Option Pricing with Stochastic Volatility Models

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Abstract

A general class of models for derivative pricing with stochastic volatility is analized. We include the possibility of jumps for the paths of the asset's price and for those of its volatility. We also consider the case of correlation between the process of the asset's price and that of its volatility. In this way we are able to give a unifying view on most of the models studied in the literature. We will examine theoretical issues related to the market price of volatility risk, the equivalent martingale measures and the possibility to get a numerically tractable formula for contingent claim pricing. Finally, we propose some methodologies to test the behavior of stochastic volatility models when applied to market data.

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1 Introduction

This paper presents a class of models for pricing option on assets whose volatility is a stochastic process. It contains as special cases several important models including those of Hull and White [13], Heston [12] and Naik [16]. Most stochastic volatility models are modifications of the Black-Scholes [5] setting, where the asset rate of return is given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t; \tag{1.1}$$

here μ is a constant, W is a Brownian motion and the diffusion coefficient σ is called the "volatility." A strong assumption of the Black-Scholes model, often contradicted by statistical evidence and by everyday practice, is that volatility is constant and deterministic through time.

A stochastic volatility model assumes that σ is a process dependent on a risk factor M. The introduction of a second risk factor (which may also be correlated with W) leads to an incomplete market model. Therefore a parameter, usually called the "premium for volatility risk," has to be exogenously given to determine an arbitrage-free price for derivative securities written on S.

To recover an analogous of the Black-Scholes formula for European options, several authors set such a premium equal to zero, giving the economic motivation of a "diversifiable" volatility risk. Hull and White [13] modelled the squared volatility as a log-normal diffusion

$$\frac{d\sigma_t^2}{\sigma_t^2} = pdt + qd\bar{W}_t; \tag{1.2}$$

where $\bar{W}$ is a Brownian motion independent of W and p and q are given constants. They set the premium for volatility risk to zero and determine a formula for the price of a European option for which they propose an approximation based on a power series expansion. Stein and Stein [20] suggested a mean reverting diffusion process

$$d\sigma_t = -\delta(\sigma_t - \theta)dt + kd\bar{W}_t \tag{1.3}$$

and used a Fourier inverse transformation to integrate the resulting partial differential equation for the price of a European option. The assumption of no correlation between W and $\bar{W}$ was relaxed by Heston [12], who was also able to derive closed form solutions for the price of a European option. Heston [12], Ball and Roma [2] and several others proposed a square root diffusion for volatility whose form is similar to that used in a different setting to model the term structure of interest rate, that is

$$d\sigma_t^2 = \delta(\theta - \sigma_t^2)dt + k\sigma_t d\bar{W}_t. \tag{1.4}$$

For an extensive review of the most important contributions to stochastic volatility models see Ball [1].

Black and Scholes's assumption of continuity of stock price paths has also been subject to several criticisms. After Merton's celebrated paper [14], many models have been proposed to include the possibility of jumps. Jeanblanc-Picqué and Pointier [7] proposed a model with two assets following a jump-diffusion process where the jump component is given by a non-homogeneous Poisson process. The introduction of a second asset lets them recover market completeness. Bardhan and Chao [3] extended their analysis to n assets and general point processes.

Naik [16] developed a model with only one asset, whose volatility can jump from one state to another at exponentially distributed random times. When a jump of the volatility occurs, the asset price also jumps. Unlike [7] and [3], Naik's model is not complete. Therefore, the market price for volatility risk must be exogenously given.

In this paper we consider a unique asset which can be either a diffusion or a jump-diffusion process. In this way, besides determining a general no-arbitrage pricing formula, we give a unifying view to most of the models with stochastic volatility proposed in the literature. We investigate the properties of the Equivalent Martingale Measure and examine the processes for the market prices for risks. We determine under which conditions on the models one gets that the no-arbitrage price of European options can be written as an expected value (conditioned on the paths of the stochastic volatility process) of some Black-Scholes prices. In such a case it is only the probability distribution (under an equivalent martingale measure) of the path of the volatility process that matters for pricing derivatives securities. We will show how to determine an approximation of such a distribution from real market prices as a solution of some optimization problems. Moreover, we will test the behavior of this sub-class of models when applied to prices of options traded at the Milan Stock Exchange.

The paper is structured as follows. In Section 2 we define the class of market models, establish necessary and sufficient conditions to get an Equivalent Martingale Measure and give a formal interpretation of the risk price process. We then get the stochastic dynamics of the asset price with respect to an Equivalent Martingale Measure. Section 3 is focused on the problem of pricing a contingent claim. We determine sufficient conditions on the model coefficients for a pricing formula that is an expected value of Black-Scholes prices. In Section 4 we formulate some methods, based on optimization procedures, for fitting a stochastic volatility model to market prices and we apply it to some options written on the indexes MIB30 (the Italian stock index) and S&P500. In Section 5 we draw the conclusions and propose some topics for future research.

2 The class of models

We consider a continuous time economy with a finite time horizon T, and the usual assumptions of a perfect market (e.g. no frictions, taxes, transaction costs or short sales constraints) where S, a risky asset (the "stock"), and B, a riskless asset (the "money market account"), are traded.

We suppose that there are two sources of risk: W, a one-dimensional Brownian Motion, and M, a continous time martingale, independent of W. For most of the models proposed in the literature, W directly affects the stock price, while M drives the changes of the stock's volatility. We call "volatility" the sensitivity of the stock's rate of return to changes in W. It will be indicated by the random process σ. Sometimes it might be desirable to let changes in M directly (i.e. not only indirectly through σ) affect the stock price process. It might be the case, for instance, that M represents a flow of information which influences both the stock price and its volatility at the same time. In fact, it has sometimes been observed that jumps in prices are followed by a period of higher volatility. For this reason we will include the risk factor M in the stochastic differential equation of the stock S, and the sensitiveness of S to changes in M will be given by a process γ, called the "M-coefficient,"

Since we are mostly worried about stochastic volatility, and we are not looking for the greatest generality, we assume that the rate of return of the money market account is a positive real number r.

Here is a formal construction of what we have just stated: let W be a one-dimensional Brownian motion on a probability space $(\Omega^W, \mathcal{F}^W, P^W)$ endowed with the filtration $\{\mathcal{F}_t^W\}_{t\in\mathbb{R}}$ generated by W and completed by the sets of measure zero.

Analogously, we let M be a martingale on a filtered probability space $(\Omega^M, \mathcal{F}^M, P^M)$, where the filtration $\{\mathcal{F}_t^M\}_{t\in\mathbb{R}}$ is generated by M and completed by the sets of measure zero. We will allow two possible choices for the process M: either

$$M_t := \bar{W}_t \tag{2.1}$$

where $\bar{W}$ is a one-dimensional Brownian Motion, or

$$M_t := N_t - \lambda t, \tag{2.2}$$

where N is a Poisson process with intensity λ.

The probability space underlying the class of models is the product space

$$(\Omega, \mathcal{F}, P) := (\Omega^W \times \Omega^M, \mathcal{F}_T^W \bigotimes \mathcal{F}_T^M, P^W \times P^M),$$

endowed with the filtration

$$\{\mathcal{F}_t\}_{0\leq t\leq T} := \{\mathcal{F}_t^M \bigotimes \mathcal{F}_t^W\}_{0\leq t\leq T},$$

where T is some given positive constant (representing the time horizon of our model). The filtration $\mathcal{F}_t$ is generated by a two-dimensional Lèvy-process and it is completed with the set of measure zero; therefore it is right continuous (Protter [17], Theorem 1.4.31). With a minor abuse of notation, we now let W and M denote the natural extensions to the product space of the processes described above. Note that such a construction implies that M and N are independent processes.

The price of the stock at time t is indicated by S_t. We assume that S is a stochastic process whose sample paths are strictly positive between 0 and T and whose instantaneous rate of return is given by

$$\frac{dS_t}{S_{t-}} = \mu_t dt + \sigma_t dW_t + \gamma_t dM_t, \tag{2.3}$$

where

$$S_{t-} := \lim_{h\downarrow 0} S_{t-h}$$

might be different from S_t if M is a Poisson process. The "drift" μ, the "volatility" σ and the "M-coefficient" γ, are predictable processes whose dynamics are specific of any given model within the class. Of course, they have to satisfy some technical conditions to get the (strong) integrability of the stochastic differential equation (2.3)

$$\int_0^T |\mu_u| du < \infty \tag{2.4}$$

$$E^P \int_0^T \sigma_u^2 du \quad < \quad \infty \tag{2.5}$$

$$E^P \int_0^T \gamma_u^2 d[M, M]_u \quad < \quad \infty, \tag{2.6}$$

where $[M, M]$ is the quadratic variation process of M (see [17], II.6).

When M is a compensated Poisson process, a further condition has to be imposed on γ for the paths of S to remain strictly positive:

$$\gamma_t > -1, \quad 0 \leq t \leq T \quad \text{(if (2.2) holds)}. \tag{2.7}$$

An intuitive explanation of this condition is that the instantaneous rate of return of S should not fall below -1; the formal reason will be clear when the explicit formula for the stock price process will be computed (formula 2.10 below).

We remark that we have not made any assumption on the measurability of the processes σ and γ with respect to M.

The stock price S_0 at time $t = 0$ and the money market account

$$B_t := e^{rt}, \quad 0 \leq t \leq T,$$

complete the setting of the class of models.

Such a class is sufficiently general to include many models proposed by the literature as, for instance, those of Black and Scholes [5], Hull and White [13], Heston [12] (after an orthogonalization of the Brownian processes), Naik [16], as well as the "simple model" proposed by Herzel in [10]. The next result shows that it is also sufficiently tractable from a mathematical standpoint.

Proposition 2.1 *The process*

$$X_t := \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \gamma_s dM_s \tag{2.8}$$

is a semimartingale, and the stock price process S is the stochastic (Dolèans-Dade) exponential of X

$$S_t = S_0 \mathcal{E}(X)_t \tag{2.9}$$

Proof. From assumptions (2.5) and (2.6) it follows that the process

$$\int_0 \sigma_s dW_s + \int_0 \gamma_s dM_s$$

is a (square integrable) martingale. Hence the process X defined in (2.8) is the sum of a martingale and of the finite variation process $\int_0 \mu_s ds$; therefore ([17], Theorem III.14) it is a semimartingale.

Let us define $Y_t := S_t/S_0$: then, obviously, $Y_0 = 1$ and

$$dY_t = dS_t/S_0 = (S_t/S_0)(dS_t/S_t) = Y_t dX_t.$$

Hence,

$$Y_t = 1 + \int_0^t dY_s = 1 + \int_0^t Y_s dX_s,$$

and the result then follows from [17], Theorem II.36 . ♠

For future reference we write the explicit form of the Dolèans-Dade exponential

$$\mathcal{E}(X)_t := \exp\left(X_t - \frac{1}{2}[X,X]_t\right) \prod_{0<s\leq t} (1 + \Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right) \tag{2.10}$$

where

$$\Delta X_s := X_s - X_{s-}.$$

is the jump of X at time s. From (2.10) it follows that $S_t > 0$ if and only if $\Delta X_t > -1$. Since ΔX_t is either equal to zero or equal to γ_t, condition (2.7) is necessary and sufficient for the strict positiveness of the price process S.

The following definition plays a key role in the no arbitrage pricing theory of derivative securities:

Definition 2.1 *A probability measure Q is a (Local) Equivalent Martingale Measure if and only if Q is equivalent to P and the discounted stock price process S/B is a (local) martingale with respect to Q.*

Equivalent Martingale Measures have been introduced in the context of financial market models by Harrison and Kreps [8]. They are a fundamental tool to examine some features of the market model such as "viability" (i.e. absence of arbitrage opportunities) and "completeness" (i.e. replication of any contingent claim). For a study of these properties it is necessary to give a precise definition of the sets of the "trading strategies" as well as that of "contingent claims." This is a rather subtle issue and we are not going to get into it. As for the distinction between "Local" Martingale Measures and martingale Measures, we observe that most of the models proposed by the literature assume that σ and γ are bounded processes. Hence, if the discounted process is a local martingale, it is also a square integrable martingale ([17], Corollary 3, page 66). This is the reason why in the vast majority of the cases one does not have to worry about such an issue. However, the distinction is important because some pathologies may arise when the discounted price process is a strict local martingale. Sin [19] is an excellent reference for this issue and also gives examples of problems that may arise when the Martingale Measure is indeed a strictly "Local" one. Henceforth we will not pursue such a distinction and we will only use the expression "Equivalent Martingale Measure".

Now we examine necessary and sufficient conditions for the existence of an equivalent martingale measure Q. Let Q and P be two equivalent measures. Then the Radon-Nikodym derivative

$$Z_T := \frac{dQ}{dP}$$

(as well as its inverse) is in $L^1(dP)$. We define the process Z as

$$Z_t := E^P(Z_T|\mathcal{F}_t), \quad 0 \le t \le T,$$

taking the right continuous version (which exists because $\mathcal{F}_t$ satisfies the "usual hypotheses" and Z is a martingale. See Protter [17], Corollary 1.2.1). Z is a uniformly integrable P-martingale. From a version of the Martingale Representation Theorem, (see [4], Remark 3.2),

$$Z_t = 1 + \int_0^t \phi_s dW_s + \int_0^t \psi_s dM_s$$

where ϕ and ψ are predictable processes such that

$$\begin{aligned} E^P \int_0^{\tau_n} \phi_u^2 du &< \infty \\ E^P \int_0^{\tau_n} \psi_u^2 d[M,M]_u &< \infty, \end{aligned}$$

where $\{\tau_n\}_{n=0}^{\infty}$ are a sequence of stopping times converging to T. Since $P(Z_t > 0, t \in [0,T]) = 1$ it is not hard to show that the processes

$$\Phi_t := \frac{\phi_t}{Z_{t-}}$$

$$\Psi_t := \frac{\psi_t}{Z_{t-}}$$

are, respectively, W- and M- integrable. Hence Z is the solution of

$$\begin{aligned} \frac{dZ_t}{Z_{t-}} &= \Phi_t dW_t + \Psi_t dM_t \\ Z_0 &= 1, \end{aligned} \tag{2.11}$$

where Φ_t and Ψ_t are predictable processes such that

$$E^P \int_0^{\tau_n} \Phi_u^2 du < \infty \tag{2.12}$$

$$E^P \int_0^{\tau_n} \Psi_u^2 d[M,M]_u < \infty. \tag{2.13}$$

Moreover

$$\Psi_t > -1, \quad 0 \leq t \leq T, \quad \text{(if (2.2) holds)} \tag{2.14}$$

since Z is strictly positive (a.s.).

Therefore, an equivalent measure is determined once the processes Φ and Ψ have been chosen. The next result establishes a necessary and sufficient condition on Φ and Ψ for Q to be an Equivalent Martingale Measure.

Proposition 2.2 *The probability measure Q is an Equivalent Martingale Measure if and only if, when M is a Brownian Motion*

$$\mu_t - r + \sigma_t \Phi_t + \gamma_t \Psi_t = 0, \tag{2.15}$$

or, when M is a compensated Poisson process,

$$\mu_t - r + \sigma_t \Phi_t + \gamma_t \lambda \Psi_t = 0. \tag{2.16}$$

Proof. We will use the fact that, almost by definition of the process Z (see the Lemma at page 109 of [17]), the discounted process $\bar{S} := S/B$ is a Q-local martingale if and only $Z\bar{S}$ is a P-local martingale. Its stochastic differential is easily obtained using the "integration by parts" formula ([17], Corollary 2, pg. 60):

$$\begin{aligned} d(S/B)_t &= S_{t-} d(1/B)_t + (1/B_t) dS_t + d[1/B, S]_t \\ &= -(S_{t-}/B_t) r dt + (S_{t-}/B_t)(\mu_t dt + \sigma_t dW_t + \gamma_t dM_t), \end{aligned}$$

where we have used the fact that the quadratic covaration between a differentiable function (it is actually enough to be a process with continuous path with finite variation) and any stochastic process is constant.

We are now ready to compute the differential of $Z\bar{S}$. Still from the integration by parts formula:

$$d(Z\bar{S})_t \quad = \quad Z_{t-}d\bar{S}_t + \bar{S}_{t-}dZ_t + d[Z,\bar{S}]_t$$

where

$$d[Z,\bar{S}]_t = Z_{t-}\bar{S}_{t-}\sigma_t\Phi_t d[W,W]_t + S_{t-}\gamma_t\Psi_t Z_{t-}d[M,M]_t.$$

Hence,

$$d(Z\bar{S})_t \quad = \quad \bar{S}_{t-}dZ_t + Z_{t-}\bar{S}_{t-}\left(\mu_t - r + \sigma_t\Phi_t\right)dt + Z_{t-}\bar{S}_{t-}\gamma_t\Psi_t d[M,M]_t.$$

Now we will separately examine the two possible choices for M.

When M is a compensated Poisson process, $d[M,M]_t = dN_t$. By substitution (and some easy algebra) we get

$$Z_{t-}\bar{S}_{t-}\left(\mu_t - r + \sigma_t\Phi_t + \gamma_t\lambda\Psi_t\right)dt = d(Z\bar{S})_t - Z_{t-}\bar{S}_{t-}\gamma_t\Psi_t(dN_t - \lambda dt).$$

Therefore, $Z\bar{S}$ is a P-local martingale if and only if the process

$$Y := \int_0 Z_{t-}\bar{S}_{t-}\left(\mu_t - r + \sigma_t\Phi_t + \lambda\Psi_t\right)dt$$

is a P-local martingale; moreover, since Y is a finite variation process (it is pathwise differentiable), it is zero a.s. ([17], Theorem II.27, Corollary 1). Since $Z_{t-}\bar{S}_{t-}$ is strictly positive, (2.16) follows.

When M is a Brownian Motion $d[M,M]_t = dt$ and, from a similar argument, it follows that $Z\bar{S}$ is a P-local martingale if and only if (2.15) holds. ♠

The difference between the expected rate of returns on the risky and on the riskless asset is called the "risk premium," and the ratio of the "risk premium" and the "risk coefficient" is often referred to as the "market price of risk."

Formulas (2.15) or (2.16) define a relation between the processes Φ_t and Ψ_t and the "market prices for risks." In our case, in fact, there are two sources of risk, represented by W and M, with their risk coefficients given by, respectively, σ_t and γ_t.

When the stock price process is only subject to the first risk (i.e. γ is zero) and σ is strictly positive (a.s.), from (2.15) or (2.16) it follows that

$$-\Phi_t = \frac{\mu_t - r}{\sigma_t};$$

that is, $-\Phi$ is the market price for the W-risk. In such a case, which is common to several existing models, Ψ is then fixed exogenously (sometimes, a bit arbitrarily). For instance Hull and White [13] assume a "diversifiable" volatility risk (that is, the market pays no premium for taking the volatility risk) and hence set Ψ always equal to zero. Naik [16] sets $\Psi_t = f(\sigma_{t-})$, with $f(\cdot)$ a continuous real function. Also, when $\gamma = 0$, the Minimal Martingale Measure of Föllmer-Schweizer [6] always sets Ψ equal to zero. In the next section we will show why such an assumption considerably simplifies the numerical evaluation of contingent claim's prices.

The next two propositions determine the stochastic dynamics of the stock price process with respect to an Equivalent Martingale Measure.

Proposition 2.3 *Let Q be an Equivalent Martingale Measure and let M be the P-Brownian motion $\bar{W}$. Then*

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t^Q + \gamma_t d\bar{W}_t^Q \tag{2.17}$$

where

$$\begin{aligned} W^Q &= W - \int_0 \Phi_s ds \\ \bar{W}^Q &= \bar{W} - \int_0 \Psi_s ds \end{aligned}$$

are Q-independent Brownian motions.

Proof. The Girsanov-Meyer Theorem ([17], Theorem III.20) states that the process

$$L_t := W_t - \int_0^t \frac{1}{Z_s} d[Z, W]_s$$

is a Q-local martingale (note that in this case Z is a continuous path process, therefore $Z_{t-} = Z_t$).

Since

$$d[Z, W]_s = Z_s \Phi_s ds,$$

it follows that $L_t = W^Q$. In the same way we see that $\bar{W}^Q$ is also a Q local martingale.

To show that the process $(W, \bar{W})$ is a two dimensional standard Brownian motion we use Lévy Theorem ([17], II.39). First we observe that W_t^Q and $\bar{W}_t^Q$ have continuous paths. Then we compute their quadratic variation:

$$[W^Q, W^Q]_t = [\bar{W}^Q, \bar{W}^Q]_t = t,$$

and their quadratic covariation:

$$\begin{aligned} [W^Q, \bar{W}^Q] &= [W, \bar{W}] & (2.18) \\ &= 0 & (2.19) \end{aligned}$$

where (2.18) holds because $\int_0 \Phi_s ds$ and $\int_0 \Psi_s ds$ are finite variation processes and (2.19) follows from the independence of W and $\bar{W}$ under P (and the fact that the quadratic covariation process is the same under two equivalent measures). Therefore $(W^Q, \bar{W}^Q)$ is a standard two-dimensional Q-Brownian motion and, in particular, W^Q and $\bar{W}^Q$ are independent.

Finally, we note that equation (2.17) is obtained from Proposition 2.2 and a straightforward algebraic manipulation of (2.3). ♠

An analogous result can be proven when M is a compensated Poisson process:

Proposition 2.4 *Let Q be an Equivalent Martingale Measure and let M be the P-martingale $N_t - \lambda t$. Then*

$$\frac{dS_t}{S_{t-}} = rdt + \sigma_t dW_t^Q + \gamma_t dM_t^Q \tag{2.20}$$

where

$$W_t^Q = W_t - \int_0^t \Phi_s ds$$

is a Q-Brownian motion and

$$M_t^Q = N_t - \int_0^t \lambda(1 + \Psi_s) ds$$

is a Q- local martingale, with

$$[W^Q, M^Q] = 0. \tag{2.21}$$

Proof. The structure of this proof is the same as that of the previous proposition.

From the Girsanov-Meyer Theorem ([17], Theorem III.20) it follows that

$$L_t \quad := \quad M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s$$

is a Q-local martingale. Now observe that

$$\int_0^t \frac{1}{Z_s} d[Z, M]_s = \int_0^t \frac{Z_{s-}}{Z_s} \Psi_s dN_s.$$

The following relation holds at any time s when the Poisson process jumps:

$$Z_s - Z_{s-} = Z_{s-} \Psi_s,$$

that is

$$\frac{Z_{s-}}{Z_s}(1 + \Psi_s) = 1;$$

therefore, for any t,

$$\int_0^t \frac{Z_{s-}}{Z_s} \Psi_s dN_s = \int_0^t \frac{\Psi_s}{1 + \Psi_s} dN_s.$$

By substitution we get

$$L_t \quad = \quad N_t - \lambda t - \int_0^t \frac{\Psi_s}{1 + \Psi_s} dN_s$$

From the integrability condition (2.6) it follows that

$$\begin{aligned} M_t^Q \quad &:= \quad \int_0^t (1 + \Psi_s) dL_s \\ &= \quad N_t - \int_0^t \lambda(1 + \Psi_s) ds \end{aligned}$$

is also a Q-local martingale.

♠

Remark 2.1 *Note that in the Poisson case, under our general assumptions, it is only possible to prove that M^Q is a Q-local martingale. It is a square integrable martingale when the process of the change of measure Z satisfies some further conditions.*

For example, if we have that

$$E^Q \int_0^T \Psi_s ds < \infty,$$

then computing the expectation of the quadratic variation at time T, we obtain

$$\begin{aligned} E^Q[M^Q, M^Q]_T &= E^Q N_T \\ &= E^Q \int_0^T \lambda(1+\Psi_s)ds \\ &= \lambda T + \lambda E^Q \int_0^T \Psi_s ds \end{aligned}$$

and from Corollary 3 page 66 of [17] it then follows that M^Q is a square integrable martingale.

We can now give an explicit representation of the price process S. In fact, when $M = \bar{W}$ Proposition 2.3 implies that S is given by

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\int_0^t \sigma_s^2 ds - \frac{1}{2}\int_0^t \gamma_s^2 ds + \int_0^t \sigma_s dW_s^Q + \int_0^t \gamma_s d\bar{W}_s^Q \right). \tag{2.22}$$

When $M_t = N_t - \lambda t$, Proposition 2.4 gives

$$\begin{aligned} S_t &= S_0 \prod_{0\le s\le t} (1+\gamma_s)\Delta N_s \exp\left(rt - \frac{1}{2}\int_0^t \sigma_s^2 ds \right. \\ &- \left. \int_0^t \gamma_s \lambda[1+\Psi_s]ds + \int_0^t \sigma_s dW_s^Q \right). \end{aligned} \tag{2.23}$$

We will now apply these results to the problem of contingent claim pricing.

3 Pricing a contingent claim

In this section we examine the problem of no-arbitrage pricing of a derivative security X, written on S, expiring at $T > 0$, and whose payoff at time T is equal to $X(S_T)$. We assume that $X(\cdot)$ is a real, positive, measurable function and that $X(S_T)$ has a finite expectation.

Whether the contingent claim's payoff $X(S_T)$ can be replicated or not with a portfolio strategy involving just the stock and the money market account, depends not only on the specification of the particular model in our class but also on the definition of the set of feasible "trading strategies." Indeed, from Proposition 2.2, it follows that it is easy to construct a particular model within the class such that the Equivalent Martingale Measure is not unique; this implies that the market model is not complete. In any case, the the "Q-price" (at time $t = 0$) of X

$$X_0^Q := E^Q e^{-rT} X(S_T),$$

where E^Q is the expectation with respect to Q, an Equivalent Martingale Measure, will still be useful, both from a theoretical and from a practical point of view. In fact, under any meaningful definition of feasible self-financing replicating strategy the Q-price is a price that prevents arbitrage opportunities.

We want to determine under which conditions on the model the Q-price takes on a particularly convenient form. From Propositions 2.3 and 2.4,

$$S = S_0 \mathcal{E}\left(\int_0 r du + \int_0 \sigma_u dW_u^Q + \int_0 \gamma_u dM_u^Q\right);$$

therefore, from [17], Theorem II.37 and $[W^Q, M^Q] = 0$:

$$S_T = S_0 S_T^W S_T^M \tag{3.1}$$

where

$$\begin{aligned} S_T^W &= \mathcal{E}(\int_0 r du + \int_0 \sigma_u dW_u^Q)_T \\ S_T^M &= \mathcal{E}(\int_0 \gamma_u dM_u^Q)_T. \end{aligned}$$

Let $\mathcal{F}_T^{M^Q}$ be the σ-algebra generated by the paths of the process M^Q up to time T, that is

$$\mathcal{F}_T^{M^Q} := \sigma(M_u^Q, 0 \leq u \leq T).$$

When M^Q and W^Q are Q-independent and the the paths of σ and γ up to time T are measurable with respect to $\mathcal{F}_T^{M^Q}$, the distribution of S_T conditioned to $\mathcal{F}_T^{M^Q}$ is log-normal. In particular, it is the same distribution as in the Black-Scholes model where

$$\tilde{S}_0 := S_0 S_T^M$$

is the price at time 0 and

$$\tilde{\sigma} := \left[\frac{1}{T}\int_0^T \sigma_s^2 ds\right]^{\frac{1}{2}}$$

is the volatility. Therefore, in such a case, it is possible to write the Q-price as an expected value of "Black-Scholes" prices. In the following we will make such a statement more precise: to do that, we need to introduce some notation.

Let $g(\cdot)$ be a measurable function. We denote by $BS(g(\cdot), s, v, r, m)$ the no-arbitrage price assigned by the Black-Scholes model to a contingent claim whose payoff function is $g(\cdot)$, assuming that: s is the actual price of the underlying asset; v is the asset's volatility; r is the risk-free interest rate; m is the maturity date. That is

$$BS(g(\cdot), s, v, r, m) := e^{-rm} E g\left(s \exp\left(rm - \frac{1}{2}v^2 m + v\sqrt{m}\xi\right)\right)$$

where ξ is a standard normal random variable.

Proposition 3.1 *Let $M = \bar{W}$ and let σ and γ be adapted with respect to the filtration $\{\mathcal{F}_t^{M^Q}\}_{0\le t\le T}$; then*

$$X_0^Q = E^Q BS(X(\cdot), \tilde{S}_0, \tilde{\sigma}, r, T), \tag{3.2}$$

where

$$\begin{aligned} \tilde{S}_0 &:= S_0 \exp\left(-\frac{1}{2}\int_0^T \gamma_s^2 ds + \int_0^T \gamma_s d\bar{W}_s^Q\right) & (3.3) \\ \tilde{\sigma} &:= \left[\frac{1}{T}\int_0^T \sigma_s^2 ds\right]^{\frac{1}{2}}. & (3.4) \end{aligned}$$

Proof. By definition of conditional expectation,

$$\begin{aligned} X_0^Q &= E^Q e^{-rT} X(S_T) \\ &= E^Q E^Q \left[e^{-rT} X(S_T) \middle| \mathcal{F}_T^{M^Q}\right] \end{aligned}$$

From Proposition 2.3 the processes W^Q and M^Q are Q-independent, and W^Q is a Q-Brownian motion; since σ and γ are $\mathcal{F}_t^{M^Q}$-adapted, the distribution of S^T given $\mathcal{F}_T^{M^Q}$ is the same as that of an asset in a Black-Scholes economy when the initial price is $\tilde{S}_0$ and the volatility is $\tilde{\sigma}$. ♠

Note that the previous result can also be stated by modifying the interest rate r instead of the stock price S. In fact,

$$X_0^Q = E^Q BS(X(\cdot), S_0, \tilde{\sigma}, \tilde{r}, T),$$

where $\tilde{\sigma}$ is given by (3.4) and

$$\tilde{r} = r + \frac{1}{T}\left(-\frac{1}{2}\int_0^T \gamma_s^2 ds + \int_0^T \gamma_s d\bar{W}_s^Q\right).$$

The analogous result for the compensated Poisson process requires one more condition on independence between M^Q and W^Q.

Proposition 3.2 *Let M^Q be a Q- local martingale that is Q - independent of W^Q; let σ and γ be $\{\mathcal{F}_t^{M^Q}\}_{0\leq t\leq T}$-adapted; then*

$$X_0^Q = E^Q BS(X(\cdot), \tilde{S}_0, \tilde{\sigma}, r, T) \tag{3.5}$$

where

$$\tilde{S}_0 = S_0 \prod_{0\leq s\leq T}(1+\gamma_s \Delta N_s)\exp\left(-\int_0^T \gamma_s \lambda(1+\Psi_s)ds\right) \tag{3.6}$$

$$\tilde{\sigma} = \left[\frac{1}{T}\int_0^T \sigma_s^2 ds\right]^{\frac{1}{2}} \tag{3.7}$$

Proof. It follows from (2.23) and the same argument as in Proposition 3.1. ♠

In this case, too, we can state the result by leaving the initial stock price fixed while changing the risk-free spot rate, that is

$$X_0^Q = E^Q BS(X(\cdot), S_0, \tilde{\sigma}, \tilde{r}, T),$$

where $\tilde{\sigma}$ is as in (3.7) and

$$\tilde{r} = r - \frac{1}{T}\int_0^T \gamma_s \lambda(1+\Psi_s)ds + \frac{1}{T}\log \sum_{0\leq s\leq T}(1+\gamma_s \Delta N_s).$$

Assumptions on the Q-independence of W^Q and M^Q might not be very useful in practice. In fact, since time series are observed under the original probability measure P (the *phyisical* measure), it might be hard to check independence under the Equivalent Martingale Measure. The next result might be helpful in those cases: it shows that, when M is a general martingale (not just a Brownian Motion), Q-independence is related to the risk premiums. Some assumptions on the measurability of the risk premium will get the Q independence between the risk-factor processes.

Proposition 3.3 *Let Φ be adapted to the filtration generated by W and let Ψ be adapted to the filtration generated by M; then W^Q and M^Q are Q-independent.*

Proof. Let $s \leq t$ and let A and B be two measurable sets in $\mathbb{R}$; then

$$\begin{aligned} Q(W_s^Q \in A, M_t^Q \in B) &= E^P Z_t \mathbf{1}_{W_s^Q \in A} \mathbf{1}_{M_t^Q \in B} \\ &= E^P Z_t^\Phi \mathbf{1}_{W_s^Q \in A} Z_t^\Psi \mathbf{1}_{M_t^Q \in B} \end{aligned}$$

where $Z^\Phi := \mathcal{E}(\Phi W)$ and $Z^\Psi := \mathcal{E}(\Psi M)$. From the independence of W and M we get

$$\begin{aligned} Q(W_s^Q \in A, M_t^Q \in B) &= E^P \left(Z_t^\Phi \mathbf{1}_{W_s^Q \in A} \right) E^P \left(Z_t^\Psi \mathbf{1}_{M_t^Q \in B} \right) \\ &= Q(W_s^Q \in A) Q(M_t^Q \in B). \end{aligned}$$

The result then follows. ♠

Proposition 3.2 gives sufficient, but not necessary condition for independence. It is not too hard to construct examples where there is independence but the hypothesis of the Proposition are not satisfied. For an example, see the model proposed by Herzel [10].

4 Testing a set of stochastic volatility models

In this section we propose some procedures to test the behavior of some of the most widely used stochastic volatility models when applied to observed market prices. We have to emphasize that more than looking for general results, which would require a much deeper statistical analysis, here we want to suggest some numerical approaches and show examples of possible applications, with the goal of getting a feeling of the possible pros and cons to the use of such models.

We will consider a particular specification of the class of model described so far, where the Q-price of a derivative X is given by

$$X_0^Q = E^Q BS(X(\cdot), S_0, \tilde{\sigma}, r, T) \tag{4.1}$$

with

$$\tilde{\sigma} \quad = \quad \left[\frac{1}{T} \int_0^T \sigma_s^2 ds \right]^{\frac{1}{2}} . \tag{4.2}$$

Since the pricing formula (4.1) is a generalization of that obtained by Hull and White [13], therefore we will call such models "HW-like." From Propositions 3.1 and 3.2 it follows that a stochastic volatility model is an "HW-like" model if γ is zero, σ is adapted to the filtration generated by M^Q and the risk processes M^Q and W^Q are Q-independent. This is equivalent to say that the process of the volatility is independent (under the equivalent martingale measure Q) of the Brownian motion driving the dynamics of the stock. Many of the models proposed by the literature share such property; however, some instances of Heston's and Naik's model fail to satisfy the condition on γ.

Suppose that, at time $t = 0$, N derivatives with prices $P_1, P_2, \ldots, P_N$, written on S and with the same maturity T are traded. We want to determine the "HW-like" model which gives the best fitting to such prices. The first question to be answered should be on the possibility of getting a perfect fit of market prices. It is clear that, since "HW-like" model are arbitrage-free, they can only fit arbitrage-free prices. In particular, if the market prices allow *static arbitrage opportunities*, i.e. an arbitrage that can be obtained through a static portfolio strategy (like, for instance a Put-Call parity violation), such prices can not be obtained by any arbitrage-free model. For further discussions on this point and for a general method to determine whether a set of prices allows static arbitrage opportunities, see Herzel [11].

As for the thoretical fitting capabilities of HW-like models, we should remind an important result, due to Renault and Touzi [18], stating that HW-like models produce a symmetric smile for the implied volatility curve of European option prices. Therefore, when the implied volatility of the observed market prices as a function of the strike prices is not convex and symmetric, it is not possible to determine any HW-like model fitting them. Indeed, the result by Renault and Touzi shows that a perfect fitting with an HW-like model is rather hard to be attained, since it is quite unusual that a perfectly symmetric U-shaped curve connects all the observed implied volatility points. For this reason, we will assume that prices are observed with an error and we will fit the stochastic volatility model that minimizes such error.

Let us define the random variable v as

$$v(\omega) := \left(\frac{1}{T} \int_0^T \sigma_s^2(\omega) ds \right)^{1/2},$$

and let $F_v(x)$ be its distribution function. Let X be a traded security, written on S and let $J_X(v)$ be its Black-Scholes price at time 0 when the volatility is v. From the theory developed in the previous sections it follows that the Q price assigned to X by an HW-like model is

$$X_0^Q = \int_0^\infty J_X(v) dF_v.$$

We would like to determine a distribution function $F_v(x)$ such that

$$\int_0^\infty J_i(v) dF_v = P_i, \quad i = 1, \ldots, N$$

where J_i is the Black-Scholes price of security i, when the volatility is v. Approximating the integrals with a finite sum and introducing the errors ϵ_i, we get

$$\sum_{k=1}^{M} J_i(v_k) p(v_k) = P_i + \epsilon_i, \quad i = 1, \ldots, N,$$

where $p(v_k)$ is a discrete approximation approximation of the probability density function of v.

Hence, the optimal HW-like model (in the sense of least squares) is determined by solving the following constrained least square problem:

$$\begin{aligned} \min_{\mathbf{x}} \|J\mathbf{x} - \mathbf{P}\|_2 \\ \sum_{i=1}^{M} x_i = 1 \\ \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{4.3}$$

where $\mathbf{x}$ is an M-dimensional array representing the probability function of the random variable v, J is an M by N matrix whose entries are

$$J_{ik} := J_i(v_k),$$

and $\mathbf{P}$ is the N-vector of market prices.

As a first test of the capability of HW-like models to explain market data, we find the best fit to some options traded on the Milan Stock Exchange on March 19, 1999. The options were both Put and Call, written on the index Mib30 and expiring on April 1999. The data have been provided by Reuter.

Let us start by examining the closing prices. It is apparent from the top part of Figure 1 that the reported prices have some inconsistency, in fact there are several convexity violations, both among the put and the call prices. This implies that there are static arbitrage opportunities. Therefore, no perfect fit of the data with a no-arbitrage model has to be expected.

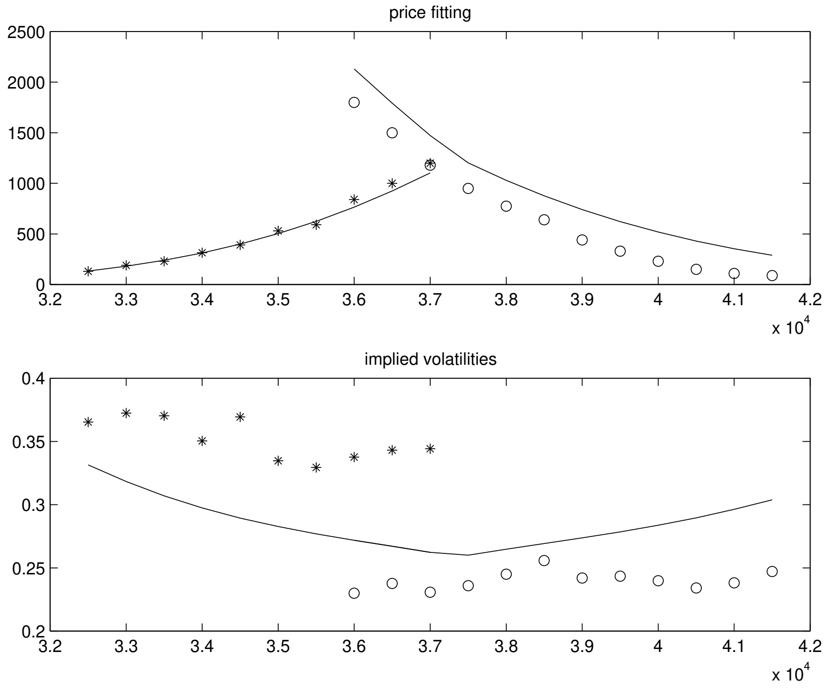

Figure 1: Price fitting of the best HW-like model (in the sense of least square) for option closing prices on MIB 30 (top) and corresponding implied volatilities (bottom). March 19, 1999. Milan Stock Exchange. Source: Reuter.

The numerical solution of the least square problem (4.3) was performed with an optimization routine written in MATLAB[1]. The model prices are also shown in Figure 1: it is interesting to observe that the model constantly underestimates call prices and overestimates put prices. From the bottom part of the same figure, we note that the shape of implied volatilities (computed with underlying price $S_0 = 37220$, spot interest rate $r = \log(1 + 0.05)$, maturity $T = 1/12$) is certainly not U-shaped: it seems that the optimal model produces the best approximating implied volatility curve. The relative error in pricing is

$$\frac{\|J\mathbf{x}^* - \mathbf{P}\|_2}{\|\mathbf{P}\|_2} = 0.1826,$$

where $\mathbf{x}^*$ is the optimal probability density function. A graph of the optimal density function $\mathbf{x}^*$ is reported in Figure 4: we note that the optimal model is rather close to a constant volatility model (i.e. a Black-Scholes model), with a value for the volatility parameter close to the mean of the Black-Scholes implied volatilities.

Let us now consider a more realistic situation and examine actual bid-ask prices. It is well known that, to prevent arbitrage opportunities, bid and ask prices have to be separated by some *martingale prices*, that is by a set of model prices produced by an equivalent martingale measure.

In this context we will formulate a numerical approach to investigate the existence of an HW-like model that separates market prices. If such a model does not exist we will determine one by stretching market prices, moving the ask prices up by adding a positive vector ϵ^a and the bid prices down by subtracting a positive vector ϵ^b . Of course, our optimization procedure will try to minimize the error vectors ϵ^a and ϵ^b. We will solve the following Linear Programming problem:

$$\begin{aligned} \min_{\mathbf{x},\epsilon^a,\epsilon^b} \sum_{i=1}^{N} \left(\epsilon_i^a + \epsilon_i^b\right) \\ \mathbf{P}^b - \epsilon^b \le J\mathbf{x} \le \mathbf{P}^a + \epsilon^a \\ \sum_{i=1}^{M} x_i = 1 \\ \mathbf{x}, \epsilon^a, \epsilon^b \ge 0, \end{aligned} \tag{4.4}$$

[1] This and any other code used to get the results showed in this paragraph can be obtained by the author upon request.

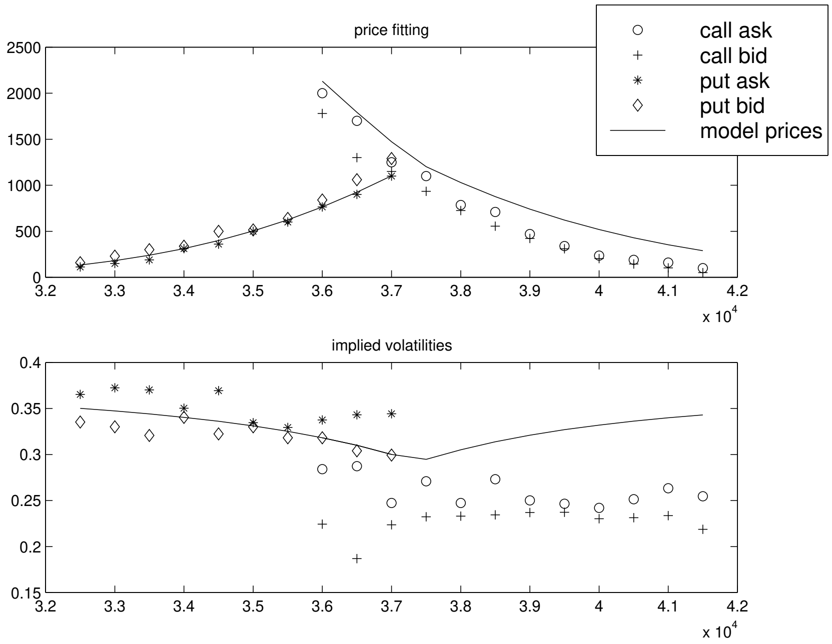

Figure 2: Price fitting of the best HW-like model (in the sense of the linear programming 4.4) for bid and ask prices on options on MIB 30 (top) and corresponding implied volatilities (bottom). March 19, 1999, 16:34. Milan Stock Exchange. Source: Reuter.

where the notation is the same as in (4.3), with superscript a and b to indicate ask and bid quantities.

The result of the optimization procedure is exhibited in Figure 2. It is worth remarking that, at the optimal solution, only the ask prices for call options are moved upward, while all other prices are left unchanged. In particular, the optimal HW-like model separates the bid and ask prices of the Put options without the need of any stretching. The explanation of such a behavior should be found, again, in the particular shape of the implied volatilities: in fact, the implied volatilities of the call options are much lower than those of the puts, and the optimal HW-model has to sit somewhere between them.

The average amount of upward shifting for call ask prices is ITL 201.48, which may be acceptable when referred to very high price options, but it is rather high for far out of the money options. Of course, the fact that it is necessary to raise the call ask prices, does not mean that the original prices are not arbitrage-free. In fact, by using the linear programming approach explained in [11] we have checked that this set of prices is in fact separable, that is, they are arbitrage-free. However, the solution of (4.4) shows that there does not exist any model from the HW-like class which separates them.

The optimal solution of the LP problem (4.4) also gives the probability density function of the random variable v, shown in Figure 4. In this case, too, it seems that the best stochastic volatility model is one with an (almost) constant volatility!

As a further test of fitting capabilities on a more mature option market, we have performed the same procedure on prices of options on the S&P500 index, expiring on September 1999 , traded at CBOE at 11:26 of August 3, 1999, data provided by Bloomberg. After having checked that also in such a case the prices are arbitrage-free, we have applyied the linear programming (4.4). The result is shown in Figure 3. In this case, too, no HW-like separating model exists. The fact that the implied volatility is a decreasing function of the strike price is again a possible explanation of the overpricing of options with higher strike prices. Note that also in this case the optimal density function has a peak close to the mean of the implied volatilities, see Figure 4.

The main impression that I have received from this first, partial, test on HW-like stochastic volatility models, is that their major plus is their easy numerical implementation, while their major drawback is in their rather poor fitting capabilities. However, it must be emphasized that here we only wanted to propose a methodology and get a feeling for some possible future investigation which will require a deeper statistical analysis.

5 Conclusions

We have examined the problem of option pricing under a stochastic volatility model. We have seen that it is possible to take a unifying view on most of the major models presented by the modern literature. By looking at the same time to processes with or without jumps, we have been able to outline the consequences on the market model of the two assumptions. We proved, in both cases, a relation between equivalent martingale measures and market prices of risks in the context of stochastic volatility models. Such a relation turned out to be very useful when we tackled the problem of contingent claim

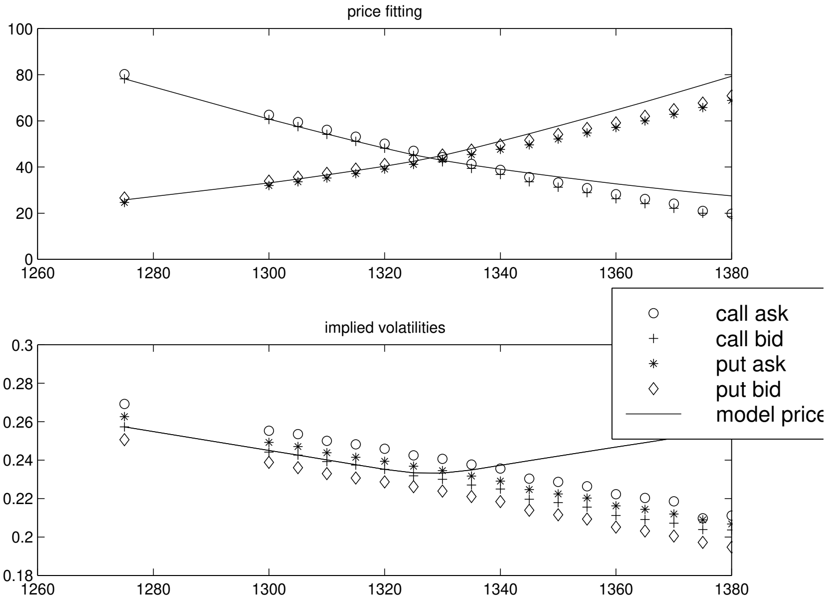

Figure 3: Price fitting of the best HW-like model (in the sense of the linear programming 4.4) for bid and ask prices on S&P500 options expiring at September 1999 (top) and corresponding implied volatilities (bottom). August 3, 1999, 11:26. CBOE. Source: Bloomberg.

pricing. In fact we were able to determine under which hypotheses on the market prices of risks it is possible to recover a rather easy (from a numerical point of view) formula for contingent claim pricing. This is an important point when stochastic volatility models have to be applied to market data. Moreover, we proved some sufficient conditions on the market prices of risks to get a numerically convenient pricing formula.

We have applied some of the theory to the problem of fitting market prices. By solving a constrained least square problem we determined the HW-like model (a very widely used subclass of stochastic volatility models) that offers the best fitting to some prices on MIB30 and SP500 options. Without questioning about the goodness of the fitting, for which a deeper statistical analysis is required, we observed that the optimal model within the HW-like class is the one that better approximates the market implied volatilities. In all the cases we examined the market implied volatility was a

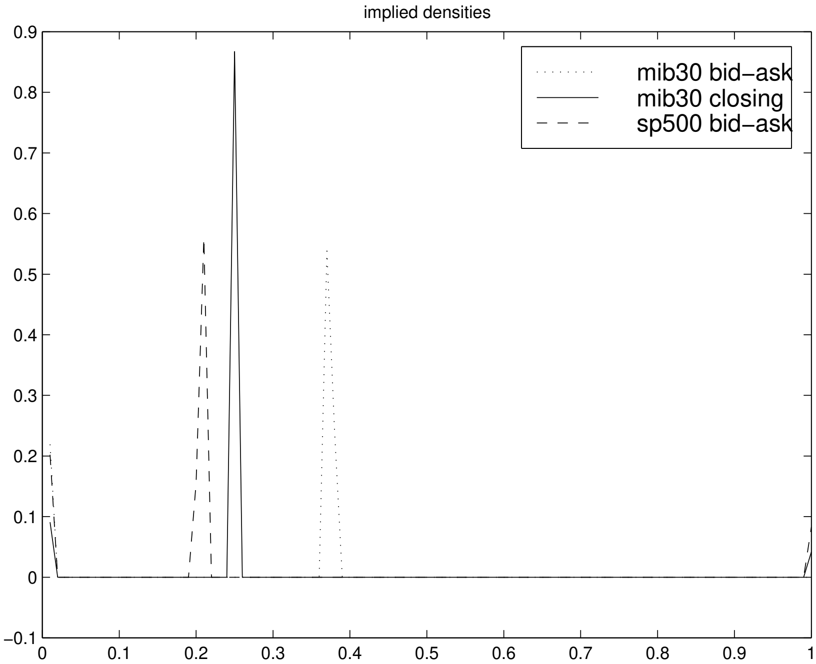

Figure 4: The optimal probability functions of the random variable v for the three tests examined in the paper. They all seem to suggest that the optimal stochastic volatility model is *almost* a constant volatility one (i.e. Black-Scholes

decreasing function of the strike price K. We know that the HW-like implied volatility has always to be U-shaped. Therefore the model implied volatility were lower than the market one for small strikes and higher for great strikes. Obviously, the effect on price fitting is that options with a great strike prices are overvalued by the model. It should be possible (and it is left for future research) to make the last statement more rigorous, by formally showing how to translate the optimization problem for prices into one for implied volatilities.

From the observation of the optimal density functions (in all of the cases examined), it seems that the optimal HW-like model is actually very close to a constant volatility model. We certainly do not have enough elements to draw any conclusion from such an observation, but, has it been confirmed by a more thorough statistical test, it would mean that HW-like models are not much better, when one has to fit market data, than a good old constant volatility (i.e. Black-Scholes) model. Therefore it may be sometimes necessary to use more sophisticated models, probabily those with a positive correlation between price level and volatility. Models of this kind are contained in the general class we introduced in this paper. An interesting research topic, with many practical implications, is the elaboration of a test of the fitting capabilities of this more general class of models.

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