Extending the Cox-Ross-Rubinstein Algorithm for Pricing Options with an Exponential Boundary

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Abstract

It is well known that the Cox, Ross, and Rubinstein algorithm [3] produces consistently biased prices in the case of barrier options with time-varying boundaries, even when a large number of steps is used in price computations. To overcome this drawback, we extend the standard CRR model by building-up a binomial tree for the underlying asset price dynamics, characterized by sets of nodes that mirror the barrier evolution. As a result, a very easy algorithm is derived that produces accurate prices with respect to the corresponding continuous-time values.

1 Introduction

In the traditional binomial method of Cox, Ross and Rubinstein (CRR) [3] for pricing European options, the underlying asset price dynamics is modeled by a random walk. This means that, if $S(t)$ is the asset value at time t then, after one period, at time $t+1$, it can rise to uS or decrease to dS, where u and d represent, respectively, the magnitude of one up-step and the magnitude of one down-step. As a consequence, the underlying asset price evolution can be represented by a binomial tree where each node corresponds to one possible value of the asset price. The usual condition $ud = 1$ implies that the binomial tree is characterized by sets of horizontal nodes.

In 1994 Boyle and Lau [2] pointed out that a naïve application of the traditional CRR algorithm can lead to consistent bias in pricing options with one flat barrier. The way to overcome this drawback is to build up a binomial tree with a set

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of nodes that is as close as possible to the barrier and mirrors the barrier evolution. In this case the problem of defining a binomial tree with a set of nodes that mimics the barrier dynamics is solved automatically because the barrier is constant and we consider trees with horizontal set of nodes. So we only need to choose a suitable number of steps in price computations such that there exists one layer of the lattice as close as possible to the barrier.

In the case of options with time-varying barriers, this approach fails because of the fact that the sets of horizontal nodes in the binomial tree cannot mirror the barrier evolution. As a consequence, the CRR algorithm computes consistently biased prices that converge very slowly towards the corresponding values obtained in a continuous-time setting. To price options with curved boundaries in a binomial setting we need to construct different trees characterized by set of nodes that mimic the barrier dynamics. This can be done only by relaxing the usual condition $ud = 1$ and by choosing a particular size for up-moves and down-moves so that a layer of nodes exactly mirrors the moving-time barrier. Following this way we construct a very simple algorithm that produces accurate prices with respect to continuous-time values illustrated in [8].

The remainder of the paper is organized as follows. In Section 2 we illustrate the numerical results of the standard binomial algorithm for pricing options with an exponential boundary. In Section 3 we present the revised binomial model and in Section 4 we illustrate the numerical results of the revised binomial algorithm.

2 Pricing barrier options with the standard CRR algorithm

Let consider a European down-and-out call option with strike price K and time to maturity T, written at time t on an asset with price $S(t) = S$. The option vanishes if the underlying asset price reaches before expiration the barrier $He^{\delta\tau}$ ($\delta \in \mathbb{R}, \quad 0 \leq \tau \leq T$).

In a continuous-time setting, under the usual assumptions of the Black-Scholes [1] analysis, the price of this option with $K > He^{\delta\tau}$ can be obtained, via the in-out parity, as the difference between the value of a standard European call and the value of a down-and-in European call with one exponential boundary (see [8])

$$\mathrm{DOC}^c(t) = \mathrm{C_{bs}}(S, K, T, r, \sigma) - (H/S)^{2v/\sigma^2}\mathrm{C_{bs}}(H^2/S, K, T, r, \sigma) \qquad (1)$$

where $\mathrm{C_{bs}}(S, K, T, r, \sigma)$ is the Black-Scholes formula for a standard European call option, σ is the volatility of the underlying asset price process, r is the risk-free interest rate (continuously compounded) and $v = r - \delta - \sigma^2/2$.

The most popular and diffused algorithm for pricing options in a discrete-time world is the CRR binomial method [3]. According to this model, at the end of each period, the underlying asset price rise by a factor $u = \exp(\sigma\sqrt{h})$ or decrease by a factor $d = u^{-1}$, where h is the size of the time interval between two successive steps. The probability of an up-step is the risk-neutral probability $p = (\exp(rh)-d)/(u-d)$ and the probability of a down-step is $q = 1 - p$. In the case of options with a time-varying barrier it is very difficult to derive a closed-form solution for the option price that, in general, is computed by solving the dynamic programming equation

$$C(n, S(n)) = e^{-rh}[pC(n+1, uS(n))I_{(He^{(n+1)\delta h},\infty)}+qC(n+1, dS(n))I_{(He^{(n+1)\delta h},\infty)}] \quad (2)$$

where $C(n, S(n))$ is the option price after n time-steps from inception and $I_{(He^{\delta\tau},\infty)}$ is the indicator function whose value is 1 if the underlying asset price is in the interval $(He^{\delta\tau}, \infty)$ and 0 otherwise.

Table 1 below shows the numerical results of the binomial algorithm for pricing down-and-out European call options with an exponential boundary for different values of the coefficient δ. The last row illustrates the corresponding continuous-time prices ($\mathrm{DOC}^c(t)$) computed using formula (1). The parameters involved in price computations are $S = 95$, $K = 100$, $H = 90$, $\sigma = 0.25$, $r = 0.1$.

Table 1

The standard binomial algorithm

n	$\delta = -0.05$	$\delta = -0.1$	$\delta = 0.05$	$\delta = 0.1$
25	8.849	8.979	7.013	6.750
50	7.447	7.787	6.918	6.355
75	7.249	7.745	6.178	5.724
100	7.609	7.917	6.517	5.837
150	6.974	7.488	6.255	5.676
200	7.365	7.681	5.887	5.335
300	6.849	7.366	5.993	5.444
400	6.926	7.323	6.037	5.398
500	6.862	7.329	5.851	5.317
$\mathrm{DOC}^c(t)$	6.466	6.896	5.485	4.928

The performance of the standard binomial algorithm is very poor in computing prices for options with exponential boundaries. Indeed, the prices obtained with the CRR method present a consistent bias with respect to continuous-time values and moreover the algorithm seems to converge very slowly to the accurate values computed with (1).

3 The revised binomial algorithm

As shown in Table 1, the performance of the standard binomial algorithm for pricing European options with curved boundaries is very poor. The reason was implicitly given in Boyle and Lau [2]. Indeed, as they pointed out, a naïve application of the CRR model, in general, produces consistently biased prices for barrier options.

In the case of barrier options with one flat boundary they showed the way to overcome this drawback. The trick is to construct a binomial lattice with a set of nodes that mirrors the barrier evolution and moreover it is as close as possible to the barrier. The first condition is satisfied in the CRR model because it considers only sets of horizontal nodes. The second is satisfied by choosing a suitable number of time-steps in price computations.

Things are different if we consider barrier options with a time-varying boundary. In the CRR framework it is not possible to build up a binomial tree with set of nodes that mirror the barrier evolution. To do this we need to relax the usual condition $ud = 1$. The strategy is the following: first, we set the magnitude of a down-step equal to that of the CRR model, $d = \exp(-\sigma\sqrt{h})$. Second, we select a number of steps for price computation such that, after a certain number, m, of successive down-moves from inception, the underlying asset price reaches the barrier. Third, we define the size of an up-step in a way that an entire set of nodes follows the barrier dynamics.

Let consider again the down-and-out European call option with the moving-time barrier, $He^{\delta\tau}$, described in Section 2. The underlying asset price will touch the barrier after m down-steps of size h if $Sd^m = He^{m\delta h}$, i.e.

$$m \log d = \log\left(\frac{H}{S}\right) + m\delta h.$$

We choose $d = e^{-\sigma\sqrt{h}}$, plug in it in the above identity and obtain,

$$m\delta h + m\sigma\sqrt{h} + \log\left(\frac{H}{S}\right) = 0. \tag{3}$$

We solve equation (3) with respect to the unknown $\sqrt{h}$, consider only the positive solutions and determine the number of steps n' corresponding to the solutions obtained. If we construct a binomial tree with n' time-steps, after m successive down-moves from inception, the underlying asset price reaches exactly the exponential boundary. The problem is that, in general, n' is not an integer and so we consider a number of steps $n^* = \lfloor n' \rfloor$ where $\lfloor x \rfloor$ denotes the largest integer smaller than x.

The condition stated above guarantees that there exists a node of the tree close to but just beyond the barrier. In order to define an entire set of nodes that follows the barrier evolution, it suffices to impose that an up-move followed by a down-move gives rise to the same increment of the boundary value along a time interval of length $2h$. This is done by choosing the size of an up-step, u, such that

$$ud = e^{2\delta h}.$$

As a consequence, in our model the size of an up-step is

$$u = e^{\sigma\sqrt{h}+2\delta h}$$

The transition probabilities to be used in price computations are the risk-neutral probabilities. In particular the probability of an up-step is

$$p = \frac{e^{rh} - d}{u - d}$$

and, obviously, the probability of a down-step is $q = 1 - p$.

In order to derive a closed-form formula for pricing the European down-and-out call option, we need to compute the number of trajectories that do not touch the exponential boundary. To do this, we may use the reflection principle of Desiré Andrè (see [5] for a detailed description). The reflection principle is a combinatorial method that allows us to compute, in a random walk setting with one absorbing barrier, the number of trajectories of a particle that, after a certain number of time-steps, doesn't touch the barrier. The reflection principle has been originally derived for a particle that follows a symmetrical random walk with one flat boundary. The main feature of such a model is that the particle moves in a binomial tree with layers of horizontal nodes that exactly mimic the barrier evolution. The model presented here has the same structure because it considers a particle moving in a binomial tree with sets of nodes that follow the boundary dynamics. As a consequence, in this context the reflection principle can be applied to count the number of trajectories of the particle that end to any node of the tree without touching the exponential boundary.

Let $N_{n,j}$ be the number of trajectories of the underlying asset price with j up-steps and $n - j$ down-steps. This number is given by the binomial coefficient $\binom{n}{j}$. Let $N^t_{n,j}$ be the number of trajectories with j up-steps and $n - j$ down-steps that touch or cross the curved boundary. Let m be the number of successive down-steps for the underlying asset price at inception to touch or cross the barrier. The reflection principle tells us that

$$N^t_{n,j} = \binom{n}{j+m}.$$

As a consequence, the number of trajectories with j up-steps and $n-j$ down-steps that do not touch or cross the exponential barrier is

$$N^{nt}_{n,j} = \binom{n}{j} - \binom{n}{j+m}.$$

The reflection principle allows us to derive the following closed form formula for pricing the down-and-out call option with the exponential boundary described before,

$$\mathrm{DOC}^d(t) = e^{-rT} \sum_{j=a}^{n} N^{nt}_{n,j} p^j q^{n-j} (Su^j d^{n-j} - K) \tag{4}$$

where a is the minimum number of up-steps of the underlying asset price for the option to be in the money at maturity, and is computed as the smallest integer such that $Su^a d^{n-a} > K$.

The possibility of using the reflection principle in computing option prices is very important from a computational point of view. Indeed, it allows us to derive a closed form solution for option prices and, as a consequence, to consider only the nodes of the tree corresponding to the option values at maturity. The binomial algorithm described above generalizes the approach of Boyle and Lau [2]. Indeed, if we consider options with one flat boundary ($\delta = 0$), the evaluation formula (4) reduces to that of Cox, Ross and Rubinstein adapted to barrier options as proposed by Boyle and Lau.

Moreover, the revised binomial algorithm can be easily modified to price all kinds of knock-in and knock-out options with one exponential boundary.

4 Numerical results of the revised binomial algorithm

In general, discrete-time algorithms for options evaluation are defined in such a way that the first k moments of the discrete-time distribution of the underlying asset price match the first k moments of the corresponding continuous-time distribution of the underlying asset price. For example, in the case of binomial and trinomial models $k = 2$, in quadrinomial models $k = 3$ and so on.

In a risk-neutral environment, under the usual assumption of the Black-Scholes analysis, the underlying asset price is log-normally distributed and, after a time interval of length h, the first order moment is $S\exp(rh)$ and the second order moment is $S^2 \exp[(2r + \sigma^2)h]$.

Clearly, this still holds even for the revised binomial algorithm, that is,

$$pu + qd = e^{rh}, \tag{5}$$

$$pu^2 + qd^2 = e^{(2r+\sigma^2)h}. \tag{6}$$

The choice of a risk-neutral probability measure in the revised binomial algorithm guarantees that condition (5) is satisfied. Expanding the exponential in Taylor series, and ignoring terms of order $h^{3/2}$ and higher, the second order moment of the discrete-time distribution of the underlying asset price is $1 + 2rh + \sigma^2 h$ and it matches the second order moment of the log-normal random variable.

However, matching the first and the second order moments of the discrete-time and continuous-time distributions of the underlying asset price does not assure, in general, the convergence of the prices computed with the discrete-time algorithm towards the corresponding values obtained in a continuous-time setting. A proof of the convergence is illustrated in the Appendix.

Numerical results of the revised binomial model are shown in Table 2. We compute prices of European down-and-out call options, $\mathrm{DOC}^d(t)$, with the same parameters specified in Section 2.

Table 2

The revised binomial algorithm

$\delta = -0.05$			$\delta = -0.1$			$\delta = 0.05$			$\delta = 0.1$		
m	n^*	DOC$^d(t)$	m	n^*	DOC$^d(t)$	m	n^*	DOC$^d(t)$	m	n^*	DOC$^d(t)$
1	19	6.598	1	17	7.002	1	23	5.513	1	24	5.020
2	81	6.511	2	77	6.958	2	89	5.490	2	92	4.949
3	186	6.487	3	181	6.920	3	197	5.494	3	203	4.934
4	334	6.479	4	327	6.910	4	349	5.487	4	356	4.934
5	525	6.473	5	515	6.912	5	543	5.487	5	552	4.932
10	2119	6.469	10	2100	6.902	10	2156	5.485	10	2174	4.929
15	4782	6.468	15	4754	6.900	15	4838	5.485	15	4865	4.929
20	8515	6.467	20	8477	6.899	20	8589	5.485	20	8625	4.928
25	13316	6.467	25	13269	6.898	25	13408	5.485	25	13454	4.928
30	19186	6.467	30	19131	6.898	30	19297	5.485	30	19353	4.928
DOC$^c(t)$=6.466			DOC$^c(t)$=6.896			DOC$^c(t)$=5.485			DOC$^c(t)$=4.928		

5 Conclusions

In the case of options with curved boundaries, the standard binomial algorithm of Cox, Ross and Rubinstein [3] computes consistently biased prices with respect to accurate values obtained with continuous-time models. The reason is that in the CRR model it is impossible to build up a lattice with sets of nodes that follows the barrier evolution.

In order to solve this problem we extend the standard CRR model by building up a binomial tree with a set of nodes that exactly mirrors the barrier evolution. As a result, we define a very simple binomial algorithm that gives accurate prices compared with the corresponding continuous-time formulae illustrated in [8].

Appendix: proof of the convergence of the revised binomial algorithm to the continuous-time formula

Let us consider a down-and-out European call option written at time t, with time to maturity T and strike price K. Let $S(t) = S$ be the underlying asset price at inception, σ the volatility and r the risk-free interest rate (continuously compounded). The option vanishes if the underlying asset price reaches before maturity the exponential boundary $He^{\delta\tau}$, $\delta \in \mathbb{R}$ and $0 \leq \tau \leq T$.

We consider the binomial algorithm developed in the previous paragraphs with an upward step size $u = \exp(\sigma\sqrt{h} + 2\delta h)$, where $h = T/n$ and n is the number of steps used in price computation. The magnitude of a down-step is $d = \exp(-\sigma\sqrt{h})$ and the transition probabilities are

$$p = \frac{e^{rh} - d}{u - d}, \quad q = 1 - p.$$

Our goal is to prove the convergence, as $n \to \infty$, of the revised binomial formula (4) to the corresponding continuous-time formula (1).

First we recall the revised binomial formula (4)

$$\mathrm{DOC}^d(t) = e^{-rT} \sum_{j=a}^{n} N_{n,j}^{nt} p^j q^{n-j} (Su^j d^{n-j} - K),$$

that we may rewrite as

$$\mathrm{DOC}^d(t) = e^{-rT} \sum_{j=a}^{n} \binom{n}{j} p^j q^{n-j} (Su^j d^{n-j} - K) - e^{-rT} \sum_{j=a}^{n-m} \binom{n}{j+m} p^j q^{n-j} (Su^j d^{n-j} - K) \tag{7}$$

Notice that the first part of (7) is the revised binomial formula for a standard European call option while the second one is the revised binomial formula for a down-and-in European call option with the exponential boundary defined above.

Recalling that the continuous-time price for the down-and-out call has been defined in (1) and is

$$\mathrm{DOC}^c(t) = \mathrm{C}_{\mathrm{bs}}(S, K, T, r, \sigma) - (H/S)^{2v/\sigma^2} \mathrm{C}_{\mathrm{bs}}(H^2/S, K, T, r, \sigma),$$

we will prove the convergence of the down-and-out call, by showing at first the convergence of the standard European call and then that of the down-and-in call. To do this we consider the instantaneous rate of return of the underlying asset price that, at the end of each period, i, $(i = 1 \dots n)$, can be represented by the random variable X_i, defined as follows

$$X_i = \begin{cases} \log u & \text{with probability} \quad p \\ \log d & \text{with probability} \quad q \end{cases}$$

It is worth to notice that the X_i's, are independent and identically distributed random variables with mean

$$E(X_i) = p\log(u/d) + \log d$$

and variance

$$\mathrm{Var}(X_i) = pq[\log(u/d)]^2.$$

After n periods, the underlying asset return is equal to the random sum $S_n = \sum_{i=1}^{n} X_i$ with expected value

$$E(S_n) = nE(X_i) = n[p\log(u/d) + \log d]$$

and variance

$$\mathrm{Var}(S_n) = n\mathrm{Var}(X_i) = npq[\log(u/d)]^2.$$

Furthermore, the limit, as $n \to \infty$, of the mean and of the variance of S_n are

$$\lim_{n\to\infty} E(S_n) = (r - \frac{\sigma^2}{2})T, \quad \lim_{n\to\infty} \mathrm{Var}(S_n) = \sigma^2 T.$$

Part 1: convergence of the standard European call

We recall that the price of the standard European call with the revised binomial algorithm is

$$C(t) = e^{-rT} \sum_{j=a}^{n} \binom{n}{j} p^j q^{n-j} (Su^j d^{n-j} - K) = SB^c(a,n,p') - Ke^{-rT} B^c(a,n,p)$$

where

$$B^c(a,n,p) = \sum_{j=a}^{n} \binom{n}{j} p^j q^{n-j}$$

is the complementary distribution function of a binomial random variable, while $B^c(a, n, p')$ is the complementary distribution function of a binomial random variable with $p' = (u/\exp(rh))p$.

The continuous-time price of a standard European option computed with the Black-Scholes [1] model is

$$\mathrm{C}_{\mathrm{bs}}(S, K, T, r, \sigma) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where $\Phi(x)$ is the distribution function of a standardized normal random variable and

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\log(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

In order to prove the convergence of the binomial formula to the continuous-time formula, first we show that

$$B^c(a, n, p) \to \Phi(d_2) \quad \text{and then} \quad B^c(a, n, p') \to \Phi(d_1), \quad \text{as} \quad n \to \infty.$$

$B^c(a, n, p)$ can be interpreted as the probability that among n time-steps the underlying asset price has at least a up-steps; in other words

$$B^c(a, n, p) = \mathrm{Prob}[S_n \geq a\log(u/d) + n\log d] =$$

$$= 1 - \mathrm{Prob}[S_n \leq (a-1)\log(u/d) + n\log d].$$

Recalling that

$$a - 1 = \frac{\log(K/Sd^n)}{\log(u/d)} - \varepsilon, \quad \varepsilon \in (0, 1),$$

$$B^c(a, n, p) = 1 - \mathrm{Prob}[S_n \leq \log(K/S) - \varepsilon\log(u/d)] =$$

$$= 1 - \mathrm{Prob}\left[\frac{S_n - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \leq \frac{\log(K/S) - \varepsilon\log(u/d) - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}}\right].$$

Now we can apply the central limit theorem. Since

$$\lim_{n\to\infty} \frac{\log(K/S) - \varepsilon \log(u/d) - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}} = \frac{\log(K/S) - (r - \sigma^2/2)T}{\sigma\sqrt{T}},$$

then

$$B^c(a,n,p) \to 1 - \Phi\left[\frac{\log(K/S) - (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right] =$$

$$= \Phi\left[\frac{\log(S/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right] = \Phi(d_2).$$

Analogously, we can show that

$$B^c(a,n,p') \to \Phi(d_1), \quad \text{as} \quad n \to \infty.$$

Part 2: convergence of the down-and-in European call

We recall that the price of the down-and-in European call with the revised binomial algorithm is

$$\mathrm{DIC}^d(t) = e^{-rT} \sum_{j=a}^{n-m} \binom{n}{j+m} p^j q^{n-j} (Su^j d^{n-j} - K) =$$

$$= \left(\frac{q}{p}\right)^m e^{-rT} \sum_{j=a+m}^{n} \binom{n}{j} p^j q^{n-j} \left[S\left(\frac{d}{u}\right)^m u^j d^{n-j} - K\right] =$$

$$= \left(\frac{q}{p}\right)^m \left[S\left(\frac{d}{u}\right)^m B^c(a+m,n,p') - Ke^{-rT} B^c(a+m,n,p)\right].$$

In the above formula, m is the number of successive down-steps for the underlying asset price at inception to touch or cross the barrier and is equal to

$$-\frac{\log(H/S)}{\delta h + \sigma\sqrt{h}} + \xi, \quad \xi \in (0,1).$$

The continuous-time price of a down-and-in European call is

$$\mathrm{DIC}^c(t) = \left(\frac{H}{S}\right)^{2v/\sigma^2} \mathrm{C}_{\mathrm{bs}}(H^2/S, K, T, r, \sigma) =$$

$$= \left(\frac{H}{S}\right)^{2v/\sigma^2} \left[\frac{H^2}{S}\Phi(d_3) - Ke^{-rT}\Phi(d_4)\right],$$

where

$$d_3 = \frac{\log(H^2/SK) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_4 = \frac{\log(H^2/SK) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

To prove the convergence, first we note that

$$\lim_{n\to\infty} \left(\frac{q}{p}\right)^m = \left(\frac{H}{S}\right)^{2v/\sigma^2} \quad \text{and} \quad \lim_{n\to\infty} S\left(\frac{d}{u}\right)^m = \frac{H^2}{S}.$$

Recalling now that $B^c(a+m, n, p)$ is the probability that the underlying asset price has at least $a+m$ up-steps among n, i.e.,

$$B^c(a+m, n, p) = \mathrm{Prob}[S_n \geq (a+m)\log(u/d) + n\log d].$$

It suffices then to follow the same procedure used in the proof of the convergence of the standard European call to obtain that

$$B^c(a+m, n, p) \to \Phi\left(\frac{\log(H^2/SK) + (r - \sigma^2/2)}{\sigma\sqrt{T}}\right) = \Phi(d_4), \quad \text{as} \quad n \to \infty.$$

An analogous proof shows that

$$B^c(a+m, n, p') \to \Phi(d_3), \quad \text{as} \quad n \to \infty.$$

$\square$

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