hedging strategy needs to be continuously rebalanced, but only in the number of shares of the used assets: actually, because the K which makes non singular the matrix does not depend on t, the same assets chosen at t=0 for the replication hedge us up to the end.
I.e.: for any $K > 0$ the set $(\tilde{v}(t;T),\ \tilde{I}(t;T),\ \tilde{C}(t,K))$ is a market basis for all the time and an hedging strategy simplifies to a set of four processes
$\left(\tilde{U}^{(\tilde{B})}(s), \tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{I})}(s), \tilde{U}^{(K)}(s)\right).$ □

Remark. In the case n=2, also another proof for the p-linear independence of the g_js of (**??**) is available: it is based on the fact that the Call price function satisfies the *Fundamental Evaluation Equation* of this market model ([Te]).

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5. Appendix: the General Result of BKR

The General Result of [BKR] applied to our framework. *If for any $t < T$ there exists a t-market basis then, given any contingent claim $\tilde{H}$, we can find a SF trading strategy whose final value is $\tilde{H}$. In particular, as long as the discounted prices of the assets of any t-market basis are martingales, the discounted value process $\left(\tilde{V}(t)^d\right)_{t\in[0,T]}$ is a Q-martingale.*

Proof. Let us take a claim $\tilde{H}$. We are going to see how the basic securities allow us to construct a replicating strategy.
Let us know that at each t we have n+2 basic assets. The set of assets forming a basis can change during time, so we adopt the general structure $\left(\tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s, dK)\right)_{s\in[0,T[}$ for the strategy which is candidate in order to hedge $\tilde{H}^d$ (on the light of Proposition (**??**)).

Notations: let us sum over repeated indexes.
The index i in W^i stands for r or p.

Let t=0 to be the instant in which we sell the payoff $\tilde{H}$.
We want to hedge $\tilde{H} \in L^2(F_T, Q)$ using as risky assets the primary assets $\tilde{v}(t;T), \tilde{C}(t;K), K \in \mathbb{R}_+$.
Thanks to Proposition **??**, we are looking for a couple $\mathbf{U}$=$(\tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s, dK))$ with switably integrable components and s.t. $\exists\ \Pi \in F_0$ s.t. the consequent discounted portfolio value is $\tilde{H}^d$ at T:

$$\tilde{V}^d(s) = \Pi + \int_0^s \tilde{U}^{(\tilde{v})}(t) d\tilde{v}^d(t;T) + \int_0^s \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK) d\tilde{C}^d(t;K)$$

$$\tilde{H}^d = \Pi + \int_{t=0}^{t=T} \tilde{U}^{(\tilde{v})}(t) d\tilde{v}^d(t;T) + \int_{t=0}^{t=T} \int_{K\in\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK) d\tilde{C}^d(t;K);$$

but, getting from (**??**) the discounted dynamics

$$\begin{aligned} d\tilde{v}^d(t;T) &= \tfrac{\tilde{\Sigma}_r(t-;T)}{\tilde{B}(t)} dW^r(t) \\ d\tilde{C}^d(t;K) &= \tfrac{\tilde{\Sigma}_i(t-;K)}{\tilde{B}(t)} dW_t^i + \tfrac{\Delta^j \tilde{C}(t-;K)}{\tilde{B}(t)} [dN^j(t) - \nu_j dt], \end{aligned} \tag{5.16}$$

the above right member is

$$\begin{aligned} \tilde{H}^d = \Pi + \textstyle\int_0^T \left[\tilde{U}^{(\tilde{v})}(t) \tfrac{\tilde{\Sigma}_r(t-;T)}{\tilde{B}(t)} + \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK) \tfrac{\tilde{\Sigma}_r(t-;K)}{\tilde{B}(t)} \right] dW^r(t) + \\ + \textstyle\int_0^T \left[\int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK) \tfrac{\tilde{\Sigma}_p(t-;K)}{\tilde{B}(t)} \right] dW^p(t) + \\ + \textstyle\sum_{j=1}^n \int_0^T \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK) \tfrac{\Delta^j \tilde{C}(t-;K)}{\tilde{B}(t)} [dN^j(t) - \nu_j dt]. \end{aligned} \tag{5.17}$$

If the requested $\tilde{\mathbf{U}}$ was found, the integrability properties of its components would make the process $\tilde{V}^d(t) = \Pi + \int_0^t d\tilde{V}^d(u),\ t \in [0,T]$ a martingale, which has final value $\tilde{H}^d$, so the process would have to coincide with $\tilde{M}_t \doteq E_t^Q[\tilde{H}^d]$.

On the other hand, $\tilde{M}_t$ is in fact a square-integrable martingale (by definition, a claim $\tilde{H}$ belongs to $L^2(Q)$), thus thanks to the (Ω, F, Q)-martingale representation theorem ([Bu]), $\tilde{M}$ has the following structure:

$$\tilde{M}_T = x + \int_0^T \tilde{\gamma}^i(t)dW^i(t) + \int_0^T \tilde{\phi}^j(t)[dN^j(t) - \nu_j dt]$$

for some predictable $L^2(dQ \times dt)$-integrable processes $\tilde{\gamma}^r$, $\tilde{\gamma}^p$, $\tilde{\phi}^1, ..., \tilde{\phi}^n$.
Thus if we put $x = \Pi$ and if we can find $(\tilde{U}^{(\tilde{v})}(s), \tilde{U}^{(\tilde{C})}(s, dK))$ s.t. for each t

$$\begin{cases} \frac{\tilde{U}^{(\tilde{v})}(t)\tilde{\Sigma}_r(t-;T)}{\tilde{B}(t)} + \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK)\frac{\tilde{\Sigma}_r(t-;K)}{\tilde{B}(t)} = \tilde{\gamma}^r(t) \\ \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK)\frac{\tilde{\Sigma}_p(t-;K)}{\tilde{B}(t)} = \tilde{\gamma}^p(t) \\ \int_{\mathbb{R}_+} \tilde{U}^{(\tilde{C})}(t, dK)\frac{\Delta^j \tilde{C}(t-;K)}{\tilde{B}(t)} = \tilde{\phi}^j(t) \;\; \forall j = 1, ..., n \end{cases} \tag{5.18}$$

holds a.s., we have finished.
A sufficient condition for this is, for each t, to find $K_1(t), ..., K_{n+1}(t)$ such that a.s.

$$\begin{cases} \frac{\tilde{U}^{(\tilde{v})}(t)\tilde{\Sigma}_r(t-;T)}{\tilde{B}(t)} + \sum_{\ell=1}^{n+1} \tilde{U}^{(\tilde{C})}(t, K_\ell)\frac{\tilde{\Sigma}_r(t-;K_\ell)}{\tilde{B}(t)} = \tilde{\gamma}^r(t) \\ \sum_{\ell=1}^{n+1} \tilde{U}^{(\tilde{C})}(t, K_\ell)\frac{\tilde{\Sigma}_p(t-;K_\ell)}{\tilde{B}(t)} = \tilde{\gamma}^p(t) \\ \sum_{\ell=1}^{n+1} \tilde{U}^{(\tilde{C})}(t, K_\ell)\frac{\Delta^j \tilde{C}(t-;K_\ell)}{\tilde{B}(t)} = \tilde{\phi}^j(t). \end{cases} \tag{5.19}$$

This reduces exactly to solve for each t a linear system into the unknowns $(\tilde{U}^{(\tilde{v})}(t), \tilde{U}^{(\tilde{C})}(t, K_\ell), \ell = 1, ..., n+1)$ and with coefficients matrix given just by the $M(p; K_1, K_2, ..., K_{n+1})$ given in the proof of Theorem **??**.
So: the existence of a t-market basis constituted of a bond and $n+1$ Calls allows us to display the proper $K_1(t), ..., K_{n+1}(t)$ such that we can solve the linear system. Because all the coefficients of the linear system are predictable, the global solution $\tilde{\mathbf{U}}$ will be predictable. $\tilde{\mathbf{U}}$ satisfies also the integrability conditions required to a trading strategy because the processes $\tilde{\gamma}^i, \tilde{\phi}^j$ belong to $L^2(dQ \times dt)$. □

6. Appendix. $\tilde{x}$ Represents the Spot Real Interest rate and $\tilde{p}$ can be interpreted as the price process of a traded asset

We assume that in our economy we can consume only one (representative) good.
Let us define the unitary *real zero coupon bond* issued at t and maturing at s as the contract that at time t guarantees to deliver one unit of the consumption good at time s. The market *real price* of such a contract is expressed in terms of units of the consumption good and it is denoted by $\underline{\tilde{v}}(t; s)$.

Remark. The basic relation which ties nominal and real amounts is:

$$\underline{\tilde{X}}(t)[units\ of\ good] \cdot \tilde{p}(t) \left[\frac{units\ of\ money}{units\ of\ good}\right] = \tilde{X}(t)[units\ of\ money] : \tag{6.20}$$

it says that, at any t, the holding of a nominal amount of $\tilde{X}(t)$ \$ is equivalent to the holding of $\underline{\tilde{X}}(t)$ units of consumption good.
It is a consequence of the Absence of Arbitrage hypothesis, in particular the Absence of Arbitrage between the real and the nominal market.

So the intertemporal equivalence among nominal and real prices is given by the following relation: if $\tilde{V}(t;\tilde{X}(s))$ is the nominal price today of the nominal amount $\tilde{X}(s)$ of s, and $\underline{\tilde{V}}(t;\underline{\tilde{X}}(s))$ is the real price of the real amount $\underline{\tilde{X}}(s)$ of s, then

$$\frac{\tilde{V}(t;\tilde{X}(s))}{\tilde{p}(t)} = \underline{\tilde{V}}(t;\underline{\tilde{X}}(s)); \tag{6.21}$$

in particular $\frac{\tilde{V}(t;\tilde{p}(s))}{\tilde{p}(t)} = \underline{\tilde{v}}(t;s)$, and because $\tilde{V}(t;\tilde{p}(s))$ is exactly $\tilde{I}(t;s)$ we have

$$\underline{\tilde{v}}(t;s) = \frac{\tilde{I}(t;s)}{\tilde{p}(t)},$$

and in this model, from (**??**), such a quantity coincides with $\tilde{v}(t;s)e^{(\tilde{r}(t)-\tilde{x}(t))(s-t)}$.

Definition. The *real spot rate*, or real instantaneous intensity of interest, is

$$\tilde{x}(t) \doteq lim_{s\to t} - \frac{log\ \underline{\tilde{v}}(t;s)}{s-t}.$$

Definition *of self-financing strategy* into securities eventually paying dividends.
$\tilde{\mathbf{U}} = (\tilde{U}^1, ..., \tilde{U}^k)$ is a SF strategy in k assets $\tilde{V}^1, ..., \tilde{V}^k$ paying dividends at respective rates $\tilde{\beta}^1, ..., \tilde{\beta}^k$ (eventually some $\tilde{\beta}^i \equiv 0$) if:
- $\tilde{\mathbf{U}}$ satisfies the proper integrability conditions and
- $d(\tilde{\mathbf{U}}\tilde{\mathbf{V}})(t) = \tilde{\mathbf{U}}(t)d\tilde{\mathbf{V}}(t) + \tilde{\mathbf{U}}(t)(\tilde{\beta}(t)\circ\tilde{\mathbf{V}}(t))dt$, where $\tilde{\mathbf{U}}\tilde{\mathbf{V}}$ denotes the scalar pro duct between the two vectors $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ and

$$\tilde{\beta}(t)\circ\tilde{\mathbf{V}}(t) \doteq (\tilde{\beta}^1(t)\tilde{V}^1(t), ..., \tilde{\beta}^k(t)\tilde{V}^k(t)).$$

So, if the strategy is (at any t) based only on one asset and it is SF, then its portfolio value $(\tilde{U}\tilde{V})(t) = \tilde{P}(t)$ follows $\frac{d\tilde{P}(t)}{\tilde{P}(t)} = \frac{d\tilde{V}(t)}{\tilde{V}(t)} + \tilde{\beta}(t)dt$.

Remark. The heuristic meaning is the following.
If an agent at t invests $\tilde{U}^i(t)$ \$ in the asset $\tilde{V}^i(t)$, i=1,...,k, he finds at t+dt in his portfolio (before rebalancing) the monetary value $\tilde{\mathbf{U}}(t)[\tilde{\mathbf{V}}(t+dt) + \tilde{\beta}(t)\circ\tilde{\mathbf{V}}(t)dt]$.
He *self-finances* himself if he reinvests all this quantity into the k assets: i.e. if he chooses $\tilde{\mathbf{U}}(t+dt)$ s.t.
$\tilde{\mathbf{U}}(t)(\tilde{\mathbf{V}}(t+dt) + \beta(t)\circ\tilde{\mathbf{V}}(t)dt) = \mathbf{U}(t+dt)\tilde{\mathbf{V}}(t+dt)$, i.e.

$$d(\tilde{\mathbf{U}}\tilde{\mathbf{V}})(t) = \tilde{\mathbf{U}}(t+dt)\tilde{\mathbf{V}}(t+dt) - \tilde{\mathbf{U}}(t)\tilde{\mathbf{V}}(t) =$$
$$= \tilde{\mathbf{U}}(t)[\tilde{\mathbf{V}}(t+dt) + \beta(t)\circ\tilde{\mathbf{V}}(t)dt] - \tilde{\mathbf{U}}(t)\tilde{\mathbf{V}}(t) = \tilde{\mathbf{U}}(t)[d\tilde{\mathbf{V}}(t) + \beta(t)\circ\tilde{\mathbf{V}}(t)dt].$$

Remark. *$\tilde{p}$ can be interpreted as the price process of an asset which pays dividends at instantaneous intensity rate $\tilde{x}(t)$, the real spot rate of interest.*

We heuristically show that there exists a self-financing strategy, into nominal ZCBs, which, starting with the initial amount p_0\$, produces a portfolio having value $\tilde{P}$ described by

$$\frac{d\tilde{P}(t)}{\tilde{P}(t)} = \frac{d\tilde{p}(t)}{\tilde{p}(t)} + \tilde{x}(t)dt:$$

this means that the return on the index $\tilde{p}$, plus an additional amount given at rate $\tilde{x}$, can be obtained as the return of an AF portfolio in some traded assets. Id est: because we can synthesize the whole process $\tilde{p}$ by the use of traded assets, we can interpret it as a traded asset.
The strategy consists on rolling over just maturing ZCBs indexed to the level of prices $\tilde{p}$.
Heuristic proof. First, let us remark that, because in our model
$\underline{\tilde{v}}(t;s) = \tilde{v}(t;s)e^{(\tilde{r}(t)-\tilde{x}(t))(s-t)}$, for each s the process $\underline{\tilde{v}}(t;s)$ has continuous paths, and by the fact that $\tilde{\Sigma}_r(t;t) = 0$, we can say that

$$\frac{d\underline{\tilde{v}}(t;t+dt)}{\underline{\tilde{v}}(t;t+dt)} = \tilde{x}(t)dt.$$

We follow the SF strategy consisting into starting at $t = 0$ with the initial amount p_0\$, and investing at each t all what we have reached untill t into the ZCB indexed to $\tilde{p}$ which matures at the immediately successive date t+dt. If at t we have reached the amount $\tilde{P}(t)$, we hold in our portfolio $\frac{\tilde{P}(t)}{\tilde{I}(t;t+dt)} \doteq \tilde{\gamma}(t)$ ZCBs $\tilde{I}(t;t+dt)$ maturing at t+dt;
our portfolio value at t is $\tilde{P}(t) = \tilde{\gamma}(t)\tilde{I}(t;t+dt)$;
at t+dt we find that our portfolio value is $\tilde{P}(t+dt) = \tilde{\gamma}(t)\cdot\tilde{I}(t+dt;t+dt) = \tilde{\gamma}(t)\tilde{p}(t+dt)$, so

$$\frac{\tilde{P}(t+dt)-\tilde{P}(t)}{\tilde{P}(t)} = \frac{\tilde{\gamma}(t)\tilde{I}(t+dt;t+dt)-\tilde{\gamma}(t)\tilde{I}(t;t+dt)}{\tilde{\gamma}(t)\tilde{I}(t;t+dt)} =$$

$$= \frac{\tilde{I}(t+dt;t+dt)-\tilde{I}(t;t+dt)}{\tilde{I}(t;t+dt)} = \left(\frac{d\tilde{I}(t;s)}{\tilde{I}(t;s)}\right)_{|s=t+dt} =$$

$$= \left(\frac{d(\underline{\tilde{v}}(t;s)\cdot\tilde{p}(t))}{\underline{\tilde{v}}(t;s)\cdot\tilde{p}(t)}\right)_{|s=t+dt} =$$

$$= \left[\frac{d\underline{\tilde{v}}(t;s)}{\underline{\tilde{v}}(t-;s)} + \frac{d\tilde{p}(t)}{\tilde{p}(t-)} + < \frac{d\underline{\tilde{v}}(t;s)}{\underline{\tilde{v}}(t-;s)}, \frac{d\tilde{p}(t)}{\tilde{p}(t-)} > + \frac{\Delta\underline{\tilde{v}}(t;s)}{\underline{\tilde{v}}(t-;s)}\frac{\Delta\tilde{p}(t)}{\tilde{p}(t-)}\right]_{|s=t+dt} =$$

$$= \left[\frac{d\underline{\tilde{v}}(t;s)}{\underline{\tilde{v}}(t;s)} + \frac{d\tilde{p}(t)}{\tilde{p}(t)} + < \frac{d\underline{\tilde{v}}(t;s)}{\underline{\tilde{v}}(t;s)}, \frac{d\tilde{p}(t)}{\tilde{p}(t)} >\right]_{|s=t+dt} =$$

because $\tilde{\Sigma}_r(t;t+dt) = 0$ we have

$$= \tilde{x}(t)dt + \frac{d\tilde{p}(t)}{\tilde{p}(t)}$$

i.e.

$$\frac{d\tilde{P}_t}{\tilde{P}(t)} = x(t)dt + \frac{d\tilde{p}(t)}{\tilde{p}(t)}.$$

□

Remark. In this model we have that $\tilde{x}(t)$ is also given by

$$\tilde{x}(t) = lim_{s\downarrow t} - \frac{\partial log\,\underline{\tilde{v}}(t;s)}{\partial s} = lim_{s\downarrow t} - \frac{1}{\underline{\tilde{v}}(t;s)}\frac{\partial\,\underline{\tilde{v}}(t;s)}{\partial s}.$$

Lemma 6 *If the processes $\tilde{r}$, $\tilde{p}$ follow dynamics*

$$d\tilde{r}(t) = k(\theta - \tilde{r}(t))dt + \sigma_r \sqrt{\tilde{r}(t)} dW^r(t) \tag{6.22}$$

$$\frac{d\tilde{p}(t)}{\tilde{p}(t-)} = \tilde{\alpha}(t)dt + \sigma_p dW^p(t) + \delta\,[dN(t) - \nu dt], \tag{6.23}$$

and we want that the family of processes

$$\{ \left(\frac{E^Q[\tilde{p}(s) \cdot e^{-\int_t^s \tilde{r}(u)du} | F(t)]}{\tilde{p}(t)} \right)_{t \in [0,s]} ; s \geq 0 \}$$

satisfies that

$$lim_{s \downarrow t} - \frac{1}{\frac{E^Q[\tilde{p}(s) \cdot e^{-\int_t^s \tilde{r}(u)du} | F(t)]}{\tilde{p}(t)}} \frac{\partial}{\partial s} \left(\frac{E^Q[\tilde{p}(s) \cdot e^{-\int_t^s \tilde{r}(u)du} | F(t)]}{\tilde{p}(t)} \right) = \tilde{x}(t),$$

then we necessarily have to take for each t

$$\tilde{\alpha}(t) =_{a.s.} \tilde{r}(t) - \tilde{x}(t).$$

Proof. Let us compute the partial derivative

$$\frac{\partial}{\partial s} \left(\frac{E^Q[\tilde{p}(s) \cdot e^{-\int_t^s \tilde{r}(u)du} | F(t)]}{\tilde{p}(t)} \right)$$

$$= \frac{1}{\tilde{p}(t)} \frac{\partial}{\partial s} E_t^Q[\tilde{p}(s) \cdot e^{-\int_t^s \tilde{r}(u)du}] =$$

p has dynamic (**??**): defined $M(s) \doteq e^{-\nu\delta s} \cdot (1+\delta)^{N(s)}$, we have

$$= \frac{1}{\tilde{p}(t)} \frac{\partial}{\partial s} E_t^Q[\tilde{p}(t) \cdot e^{\int_t^s \tilde{\alpha}(u)du - \frac{1}{2}\sigma^2(s-t) + \sigma(W^p(s) - W^p(t))} \cdot \frac{M(s)}{M(t)} \cdot e^{-\int_t^s \tilde{r}(u)du}] =$$

$$= \frac{\partial}{\partial s} E_t^Q[e^{\int_t^s -\tilde{r}(u) + \tilde{\alpha}(u) - \frac{1}{2}\sigma^2 du + \sigma(W^p(s) - W^p(t))} \cdot \frac{M(s)}{M(t)}] =$$

W^p and M are independent of $\tilde{r}$

$$= \frac{\partial}{\partial s} \left(E_t^Q[e^{\int_t^s -\tilde{r}(u) + \tilde{\alpha}(u) - \frac{1}{2}\sigma^2 du}] E_t^Q[e^{\sigma(W^p(s) - W^p(t))}] E_t^Q[\frac{M(s)}{M(t)}] \right) =$$

$$= \frac{\partial}{\partial s} \left(E_t^Q[e^{\int_t^s -\tilde{r}(u) + \tilde{\alpha}(u) - \frac{1}{2}\sigma^2 du}] e^{\frac{1}{2}\sigma^2(s-t)} \right) =$$

$$= \frac{\partial}{\partial s} E_t^Q[e^{\int_t^s -\tilde{r}(u) + \tilde{\alpha}(u)du}]$$

$$E_t^Q[\frac{\partial}{\partial s} e^{\int_t^s -\tilde{r}(u) + \tilde{\alpha}(u)du} =$$

$$= E_t^Q[e^{\int_t^s -\tilde{r}(u)+\tilde{\alpha}(u)du}(-\tilde{r}(s)+\tilde{\alpha}(s))] =$$

coming back

$$= E_t^Q[e^{\int_t^s -\tilde{r}(u)+\tilde{\alpha}(u)-\frac{1}{2}\sigma^2 du}(-\tilde{r}(s)+\tilde{\alpha}(s))\cdot e^{\sigma(W^p(s)-W^p(t)}\cdot\frac{M(s)}{M(t)}]\frac{\tilde{p}(t)}{\tilde{p}(t)} =$$

$$= E_t^Q[e^{\int_t^s -\tilde{r}(u)du}\tilde{p}(s)(-\tilde{r}(s)+\tilde{\alpha}(s))]\frac{1}{\tilde{p}(t)};$$

so:

$$-\frac{1}{\frac{E^Q[\tilde{p}(s)\cdot e^{-\int_t^s \tilde{r}(u)du}|F(t)]}{\tilde{p}(t)}}\frac{\partial}{\partial s}\left(\frac{E^Q[\tilde{p}(s)\cdot e^{-\int_t^s \tilde{r}(u)du}|F(t)]}{\tilde{p}(t)}\right) =$$

$$= -\frac{\tilde{p}(t)}{E_t^Q[e^{\int_t^s -\tilde{r}(u)du}\tilde{p}(s)]}E_t^Q[e^{\int_t^s -\tilde{r}(u)du}\tilde{p}(s)(-\tilde{r}(s)+\tilde{\alpha}(s))]\frac{1}{\tilde{p}(t)}$$

and the limit for $s\downarrow t$ is

$$= lim_{s\downarrow t} - \frac{E_t^Q[e^{\int_t^s -\tilde{r}(u)du}\tilde{p}(s)(-\tilde{r}(s)+\tilde{\alpha}(s))]}{E_t^Q[e^{\int_t^s -\tilde{r}(u)du}\tilde{p}(s)]} =$$

$$= -\frac{\tilde{p}(t)(-\tilde{r}(t)+\tilde{\alpha}(t))}{\tilde{p}(t)} = \tilde{r}(t)-\tilde{\alpha}(t);$$

so if we want that this limit coincides with $\tilde{x}(t)$ we necessarily have to choice $\tilde{\alpha}(t)$ s.t. $\tilde{r}(t)-\tilde{\alpha}(t)=\tilde{x}(t)$ for each t. □

7. Appendix: the European Call Option's price

Here we compute explicitely the t-price

$$\tilde{C}(t;K) = E^Q[(\tilde{p}(T)-K)_+\cdot e^{-\int_t^T \tilde{r}(u)du}|F_t]$$

in our framework.
For simplicity let us show the computations in the case n=1.
The following Lemma is known.

Lemma 7 *Let $\tilde{S}$ be a stochastic process s.t.*

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} =_Q \rho dt + \sigma_p dW(t) \tag{7.24}$$

with ρ, σ_p being constant numbers. Then

$$E^Q[\tilde{S}(T)I_{\{\tilde{S}(T)>K\}}|\tilde{S}(t)\equiv S] = Se^{\rho(T-t)}N[d_1(S,K,t)]$$

and

$$E^Q[KI_{\{\tilde{S}(T)>K\}}|\tilde{S}(t)\equiv S] = KN[d_2(S,K,t)]$$

where

$$d_1(S,K,t) = \left[log(\frac{S}{K})+(\rho+\frac{\sigma_p^2}{2})\tau\right]\frac{1}{\sigma_p\sqrt{\tau}},$$

$$d_2(S,K,t) = \left[log(\frac{S}{K}) + (\rho - \frac{\sigma_p^2}{2})\tau \right] \frac{1}{\sigma_p\sqrt{\tau}}$$

(d_i are functions of all the numbers (S,ρ,σ_p,K,τ), but we point out only the dependence on S, K and t) and N[x] is the cumulative distribution function of the Gaussian law. □

Let us set

$$F_{BS}(S,K,t) = E^Q[(\tilde{S}(T)-K)_+ e^{-\rho\tau} | \tilde{S}(t) \equiv S] =$$
$$= SN[d_1(S,K,t)] - Ke^{-\rho\tau}N[d_2(S,K,t)].$$

This function coincides with the Black-Scholes model price of a European Call Option subscribed on a stock, which does not pay dividends, when its dynamics is as (**??**) under the probability measure Q and when we take ρ as discounting constant rate.
We reach the following

Corollary 8 *Under the assumptions of Section 2 (and with the notations of section 3), the t-price of the Call Option on $\tilde{p}$ with strike price K and maturity T is*

$$\tilde{C}(t;K) = C(t,\tilde{r}(t),\tilde{p}(t),K)$$

where

$$C(t,r,p,K) = v(t,r)e^{\rho\tau} \sum_{m=0}^{+\infty} Q_m \left[F_{BS}(p(1+\delta)^m, K, t)\right], \tag{7.25}$$

$$Q_m = Q[N_T - N_t = m] = e^{-\tau\nu} \cdot \frac{(\tau\nu)^m}{m!},$$

$$\rho = \tilde{r}(t) - \tilde{x}(t) - \nu\delta,\ \tau = T - t,\ v(t,r) = A(\tau)e^{-rF(\tau)}.$$

Proof. Let us compute

$$E^Q[(\tilde{p}(T)-K)_+ \cdot e^{-\int_t^T \tilde{r}(u)du} | F_t] : \tag{7.26}$$

$\tilde{r}(t) - \tilde{x}(t)$ is a constant (with respect to ω and t), so the drift term ρ of $\tilde{p}$ is a constant:
$\tilde{p}(T) = p(0) \cdot e^{(\rho - \frac{1}{2}\sigma_p^2)T + \sigma_p W^p(T)}(1+\delta)^{N(T)}$ is therefore $F_T^{W^p,N}$-measurable; instead $(\tilde{r}(t))_t$ is F^{W^r}-measurable, and $F^{W^p,N}, F^{W^r}$ are independent, so we can factorize (**??**) into:

$$E^Q[(\tilde{p}(T)-K)_+ | F_t^{W^p,N}]v(t,r).$$

Now: $\tilde{p}$ is a Markovian process, so

$$E^Q[(\tilde{p}(T)-K)_+ | F_t^{W^p,N}] = E^Q[(\tilde{p}(T)-K)_+ | \tilde{p}(t) \equiv p]_{|_{p \equiv \tilde{p}(t)}},$$

and

$$E^Q[(\tilde{p}(T)-K)_+ | \tilde{p}(t) \equiv p] =$$
$$= E^Q[(\tilde{p}(t) \cdot e^{(\rho - \frac{1}{2}\sigma_p^2)\tau + \sigma_p(W^p(T) - W^p(t))} \cdot (1+\delta)^{N(T)-N(t)} - K)_+ | \tilde{p}(t) \equiv p] =$$

by the independence of $W^p(T) - W^p(t)$ and $N(T) - N(t)$ of F_t

$$= E^Q[(\tilde{p}(t) \cdot e^{(\rho - \frac{1}{2}\sigma_p^2)\tau + \sigma_p(W^p(T) - W^p(t))} \cdot (1+\delta)^{N(T)-N(t)} - K)_+] =$$

$$= \sum_{m\geq 0} E^Q[(\tilde{p}(t) \cdot e^{(\rho - \frac{1}{2}\sigma_p^2)\tau + \sigma_p(W^p(T) - W^p(t))} \cdot (1+\delta)^{N(T)-N(t)} - K)_+$$

$$|N(T) - N(t) = m] \cdot Q[N(T) - N(t) = m] =$$

$$= \sum_{m\geq 0} Q_m E^Q[(\tilde{p}(t)(1+\delta)^m \cdot e^{(\rho - \frac{1}{2}\sigma_p^2)\tau + \sigma_p(W^p(T) - W^p(t))} - K)_+] \, .$$

Now, fixed m:

$$E^Q[(\tilde{p}(t)(1+\delta)^m \cdot e^{(\rho - \frac{1}{2}\sigma_p^2)\tau + \sigma_p(W^p(T) - W^p(t))} - K)_+] =$$

$$e^{\rho\tau} E^Q[(\tilde{p}(t)(1+\delta)^m \cdot e^{(\rho - \frac{1}{2}\sigma_p^2)\tau + \sigma_p(W^p(T) - W^p(t))} - K)_+ e^{-\rho\tau}] \tag{7.27}$$

and the second factor is exactly the expected value
$E^Q[(\tilde{S}(T) - K)_+ e^{-\rho\tau} | \tilde{S}(t) \equiv S]$ for a process $\tilde{S}$ evolving by

$$\frac{d\tilde{S}}{\tilde{S}} = \rho dt + \sigma_p dW^p,$$

and with $S = p(1+\delta)^m$. Therefore (**??**) is

$$= e^{\rho\tau} F_{BS}(p(1+\delta)^m, K, t).$$

Summing up,

$$E^Q[(\tilde{p}(T) - K)_+ | \tilde{p}(t) \equiv p] = \sum_{m\geq 0} Q_m e^{\rho\tau} F_{BS}(p(1+\delta)^m, K, t),$$

so

$$E^Q[(\tilde{p}(T) - K)_+ \cdot e^{-\int_t^T \tilde{r}(u)du} | F_t] = e^{\rho\tau} \tilde{v}(t;T) \sum_{m\geq 0} Q_m F_{BS}(p(1+\delta)^m, K, t).$$

□

Remark. In Section 3, we have defined

$$G(t,p,K) = \sum_{m\geq 0} Q_m F_{BS}(p(1+\delta)^m, K, t).$$

Actually G is a function only of the variables t, p (and not of r), other than of the parameter K, because ρ is one of the constant parameters of the model, and G depends on ρ but does not depend on the variable r alone.
Therefore the Call price function

$$C(t,r,p,K) \doteq v(t,r) e^{\rho\tau} G(t,p,K)$$

has separated variables.